

Is $f_0(1710)$ a glueball?Stanislaus Janowski,¹ Francesco Giacosa,^{1,2} and Dirk H. Rischke¹¹*Institute for Theoretical Physics, Goethe University,**Max-von-Laue-Straße 1, 60438 Frankfurt am Main, Germany*²*Institute of Physics, Jan Kochanowski University, 25-406 Kielce, Poland*

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We study the three-flavor chirally and dilatation invariant extended linear sigma model with (pseudo) scalar and (axial-)vector mesons as well as a scalar dilaton field whose excitations are interpreted as a glueball. The model successfully describes masses and decay widths of quark-antiquark mesons in the low-energy region up to 1.6 GeV. Here we study in detail the vacuum properties of the scalar-isoscalar $J^{PC} = 0^{++}$ channel and find that (i) a narrow glueball is only possible if the vacuum expectation value of the dilaton field is (at tree level) quite large (i.e. larger than what lattice QCD and QCD sum rules suggest) and (ii) only solutions in which $f_0(1710)$ is predominantly a glueball are found. Moreover, the resonance $f_0(1370)$ turns out to be mainly $(\bar{u}u + \bar{d}d)/\sqrt{2}$ and thus corresponds to the chiral partner of the pion, while the resonance $f_0(1500)$ is mainly $\bar{s}s$.

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I. INTRODUCTION

Glueballs are mesons which are solely made of gluons. The prediction that glueballs exist dates back to the origin of quantum chromodynamics (QCD) [1]: gluons carry color charge and interact strongly with each other, so it is natural to expect that they form bound states. This expectation is confirmed by numerous simulations of lattice QCD (see e.g. Refs. [2,3] and Refs. therein), in which a full spectrum of glueballs with different quantum numbers J^{PC} (some of which are exotic) has been obtained. Although up to now no glueball state has been unambiguously identified, the search for glueballs will be in the focus of the future PANDA experiment at FAIR [4]. The hope is that the existence of (at least some of the foreseen) glueballs will be ultimately established.

The lightest glueball is predicted by lattice QCD to be a scalar state with a mass of about 1.6 GeV [2,3]. The search for this state has been, and still is, in the center of vivid activity in the framework of low-energy QCD. This state is also important because it is related to two basic phenomena of QCD: the anomalous breaking of dilatation invariance and the generation of the gluon condensate. In a widely studied phenomenological scenario two scalar-isoscalar quarkonia, $\bar{n}n = (\bar{u}u + \bar{d}d)/\sqrt{2}$ and $\bar{s}s$, and one bare glueball state mix and form the scalar resonances $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ [5–9]. Our aim is to investigate this system by using a three-flavor chiral effective approach which we describe in the following.

In Refs. [10–15] an effective model of hadrons, denoted as the extended linear sigma model (eLSM), has been developed. The mesonic part of the eLSM contains (pseudo)scalar and (axial-)vector states as well as a scalar dilaton/glueball field and is built under the requirements of chiral symmetry and dilatation invariance. Chiral symmetry

is broken explicitly (by the current quark masses) and, more importantly, spontaneously (by the chiral condensate). Dilatation invariance is explicitly broken by a logarithmic dilaton potential which mimics the trace anomaly of QCD, according to which gluonic quantum fluctuations give rise to the fundamental energy scale of QCD, Λ_{QCD} . The dilaton field, named G , develops a nonzero vacuum expectation value (vev) G_0 and, in turn, the fluctuations around the minimum represent the scalar glueball.

In this work we investigate the phenomenology of the scalar glueball in the eLSM. To this end, we extend both Ref. [12] and Ref. [13]. In Ref. [12] the dilaton has been first introduced in the eLSM but the model has been investigated only for the case of two flavors $N_f = 2$. In Ref. [13] a more complete study of the vacuum phenomenology has been performed in the three-flavor ($N_f = 3$) version of the eLSM and a good agreement with experimental data listed in Ref. [16] for both masses and decay widths has been achieved. However, the dilaton, although formally present in order to guarantee dilatation invariant interactions, was not included when calculating mixing in the scalar-isoscalar sector and the corresponding decays. In the present paper we close this gap: for $N_f = 3$ the scalar field G naturally couples to nonstrange and strange mesonic fields and, in particular, mixes with two scalar-isoscalar quarkonia states.

There are two important and quite general aspects of the physics of the scalar glueball, which need to be discussed separately.

1. Is the scalar glueball broad or narrow? This question is extremely important for the phenomenology and the assignment of the scalar glueball to an existing resonance. Yet, conflicting arguments exist: (i) In the large- N_c limit the glueball is predicted to be *narrow*. Namely, the decay of a bare glueball into two quarkonia (e.g. $G \rightarrow \pi\pi$) scales as

N_c^{-2} (for comparison, the decay of a quark-antiquark state into two quark-antiquark states scales as N_c^{-1}). Since the large- N_c limit is phenomenologically successful, the quite narrow resonances $f_0(1500)$ and $f_0(1710)$ are prime candidates for glueball states. (ii) In Ref. [17] it is shown that the decay $G \rightarrow \pi\pi$ depends on the vev G_0 of the dilaton field as G_0^{-2} . The values of G_0 can be related to the gluon condensate of QCD by assuming that the trace anomaly is saturated by the dilaton field. Using the values of the gluon condensate from either QCD sum rules or lattice QCD calculations, it turns out that the width of the decay $G \rightarrow \pi\pi$ is *very* large ($\gtrsim 500$ MeV). The authors of Ref. [17] conclude that the search for the scalar glueball may be very difficult (if not impossible) if this state is too broad. [Note that a wide glueball was also discussed in Refs. [18–20]].

In Fig. 1 we anticipate our result for the decay of a (bare, i.e. unmixed) scalar glueball into two pions as function of the vev G_0 : for values of G_0 which belong to the range obtained by QCD sum rules and lattice QCD (the vertical band), $G \rightarrow \pi\pi$ is also *very* large, in complete agreement with Ref. [17]. The two curves correspond to the cases with and without (axial-)vector states. One can see that the inclusion of (axial-)vector degrees of freedom reduces the decay width, but this effect is not sufficient to make it small enough (when G_0 is inside the vertical band). When mixing is taken into account, due to interference phenomena the strong coupling of G to pions may be reduced for the physical resonances. Yet, since the quarkonium state $\bar{n}n$ is also expected to be broad, it is not possible to obtain two narrow resonances $f_0(1500)$ and $f_0(1710)$ in a three-body mixing scenario. Thus, we realize that we cannot obtain a good description of the phenomenology of the states $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ if we impose that G_0 corresponds to the range given by QCD sum rules or lattice QCD.

2. *Assuming that the scalar glueball is narrow, is $f_0(1500)$ or $f_0(1710)$ mostly gluonic?* A consensus has

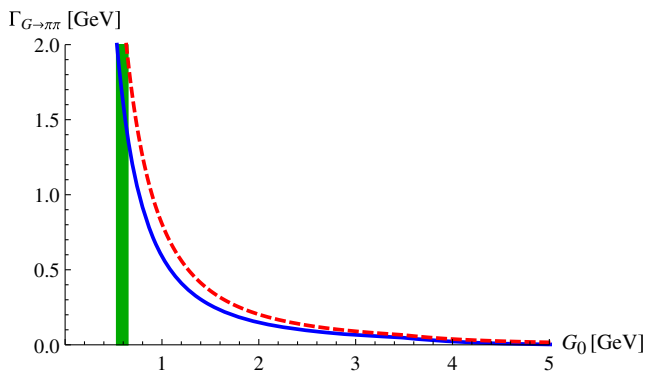


FIG. 1 (color online). Decay of the pure glueball field into $\pi\pi$ for a bare glueball mass $m_G = 1525$ MeV. Dashed (red) line: (Axial-)vector mesons are decoupled ($Z_\pi = 1$). Solid (blue) line: (Axial-)vector mesons are included ($Z_\pi \neq 1$).

grown that the light scalar mesons $f_0(500)$, $f_0(980)$, $a_0(980)$, $K_0^*(800)$ are *not* quark-antiquark states. The possible assignments are tetraquark or molecular states [21,22]. As a consequence, the scalar quark-antiquark states are located above 1 GeV: $a_0(1450)$ and $K_0^*(1430)$ represent the isovector and isodoublet $\bar{q}q$ states with $J^{PC} = 0^{++}$. This picture has been confirmed in the framework of the eLSM [10–13]. In particular, in Ref. [13] a fit to a variety of experimental data has shown that the scalar states lie between 1 and 2 GeV. Then, if the glueball is a narrow state, the main question is which of the two resonances $f_0(1500)$ and $f_0(1710)$ contains the largest gluonic amount. In our previous work [12] two solutions were found, one in which $f_0(1500)$ and one in which $f_0(1710)$ was predominantly a glueball (the former case was slightly favored). Here, we re-analyze this issue in a full three-flavor study of the eLSM and, quite remarkably, our outcome is now unique: we find that $f_0(1710)$ is predominantly the gluonic state. This result is in agreement with the original lattice study of Ref. [23], with (some of) the phenomenological solutions of Refs. [6,24] and, interestingly, with the recent lattice study of J/ψ decays in Ref. [25]. It should be stressed that the solution in which $f_0(1710)$ is a glueball is obtained only if the value of G_0 is quite large ($\gtrsim 1$ GeV). In turn, if this assignment is correct, this suggests that either the gluon condensate should be larger than what was previously believed or the dilaton field is not the only composite field which is responsible for the trace anomaly. Additional fields may change the values of the parameters in the dilaton potential and thus help to reconcile the value of G_0 with lattice QCD and QCD sum rules.

