Pion scalar form factor and another confirmation of the existence of the $f_0(500)$ meson

S. Dubnička,[1] A. Z. Dubničková,[2] and A. Liptaj[1]

[1]*Institute of Physics, Slovak Academy of Sciences, 84511 Bratislava, Slovak Republic*
[2]*Department of Theoretical Physics, Comenius University, 84248 Bratislava, Slovak Republic*
(Received 11 March 2014; revised manuscript received 24 October 2014; published 1 December 2014)

An explicit form of the pion scalar form factor $\Gamma_\pi(t)$ is constructed by using its phase representation and a correct description of the S-wave isoscalar $\pi\pi$ phase shift $\delta_0^0(t)$ data by the parametrization of $\tan\delta_0^0(t)$ in the absolute valued pion c.m. three-momentum $q = [(t - 4m_\pi^2)/4]^{1/2}$. This parametrization has been found starting from fully general considerations. Then a calculation of the corresponding integral in the framework of the theory of residua provides $\Gamma_\pi(t)$ in the form of a rational function with one zero and four poles in the q variable. Investigations of the latter poles demonstrate that two of them, to be conjugate according to the imaginary axis in the q plane, clearly correspond to complex conjugate $f_0(500)$ meson poles on the second Riemann sheet in a momentum transfer squared t variable. Another pair of poles, also conjugate according to the imaginary axis in the q plane, can be identified with two complex conjugate poles on the second Riemann sheet in the t variable, corresponding to the $f_0(980)$ meson.

DOI: 10.1103/PhysRevD.90.114003 PACS numbers: 14.40.Be, 11.55.Fv, 11.80.Et

I. INTRODUCTION

In contrast to other $SU(3)$ multiplets of hadrons, the identification of the scalar mesons is a long-standing puzzle. There are at least two reasons for the latter situation. First, more particles have been found experimentally with quantum numbers 0^{++} than can be included in one $SU(3)$ multiplet. Second, some of them have decay widths which cause a strong overlap between resonances and background. In order to avoid these problems, the present-day experimentally established scalar mesons [1] are classified into a light scalar nonet comprising the $f_0(500)$, $K_0^*(800)$, $f_0(980)$, and $a_0(980)$ mesons—not necessarily in $q\bar{q}$ states—and into a regular nonet consisting of the $f_0(1370)$, $K_0^*(1430)$, $a_0(1450)$, and $f_0(1500)$ [or $f_0(1700)$] resonances.

But the most problematic case appeared to be the lowest scalar meson, $f_0(500)$. It has been difficult to establish its parameters because of its large width and because it could not be determined by a naive Breit-Wigner form.

The $f_0(500)$ scalar meson resonance manifests itself as a pole on the second Riemann sheet of the S-wave isoscalar $\pi\pi$ scattering amplitude and many of the determinations of the mass and width of the $f_0(500)$ meson listed by the Particle Data Group (PDG) are concentrated just on the identification of the latter pole by using various models and specific parametrizations. The analytic properties of the S-wave isoscalar $\pi\pi$ scattering amplitude $t_0^0(s)$ consists of the right-hand unitary cut and the left-hand dynamical cut. Many of the aforementioned determinations neglect the left-hand cut contribution, which also contributed to the unclear situation with the parameters of the $f_0(500)$ meson. Therefore, some authors questioned the inclusion of the $f_0(500)$ meson.

As a result, this so-called σ meson was listed in PDG data as "not well established" until 1974 and one has believed in the existence of a broad and light scalar isoscalar resonance.

However, it was removed from the PDG list in 1976 because two heavier resonances, $f_0(980)$ and $f_0(1300)$, were found and σ meson could be replaced by these two heavier ones, which could complete a light $(q\bar{q})$ nonet.

The σ meson was again listed in 1996, after being absent for more than two decades, although still with an obscure denotation, $f_0(400 - 1200)$.

The $f_0(600)$ meson has been listed as "well established" since 2002, but with conservative estimate of the mass (400–1200) MeV and the width (600–1000) MeV.

A clarification of this controversial situation has been achieved only recently in Refs. [2,3], where clear arguments are presented for the existence of the lowest resonance in the spectrum of QCD, with quantum numbers 0^{++} now to be called the $f_0(500)$ scalar meson.

