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Isgur-Wise functions and unitary representations of the Lorentz group: The meson case with $j = \frac{1}{2}$ light cloud

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We pursue the group-theoretical method to study Isgur-Wise (IW) functions. We extend the general formalism, formerly applied to the baryon case $j^P = 0^+$ (for $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$), to mesons with $j^P = \frac{1}{2}^-$, i.e. $\bar{B} \to D(D^{(*)})\ell\nu$. In this case, which is more involved from the angular momentum point of view, only the principal series of unitary representations of the Lorentz group contribute. We obtain an integral representation for the IW function $\xi(w)$ with a positive measure, recover the bounds for the slope and the curvature of $\xi(w)$ obtained from the Bjorken-Uraltsev sum-rule method, and get new bounds for higher derivatives. We demonstrate also that if the lower bound for the slope is saturated, the measure is a δ function, and $\xi(w)$ is given by an explicit elementary function. Inverting the integral formula, we obtain the measure in terms of the IW function, allowing us to formulate criteria to decide if a given *Ansatz* for the Isgur-Wise function is compatible or not with the sum-rule constraints. Moreover, we obtain an upper bound on the IW function valid for any value of w. We compare these theoretical constraints to a number of forms for $\xi(w)$ proposed in the literature. The "dipole" function $\xi(w) = (\frac{2}{w+1})^{2c}$ satisfies all constraints for $c \geq \frac{3}{4}$, while the QCD sum rule result including condensates does not satisfy them. Special care is devoted to the Bakamjian-Thomas relativistic quark model in the heavy-quark limit and to the description of the Lorentz group representation that underlies this model. Consistently, the IW function satisfies all Lorentz group criteria for any explicit form of the meson Hamiltonian at rest.

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I. INTRODUCTION

The heavy-quark limit of QCD and, more generally, heavy quark effective theory (HQET), has aroused enormous interest since the 1990s, starting from the formulation of heavy-quark symmetry by Isgur and Wise (IW) [1].

Hadrons with one heavy quark such that $m_Q \gg \Lambda_{\rm QCD}$ can be thought of as a bound state of a light cloud in the color source of the heavy quark. Due to its heavy mass, the latter is unaffected by the interaction with soft gluons.

In this approximation, the decay of a heavy hadron with four-velocity v into another hadron with velocity v', for example the semileptonic decay $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell$ or $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$, occurs just by free heavy-quark decay produced by a current, and the rearrangement of the light cloud, to follow the heavy quark in the final state and constitute the final heavy hadron.

The dynamics is contained in the complicated light cloud, which concerns long-distance QCD and is not calculable from first principles. Therefore, one needs to parametrize this physics through form factors, i.e., the IW functions.

The matrix element of a current between heavy hadrons containing heavy quarks Q and Q' can thus be factorized as follows [2]:

$$\begin{aligned}
&\langle H'(v'), J'\, m' | J^{Q'Q} | H(v), J\, m \rangle \\
&= \sum_{\mu,M,\mu',M'} \left\langle \frac{1}{2}\mu', j'M' | J'm' \right\rangle \left\langle \frac{1}{2}\, \mu, jM | Jm \right\rangle \\
&\times \left\langle Q'(v'), \frac{1}{2}\, \mu' | J^{Q'Q} | Q(v), \frac{1}{2}\, \mu \right\rangle \\
&\times \langle \text{cloud}, v', j', M' | \text{cloud}, v, j, M \rangle \qquad (1)
\end{aligned}$$

where v, v' are the initial and final four-velocities, j, j', M, M' are the angular momenta and corresponding projections of the initial and final light clouds, and μ, μ' are the angular-momentum projections of the heavy quark.

The current affects only the heavy quark, and all the soft dynamics is contained in the *overlap* between the initial and final light clouds $\langle v', j', M' | v, j, M \rangle$, which follow the heavy quarks with the same four-velocity. This overlap is independent of the current heavy-quark matrix element, and depends on the four-velocities v and v'. The IW functions are given by these light-cloud overlaps.

An important hypothesis has been developed in writing the previous expression, namely neglecting *hard-gluon radiative corrections*.

As we will make explicit below, the light cloud belongs to a Hilbert space, and transforms according to a unitary representation of the Lorentz group. Then, as we have shown in Ref. [3], the whole problem of getting rigorous constraints on the IW functions amounts to decomposing unitary representations of the Lorentz group into

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irreducible ones. This allows one to obtain for the IW functions general integral formulas in which the crucial point is that *the measures are positive*.

In Ref. [3] we treated the case of a light cloud with angular momentum $j = 0$ in the initial and final states, as happens in the baryon semileptonic decay $\Lambda_b \to \Lambda_c \ell \bar{\nu}_\ell$.

A different but, as we will show below, equivalent method to that of the present paper was developed in a number of articles using *sum rules* in the heavy-quark limit, like the famous Bjorken sum rule and its generalizations [4–9].

The sum rule method is completely equivalent to the method of the present paper. Indeed, starting from the sum rules one can demonstrate that an IW function, say $\xi(v.v') = \langle v'|v\rangle$ in a simplified notation, is a function of *positive type*, and that one can construct a unitary representation of the Lorentz group $U(\Lambda)$ and a vector state $|\phi_0\rangle$ representing the light cloud at rest. The IW function is then simply (e.g. in the special case $j = 0$)

$$\xi(v.v') = \langle U(B_{v'})\phi_0|U(B_v)\phi_0\rangle \tag{2}$$

where B_v and $B_{v'}$ are the corresponding boosts.

Let us now go back to previous work on the sum-rule method. In the meson case $\bar{B} \to D^{(*)}\ell\bar{\nu}_\ell$, in the leading order of the heavy-quark expansion, the Bjorken sum rule (SR) [4,5] gives the lower bound for the derivative of the IW function at zero recoil $\rho^2 = -\xi'(1) \geq \frac{1}{4}$. A new SR was formulated by Uraltsev in the heavy-quark limit [6] that, combined with Bjorken's, gave the much stronger lower bound $\rho^2 \geq \frac{3}{4}$. A basic ingredient in deriving this bound was the consideration of the nonforward amplitude $\bar{B}(v_i) \to D^{(n)}(v') \to \bar{B}(v_f)$, allowing for general four-velocities v_i, v_f, v'.

In Refs. [7–9] we developed a manifestly covariant formalism within the operator product expansion (OPE) and the nonforward amplitude, using the whole tower of heavy meson states [2]. We did recover Uraltsev SR plus a general class of SR that allow one to bound also higher derivatives of the IW function. In particular, we found a bound on the curvature in terms of the slope ρ^2, namely

$$\xi''(1) \geq \frac{1}{5}[4\rho^2 + 3(\rho^2)^2]. \tag{3}$$

The more powerful method of the present paper will provide a new insight into the physics of QCD in the heavy-quark limit and its Lorentz group structure.

As we will see below, we obtain an integral formula for the Isgur-Wise function in terms of a *positive measure*. We will see that we recover the bound (3) and that this systematic method allows us to find bounds for higher derivatives.

We can invert this integral formula and obtain the measure corresponding to any given *Ansatz* for the IW function and we thus obtain a powerful criterium to decide if this *Ansatz* is consistent with the Lorentz group approach or, equivalently, with the generalized Bjorken-Uraltsev sum rules. The method exposed in this paper allows one to decide if a given model for the IW function is consistent with the general principles of QCD in the heavy-quark limit.

The purpose of the present paper is purely theoretical. In HQET, e.g. in $b \to c$ transitions, one can take the heavy-quark limit for both initial and final quarks while keeping the mass ratio $r = m_b/m_c$ finite. By varying the ratio r one can in principle attain any value for the variable w within the range $1 \leq w \leq \frac{1+r^2}{2r}$, and our theoretical constraints on IW functions are then valid for any value of w.

Of course, this is quite different from the physical range at finite masses, namely $1 \leq w \leq 1.4$ GeV. To perform an analysis at finite mass would require not only implementing the theoretical constraints on the IW function obtained in the present work; one would also need to perform a serious phenomenological discussion and to include $1/m_Q$ corrections, radiative corrections within the effective theory HQET, and make use of the Wilson coefficients to match with the true QCD, as has been done for the curvature of the IW function (3) by Dorsten [10]. This whole program is outside the intention of the present work, which only deals with rigorous constraints on the shape of the IW function.

The outline of the paper is as follows. Sections II and III recall necessary generalities and details on the present Lorentz group approach to IW functions, following closely Ref. [3]. In Sec. IV we apply the method exposed in detail in Ref. [3] to the present meson case, making explicit the needed unitary representations of the Lorentz group. In Sec. V we compute the irreducible IW functions in the case $j = \frac{1}{2}$ and give an integral formula expressing the IW function in terms of the latter and a positive measure. In Sec. VI we use this integral formula to get a polynomial expression for the derivatives of the IW function, and in Sec. VII we obtain lower bounds on its derivatives. Section VIII is devoted to obtaining the inversion of the integral formula for the IW function. In Sec. IX we find an *upper bound* on the IW function. In Sec. X we apply the inverted integral formula to study consistency tests of a number of models of the IW function proposed in the literature. The Bakamjian-Thomas relativistic quark model in the heavy-quark limit and the description of the Lorentz group representation that underlies this model is studied in detail in Sec. XI. In Sec. XII we discuss the theoretical and phenomenological relevance of our results, and we conclude.

II. THE LORENTZ GROUP AND THE HEAVY-QUARK LIMIT OF QCD

In the heavy-mass limit, the states of a heavy hadron H containing a heavy quark Q are described as follows [2], as we can see from Eq. (1):

$$|H(v), \mu, M\rangle = |Q(v), \mu\rangle \otimes |v, j, M\rangle \tag{4}$$

where there is a factorization into the heavy-quark state factor $|Q(v), \mu\rangle$ and a light cloud component $|v, j, M\rangle$.

The velocity v of the heavy hadron H is the same as the velocity of the heavy quark Q, and is unquantized. The heavy-quark Q state depends only on a spin $\mu = \pm\frac{1}{2}$ quantum number, and so belongs to a two-dimensional Hilbert space. The light component is the complicated thing, but it does not depend on the spin state μ of the heavy quark Q, nor on its mass, and this gives rise to the symmetries of the heavy-quark theory.

As advanced in the Introduction, the matrix element of a heavy-heavy current J (acting only on the heavy quark) is

$$\langle H'(v'),\mu',M'|J|H(v),\mu,M\rangle = \langle Q'(v'),\mu'|J|Q(v),\mu\rangle\langle v',j',M'|v,j,M\rangle \quad (5)$$

and the IW functions are defined as the coefficients, depending only on $v.v'$, in the expansion of the unknown scalar products $\langle v',j',M'|v,j,M\rangle$ into independent scalars constructed from v, v' and the polarization tensors describing the spin states of the light components.

Now, the crucial point in the present work is that the states of the light components make up a Hilbert space in which acts a unitary representation of the Lorentz group. In fact, this is more or less implicitly stated, and used in the literature [2].

A. Physical picture of a heavy quark

To see the point more clearly, let us go into the physical picture that is at the basis of Eq. (4). We first consider a heavy hadron *at rest*, with velocity

$$v_0 = (1,\vec{0}) \quad (6)$$

whose light component enters the interactions between the light particles, light quarks, light antiquarks and gluons, and the external chromoelectric field generated by the heavy quarks at rest. This chromoelectric field does not depend on the spin μ of the heavy quark or its mass. We shall then have a complete orthonormal system of energy eigenstates $|v_0,j,M,\alpha\rangle$ of the light component, where j and M are the angular momentum quantum numbers and α denotes other quantum numbers (like the radial excitation number),

$$\langle v_0,j',M',\alpha'|v_0,j,M,\alpha\rangle = \delta_{j,j'}\delta_{M,M'}\delta_{\alpha,\alpha'}. \quad (7)$$

Now, for a heavy hadron moving with a velocity v, the only thing that changes for the light component is that the external chromoelectric field generated by the heavy quark at rest is replaced by the external chromoelectromagnetic field generated by the heavy quark moving with velocity v. Neither the Hilbert space describing the possible states of the light component, nor the interactions between the light particles, are changed. We shall then have *a new complete orthonormal system* of energy eigenstates $|v,j,M,\alpha\rangle$, in the same Hilbert space. Then, because the color fields generated by a heavy quark for different velocities are related by Lorentz transformations, we may expect that the energy eigenstates of the light component will, for various velocities, be themselves related by Lorentz transformations acting in their Hilbert space.

B. Lorentz representation from covariant overlaps

Let us now show that such a representation of the Lorentz group does in fact underly the work of Ref. [2]. The description of spin states by polarization tensors is used.

For half-integer spin j, in which we are interested in the present paper, the polarization tensor becomes a Rarita-Schwinger tensor-spinor $\epsilon_\alpha^{\mu_1,\cdots\mu_{j-1/2}}$ subject to the constraints of symmetry, transversality and tracelessness

$$v_{\mu_1}\epsilon_\alpha^{\mu_1,\ldots\mu_{j-1/2}} = 0, \qquad g_{\mu_1\mu_2}\,\epsilon_\alpha^{\mu_1,\mu_2\ldots\mu_{j-1/2}} = 0, \quad (8)$$

and

$$(\not{v}-1)_{\alpha\beta}\epsilon_\beta^{\mu_1,\ldots,\mu_{j-1/2}} = 0, \qquad (\gamma_{\mu_1})_{\alpha\beta}\epsilon_\beta^{\mu_1,\ldots,\mu_{j-1/2}} = 0. \quad (9)$$

Then a scalar product $\langle v',j',\epsilon'|v,j,\epsilon\rangle$ is a covariant function of the vectors v and v' and of the tensors (or tensor-spinors) ϵ'^* and ϵ, bilinear with respect to ϵ'^* and ϵ, and the IW functions—functions of the scalar $v.v'$—are introduced accordingly.

The covariance property of the scalar products is explicitly expressed by the equality

$$\langle\Lambda v',j',\Lambda\epsilon'|\Lambda v,j,\Lambda\epsilon\rangle = \langle v',j',\epsilon'|v,j,\epsilon\rangle \quad (10)$$

which is valid for any Lorentz transformation Λ, with the transformation of a tensor-spinor given by

$$(\Lambda\epsilon)_\alpha^{\mu_1,\ldots,\mu_{j-1/2}} = \Lambda^{\mu_1}_{\nu_1}\ldots\Lambda^{\mu_{j-1/2}}_{\nu_{j-1/2}}D(\Lambda)_{\alpha\beta}\,\epsilon_\beta^{\nu_1,\ldots,\nu_{j-1/2}}. \quad (11)$$

Then, let us *define* the operator $U(\Lambda)$, in the space of the light cloud states, by

$$U(\Lambda)|v,j,\epsilon\rangle = |\Lambda v,j,\Lambda\epsilon\rangle \quad (12)$$

where here v is a fixed, arbitrarily chosen velocity. Equation (10) implies that $U(\Lambda)$ is a *unitary operator*, as demonstrated in Ref. [3].

C. From a Lorentz representation to Isgur-Wise functions

A unitary representation of the Lorentz group emerges from the usual treatment of heavy hadrons in the heavy-quark theory. For the present purpose, we need to go in the opposite direction, namely, to show how—starting from a unitary representation of the Lorentz group—the usual treatment of heavy hadrons and the introduction of the IW functions emerge. What follows is not restricted to the $j = \frac{1}{2}$ case, but rather concerns any IW function.

So, let us consider some unitary representation $\Lambda \to U(\Lambda)$ of the Lorentz group [or more precisely of the group $SL(2,C)$] in a Hilbert space $\mathcal{H}$, where we have to identify states in $\mathcal{H}$ depending on a velocity v. As explained in Ref. [3], we have in $\mathcal{H}$ an additional structure, namely the energy operator of the light component *for a heavy quark at rest*, with $v_0 = (1,0,0,0)$. Since this energy operator is invariant under rotations, we consider the subgroup SU(2) of $SL(2,C)$. By restriction, the representation in $\mathcal{H}$ of $SL(2,C)$ gives a representation $R \to U(R)$ of SU(2), and its decomposition into irreducible representations of SU(2) is needed. We then have the eigenstates $|v_0, j, M\rangle$ of the energy operator, which are classified by the angular momentum number j of the irreducible representations of SU(2) and *associated with the rest velocity* v_0, since their physical meaning is to describe the energy eigenstates of the light component for a heavy quark at rest.