This paper is organized as follows. In Sec. II we present the chiral Lagrangian of our model: the eLSM with a scalar glueball. In Sec. III we discuss our results for the masses and decay widths as well as the three-body mixing of the resonances $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$. Finally, in Sec. IV we present our conclusions and an outlook for future work.

II. THE MODEL

As mentioned in the introduction, the aim of this work is to study the structure of the three scalar-isoscalar resonances $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$. To this end we use the chiral Lagrangian of the eLSM developed in Refs. [10–13].

A. The dilaton potential

An essential feature of the eLSM is dilatation invariance together with its anomalous breaking, which we briefly discuss in the following. The pure Yang-Mills (YM) Lagrangian reads:

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} \quad \text{with} \quad (1)$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c,$$

where A_μ^a is the gluon field with $a = 1, \dots, N_c^2 - 1 = 8$, $G_{\mu\nu}^a$ is the gluon field-strength tensor, and g is the QCD coupling constant. This Lagrangian is classically invariant under dilatation transformations $x^\mu \rightarrow \lambda^{-1}x^\mu$, together with $A_\mu^a(x) \rightarrow \lambda A_\mu^a(\lambda x)$. However, when quantum fluctuations are included and renormalization is carried out, the coupling constant becomes $g \rightarrow g(\mu)$, where $g(\mu)$ is the renormalized running coupling which is a function of the energy scale μ . As a consequence, the divergence of the dilatation (Noether) current does not vanish:

$$\partial_\mu J_{\text{YM,dil}}^\mu = T_{\text{YM},\mu}^\mu = \frac{\beta(g)}{4g} G_{\mu\nu}^a G^{a,\mu\nu} \neq 0, \quad (2)$$

where $T_{\text{YM}}^{\mu\nu}$ is the energy-momentum tensor of the YM Lagrangian and the β -function is given by $\beta(g) = \partial g / \partial \ln \mu$. At the one-loop level $\beta(g) = -bg^3$ with $b = 11N_c / (48\pi^2)$. This implies $g^2(\mu) = [2b \ln(\mu/\Lambda_{\text{YM}})]^{-1}$, where $\Lambda_{\text{YM}} \approx 200$ MeV is the YM scale (dimensional transmutation). The expectation value of the trace anomaly does not vanish and represents the so-called gluon condensate

$$\langle T_{\text{YM},\mu}^\mu \rangle = -\frac{11N_c}{48} \left\langle \frac{\alpha_s}{\pi} G_{\mu\nu}^a G^{a,\mu\nu} \right\rangle = -\frac{11N_c}{48} C^4, \quad (3)$$

where

$$C^4 \approx (0.3\text{--}0.6 \text{ GeV})^4. \quad (4)$$

The numerical values have been obtained through QCD sum rules (lower range of the interval) [26] and lattice QCD simulations (higher range of the interval) [27,28]. In particular, in Ref. [28] the value $C \approx 0.61$ GeV has been found in the quenched approximation.

At the composite level one can build an effective theory of the YM sector of QCD by introducing a scalar dilaton field G which describes the trace anomaly. The dilaton Lagrangian reads [29,30]

$$\mathcal{L}_{\text{dil}} = \frac{1}{2} (\partial_\mu G)^2 - \frac{1}{4} \frac{m_G^2}{\Lambda^2} \left(G^4 \ln \left| \frac{G}{\Lambda} \right| - \frac{G^4}{4} \right). \quad (5)$$

The minimum G_0 of the dilaton potential is realized for $G_0 = \Lambda$. Upon shifting $G \rightarrow G_0 + G$, a particle with mass m_G emerges, which is interpreted as the scalar glueball. The numerical value has been evaluated in lattice QCD and reads $m_G^{\text{lat}} \approx (1.5\text{--}1.7)$ GeV [3]. The logarithmic term of the potential explicitly breaks the invariance under a dilatation transformation. The divergence of the corresponding current reads

$$\partial_\mu J_{\text{dil}}^\mu = T_{\text{dil},\mu}^\mu = -\frac{1}{4} \frac{m_G^2}{\Lambda^2} G^4 \longrightarrow -\frac{1}{4} m_G^2 \Lambda^2, \quad (6)$$

where for the last expression we have set G equal to the minimum $G_0 = \Lambda$ of the potential.

If we now require that the dilaton field saturates the trace of the dilatation current, we equate Eq. (3) with Eq. (6) and obtain:

$$\Lambda \stackrel{!}{=} \frac{\sqrt{11} C^2}{2m_G}. \quad (7)$$

Using $m_G \approx (1.5\text{--}1.7)$ GeV and $C \approx 0.61$ GeV [28] implies $\Lambda \approx 0.4$ GeV. [Note that Eq. (5) describes only the gluonic sector of QCD, thus the comparison with lattice-QCD results should be done in the quenched approximation, such as the one of Ref. [28]]. As already shown in Fig. 1, if the value $\Lambda \approx 0.4$ GeV would hold, the glueball would be too wide when the coupling to ordinary quarkonia mesons is switched on. A phenomenology with a narrow glueball is possible only if $\Lambda \gtrsim 1$ GeV, see Sec. III and the related discussion.

B. The eLSM Lagrangian

The Lagrangian of the eLSM is built by requiring global chiral $U(3)_R \times U(3)_L$ symmetry, dilatation invariance, as well as the discrete symmetries charge conjugation C , parity P , and time reversal T :

$$\begin{aligned} \mathcal{L} = & \mathcal{L}_{\text{dil}} + \text{Tr}[(D^\mu \Phi)^\dagger (D_\mu \Phi)] - \text{Tr} \left\{ \left[m_0^2 \left(\frac{G}{G_0} \right)^2 + E \right] \Phi^\dagger \Phi \right\} - \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 - \lambda_2 \text{Tr}[(\Phi^\dagger \Phi)^2] + c_1 (\det \Phi - \det \Phi^\dagger)^2 \\ & + \text{Tr}[H(\Phi^\dagger + \Phi)] + \text{Tr} \left\{ \left[\frac{m_1^2}{2} \left(\frac{G}{G_0} \right)^2 + \Delta \right] (L_\mu^2 + R_\mu^2) \right\} - \frac{1}{4} \text{Tr}(L_{\mu\nu}^2 + R_{\mu\nu}^2) + \frac{h_1}{2} \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(L_\mu L^\mu + R_\mu R^\mu) \\ & + h_2 \text{Tr}(\Phi^\dagger L_\mu L^\mu \Phi + \Phi R_\mu R^\mu \Phi^\dagger) + 2h_3 \text{Tr}(\Phi R_\mu \Phi^\dagger L^\mu) + \dots \end{aligned} \quad (8)$$

where $D^\mu \Phi = \partial^\mu \Phi - ig_1(L^\mu \Phi - \Phi R^\mu)$ is the covariant derivative and

$$\Phi = \sum_{i=0}^8 (S_i + iP_i) T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\sigma_N + a_0^0 + i(\eta_N + \pi^0)}{\sqrt{2}} & a_0^+ + i\pi^+ & K_0^{*+} + iK^+ \\ a_0^- + i\pi^- & \frac{\sigma_N - a_0^0 + i(\eta_N - \pi^0)}{\sqrt{2}} & K_0^{*0} + iK^0 \\ K_0^{*-} + iK^- & \bar{K}_0^{*0} + i\bar{K}^0 & \sigma_S + i\eta_S \end{pmatrix} \quad (9)$$

is the multiplet of the ordinary scalar (S) and pseudoscalar (P) mesons including the bare nonstrange $\sigma_N \cong (\bar{u}u + \bar{d}d)/\sqrt{2}$ and strange $\sigma_S \cong \bar{s}s$ fields. Under $U(3)_R \times U(3)_L$ chiral transformations Φ transforms as $\Phi \rightarrow U_L \Phi U_R^\dagger$. The quantities $L^\mu = \sum_{i=0}^8 (V_i^\mu + A_i^\mu) T_i$ and $R^\mu = \sum_{i=0}^8 (V_i^\mu - A_i^\mu) T_i$ are the left- and the right-handed vector matrices, which are linear combinations of the vector and axial-vector multiplets V^μ and A^μ . Under chiral transformations, $L_\mu \rightarrow U_L L_\mu U_L^\dagger$ and $R_\mu \rightarrow U_R R_\mu U_R^\dagger$.