In this paper we demonstrate another confirmation of the existence of the $f_0(500)$ scalar meson resonance by the construction of an explicit form of the pion scalar form factor $\Gamma_\pi(t)$, which does not possess the left-hand cut, so we automatically avoid any approximations or omission of the left-hand cut contribution considered in the S-wave isoscalar partial wave $\pi\pi$ scattering amplitude $t_0^0(s)$ to be used for identification of the pole on the second sheet, corresponding to the $f_0(500)$ scalar meson resonance. Starting with the analytic properties of $\Gamma_\pi(t)$ and by an application of the Cauchy formula, one comes to dispersion relations, which in combination with the pion scalar form factor elastic unitarity condition give the Muskelishvili-Omnès integral equations. Solutions of these equations are phase representations of $\Gamma_\pi(t)$, where, under the corresponding integral S-wave isoscalar $\pi\pi$ phase shift, δ_0^0 appeared. Then the whole problem of finding the concrete form of $\Gamma_\pi(t)$ is reduced to the description of existing data on δ_0^0 and finally to an explicit (not numerical) calculation of the corresponding integral.

With such a fully solvable mathematical problem, we do not carry out any analytic extrapolations of experimental information (in the sense of the original Sorin Ciulli idea) into the complex plane. Even in advance we do not specify the number of zeros and poles of the pion scalar form factor $\Gamma_\pi(t)$, and all of them appear naturally in the specific parametrization of $\tan\delta_0^0$ in the absolute valued pion c.m. three-momentum q to be related with momentum transfer squared t by the relation $t = 4(q^2 + m_\pi^2)$, the correct description of existing data on $\delta_0^0(t)$, and an explicit calculation of the corresponding integral to be carried out in the q variable. The result of this fully solvable mathematical problem is the pion scalar form factor in the form of a rational function in the q variable with one zero and four poles.

Investigations of these poles finally demonstrate that one pair of them is conjugate according to the imaginary axis in the q plane and clearly corresponds to the complex conjugate $f_0(500)$ meson poles on the second Riemann sheet in the momentum transfer squared t variable.

This result alone can be considered another confirmation of the existence of the $f_0(500)$ scalar meson.

II. THE PION SCALAR FORM FACTOR

The pion scalar form factor (FF) $\Gamma_\pi(t)$ is defined by the matrix element of the scalar quark density

$$\langle \pi^i(p_2)|\hat{m}(\bar{u}u + \bar{d}d)|\pi^j(p_1)\rangle = \delta^{ij}\Gamma_\pi(t), \tag{1}$$

where $t = (p_2 - p_1)^2$, $\hat{m} = \frac{1}{2}(m_u + m_d)$ and it has similar properties to the pion electromagnetic FF [4].

The pion scalar FF $\Gamma_\pi(t)$ is analytic in the whole complex t plane, except for a cut along the positive real axis, starting at $t = 4m_\pi^2$.

For real values, $t < 4m_\pi^2$ $\Gamma_\pi(t)$ is real. The latter implies the so-called reality condition $\Gamma_\pi^*(t) = \Gamma_\pi(t^*)$, i.e., that the values of FF above and below the cut are complex conjugate to each other.

At $t = 0$ the pion scalar FF $\Gamma_\pi(t)$ coincides with the pion σ term [5] $\Gamma_\pi(0) = (0.99 \pm 0.02)m_\pi^2$ to be evaluated in the framework of chiral perturbation theory (χPT). Further, the pion scalar FF will be normalized exactly to m_π^2.

The FF $\Gamma_\pi(t)$ is not a directly measurable quantity and it enters, e.g., the matrix element for the decay of the Higgs boson into two pions. However, the contribution to the decay rate seems to be negligibly small [6,7].

If $\Gamma_\pi(t)$ is evaluated on the upper boundary of the cut, one finds that the following unitarity condition is obeyed:

$$\mathrm{Im}\Gamma_\pi(t) = \sum_n \langle \pi(p')\pi(p)|T|n\rangle\langle n|\hat{m}(\bar{u}u + \bar{d}d)|0\rangle, \tag{2}$$

where T is the T operator and the sum runs over a complete set of the allowed states $2\pi, 4\pi, \ldots, K\bar{K}, \ldots$, which create branch points on the positive real axis of the t plane between $4m_\pi^2$ and ∞.

In the elastic region $4m_\pi^2 \le t \le 16m_\pi^2$, only the first term on the right-hand side of (2) contributes, then

$$\mathrm{Im}\Gamma_\pi(t) = \Gamma_\pi(t)(\sigma T_0^0)^*, \tag{3}$$

where σT_0^0 is the S-wave isoscalar $\pi\pi$ scattering amplitude

$$M_0^0 = \sigma T_0^0 = \frac{1}{2\,i}(e^{2\,i\delta} - 1); \tag{4}$$

$\delta = \delta_0^0 + i\varphi$; δ_0^0, φ real, where δ_0^0 stands for the S-wave isoscalar $\pi\pi$ phase shift and $\varphi > 0$ measures the inelasticity.