We now need to express the states $|v, j, \epsilon\rangle$ in terms of the states $|v_0, j, M\rangle$. We begin with $v = v_0$. For fixed j and α, the states $|v_0, j, M\rangle$ constitute, for $-j \leq M \leq j$, a standard basis of a representation j of SU(2):

$$U(R)\,|v_0, j, M\rangle = \sum_{M'} D^j_{M',M}(R)|v_0, j, M'\rangle \qquad (13)$$

where the rotation matrix elements $D^j_{M',M}$ are defined by

$$D^j_{M',M} = \langle j, M'|U_j(R)|j, M\rangle, \qquad R \in \mathrm{SU}(2). \qquad (14)$$

On the other hand, the states $|v_0, j, \epsilon\rangle$ constitute [when ϵ goes over all polarization tensors (or tensor-spinors)] the whole space of a representation j of SU(2). As emphasized in Ref. [3], this representation of SU(2) in the space of 3-tensors (or 3-tensor-spinors) is not irreducible, but it contains an irreducible subspace of spin j, which is precisely the polarization 3-tensor (or 3-tensor-spinor) space selected by the other constraints (8) and (9) for the velocity v_0.

We may then introduce a standard basis $\epsilon^{(M)}$, $-j \leq M \leq j$, for the SU(2) representation of spin j in the space of polarization 3-tensors (or 3-tensor-spinors). As demonstrated in Ref. [3], the states $|v, j, \epsilon\rangle$ are given by

$$|v, j, \epsilon\rangle = \sum_M (\Lambda^{-1}\epsilon)_M\, U(\Lambda)|v_0, j, M\rangle \qquad (15)$$

for any Λ such that $\Lambda v_0 = v$, with v_0 given by Eq. (6), and where $(\Lambda^{-1}\epsilon)_M$ is the component of the velocity v_0 polarization tensor $\Lambda^{-1}\epsilon$ in the standard basis.

Equation (15) is *our final result*, defining—in the Hilbert space $\mathcal{H}$ of a unitary representation of $SL(2,C)$—the states $|v, j, \epsilon\rangle$ which transform as Eq. (12) and whose scalar products define the IW functions, in terms of $|v_0, j, M\rangle$ which occur as SU(2) multiplets in the restriction to SU(2) of the $SL(2,C)$ representation. Furthermore, these states $|v, j, \epsilon\rangle$ defined by Eq. (15) do indeed transform as Eq. (12).

III. DECOMPOSITION INTO IRREDUCIBLE REPRESENTATIONS

In the case of a compact group [such as SU(2)], any unitary representation can be written as a direct sum of irreducible ones. In the present case of $SL(2,C)$ (a noncompact group), the more general notion of a direct integral is required [11]. Let us denote by X the set of irreducible unitary representations of $SL(2,C)$, by $\mathcal{H}_\chi$ the Hilbert space of a representation $\chi \in X$, and by $U_\chi(\Lambda)$ the unitary operator acting in $\mathcal{H}_\chi$ which corresponds to any $\Lambda \in SL(2,C)$. Then, for any unitary representation of $SL(2,C)$, the Hilbert space $\mathcal{H}$ can be written in the form

$$\mathcal{H} = \int_X^{\oplus} \oplus_{n_\chi} \mathcal{H}_\chi\, d\mu(\chi) \qquad (16)$$

where μ is an arbitrary *positive* measure on the set X, and n_χ is a function on X with ≥ 1 integer values or possibly ∞. Explicitly, an element $\psi \in \mathcal{H}$ is a function

$$\psi\colon \chi \in X \to \psi_\chi = (\psi_{1,\chi}, \ldots, \psi_{n_\chi,\chi}) \in \oplus_{n_\chi} \mathcal{H}_\chi \qquad (17)$$

which assigns to each $\chi \in X$ an element $\psi_\chi \in \oplus_{n_\chi} \mathcal{H}_\chi$, and which is μ-measurable and square μ-integrable. The scalar product in $\mathcal{H}$ is given by

$$\langle \psi'|\psi\rangle = \int_X \langle \psi'_\chi|\psi_\chi\rangle d\mu(\chi) \qquad (18)$$

and the operator $U(\Lambda)$ of the representation in the space $\mathcal{H}$ is given by

$$(U(\Lambda)\psi)_{k,\chi} = U_\chi(\Lambda)\psi_{k,\chi}. \qquad (19)$$

Now let us look at the consequences for the IW functions. For simplicity, we take here the case of a spinor ($j = \frac{1}{2}$) light component. For the hadron at rest, the light component will be described by *some* element $\psi_{\frac{1}{2}} \in \mathcal{H}$ *which is a spinor* for the subgroup SU(2) of $SL(2,C)$. Then, according to the transformation law (19), requiring that $\psi_{\frac{1}{2}}$ is a spinor under rotations is the same as requiring that $\psi_{\frac{1}{2},k,\chi}$ is a spinor under rotations for all χ's and all $k = 1, \ldots, n_\chi$. More generally, the decomposition of the irreducible representations of $SL(2,C)$ into irreducible representations of SU(2) is known (see the next section). Since SU(2) is compact, the decomposition is by a direct sum, and therefore each $\mathcal{H}_\chi$ admits an orthonormal basis adapted to SU(2). Moreover, it turns out that each representation j of SU(2) appears with multiplicity 0 or 1. Then, there is a subset $X_0 \subset X$ of irreducible representations of $SL(2,C)$ containing a nonzero SU(2) spinor subspace and, for $\chi \in X_0$, there is a unique (up to a phase) normalized SU(2) scalar element in $\mathcal{H}_\chi$, which we denote as $\phi_{\frac{1}{2},\chi}$. Each scalar element in $\mathcal{H}_\chi$ is then proportional to $\phi_{\frac{1}{2},\chi}$. So, one has

$$\psi_{\frac{1}{2},\chi} = (c_{1,\chi}\,\phi_{\frac{1}{2},\chi}, \ldots, c_{n_\chi,\chi}\,\phi_{\frac{1}{2},\chi}) \tag{20}$$

with some coefficients $c_{1,\chi}, \ldots, c_{n_\chi,\chi}$. From the scalar product (18) in $\mathcal{H}$, one sees that the normalization $\langle\psi_{\frac{1}{2}}|\psi_{\frac{1}{2}}\rangle = 1$ of the light component amounts to

$$\int_{X_0} \sum_{k=1}^{n_\chi} |c_{k,\chi}|^2\, d\mu(\chi) = 1. \tag{21}$$

IV. LORENTZ-GROUP-IRREDUCIBLE UNITARY REPRESENTATIONS AND THEIR DECOMPOSITION UNDER ROTATIONS

A. Explicit form of the principal series of irreducible unitary representations of the Lorentz group

We have described in Ref. [3] an explicit form of the irreducible unitary representations of $SL(2,C)$. Their set X is divided into three sets: the set X_p of representations of the principal series, the set X_s of representations of the supplementary series, and the one-element set X_t made up of the trivial representation [12].

Actually, for the $j = \frac{1}{2}$ case, only the principal series is relevant. For the moment, let us however consider the principal series, leaving j completely general.

A representation $\chi = (n, \rho)$ in the principal series is labeled by an integer $n \in Z$ and a real number $\rho \in R$. Actually, the representations (n, ρ) and $(-n, -\rho)$ (as given below) turn out to be equivalent so that, in order to have each representation only once, n and ρ will be restricted as follows [12]:

$$\begin{aligned} n &= 0, \qquad \rho \geq 0, \\ n &> 0, \qquad \rho \in R. \end{aligned} \tag{22}$$

Notice that we keep the standard notation ρ used in mathematical books to label the irreducible Lorentz-group representations. This parameter should not be confused with the standard notation in HQET for the slope of the IW function ρ^2.

The Hilbert space $\mathcal{H}_{n,\rho}$ is made up of functions of a complex variable z with the standard scalar product

$$\langle\phi'|\phi\rangle = \int \overline{\phi'(z)}\,\phi(z)\, d^2z \tag{23}$$

with the measure d^2z in the complex plane being simply $d^2z = d(\mathrm{Re}\, z)d(\mathrm{Im}\, z)$. So, $\mathcal{H}_{n,\rho} = L^2(C, d^2z)$.

The unitary operator $U_{n,\rho}(\Lambda)$ is given by

$$(U_{n,\rho}(\Lambda)\phi)(z) = \left(\frac{\alpha - \gamma z}{|\alpha - \gamma z|}\right)^n |\alpha - \gamma z|^{2i\rho - 2}\, \phi\left(\frac{\delta z - \beta}{\alpha - \gamma z}\right) \tag{24}$$

where α, β, γ, δ are complex matrix elements of $\Lambda \in SL(2,C)$:

$$\Lambda = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \qquad \alpha\delta - \beta\gamma = 1. \tag{25}$$

B. Decomposition under the rotation group

Next we need the decomposition of the restriction to the subgroup SU(2) of each irreducible unitary representation of $SL(2,C)$.

Since SU(2) is compact, the decomposition is by a direct sum so that for each representation $\chi \in X$ we have an *orthonormal basis* $\phi^\chi_{j,M}$ of $\mathcal{H}_\chi$ adapted to SU(2). Having in mind the usual notation for the spin of the light component of a heavy hadron, here we denote by j the spin of an irreducible representation of SU(2). It turns out [12] that each representation j of SU(2) appears in χ with multiplicity 0 or 1, so that $\phi^\chi_{j,M}$ needs no other indices, and that the values taken by j are a part of the integer and half-integer numbers. For fixed j, the functions $\phi^\chi_{j,M}$, $-j \leq M \leq j$ are chosen as a standard basis of the representation j of SU(2).

It turns out [12] that the functions $\phi^\chi_{j,M}(z)$ are expressed in terms of the rotation matrix elements $D^j_{M',M}$ defined by Eq. (14). For a matrix $R \in \mathrm{SU}(2)$ of the form

$$R = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \qquad |a|^2 + |b|^2 = 1 \tag{26}$$

we shall also consider $D^j_{M',M}$ as a function of a and b, satisfying $|a|^2 + |b|^2 = 1$.

We can now give explicit formulas for the orthonormal basis $\phi^\chi_{j,M}$ of $\mathcal{H}_\chi$.

The spins j which appear in a representation $\chi = (n, \rho)$ are [3]

$$\text{all integers} \quad j \geq \frac{n}{2} \quad \text{for } n \text{ even}, \tag{27}$$

$$\text{all half-integers} \quad j \geq \frac{n}{2} \quad \text{for } n \text{ odd}. \tag{28}$$

Such a spin appears with multiplicity 1.

The basis functions $\phi^{n,\rho}_{j,M}(z)$ are given by the expression [3]

$$\phi^{n,\rho}_{j,M}(z) = \frac{\sqrt{2j+1}}{\sqrt{\pi}}(1+|z|^2)^{i\rho-1}D^j_{n/2,M} \times \left(\frac{1}{\sqrt{1+|z|^2}}, -\frac{z}{\sqrt{1+|z|^2}},\right) \quad (29)$$

or, using an explicit formula for $D^j_{n/2,M}$,

$$\phi^{n,\rho}_{j,M}(z) = \frac{\sqrt{2j+1}}{\sqrt{\pi}}(-1)^{n/2-M} \times \sqrt{\frac{(j-n/2)!(j+n/2)!}{(j-M)!(j+M)!}}(1+|z|^2)^{i\rho-j-1} \times \sum_k (-1)^k \binom{j+M}{k} \times \binom{j-M}{j-n/2-k} z^{n/2-M+k}\,\bar{z}^k \quad (30)$$

where the range for k can be limited to $0 \le k \le j-n/2$ due to the binomial factors.

V. IRREDUCIBLE ISGUR-WISE FUNCTIONS FOR $j = \frac{1}{2}$

For $j = \frac{1}{2}$, one has a fixed value for n,

$$j = \frac{1}{2} \Rightarrow n = 1, \qquad \rho \in R \quad (31)$$

and we are thus in the case (28).

From now on we delete the fixed indices $j = \frac{1}{2}$ and $n = 1$, and we apply the explicit formula (30) to this case, so that the nonvanishing functions (30) read

$$\phi^{\rho}_{+\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi}}(1+|z|^2)^{i\rho-\frac{3}{2}}, \quad (32)$$

$$\phi^{\rho}_{-\frac{1}{2}}(z) = -\sqrt{\frac{2}{\pi}}z(1+|z|^2)^{i\rho-\frac{3}{2}}. \quad (33)$$

Let us now specialize the $SL(2,C)$ matrix (25) to a boost in the z direction,

$$\Lambda_\tau = \begin{pmatrix} e^{\frac{\tau}{2}} & 0 \\ 0 & e^{-\frac{\tau}{2}} \end{pmatrix}, \qquad w = \cosh(\tau) \quad (34)$$

and, following the $j = 0$ case studied at length in Ref. [3], let us consider the following objects:

$$\xi^{\rho}_{+\frac{1}{2},+\frac{1}{2}}(w) = \langle \phi^{\rho}_{+\frac{1}{2}} | U^{\rho}(\Lambda_\tau)\phi^{\rho}_{+\frac{1}{2}}\rangle, \quad (35)$$

$$\xi^{\rho}_{-\frac{1}{2},-\frac{1}{2}}(w) = \langle \phi^{\rho}_{-\frac{1}{2}} | U^{\rho}(\Lambda_\tau)\phi^{\rho}_{-\frac{1}{2}}\rangle. \quad (36)$$

From the transformation law (24) and the explicit forms (32) and (33), one gets

$$\left(U^{\rho}(\Lambda_\tau)\phi^{\rho}_{+\frac{1}{2}}\right)(z) = \sqrt{\frac{2}{\pi}}e^{(i\rho-1)\tau}(1+e^{-2\tau}|z|^2)^{i\rho-\frac{3}{2}}, \quad (37)$$

$$\left(U^{\rho}(\Lambda_\tau)\phi^{\rho}_{-\frac{1}{2}}\right)(z) = -\sqrt{\frac{2}{\pi}}e^{(i\rho-1)\tau}e^{-\tau}z(1+e^{-2\tau}|z|^2)^{i\rho-\frac{3}{2}} \quad (38)$$

and therefore, from these expressions and Eqs. (35) and (36), one obtains

$$\xi^{\rho}_{+\frac{1}{2},+\frac{1}{2}}(w) = \frac{2}{\pi}\int (1+|z|^2)^{-i\rho-\frac{3}{2}}e^{(i\rho-1)\tau} \times (1+e^{-2\tau}|z|^2)^{i\rho-\frac{3}{2}}d^2z, \quad (39)$$

$$\xi^{\rho}_{-\frac{1}{2},-\frac{1}{2}}(w) = \frac{2}{\pi}\int e^{-\tau}|z|^2(1+|z|^2)^{-i\rho-\frac{3}{2}}e^{(i\rho-1)\tau} \times (1+e^{-2\tau}|z|^2)^{i\rho-\frac{3}{2}}d^2z. \quad (40)$$

We must now extract the Lorentz-invariant Isgur-Wise function $\xi(w)$. To do this, we must decompose the matrix elements (39) and (40) into invariants using the spin-$\frac{1}{2}$ spinors of the light cloud $u_{\pm\frac{1}{2}}$. We have not introduced parity into our formalism. Therefore, we will have the following decomposition:

$$\xi^{\rho}_{+\frac{1}{2},+\frac{1}{2}}(w) = (\bar{u}_{+\frac{1}{2}}(v')u_{+\frac{1}{2}}(v))\xi^{\rho}(w) + (\bar{u}_{+\frac{1}{2}}(v')\gamma_5 u_{+\frac{1}{2}}(v))\tau^{\rho}(w), \quad (41)$$

$$\xi^{\rho}_{-\frac{1}{2},-\frac{1}{2}}(w) = (\bar{u}_{-\frac{1}{2}}(v')u_{-\frac{1}{2}}(v))\xi^{\rho}(w) + (\bar{u}_{-\frac{1}{2}}(v')\gamma_5 u_{-\frac{1}{2}}(v))\tau^{\rho}(w) \quad (42)$$

where $\xi^{\rho}(w)$ is an irreducible $\frac{1}{2}^- \to \frac{1}{2}^-$ elastic IW function, labeled by the index ρ, and $\tau^{\rho}(w)$ is a function corresponding to the flip of parity $\frac{1}{2}^- \to \frac{1}{2}^+$.