The assignment of the quark-antiquark fields of our model to the resonances listed by the PDG [16] is as follows. (i) In the pseudoscalar sector we assign the fields $\bar{\pi}$ and K to the physical pion isotriplet and the kaon isodoublets [16]. The bare fields $\eta_N \cong i(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d)/\sqrt{2}$ and $\eta_S \cong i\bar{s}\gamma_5 s$ are the nonstrange and strange contributions to the physical states η and $\eta'(958)$ [16]

$$\begin{aligned} \eta &= \eta_N \cos \varphi_\eta + \eta_S \sin \varphi_\eta, \\ \eta' &= -\eta_N \sin \varphi_\eta + \eta_S \cos \varphi_\eta, \end{aligned} \quad (10)$$

where $\varphi_\eta \approx -44.6^\circ$ is the pseudoscalar mixing angle [13]. (ii) As shown in the comprehensive study of Ref. [13], the scalar $\bar{q}q$ states lie above 1 GeV [in turn, the scalar states below 1 GeV should not be interpreted as $\bar{q}q$ states but as tetraquarks and/or mesonic molecular states, see Refs. [21,22,31,32]]. Hence, in the scalar sector we assign the field \bar{a}_0 to the physical isotriplet resonance $a_0(1450)$ and the scalar kaon isodoublet field K_0^* to the resonance $K_0^*(1430)$ [16]. The least clear assignment occurs in the scalar-isoscalar channel because in the region from 1 to 2 GeV there are three resonances which are listed in Ref. [16]: $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$. Only two of them can be interpreted as predominantly $\bar{q}q$ states while the third one is probably predominantly a glueball state G . The determination of the mixing matrix is carried out later. (iii) The assignment of the (axial-)vector fields of the model is straightforward and is presented, together with the corresponding multiplets, in Appendix A. Notice that there are other interpretations in which $f_0(1370)$ and $f_0(1710)$ are described as dynamically generated resonances out of vector-vector interactions [33].

Chiral symmetry is spontaneously broken when $m_0^2 < 0$. Dilatation symmetry is explicitly broken by the logarithmic term in Eq. (5). The quantity G_0 is the vev of the G field, which, in the full version of the model (8), is (slightly) larger than Λ appearing in Eq. (5). Moreover, both chiral and dilatation transformations are explicitly broken by the

terms which describe the nonzero bare quark masses of the mesons, which are proportional to $H = \text{diag}\{h_N, h_N, h_S\}$, $\Delta = \text{diag}\{0, 0, \delta_S\}$, and $E = \text{diag}\{0, 0, \epsilon_S\}$. Note that the latter term was not included in Ref. [13] because it represents a next-to-leading order correction in the expansion in terms of the quark mass. However, due to the fact that the current mass of the strange quark is not small, this term is important in our study of the quark-antiquark scalar state σ_S . The axial anomaly is described by the determinant term which is invariant under $SU(3)_R \times SU(3)_L$ but breaks $U_A(1)$. This term which breaks dilatation symmetry and originates also from the gluon dynamics is responsible for the large mass splitting of η and η' . Note that in the chiral limit (in which $H = \Delta = E = 0$) and neglecting the chiral anomaly, the requirement of dilatation invariance and analyticity in G ensures that only a finite number of terms is allowed in our chiral Lagrangian (8).

Finally, the dots in Eq. (8) indicate further terms which do not affect the calculations of this work and are therefore neglected, and additional degrees of freedom which can be studied in the framework of the eLSM, e.g. a pseudoscalar glueball \tilde{G} , $J^{PC} = 0^{-+}$, which couples to the ordinary scalar and pseudoscalar mesons. The origin of the corresponding chiral Lagrangian, $\mathcal{L}_{\tilde{G}} = ic_{\tilde{G}\Phi} \tilde{G}(\det \Phi - \det \Phi^\dagger)$, comes from the axial anomaly in the pseudoscalar-isoscalar sector, see details and predictions for branching ratios in Ref. [34]. An extension of the eLSM to four flavors allows to describe quite successfully charmed meson masses and decays as well [35].

C. Lagrangian, masses, and mixing matrix of the scalar-isoscalar fields

The three scalar-isoscalar fields $\sigma_N \cong (\bar{u}u + \bar{d}d)/\sqrt{2}$, $\sigma_S \cong \bar{s}s$, and G are the only fields of the model with quantum numbers of the vacuum, $J^{PC} = 0^{++}$. In order to study the vev's and the mixing behavior of these fields we set all the other fields of the chiral Lagrangian (8) to zero and obtain the scalar-isoscalar Lagrangian

$$\begin{aligned} \mathcal{L}_{\sigma_N \sigma_S G} &= \mathcal{L}_{\text{dil}} + \frac{1}{2}(\partial_\mu \sigma_N)^2 + \frac{1}{2}(\partial_\mu \sigma_S)^2 - \frac{m_0^2}{2} \left(\frac{G}{G_0} \right)^2 \\ &\quad \times (\sigma_N^2 + \sigma_S^2) - \lambda_1 \left(\frac{\sigma_N^2}{2} + \frac{\sigma_S^2}{2} \right)^2 \\ &\quad - \frac{\lambda_2}{4} \left(\frac{\sigma_N^4}{2} + \sigma_S^4 \right) + h_{\sigma_N} \sigma_N + h_{\sigma_S} \sigma_S - \frac{1}{2} \epsilon_S \sigma_S^2. \end{aligned} \quad (11)$$

Now we perform the shifts of the $J^{PC} = 0^{++}$ fields by their vev's, $\sigma_N \rightarrow \sigma_N + \phi_N$, $\sigma_S \rightarrow \sigma_S + \phi_S$, and $G \rightarrow G + G_0$, in order to obtain their bare masses and the bilinear mixing terms $\propto \sigma_N \sigma_S$, $\propto \sigma_N G$, and $\propto \sigma_S G$. The bare masses of the nonstrange and strange $\bar{q}q$ fields read

$$\begin{aligned} m_{\sigma_N}^2 &= C_1 + 2\lambda_1 \phi_N^2 + \frac{3}{2} \lambda_2 \phi_N^2, \\ m_{\sigma_S}^2 &= C_1 + 2\lambda_1 \phi_S^2 + 3\lambda_2 \phi_S^2 + \epsilon_S, \end{aligned} \quad (12)$$

where

$$C_1 = m_0^2 + \lambda_1 (\phi_N^2 + \phi_S^2) \quad (13)$$

is a constant [13] (see Table I),

$$\phi_N = Z_\pi f_\pi, \quad \phi_S = \frac{2Z_K f_K - \phi_N}{\sqrt{2}}, \quad (14)$$

are the condensates of the nonstrange and strange quark-antiquark states, where $Z_{\pi/K}$ are the wave-function renormalization constants given in Eq. (A7) in Appendix A, and $f_{\pi/K}$ are the vacuum decay constants. The bare mass of the scalar glueball reads

$$M_G^2 = \frac{m_0^2}{G_0^2} (\phi_N^2 + \phi_S^2) + \frac{m_G^2 G_0^2}{\Lambda^2} \left(1 + 3 \ln \left| \frac{G_0}{\Lambda} \right| \right). \quad (15)$$

Note that the bare glueball mass also depends on the quark condensates ϕ_N and ϕ_S , but correctly reduces to m_G in the limit $m_0^2 = 0$ (when quarkonia and the glueball decouple). When quarkonia couple to the glueball, $m_0^2 \neq 0$, the vev G_0 is given by the equation

$$-\frac{m_0^2 \Lambda^2}{m_G^2} (\phi_N^2 + \phi_S^2) = G_0^4 \ln \left| \frac{G_0}{\Lambda} \right|. \quad (16)$$

This equation shows that $G_0 \gtrsim \Lambda$. For large values of Λ one has $G_0 \simeq \Lambda$, while for small values G_0 can be somewhat larger than Λ . This is indeed the reason why the band in Fig. 1 is slightly shifted to the right when compared to the range of Λ determined from Eqs. (4) and (7).

The contribution to the tree-level potential, which is of second order in the fields, reads

TABLE I. Values of the parameters from Ref. [13].

Parameter	Value	Parameter	Value
C_1	$-0.918 \times 10^6 \text{ MeV}^2$	C_2	$0.413 \times 10^6 \text{ MeV}^2$
c_1	$450 \times 10^{-6} \text{ MeV}^{-2}$	δ_S	$0.151 \times 10^6 \text{ MeV}^2$
g_1	5.84	λ_2	68.3
h_2	9.88	h_3	3.87
ϕ_N	164.6 MeV	ϕ_S	126.2 MeV

$$\begin{aligned} V^{(2)} &= \frac{1}{2} \Sigma^T M \Sigma, \\ M &\equiv \begin{pmatrix} m_{\sigma_N}^2 & 2\lambda_1 \phi_N \phi_S & 2m_0^2 \phi_N G_0^{-1} \\ 2\lambda_1 \phi_N \phi_S & m_{\sigma_S}^2 & 2m_0^2 \phi_S G_0^{-1} \\ 2m_0^2 \phi_N G_0^{-1} & 2m_0^2 \phi_S G_0^{-1} & M_G^2 \end{pmatrix}, \\ \Sigma &\equiv \begin{pmatrix} \sigma_N \\ \sigma_S \\ G \end{pmatrix}. \end{aligned} \quad (17)$$