Though φ vanishes exactly only below $t = 16m_\pi^2$, the phenomenological analysis of the $\pi\pi$ interactions [8] shows that final states containing more than two particles start playing a significant role only well above $4m_K^2 \approx 1$ GeV^2, where the inelastic two-body channel $\pi\pi \to K\bar{K}$ opens.

Then,

$$\mathrm{Im}\Gamma_\pi(t) = \Gamma_\pi(t)e^{-i\delta_0^0}\sin\delta_0^0 \tag{5}$$

for $4m_\pi^2 \le t \le 1$ GeV^2.

From the relation (5), it follows that the phase δ_Γ of $\Gamma_\pi(t)$ coincides with δ_0^0 and that this identity alone enables us to obtain the pion scalar FF $\Gamma_\pi(t)$ behavior that is valid at the elastic interval $4m_\pi^2 \le t \le 1$ GeV^2.

The asymptotic behavior

$$\Gamma_\pi(t)_{|t|\to\infty} \sim \frac{1}{t} \tag{6}$$

is predicted by the quark counting rules.

Starting with the unitarity condition for the S-wave isoscalar $\pi\pi$ scattering amplitude

$$\mathrm{Im}M_0^0 = -|M_0^0|^2, \tag{7}$$

one can do analytic continuation of M_0^0 through the upper and lower boundaries of the unitary elastic cut and can prove in this way that the singularity at $t = 4m_\pi^2$ is a square root branch point. As a result, one gets

$$(M_0^0)^{II} = \frac{(M_0^0)^I}{1 - 2\,i(M_0^0)^I}. \tag{8}$$

The same can be done with the pion scalar FF and, as a result, one gets the expression

$$(\Gamma_\pi)^{II} = \frac{(\Gamma_\pi)^I}{1 - 2\,i(M_0^0)^I}, \tag{9}$$

relating the pion scalar FF on the second Riemann sheet with the pion scalar FF and the S-wave isoscalar $\pi\pi$

scattering amplitude on the first Riemann sheet, demonstrating in this way that the singular point of the pion scalar FF at $t = 4m_\pi^2$ is the square root branch point, generating the two-sheeted Riemann surface on which the pion scalar FF is defined.

Moreover, by a comparison of (8) with (9), one can see that both expressions have identical denominators, from which it automatically follows that if there are 0^{++} scalar mesons in the form of the poles of the S-wave isoscalar $\pi\pi$ scattering amplitude on the second Riemann sheet, then they also appear as poles on the second Riemann sheet of the pion scalar FF.

Now, by an application of the conformal mapping

$$q = [(t-4)/4]^{1/2}, \quad m_\pi = 1, \tag{10}$$

the two-sheeted Riemann surface of $\Gamma_\pi(t)$ in the t variable is mapped into one absolute valued pion c.m. three-momentum q plane and the elastic cut disappears. Neglecting branch points beyond 1 GeV^2, there are only poles and zeros of $\Gamma_\pi(t)$ in the q plane and, as a consequence, the pion scalar FF can be represented by a Padé-type approximation,

$$\Gamma_\pi(t) = \frac{\sum_{n=0}^{M} a_n q^n}{\prod_{i=1}^{N}(q-q_i)}. \tag{11}$$

Because $\Gamma_\pi(t)$ is a real analytic function, the coefficients a_n in (11) with M even (odd) are real (purely imaginary), and the poles q_i either can appear on the imaginary axis or the two of them are placed symmetrically along it.

If one multiplies both the numerator and the denominator of (11) by the complex conjugate factor $\prod_{i=1}^{N}(q-q_i)^*$, the new denominator is a polynomial with real coefficients already and a tangent of the pion scalar FF phase $\delta_\Gamma(t)$ is given simply by the ratio of the imaginary part to the real part of the new numerator, as follows:

$$\tan\delta_\Gamma(t) = \frac{\mathrm{Im}[\prod_{i=1}^{N}(q-q_i)^* \sum_{n=1}^{M} a_n q^n]}{\mathrm{Re}[\prod_{i=1}^{N}(q-q_i)^* \sum_{n=1}^{M} a_n q^n]}. \tag{12}$$

Further, by using the identity $\delta_\Gamma = \delta_0^0$ following from (5) and the threshold behavior of δ_0^0, the following parametrization,

$$\tan\delta_0^0(t) = \frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots}, \tag{13}$$

or the equivalent relation

$$\delta_0^0(t) = \frac{1}{2\,i}\ln\frac{(1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots) + i(A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots)}{(1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots) - i(A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots)} \tag{14}$$

is obtained from (12), where A_i are all real new coefficients. The parameter A_1 is exactly equal to the S-wave isoscalar $\pi\pi$ scattering length a_0^0.