The notation for the function $\tau^{\rho}(w)$ has to be distinguished from that for the boost parameter τ introduced in Eq. (34).

Let us now compute the spinor bilinears of Eqs. (41) and (42). From the expression

$$u_{\pm\frac{1}{2}}(v) = \sqrt{\frac{v^0+1}{2}}\begin{pmatrix} \chi_{\pm\frac{1}{2}} \\ \frac{\sigma.\mathbf{v}}{v^0+1}\chi_{\pm\frac{1}{2}} \end{pmatrix}, \qquad \bar{u}_{\pm\frac{1}{2}}(v)u_{\pm\frac{1}{2}}(v) = 1 \quad (43)$$

one gets

$$\bar{u}_{+\frac{1}{2}}(v')u_{+\frac{1}{2}}(v) = \bar{u}_{-\frac{1}{2}}(v')u_{-\frac{1}{2}}(v) = \sqrt{\frac{w+1}{2}}, \quad (44)$$

$$\bar{u}_{+\frac{1}{2}}(v')\gamma_5 u_{+\frac{1}{2}}(v) = -\bar{u}_{-\frac{1}{2}}(v')\gamma_5 u_{-\frac{1}{2}}(v) = \frac{1}{\sqrt{2}}\sqrt{\frac{w-1}{w+1}} \quad (45)$$

and therefore we obtain

$$\xi^\rho(w) = \sqrt{\frac{2}{w+1}}\frac{1}{2}[\xi^\rho_{+\frac{1}{2},+\frac{1}{2}}(w) + \xi^\rho_{-\frac{1}{2},-\frac{1}{2}}(w)], \quad (46)$$

$$\tau^\rho(w) = \sqrt{2}\sqrt{\frac{w+1}{w-1}}\frac{1}{2}[\xi^\rho_{+\frac{1}{2},+\frac{1}{2}}(w) - \xi^\rho_{-\frac{1}{2},-\frac{1}{2}}(w)]. \quad (47)$$

Finally, from Eqs. (39) and (40) one gets

$$\xi^\rho(w) = \frac{1}{1+\cosh(\tau)}\frac{1}{\sinh(\tau)} \times \frac{1}{2}\left[\frac{e^{(i\rho-\frac{1}{2})\tau} - e^{-(i\rho-\frac{1}{2})\tau}}{i\rho-\frac{1}{2}} + \frac{e^{(i\rho+\frac{1}{2})\tau} - e^{-(i\rho+\frac{1}{2})\tau}}{i\rho+\frac{1}{2}}\right] \quad (48)$$

or

$$\xi^\rho(w) = \frac{1}{1+\cosh(\tau)}\frac{1}{\sinh(\tau)}\frac{4}{4\rho^2+1} \times \left[\sinh\left(\frac{\tau}{2}\right)\cos(\rho\tau) + 2\rho\cosh\left(\frac{\tau}{2}\right)\sin(\rho\tau)\right]. \quad (49)$$

This is the expression for the elastic $\frac{1}{2}^- \to \frac{1}{2}^-$ irreducible IW functions we were looking for, parametrized by the real parameter ρ, which satisfies

$$\xi^\rho(1) = 1. \quad (50)$$

Like in the case $j = 0$, which was analyzed in great detail in Ref. [3], the elastic $\frac{1}{2}^- \to \frac{1}{2}^-$ IW function $\xi(w)$ will be given by the integral over a positive measure $d\nu(\rho)$:

$$\xi(w) = \int_{]-\infty,\infty[} \xi^\rho(w)\, d\nu(\rho) \quad (51)$$

where the measure is normalized according to

$$\int_{]-\infty,\infty[} d\nu(\rho) = 1. \quad (52)$$

Notice that the range $]-\infty,\infty[$ for the parameter ρ that labels the irreducible representations follows from the fact that in the $j = \frac{1}{2}$ case one has $n = 1$ and $\rho \in R$ [Eq. (31)]. Notice also that the IW irreducible function (49) is even in ρ, $\xi^\rho(w) = \xi^{-\rho}(w)$. (This seems to contradict the nonequivalence of the irreducible representations labeled by ρ and $-\rho$, but this can be resolved by considering the Lorentz-plus-parity group.)

The irreducible IW functions (49), parametrized by some value of $\rho = \rho_0$, are legitimate IW functions since the corresponding measure is given by a delta function,

$$d\nu(\rho) = \delta(\rho - \rho_0)\, d\rho. \quad (53)$$

In the case of the irreducible representation $\rho_0 = 0$ one finds

$$\xi^0(w) = \frac{4\sinh(\frac{\tau}{2})}{(1+\cosh(\tau))\sinh(\tau)} = \left(\frac{2}{1+w}\right)^{\frac{3}{2}} \quad (54)$$

which saturates the lower bound for the slope $-\xi'(1) \geq \frac{3}{4}$. This is the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) limit of the IW function, which was considered previously using different theoretical arguments [13,14].

VI. INTEGRAL FORMULA FOR THE IW FUNCTION $\xi(w)$ AND POLYNOMIAL EXPRESSION FOR ITS DERIVATIVES

From the normalization of the norm (52) and the normalization of the irreducible IW functions (50) one gets the correct value of the IW function at zero recoil,

$$\xi(1) = 1. \quad (55)$$

The integral formula (49) and (51) is, explicitly,

$$\xi(w) = \frac{1}{1+\cosh(\tau)}\frac{1}{\sinh(\tau)} \times \int_{]-\infty,\infty[} \frac{4}{4\rho^2+1} \times \left[\sinh\left(\frac{\tau}{2}\right)\cos(\rho\tau) + 2\rho\cosh\left(\frac{\tau}{2}\right)\sin(\rho\tau)\right] d\nu(\rho) \quad (56)$$

from which one can find the following polynomial expression for its derivatives:

$$\xi^{(n)}(1) = (-1)^n \frac{1}{2^{2n}(2n+1)!!} \times \prod_{i=1}^{n}\langle[(2i+1)^2 + 4\rho^2]\rangle \quad (n \geq 1) \quad (57)$$

where the mean value in Eq. (57) is defined as

$$\langle f(\rho)\rangle = \int_{]-\infty,\infty[} f(\rho)\, d\nu(\rho). \quad (58)$$

Equation (57) can be demonstrated along the same lines as the corresponding equation in the baryon case

(Appendix D of Ref. [3]) by using the following integral representation of the irreducible IW function (48) or (49):

$$\xi^{\rho}(w) = \frac{2}{\rho^2+\frac{1}{4}} \frac{\cosh(\pi\rho)}{\pi} \int_0^\infty x^{-i\rho+\frac{1}{2}} \frac{1+x}{(1+2wx+x^2)^2} dx. \tag{59}$$

VII. BOUNDS ON THE DERIVATIVES OF THE IW FUNCTION

A. Lower bounds on the derivatives

Bounds on the successive derivatives of the IW function are important. Indeed, the extrapolation at zero recoil to obtain $|V_{cb}|$ from the semileptonic exclusive data is sensitive to high derivatives (curvature and third derivative, at least) because the data points are more precise at large recoil than at low recoil [8].

From the expression (57) one gets immediately the lower bounds on the derivatives

$$(-1)^n \xi^{(n)}(1) \geq \frac{(2n+1)!!}{2^{2n}} \tag{60}$$

obtained in Ref. [8], which reduces for the slope and the curvature to the bounds

$$-\xi'(1) \geq \frac{3}{4}, \qquad \xi''(1) \geq \frac{15}{16}. \tag{61}$$

B. Improved bounds on the derivatives

To get improved bounds on the derivatives we must (like in Ref. [3]) express the derivatives in terms of moments of the *positive* variable ρ^2, which we can read from Eq. (57). Calling the moments

$$\mu_n = \langle \rho^{2n} \rangle \geq 0 \quad (n \geq 0) \tag{62}$$

one gets the successive derivatives in terms of moments:

$$\begin{aligned}
\xi(1) &= \mu_0 = 1,\\
\xi'(1) &= -\left(\frac{3}{4} + \frac{1}{3}\mu_1\right),\\
\xi''(1) &= \frac{15}{16} + \frac{17}{30}\mu_1 + \frac{1}{15}\mu_2,\\
\xi^{(3)}(1) &= -\left(\frac{105}{64} + \frac{1891}{1680}\mu_1 + \frac{83}{420}\mu_2 + \frac{1}{105}\mu_3\right),\\
\xi^{(4)}(1) &= \frac{945}{256} + \frac{4561}{1680}\mu_1 + \frac{4307}{7560}\mu_2 + \frac{41}{945}\mu_3 + \frac{1}{945}\mu_4,
\end{aligned} \tag{63}$$

etc.

Notice that the lowest bounds (60) and (61) are found in the limit $\mu_n = 0$ $(n \geq 1)$.

The equations (63) can be solved step by step, and the moment μ_n is expressed as a combination of the derivatives $\xi(1)$, $\xi'(1)$, $\ldots\xi^{(n)}(1)$:

$$\begin{aligned}
\mu_0 &= \xi(1) = 1,\\
\mu_1 &= -\frac{3}{4}[3 + 4\xi'(1)],\\
\mu_2 &= \frac{3}{16}[27 + 136\xi'(1) + 80\xi''(1)],\\
\mu_3 &= -\frac{3}{64}[243 + 3724\xi'(1) + 6640\xi''(1) + 2240\xi^{(3)}(1)],\\
\mu_4 &= \frac{3}{256}[2187 + 96016\xi'(1) + 399840\xi''(1)\\
&\quad + 367360\xi^{(3)}(1) + 80640\xi^{(4)}(1)],
\end{aligned} \tag{64}$$

etc.

Since ρ^2 is a positive variable, one can obtain improved bounds on the derivatives from the following set of constraints. For any $n \geq 0$, one has [3]

$$\det[(\mu_{i+j})_{0\leq i,j\leq n}] \geq 0, \tag{65}$$

$$\det[(\mu_{i+j+1})_{0\leq i,j\leq n}] \geq 0. \tag{66}$$

Since each moment μ_k is a combination of the derivatives $\xi(1)$, $\xi'(1)$, $\ldots\xi^{(k)}(1)$, the constraints on the moments translate into constraints on the derivatives.

Here we shall treat in detail only the constraints on μ_1, μ_2, and μ_3, which are given by Eqs. (66) $(n=0)$, (65) $(n=1)$, and (66) $(n=1)$, respectively:

$$\mu_1 \geq 0, \tag{67}$$

$$\det\begin{pmatrix} 1 & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix} = \mu_2 - \mu_1^2 \geq 0, \tag{68}$$

$$\det\begin{pmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{pmatrix} = \mu_1\mu_3 - \mu_2^2 \geq 0, \tag{69}$$

$$\det\begin{pmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{pmatrix} = (\mu_2 - \mu_1^2)\mu_4 - (\mu_3^2 - 2\mu_1\mu_2\mu_3 + \mu_2^3) \geq 0, \tag{70}$$

etc.

Clearly, each moment μ_k is bounded from below, and the lower bound is given by Eq. (65) for even k and by Eq. (66) for odd k in terms of the lower moments. So Eqs. (67)–(70) give

$$\mu_1 \geq 0, \tag{71}$$

$$\mu_2 \geq \mu_1^2, \tag{72}$$

$$\mu_3 \geq \frac{\mu_2^2}{\mu_1}, \tag{73}$$

$$\mu_4 \geq \frac{-\mu_2^3 + 2\mu_1\mu_2\mu_3 - \mu_3^2}{\mu_1^2 - \mu_2}, \tag{74}$$

etc.

The constraints (71)–(73) imply, respectively, in terms of the derivatives

$$-\xi'(1) \geq \frac{3}{4}, \tag{75}$$

$$\xi''(1) \geq \frac{1}{5}[-4\xi'(1) + 3\xi'(1)^2], \tag{76}$$

$$-\xi^{(3)}(1) \geq \frac{5}{28}\frac{-12\xi'(1) + 9\xi'(1)^2 - 39\xi''(1) - 12\xi'(1)\xi''(1) + 16\xi''(1)^2}{-3 - 4\xi'(1)} \tag{77}$$

and from Eq. (74) we find a lower bound on $\xi^{(4)}(1)$, etc.

We see that we recover the bounds obtained using the SR method.

The lower bound of the third derivative (77) is apparently singular for the lower bound (75) of the first derivative $-\xi'(1)$. However, using the lower bound (76) to eliminate $\xi''(1)$ we find the less restrictive lower bound

$$-\xi^{(3)}(1) \geq \frac{-\xi'(1)[10 - 3\xi'(1)][4 - 3\xi'(1)]}{35}. \tag{78}$$

VIII. INVERSION OF THE INTEGRAL REPRESENTATION OF THE ISGUR-WISE FUNCTION

Let us now show that the integral formula for the IW function (51) can be inverted, giving the positive measure $d\nu(\rho)$ in terms of the IW function $\xi(w)$. This will allow us to formulate criteria to test the validity of a given phenomenological *Ansatz* of $\xi(w)$.

Let us define

$$\hat{\xi}(\tau) = (\cosh(\tau) + 1)\sinh(\tau)\xi(\cosh(\tau)) \tag{79}$$

and similarly for the irreducible IW function

$$\hat{\xi}^\rho(\tau) = (\cosh(\tau) + 1)\sinh(\tau)\xi_\rho(\cosh(\tau)). \tag{80}$$

The integral formula (51) then reads

$$\hat{\xi}(\tau) = \int \hat{\xi}^\rho(\tau) d\nu(\rho). \tag{81}$$

It is convenient to use the form (48) for the irreducible IW function. One finds, for its derivative, the simple formula

$$\frac{d}{d\tau}\hat{\xi}^\rho(\tau) = 2\cos(\rho\tau)\cosh\left(\frac{\tau}{2}\right). \tag{82}$$

We now assume that the general measure $d\nu(\rho)$ is even, i.e. like the measure $d\rho$, without loss of generality because $\xi^\rho(w)$ is even in ρ. This means that $\int f(\rho)d\nu(\rho) = \int f(-\rho)d\nu(\rho)$ for any function $f(\rho)$.

Defining the function

$$\eta(\tau) = \frac{1}{2\cosh(\frac{\tau}{2})}\frac{d}{d\tau}\hat{\xi}(\tau) \tag{83}$$

one sees, from Eq. (82), that the integral formula (51) reads

$$\eta(\tau) = \int_{]-\infty,\infty[} \cos(\rho\tau)d\nu(\rho) = \int_{]-\infty,\infty[} e^{-i\rho\tau}d\nu(\rho). \tag{84}$$

Computing the Fourier transform

$$\tilde{\eta}(\rho) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\tau\rho}\, d\tau\, \eta(\tau) = \int_{]-\infty,\infty[} \delta(\rho - \rho')\, d\nu(\rho') \tag{85}$$

and defining the function

$$\mu(\rho) = \frac{d\nu(\rho)}{d\rho} \tag{86}$$

one finds

$$\tilde{\eta}(\rho) = \mu(\rho). \tag{87}$$

The function (86) is even

$$\mu(\rho) = \mu(-\rho) \tag{88}$$

and one finally finds

$$d\nu(\rho) = \tilde{\eta}(\rho) d\rho \tag{89}$$

or

$$\frac{d\nu(\rho)}{d\rho} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\tau\rho}\, d\tau\, \frac{1}{2\cosh(\frac{\tau}{2})} \times \frac{d}{d\tau}[(\cosh(\tau)+1)\sinh(\tau)\xi(\cosh(\tau))]. \tag{90}$$

This completes the inversion of the integral representation. Equation (90) is the master formula expressing the measure in terms of a given *Ansatz* for the Isgur-Wise function.