Following the usual diagonalization procedure, an orthogonal matrix B is introduced such that the matrix $M' = BMB^T$ is diagonal. As a consequence, B links the bare scalar-isoscalar fields to the physical resonances:

$$\begin{pmatrix} f_0(1370) \\ f_0(1500) \\ f_0(1710) \end{pmatrix} \equiv \Sigma' = \begin{pmatrix} \sigma'_N \\ \sigma'_S \\ G' \end{pmatrix} = B \Sigma = B \begin{pmatrix} \sigma_N \\ \sigma_S \\ G \end{pmatrix}. \quad (18)$$

D. Parameters of the model

In Ref. [13] a global fit was performed, in which 21 experimental quantities were fitted to eleven parameters of the eLSM. Due to their ambiguous status, scalar-isoscalar mesons were not part of the fit, which allowed to exclude the coupling constants λ_1 and h_1 from the fit. Since we are now explicitly interested in the scalar-isoscalar resonances, these two coupling constants must be considered, which brings the number of parameters to 13. Furthermore, in the fit of Ref. [13], the glueball was considered to be frozen. This approximation is justifiable in the large- N_c limit because the coupling of one scalar glueball to m ordinary mesons scales as $\sim N_c^{-m/2}$. In this study the scalar glueball is present, which introduces two additional parameters Λ and m_G , so that we have 15 parameters. Moreover, there is an additional mass term $\propto \epsilon_S$ not present in the study of Ref. [13], and thus our chiral Lagrangian (8) contains 16 parameters. However, since the parameter g_2 (which is contained in the dots in Eq. (8)) does not play any role in the present study, we can omit it in the following, bringing the total number of relevant parameters to be fitted to 15: Λ , m_G , m_0 , m_1 , λ_1 , λ_2 , h_1 , h_2 , h_3 , g_1 , c_1 , h_{0N} , h_{0S} , δ_S , ϵ_S . For the calculations in this work we use the values of the parameters entering the Lagrangian (8), i.e. λ_2 , h_2 , h_3 , g_1 , c_1 , h_{0N} , h_{0S} , δ_S , as well as the two combinations C_1 [see Eq. (13)], $C_2 = m_1^2 + \frac{h_1}{2} (\phi_N^2 + \phi_S^2)$ determined in Ref. [13] and shown in Table I.

We will perform a fit by using the remaining five free parameters entering into the model: Λ , m_G , λ_1 , h_1 , ϵ_S .

III. RESULTS AND DISCUSSION

A. Input and results of the χ^2 analysis

Using the χ^2 analysis,

$$\chi^2 \equiv \chi^2(x_i) = \sum_{j=1}^8 \left(\frac{Q_j^{th}(x_i) - Q_j^{ex}}{\Delta Q_j^{ex}} \right)^2, \quad \text{with} \quad (19)$$

$$i = 1, \dots, 5,$$

we fit eight experimental quantities to the five parameters $x_i = \Lambda, m_G, \lambda_1, h_1, \epsilon_S$ of our chiral model summarized in Tables. II and III.

For the mass of $f_0(1370)$ we use the value $M_{f_0(1370)} = (1350 \pm 150)$ MeV and we increase the experimental errors of $M_{f_0(1500)} = (1505 \pm 6)$ MeV and $M_{f_0(1710)} = (1720 \pm 6)$ [16] to 5% of their physical values. This procedure was also applied in Ref. [13], arguing that the precision of our model cannot be better than 5% since it does not account e.g. for isospin breaking effects. Moreover, in order to better constrain the fit we use the value $\Gamma_{f_0(1370) \rightarrow \pi\pi} = 325$ MeV [36] together with an estimated uncertainty [not given in [36]] of about 100 MeV. The parameters in Table II, for which $\chi^2/\text{d.o.f.} \approx 0.35$ was achieved, and the masses as well as the decay widths of the scalar-isoscalar resonances in Table III correspond to the solution in which $\sigma'_N \equiv f_0(1370) \cong (\bar{u}u + \bar{d}d)/\sqrt{2}$ is predominantly a nonstrange, $\sigma'_S \equiv f_0(1500) \cong \bar{s}s$ predominantly a strange $\bar{q}q$ state, and $G' \equiv f_0(1710)$ predominantly a glueball state.

The bare fields $\sigma_N \cong (\bar{u}u + \bar{d}d)/\sqrt{2}$, $\sigma_S \cong \bar{s}s$, and G generate the resonances $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$, where the corresponding mixing matrix B , cf. Eq. (18), is given by

$$B = \begin{pmatrix} -0.91 & 0.24 & -0.33 \\ 0.30 & 0.94 & -0.17 \\ -0.27 & 0.26 & 0.93 \end{pmatrix}, \quad (20)$$

which implies the following admixtures of the bare fields to the resonances:

TABLE II. Parameters obtained from the fit with the solution: $\{\sigma'_N, \sigma'_S, G'\} \equiv \{f_0(1370), f_0(1500), f_0(1710)\}$.

Parameter	Value
Λ	3297 [MeV]
m_G	1525 [MeV]
λ_1	6.25
h_1	-3.22
ϵ_S	0.4212×10^6 [MeV ²]

TABLE III. Fit with the solution: $\{\sigma'_N, \sigma'_S, G'\} \equiv \{f_0(1370), f_0(1500), f_0(1710)\}$.

Quantity	Fit [MeV]	Experiment [MeV]
$M_{f_0(1370)}$	1444	1200–1500
$M_{f_0(1500)}$	1534	1505 ± 6
$M_{f_0(1710)}$	1750	1720 ± 6
$f_0(1370) \rightarrow \pi\pi$	423.6	...
$f_0(1500) \rightarrow \pi\pi$	39.2	38.04 ± 4.95
$f_0(1500) \rightarrow K\bar{K}$	9.1	9.37 ± 1.69
$f_0(1710) \rightarrow \pi\pi$	28.3	29.3 ± 6.5
$f_0(1710) \rightarrow K\bar{K}$	73.4	71.4 ± 29.1

$$\begin{aligned} f_0(1370): & 83\% \sigma_N, & 6\% \sigma_S, & 11\% G, \\ f_0(1500): & 9\% \sigma_N, & 88\% \sigma_S, & 3\% G, \\ f_0(1710): & 8\% \sigma_N, & 6\% \sigma_S, & 86\% G. \end{aligned} \quad (21)$$

The parameters λ_1 and h_1 are small, in agreement with the large- N_c expectation: they scale as $1/N_c^2$ and not as $1/N_c$. The numerical value $\Lambda \approx 3.3$ GeV suppresses the quarkonium-gluon mixing: this is why the admixtures in Eq. (21) are small.

In the pure YM sector the vev of the dilaton field G is given by $G_0 = \Lambda$. The numerical value $\Lambda \approx 3.3$ GeV implies that the resulting gluon condensate in pure YM, which is parametrized by the constant C defined in Eq. (3), reads $C \approx 1.8$ GeV, which is a factor 3 larger than the lattice value $C \approx 0.61$ GeV obtained in Ref. [28] in the quenched-approximation. When quarks are included, the value of G_0 is such that $G_0 \approx \Lambda$ to a very good level of precision, see Eq. (16). Similarly, using Eq. (15) the value of the bare glueball mass in the presence of quarks reads $M_G \approx m_G$. The fact that $G_0 \approx \Lambda$ and $M_G \approx m_G$ is also a consequence of the large value of Λ . (For small $\Lambda \lesssim 0.6$ GeV the differences are larger.)

Our determination of the parameter C is based on the assumption that the glueball is narrow, see Fig. 1 and the discussion in the introduction. If this assumption does not hold, the glueball is very broad (and would probably remain undetected). If, however, the narrow-gluon hypothesis is correct, our results imply that either (i) the value of the constant C cannot be directly compared to the corresponding one appearing in lattice QCD or QCD sum rules (which is entirely possible because there may be corrections to the tree-level Lagrangian (5) arising from renormalization), or (ii) that it is *not* allowed to assume that the dilaton field saturates the trace anomaly. In turn, Eq. (6) would not hold and other contributions should appear in order to reconcile the mismatch.

The stability of the fit has been also tested by repeating the minimum search for different values of the parameters, by increasing or reducing the errors in some channels

TABLE IV. Consequences of the fit with the solution: $\{\sigma'_N, \sigma'_S, G'\} \equiv \{f_0(1370), f_0(1500), f_0(1710)\}$.

Decay channel	Our value [MeV]	Experiment [MeV]
$f_0(1370) \rightarrow K\bar{K}$	117.5	...
$f_0(1370) \rightarrow \eta\eta$	43.3	...
$f_0(1370) \rightarrow \rho\rho \rightarrow 4\pi$	13.8	...
$f_0(1500) \rightarrow \eta\eta$	4.7	5.56 ± 1.34
$f_0(1500) \rightarrow \rho\rho \rightarrow 4\pi$	0.2	$> 54.0 \pm 7.1$
$f_0(1710) \rightarrow \eta\eta$	57.9	34.3 ± 17.6
$f_0(1710) \rightarrow \rho\rho \rightarrow 4\pi$	0.5	...

and by including and/or removing some experimental quantities. The same pattern has always been found: in all solutions the resonance $f_0(1710)$ is (by far) predominantly a glueball, while $f_0(1370)$ and $f_0(1500)$ are predominantly $(\bar{u}u + \bar{d}d)/\sqrt{2}$ and $\bar{s}s$ quark-antiquark states, respectively.