One can see directly from (13) that if the degree of the numerator is higher than the degree of its denominator, then

$$\lim_{q\to\infty}\delta_0^0(t) = \frac{\pi}{2}. \tag{15}$$

However, if the degree of the numerator in (13) is lower than the degree of its denominator, then

$$\lim_{q\to\infty}\delta_0^0(t) = 0. \tag{16}$$

The above-mentioned asymptotic behaviors cannot be solved beforehand and only a comparison of (13) with data on $\delta_0^0(t)$ can determine what type of pion scalar FF phase representations—ones derived from either the dispersion relation with one subtraction or the dispersion relation without subtractions—will be suitable in our further considerations.

III. ANALYSIS OF S-WAVE ISOSCALAR $\pi\pi$ SCATTERING PHASE SHIFT DATA

In order to obtain an explicit algebraic form of the S-wave isoscalar $\pi\pi$ scattering phase shift $\delta_0^0(t)$ in the q variable with the final number of real coefficients in (13), one has to compare the corresponding relation with the existing experimental information on $\delta_0^0(t)$ in the elastic region $4m_\pi^2 < t < 1$ GeV^2 and to find a minimal value of χ^2/ndf. The existing data, however, in comparison, e.g., with P-wave isovector $\pi\pi$ scattering phase shift $\delta_1^1(t)$, are very scattered, and some points are even inconsistent with each other. As will be demonstrated despite using such scattered data on $\delta_0^0(t)$, one can obtain some reasonable results in the framework of the fully solvable mathematical scheme elaborated on in this paper.

The majority of experimental data on $\delta_0^0(t)$ in the approximate region from $t = 0.26$ GeV^2 up to $t = 0.94$ GeV^2 [9–11] were obtained in the 1970s. They have been supplemented at the same approximate region by the data [12] obtained at the beginning of this century by a joint analysis of the CERN-Munich experiment [9,13] with the reaction $\pi^- p \to \pi^+\pi^- n$, at 17.2 GeV$/c$, and the BNL-E852 Collaboration [14] with the reaction $\pi^- p \to \pi^0\pi^0 n$, at

18.3 GeV/c. The data in [12] are more or less consistent with the data of Estabrooks and Martin [11]; however, both set [11] and set [12] are approximately at the region $0.54\ \mathrm{GeV}^2 < t < 0.84\ \mathrm{GeV}^2$, in disagreement with data of Protopopescu [10].

An invaluable set of data on $\delta_0^0(t)$ at the threshold region [15,16] has been obtained by measuring the difference $\delta \equiv \delta_0^0 - \delta_1^1$ at the K_{e4} $[K^\pm \to \pi^+\pi^- e^\pm \nu_e(\bar{\nu}_e)]$ decays. Then precise experimental points with isospin correction obtained by the NA48/2 Collaboration [16] appear to be especially important in obtaining the correct value of the S-wave isoscalar $\pi\pi$ scattering length a_0^0 [in our parametrization (13) corresponding to the coefficient A_1] in order to coincide with the very precise theoretical prediction in the framework of the χPT, as well as the determination of the correct value of the $f_0(500)$ parameters. The latter was demonstrated in detail by I. Caprini in [17] by an analytic extrapolation of the data on δ_0^0 into the region of $f_0(500)$ meson resonance.

The data on $\delta_0^0(t)$ from the measured difference $\delta_0^0(t) - \delta_1^1(t)$ [15,16] at K_{e4} decays have been obtained by using the best fit of the $\delta_1^1(t)$ data in [4].

As a result, the set of 95 experimental points in the elastic region $4m_\pi^2 < t < 1\ \mathrm{GeV}^2$ has been collected and used for a determination of a final number of real coefficients in the relation

$$\delta_0^0(t) = \arctan \frac{A_1 q + A_3 q^3 + A_5 q^5 + A_7 q^7 + \ldots}{1 + A_2 q^2 + A_4 q^4 + A_6 q^6 + \ldots}, \qquad (17)$$

following from (13), by means of the χ^2 minimization method in the framework of the computer program MINUIT.

The analysis has been carried out successively, starting with the first nonzero coefficient A_1 and then repeating the optimal description of the data, always adding the next coefficient to be different from zero. So, the data with a one, two, three, etc., parameter expression (17) have been analyzed up to the moment when the minimum of χ^2/ndf is achieved.