We can now apply this formula to check if a given phenomenological formula for the IW function $\xi(w)$ satisfies the constraint that the corresponding measure $d\nu(\rho)$ must be positive. This provides a powerful consistency test for any proposed *Ansatz*. Notice also that, as a necessary condition on $\xi(w)$, the function $\eta(\rho)$, defined by Eqs. (79) and (83) in terms of $\xi(w)$, *must be bounded* by 1.

IX. AN UPPER BOUND ON THE ISGUR-WISE FUNCTION

Also, an upper bound on the whole IW function $\xi(w)$ can be obtained from the integral formula obtained above.

Defining the function

$$\eta^\rho(\tau) = \frac{1}{2\cosh(\frac{\tau}{2})}\frac{d}{d\tau}\hat{\xi}^\rho(\tau) \tag{91}$$

we have obtained, from Eqs. (82) and (83),

$$\eta^\rho(\tau) = \cos(\rho\tau) \tag{92}$$

and it follows that

$$-1 \le \eta^\rho(\tau) \le 1 \tag{93}$$

which gives

$$-2\cosh\left(\frac{\tau}{2}\right) \le \frac{d}{d\tau}\hat{\xi}^\rho(\tau) \le 2\cosh\left(\frac{\tau}{2}\right). \tag{94}$$

Integrating this inequality from 0, one gets

$$-4\sinh\left(\frac{\tau}{2}\right) \le \hat{\xi}^\rho(\tau) \le 4\sinh\left(\frac{\tau}{2}\right) \tag{95}$$

and since

$$\hat{\xi}^0(\tau) = 4\sinh\left(\frac{\tau}{2}\right) \tag{96}$$

one finds the inequalities

$$-\hat{\xi}^0(\tau) \le \hat{\xi}^\rho(\tau) \le \hat{\xi}^0(\tau) \tag{97}$$

and by simplifying common factors dependent on τ we get

$$-\xi^0(\tau) \le \xi(\tau) \le \xi^0(\tau). \tag{98}$$

Since $\xi^0(\tau)$ is given by the expression (54), we finally obtain

$$|\xi(w)| \le \left(\frac{2}{1+w}\right)^{\frac{3}{2}}. \tag{99}$$

This inequality is a strong result because it holds for any value of w.

X. CONSISTENCY TESTS FOR ANY *Ansatz* OF THE IW FUNCTION: PHENOMENOLOGICAL APPLICATIONS

In this section we examine a number of phenomenological formulas proposed in the literature.

We will compare these *Ansätze* with the theoretical criteria formulated in the two preceding sections, concerning, respectively, the lower bounds on derivatives at zero recoil (Sec. VII), the upper bound obtained for the whole IW function (Sec. IX), and the inversion of the integral formula for the IW function, and check the positivity of the measure (90). For the bounds on the derivatives, we will limit the test up to the third derivative [Eqs. (75)–(77)], although the method can be generalized to any higher derivative in a straightforward way.

We must emphasize that the satisfaction of the bounds on the derivatives and of the upper bound on the whole IW function are *necessary conditions*, while the criterium of the positivity of the measure is a *necessary and sufficient condition* to establish if a given *Ansatz* of the IW function satisfies the Lorentz-group criteria of the present paper.

To illustrate the methods exposed in this paper, we use a number of proposed phenomenological models for the IW function. Some of these functions could happen to be rather close numerically in the physical range at finite mass $1 \le w \le w_{\max} \simeq 1.4$ GeV. However, as emphasized in the Introduction, our purpose is mainly theoretical and has the interest of giving theoretical criteria as to whether a given model for the IW function satisfies or does not satisfy the general principles of QCD in the heavy-quark limit.

A. The exponential *Ansatz*

$$\xi(w) = \exp[-c(w-1)]. \tag{100}$$

This form corresponds to the nonrelativistic limit for the light quark with the harmonic-oscillator potential [16].

1. Bounds on the derivatives

The bound for the slope (75) is satisfied for $c \geq \frac{3}{4}$, the bound for the second derivative (76) is satisfied for $c \geq 2$, while the bound for the third derivative (77) is violated for any value of c.

Therefore, this phenomenological *Ansatz* on the IW function is invalid on the theoretical grounds of Sec. VII.

2. Upper bound on the IW function

The exponential *Ansatz* (100) nevertheless satisfies the upper bound (99), $\xi(w) \leq (\frac{2}{1+w})^{\frac{3}{2}}$.

3. Positivity of the measure

Let us now examine the criterium based on the positivity of the measure.

One needs to compute

$$\eta(\tau) = \frac{1}{c}\left(-\frac{d^2}{d\tau^2} + \frac{1}{4}\right)\cosh\left(\frac{\tau}{2}\right)\exp\left[-c(\cosh(\tau) - 1)\right]. \tag{101}$$

The function $\eta(\tau)$ is bounded for any value of c (Fig. 1).

The Fourier transform of this function gives, from Eq. (90),

$$d\nu(\rho) = \frac{e^c}{2\pi}\frac{1}{c}\left(\rho^2 + \frac{1}{4}\right)[K_{i\rho+\frac{1}{2}}(\rho) + K_{-i\rho+\frac{1}{2}}(\rho)]d\rho. \tag{102}$$

This function is not positive for any value of c, as we illustrate in Fig. 2.

Therefore, the exponential *Ansatz* for the IW function violates the consistency criteria exposed in Secs. VII and VIII.

B. The "dipole"

The following shape has been proposed in the literature (see for example Refs. [17,18]):

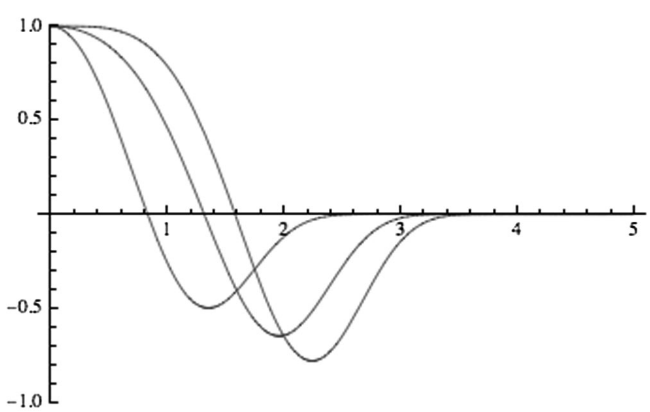

FIG. 1. $\eta(\tau)$ [Eq. (83)] for the exponential *Ansatz* $c = 3/4, 1, 2$ (higher to lower curves).

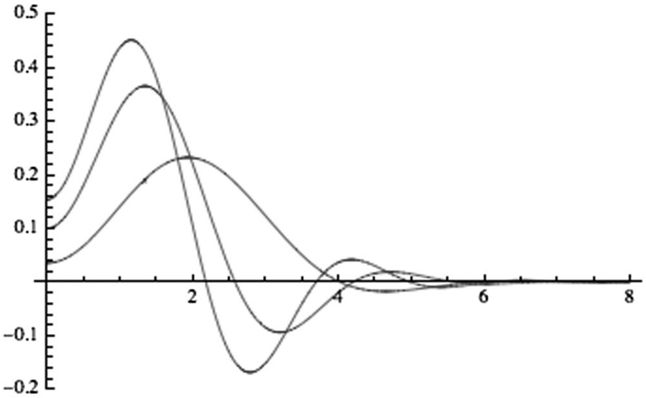

FIG. 2. $\frac{d\nu(\rho)}{d\rho}$ [Eq. (90)] for the exponential *Ansatz*, for $c = 3/4, 1, 2$ (higher to lower curves).

$$\xi(w) = \left(\frac{2}{1+w}\right)^{2c}. \tag{103}$$

1. Bounds on the derivatives

The bound for the slope (75) is satisfied for $c \geq \frac{3}{4}$, while the bounds for the second derivative (76) and third derivative (77) are also satisfied for $c \geq 3/4$. The "dipole" *Ansatz* is thus valid for any value of c.

2. Upper bound on the IW function

Of course, the "dipole" satisfies the upper bound (99), $\xi(w) \leq (\frac{2}{1+w})^{\frac{3}{2}}$, for $c \geq 3/4$.

3. Positivity of the measure

Let us verify this result in all generality by computing the measure (90).

One needs first to compute

$$\eta(\tau) = -4(c-1)\left[\cosh\left(\frac{\tau}{2}\right)\right]^{-4c+3} + (4c-3)\left[\cosh\left(\frac{\tau}{2}\right)\right]^{-4c+1}. \tag{104}$$

Since one needs the function $\eta(\tau)$ to be bounded, the parameter c must satisfy

$$c \geq \frac{3}{4}. \tag{105}$$

We realize that in this particular case

$$c = \frac{3}{4} \to \eta(\tau) = 1 \to d\nu(\rho) = \delta(\rho)\,d\rho. \tag{106}$$

Therefore, one gets in this case a delta function for the measure, which is positive and corresponds to the explicit formula (54) for the IW function in the BPS limit given above.

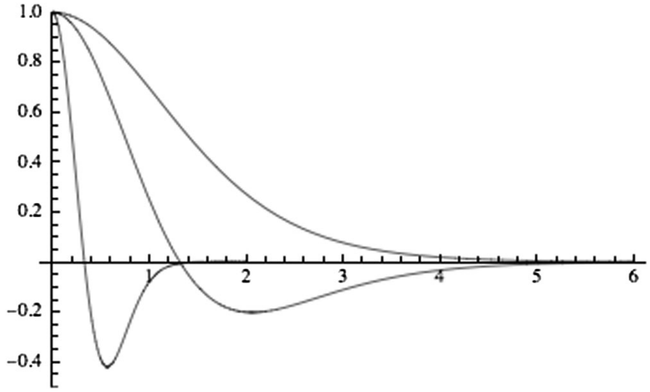

FIG. 3. $\eta(\tau)$ [Eq. (83)] for the "dipole" *Ansatz* $c = 1., 1.5, 2.$ (from higher to lower curves).

On the other hand, one sees from the lower bound (105), that the so-called meson-dominance IW proposal [19]

$$\xi_{\mathrm{MD}}(w) = \frac{2}{w+1} \tag{107}$$

does not satisfy our general constraints because in this case $c = \frac{1}{2}$.

For $c > \frac{3}{4}$ one obtains a function $\eta(\tau)$ that is bounded, as we can see from Eq. (108) and Fig. 3.

Computing its Fourier transform (90) one gets the measure

$$d\nu(\rho) = \frac{2^{4c-1}}{2\pi}(4c-3)\left(\rho^2 + \frac{1}{4}\right) \times \frac{\Gamma(i\rho + 2c - \frac{3}{2})\Gamma(-i\rho + 2c - \frac{3}{2})}{\Gamma(4c-1)} d\rho \tag{108}$$

which is positive (Fig. 4).

In conclusion, from Eqs. (106) and (108), we see that the measure $d\nu(\rho)$ for the "dipole" *Ansatz* is positive for $c \geq \frac{3}{4}$. Therefore, the "dipole" form satisfies all the consistency criteria.

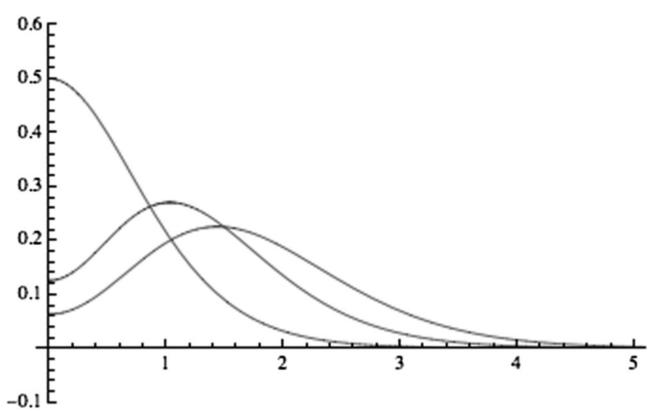

FIG. 4. $\frac{d\nu(\rho)}{d\rho}$ for the "dipole" *Ansatz* for $c = 1., 1.5, 2.$ (from higher to lower curves).

C. Kiselev's *Ansatz*

Kiselev [19] proposed the following shape:

$$\xi(w) = \sqrt{\frac{2}{w^2+1}} \exp\left(-\beta \frac{w^2-1}{w^2+1}\right), \tag{109}$$

where $\beta = \frac{m_{sp}^2}{\omega^2}$ and the slope is given by $\xi'(1) = -\frac{1}{2} - \beta$.

1. Bounds on the derivatives

The bound for the slope (75) is satisfied for $\beta \geq \frac{1}{4}$, the bound for the second derivative (76) is satisfied for $\beta \geq 0.4$, while the bound for the third derivative (77) is satisfied for $\beta \geq 1.5$. One can suspect that bounds for higher derivatives will only be satisfied for higher values of β.

2. Upper bound on the IW function

The Kiselev formula (109) does not satisfy the upper bound (99), $\xi(w) \leq (\frac{2}{1+w})^{\frac{3}{2}}$, for any value of β because, as we can see, in the limit of large w it becomes $\sqrt{\frac{2}{w^2+1}} e^{-\beta}$.

3. Positivity of the measure

One finds for the function $\eta(\tau)$ [Eq. (83)]

$$\eta(\tau) = \frac{1}{2}\cosh\left(\frac{\tau}{2}\right) \exp\left[-2\beta \frac{\sinh^2(\tau)}{3+\cosh(2\tau)}\right]\left[\frac{1}{3+\cosh(2\tau)}\right]^{\frac{5}{2}} \times [-38 + 2(53+8\beta)\cosh(\tau) - 24\cosh(2\tau) + 21\cosh(3\tau) - 16\beta\cosh(3\tau) - 2\cosh(4\tau) + \cosh(5\tau)]. \tag{110}$$

Independently of any value of the parameter $\beta = \frac{m_{sp}^2}{\omega^2}$, this function is not bounded since it blows up for $\tau \to \pm\infty$. Therefore the *Ansatz* (109) for the IW function does not satisfy the general Lorentz-group criteria formulated in the present paper.

D. BSW formula for the IW function

Using the relativistic oscillator wave functions of Bauer, Stech and Wirbel (BSW) [20] one finds the IW function [21]

$$\xi_{\mathrm{BSW}}(w) = \sqrt{\frac{2}{w+1}} \frac{1}{w} \exp\left(-c^2 \frac{w-1}{2w}\right) \frac{F\left(c\sqrt{\frac{w+1}{2w}}\right)}{F(c)} \tag{111}$$

with $c = \frac{\alpha}{\omega}$ in the notation of Ref. [20], and

$$F(x) = \int_{-x}^{+\infty} dz(z+x)e^{-z^2} = \frac{1}{2}[e^{-x^2} + \sqrt{\pi}x(1+\mathrm{erf}(x))]. \tag{112}$$

As we will see, this *Ansatz* for the IW function allows us to illustrate in detail the consistency criteria developed in this paper.

1. Bounds on the derivatives

First, the bound for the slope (75) is satisfied for any value of c [for $c = 0$, the slope is $-\xi'_{\rm BSW}(1) = \frac{5}{4}$], while the bounds for the second derivative (76) and third derivative (77) are satisfied for any value of c. Thus, up to this third derivative the BSW *Ansatz* seems valid for any value of c.

2. Upper bound on the IW function

The BSW formula (111) satisfies the upper bound (96), $\xi(w) \le (\frac{2}{1+w})^{\frac{3}{2}}$, for any value of the parameter c.

3. Positivity of the measure

We will check now that this is true in all generality, for any derivative, using the criterium of positivity of the measure $d\nu(\rho)$ [Eq (90)].

Computing the function $\eta(\tau)$ [Eq. (83)] for the BSW *Ansatz* (111) one finds, numerically, the functions $\eta_{\rm BSW}(\tau)$ of Fig. 5.