In the future, one should also go beyond the present two-step fit and perform a unique fit in which all 15 parameters are determined at once. However, we do not expect large variations for the parameters determined in Ref. [13] and listed in Table I. Otherwise the agreement with mesonic masses and decays calculated in Ref. [13] would inevitably be spoiled.

B. Consequences of the χ^2 analysis

As a consequence of our fit we calculate the decay processes given in Table IV. We discuss our results in the following:

- (a) At present, the different decay channels of the resonance $f_0(1370)$ are experimentally not yet well known because conflicting experimental results exist [16]. Only the full decay width is listed in Ref. [16]: $\Gamma_{f_0(1370)}^{\text{exp}} = (200\text{--}500)$ MeV. In our solution the dominant decay channel of $f_0(1370)$ is the one into two pions with a decay width of about 400 MeV. This corroborates that $f_0(1370)$ is predominantly a non-strange $\bar{q}q$ state as also found in Refs. [11–13]. The total decay width of $f_0(1370)$ obtained with the parameters of Table II is 598 MeV. In addition, we found non-negligible contributions from the decays $f_0(1370) \rightarrow \eta\eta$ and $f_0(1370) \rightarrow \rho\rho \rightarrow 4\pi$ (where in the latter case we have integrated over the corresponding ρ spectral function). These results are in qualitative agreement with the experimental analysis of Ref. [36], where $\Gamma_{f_0(1370) \rightarrow \pi\pi} = 325$ MeV, $\Gamma_{f_0(1370) \rightarrow 4\pi} \approx 50$ MeV, and $\Gamma_{f_0(1370) \rightarrow \eta\eta} / \Gamma_{f_0(1370) \rightarrow \pi\pi} = 0.19 \pm 0.07$. Note that the channel $f_0(1370) \rightarrow f_0(500)f_0(500) \rightarrow 4\pi$ is not included in our model, so our determination of the 4π -decay mode is not complete.
- (b) When omitting the quantity $\Gamma_{f_0(1370) \rightarrow \pi\pi}$ from the fit, a solution with a similar phenomenology is found.

However, the state $f_0(1370)$ would be somewhat too wide (≈ 700 MeV.) This is why we have decided to include the quantity $\Gamma_{f_0(1370) \rightarrow \pi\pi} = 325$ MeV [36] in the fit.

- (c) The decay channel $f_0(1500) \rightarrow \eta\eta$ turns out to be in good agreement with the experiment.
- (d) Experimentally, there is also a sizable contribution of the channel $f_0(1500) \rightarrow 4\pi$: $\Gamma_{f_0(1500) \rightarrow 4\pi}^{\text{exp}} = (54.0 \pm 7.1)$ MeV. We have calculated the decay of $f_0(1500)$ into 4π only through the intermediate $\rho\rho$ state (as in the case of $f_0(1370)$ and $f_0(1710)$, respectively, including the ρ spectral function). We found that this decay channel is strongly suppressed. However, we expect a further (and much larger) contribution to this decay channel through the intermediate state of two $f_0(500)$ resonances, but $f_0(500)$ is not implemented in the present model, see outlook I in Sec. IV.
- (e) The decay channel $f_0(1710) \rightarrow \eta\eta$ is slightly larger than the experiment.
- (f) In comparison with the $N_f = 2$ results of Ref. [12], we now find that the decay channel $f_0(1710) \rightarrow \rho\rho \rightarrow 4\pi$ is strongly suppressed. The reason is the scaling $\Gamma_{f_0(1710) \rightarrow \rho\rho \rightarrow 4\pi} \propto 1/G_0$. This is indeed an important point: in Ref. [12] two scenarios were phenomenologically acceptable, one in which $f_0(1500)$ and one in which $f_0(1710)$ is predominantly a glueball. The latter case was, however, slightly disfavored because $\Gamma_{f_0(1710) \rightarrow \rho\rho \rightarrow 4\pi}$ was too large in virtue of the vev $G_0 \sim \Lambda$, which was much smaller in that case. A solution of that type was possible because only one quarkonium existed and less experimental information was taken into account.

IV. CONCLUSIONS AND OUTLOOK

A. Conclusions

In the present paper, the scalar glueball state of the extended linear sigma model, which was considered to be frozen in Ref. [13], was elevated to a dynamical degree of freedom. We then studied a three-state mixing scenario in the scalar-isoscalar sector, where a nonstrange and a strange quark-antiquark state mix with the glueball to produce the physical resonances $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$. We have found that the resonance $f_0(1710)$ is predominantly a glueball state, as was also obtained in Refs. [6,23–25]. Moreover, we find that the state $f_0(1370)$ is predominantly a nonstrange quarkonium $(\bar{u}u + \bar{d}d)/\sqrt{2}$ and $f_0(1500)$ a strange quarkonium $\bar{s}s$. Our solution implies that the gluon condensate G_0 arising from the tree-level dilaton potential (5) is about a factor 3 larger than the one obtained in lattice QCD and QCD sum rule calculations. As already noticed in Ref. [17], this is quite natural if one wants to obtain a narrow glueball state.

B. Outlook

1. Inclusion of light tetraquark fields

One should include the nonet of light scalar states $f_0(500)$, $f_0(980)$, $a_0(980)$, and $K_0^*(800)$, which then allows to describe all scalar states up to 1.7 GeV. Indeed, in the two-flavor case the resonance $f_0(500)$ as a tetraquark/molecular field has been already included in a simplified version of the eLSM [37], in which chiral symmetry restoration at nonzero temperature has been studied, and in the extension of the eLSM to the baryonic sector [38]. The role of $f_0(500)$ is important because it induces a strong attraction between nucleons and affects the properties of nuclear matter at nonzero density.

In the three-flavor case chiral models with tetraquark fields but without (axial-)vector mesons were studied [31,32,39]. The isovector resonances $a_0(1450)$ and $a_0(980)$ arise as a mixing of a bare quark-antiquark and a bare tetraquark/molecular field configuration. A similar situation holds in the isodoublet sector for $K_0^*(1430)$ and $K_0^*(800)$. The mixing angle turns out to be small [32]. In the scalar-isoscalar sector one has a mixing of five bare fields, which leads to the five resonances $f_0(500)$, $f_0(980)$, $f_0(1370)$, $f_0(1500)$, and $f_0(1710)$ [31].

In the framework of the eLSM, the inclusion of the light scalars should also contain their coupling to (axial-)vector degrees of freedom as well as to the dilaton field. A variety of decays, such as the decays of the light scalars ($f_0(500) \rightarrow \pi\pi$, $f_0(980) \rightarrow KK$, etc.) as well as decays into them ($a_1(1230) \rightarrow f_0(500)\pi$, $f_0(1500) \rightarrow f_0(500)f_0(500)$, etc.) can be studied. Moreover, the mixing in

the isovector, isodoublet, and—most importantly—in the isoscalar sector can be investigated in such a framework.

2. Inclusion of other glueball fields

In Ref. [34] the pseudoscalar glueball has been coupled to the eLSM and its branching ratios have been calculated. The mass of the pseudoscalar glueball is about 2.6 GeV [2], which is already in the reach of the PANDA experiment [4]. Lattice QCD predicts a full tower of heavier gluonic states with various quantum numbers, such as $J^{PC} = 1^{--}, 1^{+-}, 2^{++}, \dots$ [2,3]. These glueball states can be easily implemented in the eLSM in a chirally invariant way: the decays can be evaluated, thus giving useful information about the properties of these (still hypothetical) glueballs. The search for these states could be simplified if clear theoretical input about their decay pattern is known.

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APPENDIX A: DETAILS OF THE EXTENDED LINEAR SIGMA MODEL

1. Vector and (axial-)vector multiplets and renormalization constants

The left-handed and right-handed (axial-)vector fields of the eLSM are contained in the multiplets [13]

$$L^\mu = \sum_{i=0}^8 (V_i^\mu + A_i^\mu) T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu + a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu+} + a_1^{\mu+} & K^{*\mu+} + K_1^{\mu+} \\ \rho^{\mu-} + a_1^{\mu-} & \frac{\omega_N^\mu - \rho^{\mu 0}}{\sqrt{2}} + \frac{f_{1N}^\mu - a_1^{\mu 0}}{\sqrt{2}} & K^{*\mu 0} + K_1^{\mu 0} \\ K^{*\mu-} + K_1^{\mu-} & \bar{K}^{*\mu 0} + \bar{K}_1^{\mu 0} & \omega_S^\mu + f_{1S}^\mu \end{pmatrix}, \quad (\text{A1})$$

and

$$R^\mu = \sum_{i=0}^8 (V_i^\mu - A_i^\mu) T_i = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\omega_N^\mu + \rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu + a_1^{\mu 0}}{\sqrt{2}} & \rho^{\mu+} - a_1^{\mu+} & K^{*\mu+} - K_1^{\mu+} \\ \rho^{\mu-} - a_1^{\mu-} & \frac{\omega_N^\mu - \rho^{\mu 0}}{\sqrt{2}} - \frac{f_{1N}^\mu - a_1^{\mu 0}}{\sqrt{2}} & K^{*\mu 0} - K_1^{\mu 0} \\ K^{*\mu-} - K_1^{\mu-} & \bar{K}^{*\mu 0} - \bar{K}_1^{\mu 0} & \omega_S^\mu - f_{1S}^\mu \end{pmatrix}. \quad (\text{A2})$$

The assignment of the fields in Eq. (A1) and (A2) to the physical resonances is as follows. In the $J^{PC} = 1^{--}$ sector the nonstrange ω_N^μ and the strange ω_S^μ field represent the resonance $\omega(782)$ and $\phi(1020)$, respectively. The isotriplet field $\vec{\rho}^\mu$ and the isodoublet fields $K^{*\mu}$ correspond to the resonance $\rho(770)$ and $K^*(1410)$, respectively. In the $J^{PC} = 1^{++}$ sector the nonstrange f_{1N}^μ and the strange f_{1S}^μ field are assigned to the resonance $f_1(1285)$ and $f_1(1420)$. The isotriplet field \vec{a}_1^μ is identified with the resonance $a_1(1260)$. Finally, the isodoublet fields K_1 corresponds to a mixture of $K_1(1270)$ and $K_1(1400)$, for details see Ref. [40].