We observe that for $\tau \to \infty$, the function $\eta(\tau)$ tends to a constant, which is found to be

$$\eta^{(\infty)} = \lim_{\tau\to\infty} \eta(\tau) = \frac{2 + c\sqrt{2\pi}\exp(\frac{c^2}{2})[1 + {\rm erf}(\frac{c}{\sqrt{2}})]}{4 + 4c\sqrt{\pi}\exp(c^2)[1 + {\rm erf}(c)]}. \quad (113)$$

Since the function $\eta(\tau)$ tends to a constant, its Fourier transform, which gives the measure (89), will contain a δ function. Substracting the constant (113), we define a new function

$$\eta^{(0)}_{\rm BSW}(\tau) = \eta_{\rm BSW}(\tau) - \eta^{(\infty)}. \quad (114)$$

We plot this function in Fig. 6 for some values of c, and observe that it is bounded.

Defining [like in Eq. (85)] its Fourier transform by

$$\tilde{\eta}^{(0)}_{\rm BSW}(\rho) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\tau\rho}\, \eta^{(0)}_{\rm BSW}(\tau) d\tau \quad (115)$$

we obtain the functions of Fig. 7.

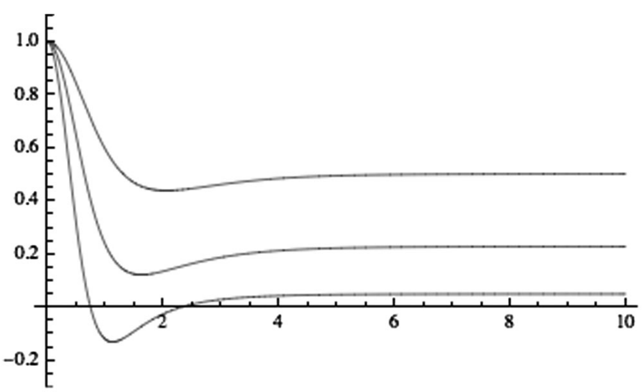

FIG. 5. The function $\eta_{\rm BSW}(\tau)$ [Eq. (83)] for $c = 0, 1, 2$ (from higher to lower curves).

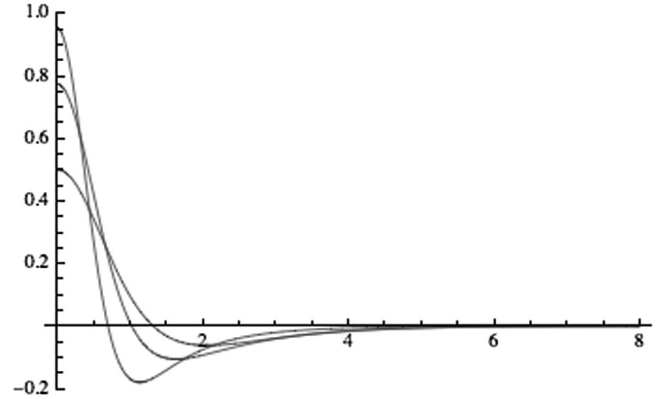

FIG. 6. The function $\eta^{(0)}_{\rm BSW}(\tau)$ [Eq. (114)].

Finally, the total measure will be given by

$$d\nu_{\rm BSW}(\rho) = \tilde{\eta}^{(0)}_{\rm BSW}(\rho) d\rho + \eta^{(\infty)}\delta(\rho) d\rho \quad (116)$$

with $\tilde{\eta}^{(0)}_{\rm BSW}(\rho)$ given in Fig. 3 and the constant $\eta^{(\infty)}$ is given by Eq. (113).

The conclusion is that the BSW *Ansatz* for the IW function is consistent. It satisfies the theoretical criteria since both pieces of the measure (116), $\tilde{\eta}^{(0)}_{\rm BSW}(\rho) d\rho$ and $\eta^{(\infty)}\delta(\rho) d\rho$, are positive. Therefore, the BSW *Ansatz* is thus valid for any value of c. However, this conclusion is only based on numerical calculation. We do not have a complete proof at this time.

E. Relativistic harmonic oscillator

The following shape follows from a relativistic quark model with a harmonic oscillator wave function [17]:

$$\xi(w) = \frac{2}{w+1}\exp\left(-\beta\frac{w-1}{w+1}\right) \quad (117)$$

where the parameter β is related to the slope by $\beta = -2\xi'(1) - 1$.

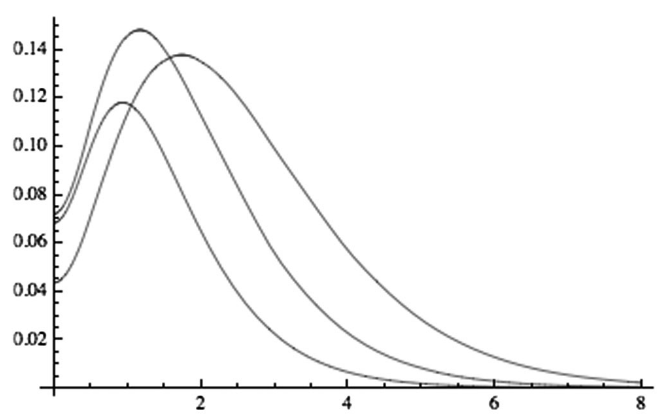

FIG. 7. Fourier transform $\tilde{\eta}^{(0)}_{\rm BSW}(\rho)$ of the function $\eta^{(0)}_{\rm BSW}(\tau)$.

1. Bounds on the derivatives

We find that the first and second derivatives satisfy the bounds of Sec. VII for $\beta \geq \frac{1}{2}$, while the third derivative satisfies the constraint (77) for $\beta > 0.73$.

2. Upper bound on the IW function

The formula (117) does not satisfy the upper bound (96), $\xi(w) \leq (\frac{2}{1+w})^{\frac{3}{2}}$, for any value of β because, as we can see, in the limit of large w it becomes a pole.

3. Positivity of the measure

This *Ansatz* for the IW function does not satisfy the general consistency criterium of Sec. VIII for any value of β.

One finds for the function $\eta(\tau)$ [Eq. (83)]

$$\eta(\tau) = \frac{1}{4\cosh^3(\frac{\tau}{2})} \exp\left[-\beta \tanh^2\left(\frac{\tau}{2}\right)\right] \times [1 + 4\beta + (2 - 4\beta)\cosh(\tau) + \cosh(2\tau)]. \quad (118)$$

This function is unbounded for any value of the parameter β, and therefore the proposal (117) does not satisfy the general criteria.

This means that bounds on some higher derivatives, as can be generalized following Sec. VII, are not satisfied for any given value of β.

F. The IW function in the QCD sum rules approach

The QCD sum rules (QCDSR) approach yields the following result for the IW function, *switching off the hard-gluon radiative corrections* [17,22]:

$$\xi_{\mathrm{QCDSR}}(w) = \frac{\frac{3}{8\pi^2}(\frac{2}{w+1})^2 I(\sigma(w)\frac{\delta}{\Lambda}) + C(\Lambda, w)}{\frac{3}{8\pi^2} I(\frac{\delta}{\Lambda}) + C(\Lambda, 1)} \quad (119)$$

where

$$C(\Lambda, w) = \left\{-\frac{\langle \bar{q}q \rangle}{\Lambda^3}\left[1 - \frac{1}{6}(w-1)\frac{4\lambda^2}{\Lambda^2}\right] + \left(\frac{w-1}{w+1}\right)\frac{\langle \alpha_s GG \rangle}{48\pi\Lambda}\right\} \exp\left[-\frac{(w+1)}{2}\frac{4\lambda^2}{\Lambda^2}\right] \quad (120)$$

and

$$I(x) = \int_0^x dy y^2 e^{-y} = 2 - (x^2 + 2x + 2)\, e^{-x}. \quad (121)$$

On the other hand, the function $\sigma(w)$ satisfies $\sigma(1) = 1$ and is bounded by

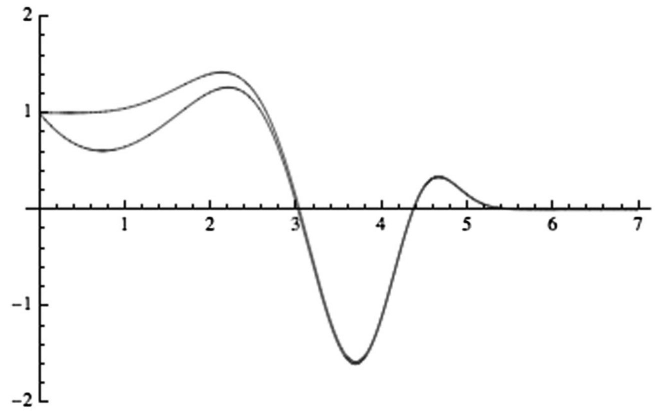

FIG. 8. The function $\eta_{\mathrm{QCDSR}}(\tau)$ [Eq. (83)] for the QCDSR formula (119) and (120) for the IW function in the cases $\sigma(x) = 1$ and $\sigma(x) = \frac{1}{2}(x + 1 - \sqrt{x^2 - 1})$ (upper and lower curves, respectively).

$$\frac{1}{2}(x + 1 - \sqrt{x^2 - 1}) \leq \sigma(x) \leq 1. \quad (122)$$

Let us now compute the functions $\eta(\tau)$ [Eq. (83)] and $d\nu(\rho)/d\rho$ [Eq. (90)].

For the parameters in the above formula we adopt the values within the QCDSR approach [17]: $\delta \simeq 1.9$ GeV, $\Lambda \simeq 0.65$–1.0 GeV, $\lambda \simeq -0.2$ GeV, $\langle \bar{q}q \rangle \simeq -\lambda^3$, $\langle \alpha_s GG \rangle \simeq 0.12$ GeV^4. For the function $\sigma(x)$ we consider the two limiting cases: $\sigma(w) = 1$ and $\sigma(w) = \frac{1}{2}(w + 1 - \sqrt{w^2 - 1})$ (Figs. 8 and 9).

1. Bounds on the derivatives

For the case $\sigma(w) = 1$ we find that the lower bounds for the slope and the curvature [Eqs. (75) and (76)] are satisfied, but the bound on the third derivative (77) is violated. For the case $\sigma(w) = \frac{1}{2}(w + 1 - \sqrt{w^2 - 1})$ we find

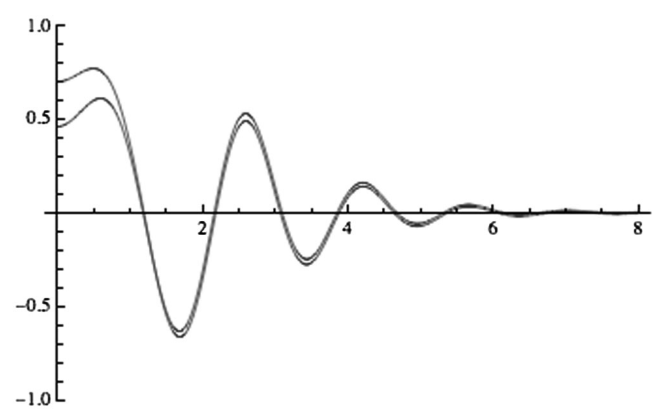

FIG. 9. $d\nu_{\mathrm{QCDSR}}(\rho)/d\rho$ in the cases $\sigma(x) = 1$ and $\sigma(x) = \frac{1}{2}(x + 1 - \sqrt{x^2 - 1})$ (upper and lower curves at low ρ, respectively).

that the derivatives diverge at $w = 1$, and the lower bounds on the derivatives are trivially satisfied.

2. Upper bound on the IW function

We find that in general the QCDSR expression for the IW function (119) does not satisfy the upper bound (99), $\xi(w) \leq (\frac{2}{1+w})^{\frac{3}{2}}$. Although for the limiting case $\sigma(w) = \frac{1}{2}(w+1-\sqrt{w^2-1})$ we find that it is satisfied, the bound is violated for the other limiting case $\sigma(w) = 1$.

3. Positivity of the measure

We see that the function $\eta_{\mathrm{QCDSR}}(\tau)$ remains bounded, but not by 1 (Fig. 8), and we can compute its Fourier transform, which gives the measure $d\nu_{\mathrm{QCDSR}}(\rho)/d\rho$ (Fig. 9).

XI. BAKAMJIAN-THOMAS RELATIVISTIC QUARK MODEL

The Bakamjian-Thomas relativistic quark model [23–26] is a class of models with a fixed number of constituents in which the states are covariant under the Poincaré group. The model relies on an appropriate Lorentz boost of the eigenfunctions of a Hamiltonian describing the hadron spectrum at rest. From now on we use the abbreviation BT for the Bakamjian-Thomas model, not to be confused with the Buchmüller-Tye quarkonium potential model.

We have proposed a formulation of this scheme for the meson ground states [27] and demonstrated the important feature that, in the heavy-quark limit, the current matrix elements, when the current is coupled to the heavy quark, are *covariant*. We have extended this scheme to P-wave excited states [28].

Moreover, these matrix elements in the heavy-quark limit exhibit IW scaling [1]. As demonstrated in Refs. [27,28], given a Hamiltonian describing the spectrum, the model provides an unambiguous result for the Isgur-Wise functions: the elastic $\xi(w)$ [1] and the inelastic to P-wave states $\tau_{1/2}(w)$, $\tau_{3/2}(w)$ [5].

On the other hand, the sum rules in the heavy-quark limit of QCD, like Bjorken [4,5] and Uraltsev SR [6] are analytically satisfied in the model [15,29,30], as well as SR involving higher derivatives of $\xi(w)$ at zero recoil [7–9].

In Ref. [18], we chose the Godfrey-Isgur Hamitonian [31], which gives a very complete description of the light $q\bar{q}$ and heavy $Q\bar{q}$ meson spectra in order to predict within the BT scheme the corresponding IW functions for the ground state and the excited states.

A. Isgur-Wise function and positivity of the measure

Let us now demonstrate that in the Bakamjian-Thomas relativistic quark model, the IW function implies a positive measure independently of the potential.

In this scheme, the IW function is given by the expression

$$\xi(v.v') = \frac{1}{1+v.v'} \int \frac{d\vec{p}}{p^0} \frac{m(v.v'+1) + p.(v+v')}{\sqrt{(p.v+m)(p.v'+m)}} \times \varphi(\sqrt{(p.v')^2 - m^2})^* \varphi(\sqrt{(p.v)^2 - m^2}) \quad (123)$$

with the wave function normalized according to

$$\int \frac{d\vec{p}}{p^0} |\varphi(|\vec{p}|)|^2 = 1. \quad (124)$$

A change in formula (123) with respect to the formula (31) in the original paper [27] is due to using here the scalar product (124) instead of (32).

Let us first transform this expression into a convenient form [Eq. (135) below] that will allow us to compute the measure $d\nu(\rho)$ [Eq. (90)] of the decomposition of $\xi(w)$ in terms of irreducible IW functions $\xi^\rho(w)$ [Eq. (48) or Eq. (49)].

Let us perform a change of integration variables:

$$(p^1, p^2, p^3) \to (p^1, x = v.p, x' = v.p'). \quad (125)$$

In this way, the arguments of φ will not depend on v and v'. Using the invariance of Eq. (123), we express v, v' in terms of the variable τ [Eq. (34)] as follows:

$$v = (\cosh(\tau/2), 0, 0, \sinh(\tau/2)),$$
$$v' = (\cosh(\tau/2), 0, 0, -\sinh(\tau/2)). \quad (126)$$

One has $v.v' = \cosh(\tau)$ and

$$x = \cosh(\tau/2)p^0 - \sinh(\tau/2)p^3,$$
$$x' = \cosh(\tau/2)p^0 + \sinh(\tau/2)p^3. \quad (127)$$

The Jacobian reads

$$\frac{d\vec{p}}{p^0} = \frac{1}{\sinh(\tau)} \frac{1}{p^2} dp^1 dx dx' \quad (128)$$

and Eq. (123) becomes (expression to be corrected below)

$$\xi(\cosh(\tau)) = \frac{1}{\cosh(\tau)+1} \frac{1}{\sinh(\tau)} \int \frac{dp^1}{|p^2|} dx dx' \times \frac{m(\cosh(\tau)+1) + x + x'}{\sqrt{(x+m)(x'+m)}} \times \varphi\left(\sqrt{x'^2 - m^2}\right)^* \varphi\left(\sqrt{x^2 - m^2}\right). \quad (129)$$

Using now Eq. (127) and $(p^2)^2 = (p^0)^2 - (p^3)^2 - (p^1)^2 - m^2$ one gets the integration domain

$$0 \leq (x' - e^{-\tau}x)(e^\tau x - x') - \sinh^2(\tau)m^2, \quad (130)$$

$$|p^1| \leq \frac{\sqrt{(x' - e^{-\tau}x)(e^\tau x - x') - \sinh^2(\tau)m^2}}{\sinh(|\tau|)}, \quad (131)$$

$$p^2 = \pm \frac{\sqrt{(x' - e^{-\tau}x)(e^{\tau}x - x') - \sinh^2(\tau)((p^1)^2 - m^2)}}{\sinh(|\tau|)}. \tag{132}$$

Let us first remark that Eq. (132) gives two values for p^2, and hence the integral (129) has to be multiplied by a factor of 2 since both domains $p^2 \leq 0$ and $p^2 \geq 0$ correspond to the domain of (p^1, x, x') given by Eqs. (130) and (131).

On the other hand, Eqs. (131) and (132) have the form $|p^1| \leq A$, $p^2 = \pm\sqrt{A^2 - (p^1)^2}$, where A can be read from Eq. (131) and hence one can compute the integral $\int \frac{dp^1}{|p^2|} = \int_{-A}^{A} \frac{dp^1}{\sqrt{A^2-(p^1)^2}} = \pi$. Using this value and multiplying Eq. (129) by the missing factor of 2, we have

$$\begin{aligned}\xi(\cosh(\tau)) = 2\pi \frac{1}{\cosh(\tau)+1}\frac{1}{\sinh(|\tau|)} &\times \int \chi(0 \leq (x' - e^{-\tau}x)(e^{\tau}x - x') \\ &- \sinh^2(\tau)m^2)dxdx' \\ &\times \frac{m(\cosh(\tau)+1) + x + x'}{\sqrt{(x+m)(x'+m)}} \\ &\times \varphi\left(\sqrt{x'^2 - m^2}\right)^* \varphi\left(\sqrt{x^2 - m^2}\right)\end{aligned} \tag{133}$$

where the characteristic function $\chi(\mathcal{D})$ of a certain domain $\mathcal{D}$ is defined to be equal to 1 within the domain, and 0 outside.

The equation (133) simplifies if we replace the variables of integration x, x' by

$$x = m\cosh(\alpha), \qquad x' = m\cosh(\alpha') \tag{134}$$

since the constraint on x, x' becomes $0 \leq (\cosh(\tau) - \cosh(\alpha' - \alpha))(\cosh(\alpha' + \alpha) - \cosh(\tau))$, or $|\alpha' - \alpha| \leq |\tau| \leq \alpha' + \alpha$ and Eq. (133) becomes

$$\begin{aligned}\xi(\cosh(\tau)) = 2\pi m^2 \frac{1}{\cosh(\tau)+1}\frac{1}{\sinh(|\tau|)} &\times \int_0^\infty \int_0^\infty \chi(|\alpha' - \alpha| \leq |\tau| \leq \alpha' + \alpha) d\alpha d\alpha' \\ &\times (\cosh(\tau) + \cosh(\alpha) + \cosh(\alpha') + 1) \\ &\times f(\alpha')^* f(\alpha)\end{aligned} \tag{135}$$

where

$$f(\alpha) = \frac{\sinh(\alpha)\,\varphi(m\sinh(\alpha))}{\sqrt{\cosh(\alpha)+1}}. \tag{136}$$

The normalization of the wave function $\varphi(\vec{p})$ [Eq. (124)] translates into the condition for the function $f(\alpha)$:

$$4\pi m^2 \int_0^\infty (\cosh(\alpha)+1)|f(\alpha)|^2 d\alpha = 1. \tag{137}$$

To compute the measure we need to go through Eqs. (79), (83), and (90). We have first that

$$\begin{aligned}\hat{\xi}(\tau) = 2\pi m^2 \mathrm{sgn}(\tau) &\times \int_0^\infty \int_0^\infty d\alpha d\alpha' \chi(|\alpha' - \alpha| \leq |\tau| \leq \alpha' + \alpha) \\ &\times (\cosh(\tau) + \cosh(\alpha) + \cosh(\alpha') + 1) f(\alpha')^* f(\alpha)\end{aligned} \tag{138}$$

and its derivative is given by

$$\begin{aligned}\frac{d}{d\tau}\hat{\xi}(\tau) = 2\pi m^2 \int_0^\infty \int_0^\infty d\alpha d\alpha'\, f(\alpha')^* f(\alpha) &\times ((\delta(|\alpha' - \alpha| - |\tau|) - \delta(\alpha' + \alpha - |\tau|))(\cosh(\tau) \\ &+ \cosh(\alpha) + \cosh(\alpha') + 1) \\ &+ \sinh(|\tau|)\chi(|\alpha' - \alpha| \leq |\tau| \leq \alpha' + \alpha))).\end{aligned} \tag{139}$$

This expression simplifies to

$$\begin{aligned}\frac{d}{d\tau}\hat{\xi}(\tau) = 2\pi m^2 \int_0^\infty \int_0^\infty d\alpha d\alpha' f(\alpha')^* f(\alpha) &\times (4\cosh(\tau/2)\,(\delta(|\alpha' - \alpha| - |\tau|) \\ &- \delta(\alpha' + \alpha - |\tau|))\cosh(\alpha'/2)\cosh(\alpha/2) \\ &+ \sinh(|\tau|)\chi(|\alpha' - \alpha| \leq |\tau| \leq \alpha' + \alpha)))\end{aligned} \tag{140}$$

and finally one gets the function

$$\begin{aligned}\eta(\tau) = 2\pi m^2 \int_0^\infty \int_0^\infty d\alpha d\alpha'\, f(\alpha')^* f(\alpha) &\times (2(\delta(|\alpha' - \alpha| - |\tau|) - \delta(\alpha' + \alpha - |\tau|)) \\ &\times \cosh(\alpha'/2)\cosh(\alpha/2) \\ &+ \sinh(|\tau|/2)\chi(|\alpha' - \alpha| \leq |\tau| \leq \alpha' + \alpha)).\end{aligned} \tag{141}$$

Now we have to compute the Fourier transform (90) of this function. Let us consider the first term of Eq. (141),

$$\begin{aligned}&\int_{-\infty}^{+\infty} e^{i\rho\tau}(\delta(|\alpha' - \alpha| - |\tau|) - \delta(\alpha' + \alpha - |\tau|))d\tau \\ &\quad = -4\sinh(i\rho\alpha)\sinh(i\rho\alpha')\end{aligned} \tag{142}$$

and the second term,

$$\int_{-\infty}^{+\infty} e^{i\rho\tau}\,\sinh(|\tau|/2)\chi(|\alpha'-\alpha|\le|\tau|\le\alpha'+\alpha)$$
$$= 2\left(\frac{1}{i\rho+\frac{1}{2}}\sinh\left(\left(i\rho+\frac{1}{2}\right)\alpha'\right)\sinh\left(\left(i\rho+\frac{1}{2}\right)\alpha\right) - \frac{1}{i\rho-\frac{1}{2}}\sinh\left(\left(i\rho-\frac{1}{2}\right)\alpha'\right)\sinh\left(\left(i\rho-\frac{1}{2}\right)\alpha\right)\right). \tag{143}$$

We finally obtain the following expression for the measure:

$$\frac{d\nu(\rho)}{d\rho} = 2m^2\int_0^\infty\int_0^\infty d\alpha d\alpha' f(\alpha')^* f(\alpha) \times\left(-4\sinh(i\rho\alpha')\cosh(\alpha'/2)\sinh(i\rho\alpha)\cosh(\alpha/2) + \frac{1}{i\rho+\frac{1}{2}}\sinh\left(\left(i\rho+\frac{1}{2}\right)\alpha'\right)\sinh\left(\left(i\rho+\frac{1}{2}\right)\alpha\right) - \frac{1}{i\rho-\frac{1}{2}}\sinh\left(\left(i\rho-\frac{1}{2}\right)\alpha'\right)\sinh\left(\left(i\rho-\frac{1}{2}\right)\alpha\right)\right). \tag{144}$$

What needs to be demonstrated now is that indeed this measure is positive, $\frac{d\nu(\rho)}{d\rho}\ge 0$. To this purpose, let us define two functions, transformed of $f(\alpha)$,

$$g_\pm(\rho) = \int_0^\infty \sinh\left(\left(i\rho\pm\frac{1}{2}\right)\alpha\right)f(\alpha)d\alpha \tag{145}$$

in terms of which Eq. (144) becomes

$$\frac{d\nu(\rho)}{d\rho} = 2m^2\left(|g_+(\rho)+g_-(\rho)|^2 - \frac{g_-(\rho)^* g_+(\rho)}{i\rho+\frac{1}{2}} + \frac{g_-(\rho)g_+(\rho)^*}{i\rho-\frac{1}{2}}\right) \tag{146}$$

and the measure can be expressed as a modulus squared

$$\frac{d\nu(\rho)}{d\rho} = |h(\rho)|^2 \tag{147}$$

where the function $h(\rho)$ is given by the expression

$$h(\rho) = -\sqrt{2}m\frac{1}{\sqrt{\rho^2+\frac{1}{4}}} \times\left(\left(i\rho-\frac{1}{2}\right)g_+(\rho) + \left(i\rho+\frac{1}{2}\right)g_-(\rho)\right). \tag{148}$$

We conclude that the measure $d\nu(\rho)/d\rho$ is positive.

Moreover, one must notice that $\frac{d\nu(\rho)}{d\rho}$ is a function, and therefore it does not contain discrete δ-function terms. This follows from the fact that, according to Eq. (145), $g_\pm(\rho)$ are Fourier transforms of functions that, from Eq. (137), are square integrable and therefore are themselves functions (square integrable).

So, not all possible IW functions $\xi(w)$ are obtained in the BT models. For instance, the so-called BPS limit for the slope $-\xi'(1)=\frac{3}{4}$, leading to the function (54) cannot be obtained.

B. Lorentz-group representation for the BT model

We will begin with a short description of what was exposed in detail for the baryon case $j=0$ [3].

The starting point is an arbitrary unitary representation U of the Lorentz group $SL(2,C)$ in an arbitrary Hilbert space $\mathcal{H}$. To have the meson states and define the Isgur-Wise functions it is moreover necessary that $\mathcal{H}$ is provided with a mass operator M that commutes with the rotations, i.e. with the subgroup SU(2) of $SL(2,C)$. The eigenvalues and eigenvectors of M will give the spectrum and eigenfunctions of the mesons at rest.

The Hilbert space $\mathcal{H}$ will describe the states of the light cloud and M will describe the effect of the heavy quark at rest on the latter. Hence, the states of the light cloud that correspond to the hadrons (for the heavy quark at rest) are the eigenstates of M.

The first step is to determine the irreducible representations of spin j of the restriction of U to SU(2), with their standard bases $|j,\mu\rangle$.

When one has the states of the light cloud of a hadron at rest $v_0=(1,\vec{0})$, the states at arbitrary velocity v are obtained from $U(\Lambda)$, with the Lorentz transformation Λ transforming v_0 into v. More specifically, we need the states $|j,v,\epsilon\rangle$ where the spin is specified by a polarization tensor ϵ, which transform in a covariant way as follows:

$$U(\Lambda)|j,v,\epsilon\rangle = |j,\Lambda v,\Lambda\epsilon\rangle. \tag{149}$$

These states are given by the following formula:

$$|j,v,\epsilon\rangle = \sum_\mu\langle\epsilon^\mu|B_v^{-1}\epsilon\rangle U(B_v)|j,\mu\rangle. \tag{150}$$

Let us emphasize that the tensors ϵ at velocity v constitute a vector space $\mathcal{E}_{j,v}$ of dimension $2j+1$, that $\Lambda\in SL(2,C)$ applies $\mathcal{E}_{j,v}$ on $\mathcal{E}_{j,\Lambda v}$ and that $\mathcal{E}_{j,v_0}$ acts on the representation j of SU(2). Then in Eq. (150) $(\epsilon^\mu)_{-j\le\mu\le j}$ is a standard basis of $\mathcal{E}_{j,v_0}$, one has $B_v^{-1}\epsilon\in\mathcal{E}_{j,v_0}$ and $\langle\epsilon^\mu|B_v^{-1}\epsilon\rangle = (B_v^{-1}\epsilon)_\mu$ are the components of $B_v^{-1}\epsilon$ on this basis. On the other hand, $B_v\in SL(2,C)$ is the boost $v_0\to v$, but Eq. (150) gives the same state $|j,v,\epsilon\rangle$ if B_v is replaced by any $\Lambda\colon\ v_0\to v$.

The second step is therefore to compute the states defined by Eq. (150). Finally, what remains is to compute the scalar products $\langle j', v', \epsilon' | j, v, \epsilon \rangle$. Because of Eq. (149) and the unitarity of U, these scalar products satisfy $\langle j', v', \epsilon' | j, v, \epsilon \rangle = \langle j', \Lambda v', \Lambda\epsilon' | j, \Lambda v, \Lambda\epsilon \rangle$, i.e. are functions of $v, \epsilon, v', \epsilon'$ that are invariant under Lorentz transformations. The Isgur-Wise functions are then the coefficients—which are functions of only $v.v'$—in the expansion of these scalar products on a basis of these invariants.

We will now apply this program to a particular representation of $SL(2,C)$ and obtain in this way the IW functions in the BT model, which were computed elsewhere. We do not need to specify the mass operator M.

1. Description of the Lorentz-group representation

The representation of $SL(2,C)$ that we consider is the one obtained from a spin-1/2 particle by restriction of the Poincaré group to the Lorentz group. The Hilbert space $\mathcal{H}$ is $L^2_{C^2}(H_m, d\mu(p))$ of the functions on the mass hyperboloid $H_m = \{p \in R^4 | p^2 = m^2, p^0 \rangle 0\}$, with values in the space C^2 of the unitary representation $D^{1/2}$ of SU(2) of spin 1/2, with the scalar product

$$\langle \psi' | \psi \rangle = \int d\mu(p) \langle \psi'(p) | \psi(p) \rangle \tag{151}$$

where $d\mu(p)$ is the invariant measure on the mass hyperboloid

$$d\mu(p) = \frac{d^3\vec{p}}{p^0} \tag{152}$$

and the action of $\Lambda \in SL(2,C)$ in $\mathcal{H}$ is given by

$$(U(\Lambda)\psi)(p) = D^{1/2}(\mathbf{R}(\Lambda, p))\psi(\Lambda^{-1}p) \tag{153}$$

where the Wigner rotation $\mathbf{R}(\Lambda, p) \in$SU(2) is

$$\mathbf{R}(\Lambda, p) = B_p^{-1} \Lambda B_{\Lambda^{-1}p} \tag{154}$$

where $B_p \in SL(2,C)$ is the boost $(m, \vec{0}) \to p$.

The check of the group law $U(\Lambda')U(\Lambda) = U(\Lambda'\Lambda)$ follows from a simple calculation, and unitarity comes from the unitarity of $D^{1/2}$ and the invariance of the measure $d\mu(p)$.

2. States j of the light cloud for the heavy quark at rest

We do not have to specify here the mass operator M (for example it can be the Hamiltonian of Godfrey-Isgur [31] in the heavy-quark limit). We need simply to describe the irreducible representations of spin j of the restriction to SU(2), with their standard bases.