Spontaneous breaking of chiral symmetry induces bilinear terms in the Lagrangian of the eLSM which can be eliminated by shifting the (axial-)vector fields as follows [13],

$$f_{1N/S}^\mu \rightarrow f_{1N/S}^\mu + Z_{\eta_{N/S}} w_{f_{1N/S}} \partial^\mu \eta_{N/S}, \quad (\text{A3})$$

$$a_1^{\mu\pm,0} \rightarrow a_1^{\mu\pm,0} + Z_\pi w_{a_1} \partial^\mu \pi^{\pm,0},$$

$$K_1^{\mu\pm,0,\bar{0}} \rightarrow K_1^{\mu\pm,0,\bar{0}} + Z_K w_{K_1} \partial^\mu K^{\pm,0,\bar{0}},$$

$$K^{*\mu\pm,0,\bar{0}} \rightarrow K^{*\mu\pm,0,\bar{0}} + Z_{K^*} w_{K^*} \partial^\mu K_0^{*\pm,0,\bar{0}}. \quad (\text{A4})$$

After performing this procedure additional kinetic terms occur. In order to remove the latter a redefinition of the (pseudo)scalar fields is required,

$$\pi^{\pm,0} \rightarrow Z_\pi \pi^{\pm,0}, \quad \eta_{N/S} \rightarrow Z_{\eta_{N/S}} \eta_{N/S}, \quad (\text{A5})$$

$$K^{\pm,0,\bar{0}} \rightarrow Z_K K^{\pm,0,\bar{0}}, \quad K_0^{*\pm,0,\bar{0}} \rightarrow Z_{K^*} K_0^{*\pm,0,\bar{0}}, \quad (\text{A6})$$

where

$$Z_\pi = Z_{\eta_N} = \frac{m_{a_1}}{\sqrt{m_{a_1}^2 - g_1^2 \phi_N^2}}, \quad (\text{A7})$$

$$Z_K = \frac{2m_{K_1}}{\sqrt{4m_{K_1}^2 - g_1^2 (\phi_N + \sqrt{2}\phi_S)^2}},$$

$$Z_{K^*} = \frac{2m_{K^*}}{\sqrt{4m_{K^*}^2 - g_1^2 (\phi_N - \sqrt{2}\phi_S)^2}}, \quad (\text{A8})$$

$$Z_{\eta_S} = \frac{m_{f_{1S}}}{\sqrt{m_{f_{1S}}^2 - 2g_1^2 \phi_S^2}}$$

are the wave-function renormalization constants and

$$w_{f_{1N}} = w_{a_1} = \frac{g_1 \phi_N}{m_{a_1}^2}, \quad w_{f_{1S}} = \frac{\sqrt{2} g_1 \phi_S}{m_{f_{1S}}^2}, \quad (\text{A9})$$

$$w_{K^*} = \frac{i g_1 (\phi_N - \sqrt{2}\phi_S)}{2m_{K^*}^2}, \quad w_{K_1} = \frac{g_1 (\phi_N + \sqrt{2}\phi_S)}{2m_{K_1}^2}. \quad (\text{A10})$$

Explicit breaking of chiral symmetry is incorporated by the following constant matrices,

$$H = H_0 T_0 + H_8 T_8 = \begin{pmatrix} \frac{h_{0N}}{2} & 0 & 0 \\ 0 & \frac{h_{0N}}{2} & 0 \\ 0 & 0 & \frac{h_{0S}}{\sqrt{2}} \end{pmatrix}, \quad (\text{A11})$$

$$E = E_0 T_0 + E_8 T_8 = \begin{pmatrix} \frac{\tilde{\epsilon}_N}{2} & 0 & 0 \\ 0 & \frac{\tilde{\epsilon}_N}{2} & 0 \\ 0 & 0 & \frac{\tilde{\epsilon}_S}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} \epsilon_N & 0 & 0 \\ 0 & \epsilon_N & 0 \\ 0 & 0 & \epsilon_S \end{pmatrix}, \quad (\text{A12})$$

$$\Delta = \Delta_0 T_0 + \Delta_8 T_8$$

$$= \begin{pmatrix} \frac{\tilde{\delta}_N}{2} & 0 & 0 \\ 0 & \frac{\tilde{\delta}_N}{2} & 0 \\ 0 & 0 & \frac{\tilde{\delta}_S}{\sqrt{2}} \end{pmatrix} \equiv \begin{pmatrix} \delta_N & 0 & 0 \\ 0 & \delta_N & 0 \\ 0 & 0 & \delta_S \end{pmatrix}, \quad (\text{A13})$$

where the terms in Eqs. (A12) and (A13) are next-to-leading corrections in the current quark masses.

APPENDIX B: DECAY WIDTHS

In this work we compute two-body decays using the well-known formula

$$\Gamma_{A \rightarrow BC} = s_f \mathcal{I} \frac{k_f}{8\pi m_A^2} | -i \mathcal{A}_{A \rightarrow BC} |^2, \quad (\text{B1})$$

where

$$k_f = \frac{1}{2m_A} \sqrt{m_A^4 + (m_B^2 - m_C^2)^2 - 2m_A^2(m_B^2 + m_C^2)} \\ \times \theta(m_A - m_B - m_C) \quad (\text{B2})$$

is the modulus of the three-momentum of one of the outgoing particles (the moduli of the momenta are equal in the rest frame of the decaying particle) and $\mathcal{A}_{A \rightarrow BC}$ is the decay amplitude. The symmetry factor s_f avoids double counting of identical Feynman diagrams and \mathcal{I} is the isospin factor which considers all subchannels of a particular decay channel. The θ function encodes the decay threshold.

All relevant expressions for the decay processes studied in this work are extracted from the Lagrangian (8) and are presented in the following.

1. Decays of the scalar-isoscalar fields into $\pi\pi$

Following the general formula (B1) we obtain for the decay widths of the scalar-isoscalar resonances into $\pi\pi$

$$\Gamma_{f_0 \rightarrow \pi\pi} = 6 \frac{\sqrt{\frac{m_{f_0}^2}{4} - m_\pi^2}}{8\pi m_{f_0}^2} | -i \mathcal{A}_{f_0 \rightarrow \pi\pi}(m_{f_0}) |^2, \quad (\text{B3})$$

where m_{f_0} is the mass of the physical f_0 resonance. The bare amplitudes (as functions of m_{f_0}) are

$$-i \mathcal{A}_{\sigma_N \rightarrow \pi\pi}(m_{f_0}) = i \left(A_{\sigma_N \pi\pi} - B_{\sigma_N \pi\pi} \frac{m_{f_0}^2 - 2m_\pi^2}{2} - C_{\sigma_N \pi\pi} m_\pi^2 \right), \quad (\text{B4})$$

$$-i \mathcal{A}_{\sigma_S \rightarrow \pi\pi}(m_{f_0}) = i \left(A_{\sigma_S \pi\pi} - B_{\sigma_S \pi\pi} \frac{m_{f_0}^2 - 2m_\pi^2}{2} \right), \quad (\text{B5})$$

$$-i\mathcal{A}_{G\rightarrow\pi\pi}(m_{f_0}) = i\left(A_{G\pi\pi} - B_{G\pi\pi}\frac{m_{f_0}^2 - 2m_\pi^2}{2}\right), \quad (\text{B6})$$

with the corresponding constants

$$A_{\sigma_N\pi\pi} = -\left(\lambda_1 + \frac{\lambda_2}{2}\right)Z_\pi^2\phi_N, \quad (\text{B7})$$

$$B_{\sigma_N\pi\pi} = -2g_1Z_\pi^2w_{a_1} + \left(g_1^2 + \frac{h_1 + h_2 - h_3}{2}\right)Z_\pi^2w_{a_1}^2\phi_N, \quad (\text{B8})$$