For a rotation $\Lambda = R \in$ SU(2), the transformation (152) reduces to

$$(U(R)\psi)(p) = D^{1/2}(R)\psi(R^{-1}p) \tag{155}$$

because the Wigner rotation is simply R

$$\mathbf{R}(R, p) = R. \tag{156}$$

This can be seen by using the following characterization of the boost:

$$\Lambda(m, \vec{0}) = p, \qquad \Lambda = \Lambda^\dagger, \qquad \Lambda > 0 \Leftrightarrow \Lambda = B_p \tag{157}$$

which implies

$$R B_{R^{-1}p} R^{-1} = B_p. \tag{158}$$

Therefore, from Eq. (155), the calculation is reduced to the combination of an orbital angular momentum L with a spin $\frac{1}{2}$ described by a Pauli spinor χ.

For each value of j one has two families of solutions $(L = j \pm \frac{1}{2})$ of opposite parity $(-1)^L$:

$$\varphi^{(L,j,\mu)}(p) = \sqrt{4\pi}\,(Y_L\chi)_j^\mu(\hat{p})\,\varphi^{(L,j)}(|\vec{p}|),$$
$$(Y_L\chi)_j^\mu(\hat{p}) = \sum_{M,\mu'} \langle j, \mu | L, M, \frac{1}{2}, \mu' \rangle Y_L^M(\hat{p})\chi^{\mu'} \tag{159}$$

which depend on the radial function $\varphi^{(L,j)}(|\vec{p}|)$ normalized by

$$\int \frac{d^3\vec{p}}{p^0} |\varphi^{(L,j)}(|\vec{p}|)|^2 = 1. \tag{160}$$

Following Eq. (150), the next step is the calculation of the wave functions of the light cloud $\varphi^{(L,j,v,\epsilon)}(p)$ for a velocity v and a polarization tensor ϵ, starting from the functions $\varphi^{(L,j,\mu)}(p)$ given by Eq. (159). This is enormously simplified if one uses a representation of $SL(2,C)$ that is *equivalent* to the preceding one, expressed in terms of spinors and Dirac matrices.

3. Representation in terms of Dirac spinors and matrices

Let us introduce now the space $\mathcal{H}'$, which is another way of describing the space $\mathcal{H}$, constituted of functions on the hyperboloid H_m, taking values at the point $p \in H_m$ in the subspace of C^4 constituted by the Dirac spinors which satisfy

$$(\not{p} - m)u = 0 \tag{161}$$

since in the BT model the light quark is on-shell. The scalar product is

$$\langle\psi'|\psi\rangle = \int d\mu(p)\,\bar{\psi}'(p)\psi(p) \qquad (\bar{\psi}(p) = \psi^\dagger(p)\gamma^0) \tag{162}$$

and the action of $\Lambda \in SL(2,C)$ is given by

$$(U(\Lambda)\psi)(p) = D(\Lambda)\psi(\Lambda^{-1}p) \tag{163}$$

where $D(\Lambda)$ is the Dirac matrix of the Lorentz transformation Λ:

$$D(\Lambda) = \frac{1}{2}\begin{pmatrix} \Lambda + \Lambda^{\dagger-1} & \Lambda - \Lambda^{\dagger-1} \\ \Lambda - \Lambda^{\dagger-1} & \Lambda + \Lambda^{\dagger-1} \end{pmatrix}. \tag{164}$$

The unitary tranformation V: $\mathcal{H} \to \mathcal{H}'$

$$\psi(p) = (V\varphi)(p) \tag{165}$$

that implements the equivalence is given by

$$(V\varphi)(p) = D(B_p)Q^\dagger\varphi(p), \tag{166}$$

$$(V^{-1}\psi)(p) = QD(B_p^{-1})\psi(p) \tag{167}$$

where the operators Q and $Q^\dagger$ make the connection between the four-component spinors and the two-component ones:

$$Q\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = \chi_1, \qquad Q^\dagger\chi = \begin{pmatrix}\chi\\ 0\end{pmatrix}. \tag{168}$$

Let us collect some identities used to establish the equivalence:

(a) $QQ^\dagger = 1$,

(b) $Q^\dagger Q = \frac{1+\gamma^0}{2}$,

(c) $(\gamma^0 - 1)Q^\dagger = 0$,

(d) $Q^\dagger D^{1/2}(R)Q = Q^\dagger QD(R) \quad [R \in SU(2)]$,

(e) $D(\Lambda')D(\Lambda) = D(\Lambda'\Lambda)$,

(f) $D(\Lambda)\not{a}D(\Lambda^{-1}) = \not{\Lambda a}$,

(g) $D(\Lambda)^\dagger = \gamma^0 D(\Lambda^{-1})\gamma^0$. (169)

Applying V^{-1} to $\psi = V\phi$ one finds, using relation (d) and then relation (a), $(V^{-1}\psi)(p) = \phi(p)$ and therefore $V^{-1}V = 1$. Next, if $\psi = V\phi$ one has $(\not{p} - m)\psi(p) = (\not{p} - m)D(B_p)Q^\dagger\phi(p) = 0$, using relations (d) and (f) and then relation (c).

Applying V to $\phi = V^{-1}\psi$, with $\psi(p)$ satisfying $(\not{p} - m)\psi(p) = 0$, one obtains $(V\phi)(p) = D(B_p)Q^\dagger QD(B_p^{-1})\psi(p) = \psi(p)$, using relation (b) and then relation (f) and finally $(\not{p} - m)\psi(p) = 0$, and one also gets $VV^{-1} = 1$.

To establish the unitarity of V one needs to show that $\langle V^{-1}\psi'|V^{-1}\psi\rangle = \langle\psi'|\psi\rangle$, where on the left one has the scalar product (151), and on the right one has the scalar product (162). One has

$$\langle V^{-1}\psi'|V^{-1}\psi\rangle = \int d\mu(p)\psi'(p)^\dagger D(B_p^{-1})^\dagger P^\dagger PD(B_p^{-1})\psi(p)$$
$$= \int d\mu(p)\bar{\psi'}(p)\psi(p) = \langle\psi'|\psi\rangle \tag{170}$$

using relation (b) and then relations (g) and (f) and finally $(\not{p} - m)\psi(p) = 0$. This establishes also that the scalar product (162) is indeed positive definite.

Finally, it remains to verify that the transformation law $VU(\Lambda)V^{-1}$ in $\mathcal{H}'$, transported from $U(\Lambda)$ in $\mathcal{H}$ [given by Eqs. (153) and (154)] by V is given by Eq. (163):

$$(VU(\Lambda)V^{-1}\psi)(p)$$
$$= D(B_p)Q^\dagger D^{1/2}(\mathbf{R}(\Lambda,p))QD(B^{-1}_{\Lambda^{-1}p})\psi(\Lambda^{-1}p)$$
$$= D(\Lambda)\frac{\not{\Lambda}^{-1}p + m}{2m}\psi(\Lambda^{-1}p) = D(\Lambda)\psi(\Lambda^{-1}p) \tag{171}$$

using relation (d), then Eq. (154) and relation (e), then relations (b) and (f), and finally $(\not{p} - m)\psi(p) = 0$.

4. States j of the light cloud in the Dirac representation

Concerning the states $|j,\mu\rangle$ in $\mathcal{H}'$, they are obtained from the states $|j,\mu\rangle$ in $\mathcal{H}$ given by Eq. (159) by applying the transformation V given by Eq. (166), i.e. $\psi^{(L,j,\mu)}(p) = V\varphi^{(L,j,\mu)}(p)$, which gives

$$\psi^{(L,j,\mu)}(p) = \sqrt{4\pi}D(B_p)\begin{pmatrix}(Y_L\chi)^\mu_j(\hat{p})\\ 0\end{pmatrix}\varphi^{(L,j)}(|\vec{p}|) \tag{172}$$

or

$$\psi^{(L,j,\mu)}(p) = \sqrt{4\pi}\frac{\not{p} + m}{\sqrt{2m(p^0+m)}}\begin{pmatrix}(Y_L\chi)^\mu_j(\hat{p})\\ 0\end{pmatrix}\varphi^{(L,j)}(|\vec{p}|) \tag{173}$$

where we have used

$$D(B_p) = \frac{m + \not{p}\gamma^0}{\sqrt{2m(p^0+m)}}. \tag{174}$$

We will see that the calculation of Eq. (150) is simple when the μ dependence of $|j,\mu\rangle$ appears in the form $(Y_{j-1/2}\chi)^\mu_j$, as is the case with Eq. (173) for $L = j - 1/2$. In the case $L = j + 1/2$ one can also express $\psi^{(L,j,\mu)}$ in terms of $(Y_{j-1/2}\chi)^\mu_j$ by using the identity

$$(Y_{j+1/2}\chi)^{\mu}_{j}(\hat{p}) = -(\vec{\sigma}.\hat{p})(Y_{j-1/2}\chi)^{\mu}_{j}(\hat{p}) \tag{175}$$

and from Eq. (175) one gets, after some algebra,

$$(\not{p}+m)\begin{pmatrix}(Y_{j+1/2}\chi)^{\mu}_{j}(\hat{p})\\ 0\end{pmatrix} = -\sqrt{\frac{p^0+m}{p^0-m}}\gamma_5(\not{p}-m)\begin{pmatrix}(Y_{j-1/2}\chi)^{\mu}_{j}(\hat{p})\\ 0\end{pmatrix} \tag{176}$$

and we have, finally

$$\psi^{(j-1/2,j,\mu)}(p) = \sqrt{4\pi}\frac{\not{p}+m}{\sqrt{2m(p^0+m)}} \times \begin{pmatrix}(Y_{j-1/2}\chi)^{\mu}_{j}(\hat{p})\\ 0\end{pmatrix}\varphi^{(j-1/2,j)}(|\vec{p}|), \tag{177}$$

$$\psi^{(j+1/2,j,\mu)}(p) = -\sqrt{4\pi}\gamma_5\frac{\not{p}-m}{\sqrt{2m(p^0-m)}} \times \begin{pmatrix}(Y_{j-1/2}\chi)^{\mu}_{j}(\hat{p})\\ 0\end{pmatrix}\varphi^{(j+1/2,j)}(|\vec{p}|). \tag{178}$$

5. States for arbitrary velocity and the polarization tensor

We can now go to the second step, the calculation using Eq. (150) of the wave functions $\psi^{(j\pm1/2,j,v,\epsilon)}(p)$ of the states $|j,v,\epsilon\rangle$ using the wave functions $\psi^{(j\pm1/2,j,\mu)}(p)$ of the states $|j,\mu\rangle$, given by Eqs. (177) and (178), with $U(\Lambda)$ given by Eq. (163). We then have to compute

$$\psi^{(j\pm1/2,j,v,\epsilon)}(p) = \sum_{\mu}\langle\epsilon^{\mu}|B_v^{-1}\epsilon\rangle D(B_v)\psi^{(j\pm1/2,j,\mu)}(B_v^{-1}p). \tag{179}$$

To do that, we need some specific information about the polarization tensors.

For half-integer j, they constitute the subspace $\mathcal{E}_{j,v}$ (dependent on the velocity v) of $(C^4)^{\otimes(j-1/2)}\otimes C^4$ of the tensors $\epsilon_{\alpha}^{\mu_1,\ldots,\mu_{j-1/2}}$ that satisfy the following conditions:

(a) symmetry under permutation of the μ indices;

(b) null trace, i.e. $g_{\mu_1,\mu_2}\epsilon_{\alpha}^{\mu_1,\ldots,\mu_{J-1/2}} = 0\left(j\geq\frac{5}{2}\right)$;

(c) $(\gamma_{\mu_1})_{\alpha,\beta}\epsilon_{\beta}^{\mu_1,\ldots,\mu_{J-1/2}}\left(j\geq\frac{3}{2}\right)$;

(d) $v_{\mu_1}\epsilon_{\alpha}^{\mu_1,\ldots,\mu_{J-1/2}}\left(j\geq\frac{3}{2}\right)$;

(e) $(\not{v}-1)_{\alpha,\beta}\epsilon_{\beta}^{\mu_1,\ldots,\mu_{J-1/2}} = 0.$ (180)

The Lorentz transformation of the polarization tensor is the following:

$$(\Lambda\epsilon)_{\alpha}^{\mu_1,\ldots,\mu_{j-1/2}} = \Lambda^{\mu_1}_{\nu_1}\ldots\Lambda^{\mu_{j-1/2}}_{\nu_{j-1/2}}D(\Lambda)_{\alpha,\beta}\epsilon_{\beta}^{\nu_1,\ldots,\nu_{j-1/2}}. \tag{181}$$

One sees that Λ transforms $\mathcal{E}_{j,v}$ into $\mathcal{E}_{j,\Lambda v}$, that $\mathcal{E}_{j,v}$ is obtained from the space at rest $\mathcal{E}_{j,v_0}$ by Λ when $\Lambda v_0 = v$, and that $\mathcal{E}_{j,v_0}$ applies to itself by rotations. Noting that in Eq. (179) the tensors ϵ^{μ} and $B_v^{-1}\epsilon$ are in $\mathcal{E}_{j,v_0}$, it is clear that the sum in Eq. (179) requires one to consider the polarization tensors at zero velocity $v_0 = (1,\vec{0})$.

For the tensors at zero velocity, the condition (d) means that any component with some $\mu = 0$ vanishes, and condition (e) means that any component where the index α is equal to 3 or 4 vanishes. Thus, keeping the other components, $\mathcal{E}_{j,v_0}$ identifies with $(C^3)^{\otimes(j-1/2)}\otimes C^2$ which, from the point of view of rotations, is the tensor product of $j-1/2$ angular momenta equal to 1 and one angular momentum 1/2. Then, conditions (a), (b) and (c) mean simply that this subspace is the one where these angular momenta add to the maximal possible value j.

For the polarization tensors one has, at rest, the following identity:

$$\sum_{\mu}\langle\epsilon^{\mu}|\epsilon\rangle(Y_{j-1/2}\chi)^{\mu}_{j}(\hat{p}') = N_{j-1/2}\frac{1}{\sqrt{4\pi}}\sum_{i_1\ldots i_{j-1/2}}\hat{p}'^{i_1}\ldots\hat{p}'^{i_{j-1/2}}\,\epsilon^{i_1\ldots i_{j-1/2}} \tag{182}$$

with

$$N_L = \frac{\sqrt{(2L+1)!}}{2^{L/2}L!}. \tag{183}$$

To demonstrate these formulas one first has to establish the relation

$$Y_L^M(\hat{p}) = \frac{N_L}{\sqrt{4\pi}}\sum_{i_1\ldots i_L}\hat{p}^{i_1}\ldots\hat{p}^{i_L}(\epsilon^M)^{i_1\ldots i_L} \tag{184}$$

where ϵ^M form a standard basis of polarization tensors (tridimensional at zero velocity) for an integer spin L. These ϵ^M are obtained by coupling L spins equal to 1 to the maximum value L.

The Clebsch-Gordan coefficients that couple two spins J and J' to the maximum value $J+J'$ are given by

$$\langle J,J',M,M'|J+J',M+M'\rangle = \frac{C(J,M)C(J',M')}{C(J+J',M+M')} \tag{185}$$

with

$$C(J,M) = \sqrt{\frac{(2J)!}{(J-M)!(J+M)!}}. \tag{186}$$

Then one gets

$$\epsilon^M = \sum_{m_1+\ldots m_L=M} \frac{C(1,m_1)\ldots C(1,m_L)}{C(L,M)} e^{m_1}\ldots e^{m_L} \quad (187)$$

where the e^m form a standard basis

$$\epsilon^{+1} = -\frac{e^1+ie^2}{\sqrt{2}}, \qquad e^0 = e^3, \qquad \epsilon^{-1} = \frac{e^1-ie^2}{\sqrt{2}}. \quad (188)$$

Let us now consider the generating function of the Y_L^M,

$$\sum_M \frac{L!}{\sqrt{(L-M)!(L+M)!}} Y_L^M(\hat{p}) s^{L+M} = \sqrt{\frac{2L+1}{4\pi}} \left(\frac{\hat{p}^1 - i\hat{p}^2}{2} + \hat{p}^3 s - \frac{\hat{p}^1+i\hat{p}^2}{2} s^2 \right)^L \quad (189)$$

and let us compute the generating function of the rhs of Eq. (184). Using Eq. (187) one finds

$$\sum_M \frac{L!}{\sqrt{(L-M)!(L+M)!}} \frac{N_L}{\sqrt{4\pi}} \times \sum_{i_1\ldots i_L} \hat{p}^{i_1}\ldots\hat{p}^{i_L} (\epsilon^M)^{i_1\ldots i_L} s^{L+M} = \frac{N_L}{\sqrt{4\pi}} \frac{L!}{\sqrt{(2L)!}} \left(\sum\nolimits_m C(1,m)(\hat{p}.e^m) s^{1+m} \right)^L \quad (190)$$

and, taking into account Eq. (188), one has

$$\sum_m C(1,m)(\hat{p}.e^m)s^{1+m} = (\hat{p}.e^{-1}) + \sqrt{2}(\hat{p}.e^0)s + (\hat{p}.e^{+1})s^2 = \sqrt{2}\left(\frac{\hat{p}^1 - i\hat{p}^2}{2} + \hat{p}^3 s - \frac{\hat{p}^1 + i\hat{p}^2}{2} s^2 \right) \quad (191)$$

and one sees that both generating functions are identical provided N_L is given by Eq. (183). This establishes the relation (184) with Eq. (183), and from it one easily obtains Eq. (182). This ends the demonstration of Eqs. (182) and (183).