$$C_{\sigma_N\pi\pi} = -g_1Z_\pi^2w_{a_1}, \quad (\text{B9})$$

$$A_{\sigma_S\pi\pi} = -\lambda_1Z_\pi^2\phi_S, \quad (\text{B10})$$

$$B_{\sigma_S\pi\pi} = \frac{h_1}{2}Z_\pi^2w_{a_1}^2\phi_S, \quad (\text{B11})$$

$$A_{G\pi\pi} = -\frac{m_0^2}{G_0}Z_\pi^2, \quad (\text{B12})$$

$$B_{G\pi\pi} = \frac{m_1^2}{G_0}Z_\pi^2w_{a_1}^2. \quad (\text{B13})$$

After performing an orthogonal transformation we obtain the amplitudes for the physical scalar-isoscalar fields $\sigma'_N \equiv f_0(1370)$, $\sigma'_S \equiv f_0(1500)$, and $G' \equiv f_0(1710)$:

$$-i\mathcal{A}_{\sigma'_N\rightarrow\pi\pi}(m_{\sigma'_N}) = i[\mathcal{A}_{\sigma_N\rightarrow\pi\pi}(m_{\sigma'_N})b_{11} + \mathcal{A}_{\sigma_S\rightarrow\pi\pi}(m_{\sigma'_N})b_{12} + \mathcal{A}_{G\rightarrow\pi\pi}(m_{\sigma'_N})b_{13}], \quad (\text{B14})$$

$$-i\mathcal{A}_{\sigma'_S\rightarrow\pi\pi}(m_{\sigma'_S}) = i[\mathcal{A}_{\sigma_N\rightarrow\pi\pi}(m_{\sigma'_S})b_{21} + \mathcal{A}_{\sigma_S\rightarrow\pi\pi}(m_{\sigma'_S})b_{22} + \mathcal{A}_{G\rightarrow\pi\pi}(m_{\sigma'_S})b_{23}], \quad (\text{B15})$$

$$-i\mathcal{A}_{G'\rightarrow\pi\pi}(m_{G'}) = i[\mathcal{A}_{\sigma_N\rightarrow\pi\pi}(m_{G'})b_{31} + \mathcal{A}_{\sigma_S\rightarrow\pi\pi}(m_{G'})b_{32} + \mathcal{A}_{G\rightarrow\pi\pi}(m_{G'})b_{33}], \quad (\text{B16})$$

where b_{ij} , $i, j = 1, 2, 3$, are the corresponding elements of the mixing matrix B from Eq. (18).

2. Decays of the scalar-isoscalar fields into KK

Following the general formula (B1) we obtain for the decay widths of the scalar-isoscalar resonances into KK

$$\Gamma_{f_0\rightarrow KK} = 2\frac{\sqrt{\frac{m_{f_0}^2}{4} - m_{KK}^2}}{8\pi m_{f_0}^2} | -i\mathcal{A}_{f_0\rightarrow KK}(m_{f_0}) |^2, \quad (\text{B17})$$

where the bare amplitudes are

$$-i\mathcal{A}_{\sigma_N\rightarrow KK}(m_{f_0}) = i\left[A_{\sigma_N KK} - (B_{\sigma_N KK} - 2C_{\sigma_N KK})\frac{m_{f_0}^2 - 2m_K^2}{2} + 2C_{\sigma_N KK}m_K^2\right], \quad (\text{B18})$$

$$-i\mathcal{A}_{\sigma_S\rightarrow KK}(m_{f_0}) = i\left[A_{\sigma_S KK} - (B_{\sigma_S KK} - 2C_{\sigma_S KK})\frac{m_{f_0}^2 - 2m_K^2}{2} + 2C_{\sigma_S KK}m_K^2\right], \quad (\text{B19})$$

$$-i\mathcal{A}_{G\rightarrow KK}(m_{f_0}) = i\left(A_{GKK} - B_{GKK}\frac{m_{f_0}^2 - 2m_K^2}{2}\right) \quad (\text{B20})$$

and the corresponding constants read

$$A_{\sigma_N KK} = \frac{Z_K^2}{\sqrt{2}}\left[\lambda_2(\phi_S - \sqrt{2}\phi_N) - 2\sqrt{2}\lambda_1\phi_N\right], \quad (\text{B21})$$

$$B_{\sigma_N KK} = \frac{g_1}{2}Z_K^2w_{K_1}\left[-2 + g_1w_{K_1}(\phi_N + \sqrt{2}\phi_S)\right] + \frac{Z_K^2w_{K_1}^2}{2}\left[(2h_1 + h_2)\phi_N - \sqrt{2}h_3\phi_S\right], \quad (\text{B22})$$

$$C_{\sigma_N KK} = \frac{g_1}{2}Z_K^2w_{K_1}, \quad (\text{B23})$$

$$A_{\sigma_S KK} = \frac{Z_K^2}{\sqrt{2}}\left[\lambda_2(\phi_N - 2\sqrt{2}\phi_S) - 2\sqrt{2}\lambda_1\phi_S\right], \quad (\text{B24})$$

$$B_{\sigma_S KK} = \frac{\sqrt{2}g_1}{2}Z_K^2w_{K_1}\left[-2 + g_1w_{K_1}(\phi_N + \sqrt{2}\phi_S)\right] + \frac{Z_K^2w_{K_1}^2}{\sqrt{2}}\left[\sqrt{2}(h_1 + h_2)\phi_S - h_3\phi_N\right], \quad (\text{B25})$$

$$C_{\sigma_S KK} = \frac{\sqrt{2}g_1}{2} Z_K^2 w_{K_1}, \quad (\text{B26})$$

$$A_{GKK} = -\frac{2m_0^2}{G_0} Z_K^2, \quad (\text{B27})$$

$$B_{GKK} = \frac{2m_1^2}{G_0} Z_K^2 w_{K_1}^2. \quad (\text{B28})$$

After performing an orthogonal transformation we obtain the amplitudes for the physical scalar-isoscalar fields

$$-i\mathcal{A}_{\sigma'_N \rightarrow KK}(m_{\sigma'_N}) = i[\mathcal{A}_{\sigma_N \rightarrow KK}(m_{\sigma'_N})b_{11} + \mathcal{A}_{\sigma_S \rightarrow KK}(m_{\sigma'_N})b_{12} + \mathcal{A}_{G \rightarrow KK}(m_{\sigma'_N})b_{13}], \quad (\text{B29})$$

$$-i\mathcal{A}_{\sigma'_S \rightarrow KK}(m_{\sigma'_S}) = i[\mathcal{A}_{\sigma_N \rightarrow KK}(m_{\sigma'_S})b_{21} + \mathcal{A}_{\sigma_S \rightarrow KK}(m_{\sigma'_S})b_{22} + \mathcal{A}_{G \rightarrow KK}(m_{\sigma'_S})b_{23}], \quad (\text{B30})$$

$$-i\mathcal{A}_{G' \rightarrow KK}(m_{G'}) = i[\mathcal{A}_{\sigma_N \rightarrow KK}(m_{G'})b_{31} + \mathcal{A}_{\sigma_S \rightarrow KK}(m_{G'})b_{32} + \mathcal{A}_{G \rightarrow KK}(m_{G'})b_{33}], \quad (\text{B31})$$

which we assign to the physical resonances as follows: $\sigma'_N \equiv f_0(1370)$, $\sigma'_S \equiv f_0(1500)$, and $G \equiv f_0(1710)$.

3. Decays of the scalar-isoscalar fields into $\eta\eta$

Following the general formula (B1) we obtain for the decay widths of the scalar-isoscalar resonances into $\eta\eta$

$$\Gamma_{f_0 \rightarrow \eta\eta} = 2 \frac{\sqrt{\frac{m_{f_0}^2}{4} - m_\eta^2}}{8\pi m_{f_0}^2} | -i\mathcal{A}_{f_0 \rightarrow \eta\eta}(m_{f_0}) |^2, \quad (\text{B32})$$

where the bare amplitudes are

$$\begin{aligned} & -i\mathcal{A}_{\sigma_N \rightarrow \eta\eta}(m_{f_0}) \\ & = i \left(A_{\sigma_N \eta\eta} - B_{\sigma_N \eta\eta} \frac{m_{f_0}^2 - 2m_\eta^2}{2} + C_{\sigma_N \eta\eta} \frac{m_{f_0}^2}{2} \right), \end{aligned} \quad (\text{B33})$$

$$\begin{aligned} & -i\mathcal{A}_{\sigma_S \rightarrow \eta\eta}(m_{f_0}) \\ & = i \left(A_{\sigma_S \eta\eta} - B_{\sigma_S \eta\eta} \frac{m_{f_0}^2 - 2m_\eta^2}{2} + C_{\sigma_S \eta\eta} \frac{m_{f_0}^2}{2} \right), \end{aligned} \quad (\text{B34})$$

$$-i\mathcal{A}_{G \rightarrow \eta\eta}(m_{f_0}) = i \left[\left(A_{G\eta_N \eta_N} - B_{G\eta_N \eta_N} \frac{m_{f_0}^2 - 2m_\eta^2}{2} \right) \cos \varphi_\eta + \left(A_{G\eta_S \eta_S} - B_{G\eta_S \eta_S} \frac{m_{f_0}^2 - 2m_\eta^2}{2} \right) \sin \varphi_\eta \right] \quad (\text{B35})$$

and the corresponding constants read

$$\begin{aligned} A_{\sigma_N \eta\eta} &= -Z_\pi^2 \phi_N \left(\lambda_1 + \frac{\lambda_2}{2} + c_1 \phi_S^2 \right) \cos^2 \varphi_\eta \\ & - Z_{\eta_S}^2 \phi_N \left(\lambda_1 + \frac{c_1}{2} \phi_N^2 \right) \sin^2 \varphi_\eta \\ & - \frac{3}{4} c_1 Z_\pi Z_{\eta_S} \phi_N^2 \phi_S \sin(2\varphi_\eta), \end{aligned} \quad (\text{B36})$$

$$\begin{aligned} B_{\sigma_N \eta\eta} &= -\frac{Z_\pi^2 w_{a_1}^2}{\phi_N} \left(m_1^2 + \frac{h_1}{2} \phi_S^2 + 2\delta_N \right) \cos^2 \varphi_\eta \\ & + \frac{h_1}{2} Z_{\eta_S}^2 w_{f_{1S}}^2 \phi_N \sin^2 \varphi_\eta, \end{aligned} \quad (\text{B37})$$

$$C_{\sigma_N \eta\eta} = g_1 Z_\pi^2 w_{a_1} \cos^2 \varphi_\eta, \quad (\text{B38})$$

$$\begin{aligned} A_{\sigma_S \eta\eta} &= -Z_{\eta_S}^2 \phi_S (\lambda_1 + \lambda_2) \sin^2 \varphi_\eta \\ & - Z_\pi^2 \phi_S (\lambda_1 + c_1 \phi_N^2) \cos^2 \varphi_\eta \\ & - \frac{1}{4} c_1 Z_\pi Z_{\eta_S} \phi_N^3 \sin(2\varphi_\eta), \end{aligned} \quad (\text{B39})$$

$$\begin{aligned} B_{\sigma_S \eta\eta} &= -\frac{Z_{\eta_S}^2 w_{f_{1S}}^2}{\phi_S} \left(m_1^2 + \frac{h_1}{2} \phi_N^2 + 2\delta_S \right) \sin^2 \varphi_\eta \\ & + \frac{h_1}{2} Z_\pi^2 w_{a_1}^2 \phi_S \cos^2 \varphi_\eta, \end{aligned} \quad (\text{B40})$$

$$C_{\sigma_S \eta\eta} = \sqrt{2} g_1 Z_{\eta_S}^2 w_{f_{1S}} \sin^2 \varphi_\eta, \quad (\text{B41})$$

$$A_{G\eta_N \eta_N} = -\frac{m_0^2}{G_0} Z_\pi^2, \quad (\text{B42})$$

$$B_{G\eta_N\eta_N} = \frac{m_1^2}{2G_0} Z_\pi^2 w_{f_{1N}}^2, \quad (\text{B43})$$

$$A_{G\eta_S\eta_S} = -\frac{m_0^2}{G_0} Z_\pi^2, \quad (\text{B44})$$

$$B_{G\eta_S\eta_S} = \frac{m_1^2}{G_0} Z_{\eta_S}^2 w_{f_{1S}}^2. \quad (\text{B45})$$

After performing an orthogonal transformation we obtain the amplitudes for the physical scalar-isoscalar fields

$$-i\mathcal{A}_{\sigma'_N \rightarrow \eta\eta}(m_{\sigma'_N}) = i[\mathcal{A}_{\sigma_N \rightarrow \eta\eta}(m_{\sigma'_N})b_{11} + \mathcal{A}_{\sigma_S \rightarrow \eta\eta}(m_{\sigma'_N})b_{12} + \mathcal{A}_{G \rightarrow \eta\eta}(m_{\sigma'_N})b_{13}], \quad (\text{B46})$$

$$-i\mathcal{A}_{\sigma'_S \rightarrow \eta\eta}(m_{\sigma'_S}) = i[\mathcal{A}_{\sigma_N \rightarrow \eta\eta}(m_{\sigma'_S})b_{21} + \mathcal{A}_{\sigma_S \rightarrow \eta\eta}(m_{\sigma'_S})b_{22} + \mathcal{A}_{G \rightarrow \eta\eta}(m_{\sigma'_S})b_{23}], \quad (\text{B47})$$

$$-i\mathcal{A}_{G' \rightarrow \eta\eta}(m_{G'}) = i[\mathcal{A}_{\sigma_N \rightarrow \eta\eta}(m_{G'})b_{31} + \mathcal{A}_{\sigma_S \rightarrow \eta\eta}(m_{G'})b_{32} + \mathcal{A}_{G \rightarrow \eta\eta}(m_{G'})b_{33}], \quad (\text{B48})$$

which we assign to the physical resonances as follows: $\sigma'_N \equiv f_0(1370)$, $\sigma'_S \equiv f_0(1500)$, and $G \equiv f_0(1710)$.

where $i = 1, 2$ and $A_{\rho\rho}$ is one of the corresponding constants

4. Decays of the scalar-isoscalar fields into $\rho\rho \rightarrow 4\pi$

The decay processes $f_0 \rightarrow \rho\rho \rightarrow 4\pi$ are on the threshold, hence we use for the calculation of the decay widths the spectral function of the ρ meson

$$d_\rho(X_{m_\rho}) = N \frac{X_{m_\rho}^2 \Gamma_{\rho \rightarrow \pi\pi}(X_{m_\rho})}{(X_{m_\rho}^2 - m_\rho^2)^2 + X_{m_\rho}^2 \Gamma_{\rho \rightarrow \pi\pi}^2(X_{m_\rho})} \theta(X_{m_\rho} - 2m_\pi), \quad (\text{B49})$$

where N is a normalization constant. Considering the polarization of the ρ mesons the general amplitude reads

$$\begin{aligned} & \overline{|-i\mathcal{A}_{f_0 \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho})|^2} \\ &= A_{\rho\rho}^2 \left[4 - \frac{X_{1,m_\rho}^2 + X_{2,m_\rho}^2}{m_\rho^2} + \frac{(m_{f_0}^2 - X_{1,m_\rho}^2 - X_{2,m_\rho}^2)^2}{4m_\rho^4} \right], \end{aligned} \quad (\text{B50})$$

$$A_{\sigma_N \rho\rho} = \frac{\phi_N}{2} (h_1 + h_2 + h_3), \quad (\text{B51})$$

$$A_{\sigma_S \rho\rho} = \frac{\phi_S}{2} h_1, \quad (\text{B52})$$

$$A_{G\rho\rho} = \frac{m_1^2}{G_0}. \quad (\text{B53})$$

The physical amplitudes of the scalar-isoscalar fields read

$$\overline{|-i\mathcal{A}_{\sigma'_N \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho})|^2} = [A_{\sigma_N \rho\rho} b_{11} + A_{\sigma_S \rho\rho} b_{12} + A_{G\rho\rho} b_{13}]^2 \left[4 - \frac{X_{1,m_\rho}^2 + X_{2,m_\rho}^2}{m_\rho^2} + \frac{(m_{f_0}^2 - X_{1,m_\rho}^2 - X_{2,m_\rho}^2)^2}{4m_\rho^4} \right], \quad (\text{B54})$$

$$\overline{|-i\mathcal{A}_{\sigma'_S \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho})|^2} = [A_{\sigma_N \rho\rho} b_{21} + A_{\sigma_S \rho\rho} b_{22} + A_{G\rho\rho} b_{23}]^2 \left[4 - \frac{X_{1,m_\rho}^2 + X_{2,m_\rho}^2}{m_\rho^2} + \frac{(m_{f_0}^2 - X_{1,m_\rho}^2 - X_{2,m_\rho}^2)^2}{4m_\rho^4} \right], \quad (\text{B55})$$

$$\overline{|-i\mathcal{A}_{G' \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho})|^2} = [A_{\sigma_N \rho\rho} b_{31} + A_{\sigma_S \rho\rho} b_{32} + A_{G\rho\rho} b_{33}]^2 \left[4 - \frac{X_{1,m_\rho}^2 + X_{2,m_\rho}^2}{m_\rho^2} + \frac{(m_{f_0}^2 - X_{1,m_\rho}^2 - X_{2,m_\rho}^2)^2}{4m_\rho^4} \right]. \quad (\text{B56})$$

The formula for the decays of the scalar-isoscalar fields into ρ mesons and 4π , respectively, reads

$$\Gamma_{f_0 \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho}) = 6 \frac{k_f(m_{f_0}, X_{i,m_\rho})}{8\pi m_{f_0}^2} \overline{|-i\mathcal{A}_{f_0 \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho})|^2} \theta(m_{f_0} - X_{1,m_\rho} - X_{2,m_\rho}), \quad (\text{B57})$$

$$\Gamma_{f_0 \rightarrow \rho\rho \rightarrow 4\pi} = \int_0^\infty \int_0^\infty \Gamma_{f_0 \rightarrow \rho\rho}(m_{f_0}, X_{i,m_\rho}) d_\rho(X_{1,m_\rho}) d_\rho(X_{2,m_\rho}) dX_{1,m_\rho} dX_{2,m_\rho}. \quad (\text{B58})$$

The scalar-isoscalar fields are assigned to the physical resonances as follows: $\sigma'_N \equiv f_0(1370)$, $\sigma'_S \equiv f_0(1500)$, and $G \equiv f_0(1710)$.

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