Taking $p' = B_v^{-1} p$, the identity (182) allows one to easily make the sum over μ in Eq. (179) for $\psi^{(j\pm 1/2,j,\mu)}(p)$ given by Eqs. (177) and (178). Indeed, Eq. (182) gives

$$\sum_\mu \langle \epsilon^\mu | B_v^{-1}\epsilon \rangle (Y_{j-1/2}\chi)_j^\mu(\hat{p}') = N_{j-1/2} \frac{1}{\sqrt{4\pi}} \sum_{i_1,\ldots i_{j-1/2}} \hat{p}^{i_1}\ldots\hat{p}^{i_{j-1/2}} (B_v^{-1}\epsilon)^{i_1\ldots i_{j-1/2}} \quad (192)$$

and using $D(\Lambda')D(\Lambda) = D(\Lambda'\Lambda)$ and

$$(B_v^{-1}p)^0 = p.v, \qquad |\overrightarrow{B_v^{-1}p}| = \sqrt{(p.v)^2 - m^2} \quad (193)$$

one gets for $\psi^{(L,j,v,\epsilon)}(p)$, omitting the spinorial index

$$\psi^{(j-1/2,j,v,\epsilon)}(p) = (-1)^{j-1/2} \frac{N_{j-1/2}}{\sqrt{2m(p.v+m)}} \times (\not{p}+m)\, p_{\mu_1}\ldots p_{\mu_{j-1/2}} \epsilon^{\mu_1\ldots\mu_{j-1/2}} \times \frac{\varphi^{(j-1/2,j)}\left(\sqrt{(p.v)^2-m^2}\right)}{\left(\sqrt{(p.v)^2-m^2}\right)^{j-1/2}} \quad (194)$$

and

$$\psi^{(j+1/2,j,v,\epsilon)}(p) = -(-1)^{j-1/2} \frac{N_{j-1/2}}{\sqrt{2m(p.v-m)}} \times \gamma_5(\not{p}-m) p_{\mu_1}\ldots p_{\mu_{j-1/2}} \epsilon^{\mu_1\ldots\mu_{j-1/2}} \times \frac{\varphi^{(j+1/2,j)}\left(\sqrt{(p.v)^2-m^2}\right)}{\left(\sqrt{(p.v)^2-m^2}\right)^{j-1/2}}. \quad (195)$$

6. *Isgur-Wise functions*

We will now consider three cases of physical interest for which, in the scalar product of states, a single IW function is involved, namely the ground-state elastic case $\{j=1/2, L=0 \to j=1/2, L=0\}$ and the ground state to $L=1$ states $j=1/2,3/2$: $\{j=1/2, L=0 \to j=1/2, L=1\}$, $\{j=1/2, L=0 \to j=3/2, L=1\}$.

Elastic case $j=1/2,\ L=0 \to j=1/2,\ L=0$

For the ground-state IW function $\xi(w)$ one must compute the overlap (for $j=1/2$ the tensor ϵ is just a spinor)

$$\langle \psi^{(0,1/2,v',\epsilon')} | \psi^{(0,1/2,v,\epsilon)} \rangle = \xi(w)\, \bar{\epsilon}'\epsilon \quad (196)$$

where

$$\psi^{(0,1/2,v,\epsilon)}(p) = \frac{1}{\sqrt{2m(p.v+m)}} (\not{p}+m)\epsilon \times \varphi^{(0,1/2)}\left(\sqrt{(p.v)^2-m^2}\right). \quad (197)$$

With the scalar product defined by Eq. (151) and with the measure (152) one obtains

$$\langle \psi^{(0,1/2,v',\epsilon')} | \psi^{(0,1/2,v,\epsilon)} \rangle = \int \frac{d^3\vec{p}}{p^0} \frac{1}{\sqrt{p.v+m}} \frac{1}{\sqrt{p.v'+m}} \bar{\epsilon}'(\not{p}+m)\epsilon \times \varphi^{(0,1/2)}\left(\sqrt{(p.v')^2-m^2}\right)^* \varphi^{(0,1/2)}\left(\sqrt{(p.v)^2-m^2}\right) \quad (198)$$

parametrizing the integrals of Eq. (198) under the form

$$A(w) = \int \frac{d^3\vec{p}}{p^0} F(p, v, v'), \tag{199}$$

$$B(w)v^\mu + C(w)v'^\mu = \int \frac{d^3\vec{p}}{p^0} F(p, v, v')\, p^\mu \tag{200}$$

where

$$F(p, v, v') = \frac{\varphi^{(0,1/2)}(\sqrt{(p.v')^2 - m^2})^* \varphi^{(0,1/2)}(\sqrt{(p.v)^2 - m^2})}{\sqrt{p.v + m}\sqrt{p.v' + m}}. \tag{201}$$

One obtains, for the scalar product (198)

$$\begin{aligned} &\langle \psi^{(0,1/2,v',\epsilon')} | \psi^{(0,1/2,v,\epsilon)} \rangle \\ &= \bar{\epsilon}'[mA(w) + B(w)\not{v} + C(w)\not{v}']\epsilon \\ &= [mA(w) + B(w) + C(w)]\bar{\epsilon}'\epsilon. \end{aligned} \tag{202}$$

On the other hand, by multiplying Eq. (200) by v_μ or v'_μ one can isolate the functions $B(w)$ and $C(w)$ and finally one gets

$$\begin{aligned} \xi(w) = &\frac{1}{w+1} \int \frac{d^3\vec{p}}{p^0} \varphi^{(0,1/2)}\left(\sqrt{(p.v')^2 - m^2}\right)^* \\ &\times \varphi^{(0,1/2)}\left(\sqrt{(p.v)^2 - m^2}\right) \\ &\times \frac{p.(v + v') + m(w + 1)}{\sqrt{(p.v + m)(p.v' + m)}} \end{aligned} \tag{203}$$

i.e. we find Eq. (123).

Case $j = 1/2,\ L = 0 \to j = 1/2,\ L = 1$

In this case the following invariant is involved:

$$\langle \psi^{(1,1/2,v',\epsilon')} | \psi^{(0,1/2,v,\epsilon)} \rangle = \zeta(w)\, \bar{\epsilon}'\gamma_5\epsilon \quad [\zeta(w) = 2\tau_{1/2}(w)] \tag{204}$$

where we quote the two notations that are currently used in the literature.

From Eq. (195), using the expressions for the $L = 1$ states

$$\begin{aligned} \psi^{(1,1/2,v,\epsilon)}(p) = &-\frac{1}{\sqrt{2m(p.v - m)}} \gamma_5(\not{p} - m)\,\epsilon \\ &\times \varphi^{(1,1/2)}\left(\sqrt{(p.v)^2 - m^2}\right) \end{aligned} \tag{205}$$

and computing the scalar product (204), the calculation is very similar to that for the ground-state IW function and we obtain, after some algebra,

$$\begin{aligned} \zeta(w) = &-\frac{1}{w-1} \int \frac{d^3\vec{p}}{p^0} \varphi^{(1,1/2)}\left(\sqrt{(p.v')^2 - m^2}\right)^* \\ &\times \varphi^{(0,1/2)}\left(\sqrt{(p.v)^2 - m^2}\right) \\ &\times \frac{1}{\sqrt{(p.v + m)(p.v' + m)}} \frac{1}{\sqrt{(p.v')^2 - m^2}} \\ &\times [(p.v') + m][(p.v') - (p.v) + m(w - 1)]. \end{aligned} \tag{206}$$

Case $j = 1/2,\ L = 0 \to j = 3/2,\ L = 1$

The following invariant is involved:

$$\begin{aligned} \langle \psi^{(1,3/2,v',\epsilon')} | \psi^{(0,1/2,v,\epsilon)} \rangle &= \tau(w)(\bar{\epsilon}'.v)\epsilon \\ &[\tau(w) = \sqrt{3}\tau_{3/2}(w)] \end{aligned} \tag{207}$$

where we quote the two notations used in the literature.

From Eq. (194) one gets

$$\begin{aligned} \psi^{(1,3/2,v,\epsilon)}(p) = &-\frac{\sqrt{3}}{\sqrt{2m(p.v + m)}} (\not{p} + m)\epsilon.p \\ &\times \frac{\varphi^{(1,3/2)}(\sqrt{(p.v)^2 - m^2})}{\sqrt{(p.v)^2 - m^2}}. \end{aligned} \tag{208}$$

The scalar product (207) is written as

$$\langle \psi^{(1,3/2,v',\epsilon')} | \psi^{(0,1/2,v,\epsilon)} \rangle = \int \frac{d^3\vec{p}}{p^0} p_\mu \bar{\epsilon}'^\mu (\not{p} + m)\epsilon\, F(p, v, v') \tag{209}$$

where now

$$\begin{aligned} F(p, v, v') = &-\frac{\sqrt{3}}{\sqrt{p.v + m}\sqrt{p.v' + m}} \\ &\times \frac{\varphi^{(1,3/2)}(\sqrt{(p.v')^2 - m^2})^*}{\sqrt{(p.v')^2 - m^2}} \\ &\times \varphi^{(0,1/2)}\left(\sqrt{(p.v)^2 - m^2}\right). \end{aligned} \tag{210}$$

We now have to compute the integrals

$$\int \frac{d^3\vec{p}}{p^0} p_\mu F(p, v, v') = A(w)v_\mu + B(w)v'_\mu, \tag{211}$$

$$\begin{aligned} &\int \frac{d^3\vec{p}}{p^0} p_\mu p_\nu\, F(p, v, v') \\ &= C(w)v_\mu v_\nu + D(w) \\ &\quad \times (v_\mu v'_\nu + v'_\mu v_\nu) + E(w)v'_\mu v'_\nu + G(w)g_{\mu\nu}. \end{aligned} \tag{212}$$

Using the Eq. (180) and conditions (c), (d), and (e) one sees from Eq. (207) that the IW function is given in terms of only three functions

$$\tau(w) = C(w) + D(w) + mA(w). \tag{213}$$

By saturating the index μ in Eq. (211) with v^μ and v'^μ one finds the equations

$$\begin{aligned} A(w) + wB(w) &= \int \frac{d^3\vec{p}}{p^0}(v.p)\, F(p,v,v'), \\ wA(w) + B(w) &= \int \frac{d^3\vec{p}}{p^0}(v'.p)\, F(p,v,v') \end{aligned} \tag{214}$$

and by saturating the indices μ, ν in Eq. (212) with the tensors $v^\mu v^\nu, v^\mu v'^\nu, \ldots g^{\mu\nu}$ one gets the set of linear equations

$$\begin{aligned} C(w) + 2wD(w) + w^2E(w) + G(w) &= \int \frac{d^3\vec{p}}{p^0}(v.p)^2\, F(p,v,v'), \\ wC(w) + (w^2+1)D(w) + wE(w) + wG(w) &= \int \frac{d^3\vec{p}}{p^0}(v.p)(v'.p)\, F(p,v,v'), \\ w^2C(w) + 2wD(w) + E(w) + G(w) &= \int \frac{d^3\vec{p}}{p^0}(v'.p)^2\, F(p,v,v'), \\ C(w) + 2wD(w) + E(w) + 4G(w) &= \int \frac{d^3\vec{p}}{p^0} m^2\, F(p,v,v'). \end{aligned} \tag{215}$$

Equations (214) and (215) allow one to compute the different functions $A(w), \ldots G(w)$. From these functions and Eq. (213) one finally gets

$$\begin{aligned} \tau(w) = &-\frac{\sqrt{3}}{2(w-1)(w+1)^2}\int \frac{d^3\vec{p}}{p^0}\varphi^{(1,3/2)}\left(\sqrt{(p.v')^2 - m^2}\right)^* \varphi^{(0,1/2)}\left(\sqrt{(p.v)^2 + m^2}\right) \\ &\times \frac{1}{\sqrt{(p.v+m)(p.v'+m)}}\frac{1}{\sqrt{(p.v')^2 - m^2}}[-3(p.v)^2 + (2w-1)(p.v')^2 \\ &+ 2(2w-1)(p.v)(p.v') + 2(w+1)(w(p.v') - (p.v))m - (w^2-1)m^2]. \end{aligned} \tag{216}$$

Taking into account differences in the definition and normalization conventions, the expressions (206) and (216) are the same asthose found in the previous papers [18,28].

XII. CONCLUSIONS

We have applied the Lorentz-group method to study the Isgur-Wise function in the case of mesons $B \to D^{(*)}\ell\nu$ where the light quark has $j = \frac{1}{2}$. We recovered the constraints obtained previously using the Bjorken-Uraltsev sum-rule method, plus a number of other results.

In particular, we have obtained an integral representation for the IW function in terms of elementary functions and a positive measure. We have inverted this representation, expressing the measure in terms of the IW function. This has allowed us to test whether a given *Ansatz* of the IW function satisfies the Lorentz or, equivalently, the generalized Bjorken-Uraltsev SR constraints.

We have compared a number of phenomenological shapes for the Isgur-Wise function with the obtained theoretical constraints. This has provided explicit illustrations of the method in a rather complete way. The different criteria based on the Lorentz group, i.e. lower limits on the derivatives at zero recoil, positivity of the measure in the inversion formula for the IW function and the upper bound for the whole IW function, have been illustrated by using different models of the IW function.

We have studied a number of models proposed in the literature: exponential shape, "dipole" form, Kiselev *Ansatz*, Bauer-Stech-Wirbel model, relativistic harmonic oscillator, QCD sum rules, Bakamjian-Thomas relativistic quark model, etc. We have shown that the "dipole," the BSW model and the BT model satisfy the theoretical constraints.

The case of the QCDSR result is particularly interesting because of its link to general principles. In the limit in which the condensates are disregarded, the predicted dipole shape satisfies all the constraints. However, switching on the condensates spoils this nice feature. Of course, one can argue that the OPE has been limited to the lower-dimension condensates. Our results show the interesting feature that in this framework one could obtain incorrect results by keeping only the lowest-dimension operators. Our study in the heavy-quark limit does not take into account the radiative corrections, which is consistent with the considered theoretical hypothesis—the factorization between the heavy-quark matrix element and the light-could overlap—in which the methods of the present paper can hold.

We have studied in detail the Bakamjian-Thomas relativistic quark model applied to mesons in the heavy-quark limit. To this aim we have described the Lorentz-group representation that underlies the model. We formulated the form of the wave functions of the light cloud for all quantum numbers, and provided the formalism to obtain the IW functions by scalar products of these states. Consistently, the elastic IW function in this model satisfies all the Lorentz-group criteria, and this feature holds for any explicit form of the Hamiltonian describing the meson spectrum at rest. Completeness in the Hilbert space implies the strong result that the full set of Bjorken-like heavy-quark-limit sum rules is automatically satisfied in the BT model at infinite mass.

In conclusion, using a method based on the Lorentz group, completely equivalent to that of the generalized Bjorken-Uraltsev sum rules, we have obtained in this paper strong constraints on the Isgur-Wise function for the ground-state mesons.

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