The results are summarized in the following table:

Number of A_i	χ^2/ndf
1	57.43
2	3.44
3	2.79
4	2.19
5	1.69
6	1.71
7	1.73

One can see from the table that the minimum of χ^2/ndf is achieved with five nonzero coefficients in (17). The value of the first coefficient is found to be 0.20820 ± 0.02491, very close to the predicted value $a_0^0 = 0.2200 \pm 0.0050$ by χPT [18]; therefore, we have fixed the first coefficient in (17) at the latter value and have repeated the minimization procedure. Then the five coefficients in (17) take the values

$$\begin{aligned} A_1 &= 0.22000 \pm 0.00500; \\ A_3 &= 0.23167 \pm 0.01268; \\ A_5 &= -.016901 \pm 0.00204 \\ A_2 &= 0.10841 \pm 0.13194; \\ A_4 &= -.02702 \pm 0.01564, \end{aligned} \qquad (18)$$

giving the description of the data on $\delta_0^0(t)$ in the elastic region presented in Fig. 1 by a solid line.

The latter result [see (15)] requires us to start a construction of the explicit form of the pion scalar FF by the dispersion relation with at least one subtraction,

$$\Gamma_\pi(t) = 1 + \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\mathrm{Im}\Gamma_\pi(t')}{t'(t'-t)} dt'. \qquad (19)$$

Of course, a dispersion relation with a larger number of substraction constants could be more effective, as we are restricted to the elastic region in our fully solvable mathematical scheme. However, as we mentioned earlier, the pion scalar FF is not an experimentally measurable quantity and we know only its normalization point from χPT, by using of which the dispersion relation (19) with one subtraction is derived.

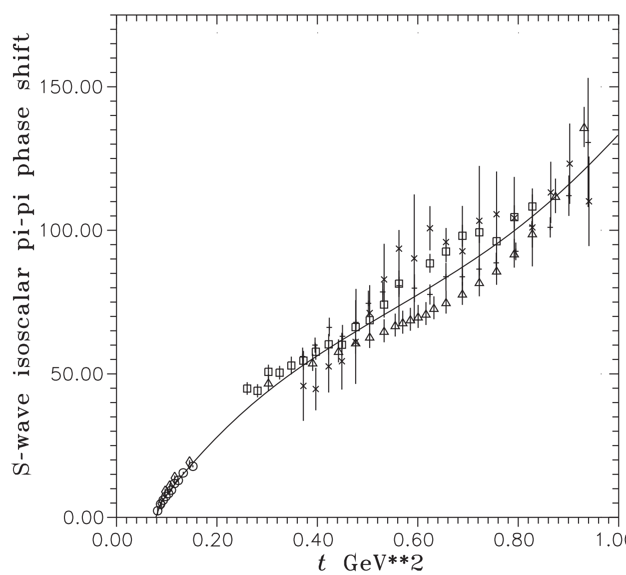

FIG. 1. Description of the S-wave isoscalar $\pi\pi$ phase shift data by the [5/4] Padé-type approximation with the values of parameters (18). (circle) J. R. Batley *et al.*, (diamond) S. Pislak *et al.*, (cross) R. Kaminski *et al.*, (square) P. Estabrooks and A. D. Martin, (triangle) S. D. Protopopescu *et al.*, and (plus) B. Hyams *et al.*

IV. THE PHASE REPRESENTATION AND EXPLICIT FORM OF THE PION SCALAR FORM FACTOR

Now, substituting the pion scalar FF elastic unitarity condition (5) into the dispersion relation with one subtraction (19), one finds the Muskhelishvili-Omnes integral equation,

$$\Gamma_\pi(t) = 1 + \frac{t}{\pi}\int_{4m_\pi^2}^{\infty} \frac{\Gamma_\pi(t')e^{-i\delta_0^0}\sin\delta_0^0}{t'(t'-t)}dt', \tag{20}$$

the solution of which is the pion scalar FF phase representation with one subtraction

$$\Gamma_\pi(t) = P_n(t)\exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta_0^0(t')}{t'(t'-t)}dt'\right], \tag{21}$$

where $P_n(t)$ is an arbitrary polynomial to be restricted with the normalization $P_n(0) = 1$ and its degree must not be higher than $\delta_0^0(\infty)/\pi$.

The substitution of $\delta_0^0(t)$ (17) in the equivalent form (14) with the five above-mentioned nonzero coefficients into the pion scalar FF phase representation (21) leads to the expression

$$\Gamma_\pi(t) = P_n(t)\exp\frac{(q^2+1)}{\pi i} \times \int_0^\infty \frac{q'\ln\frac{(1+A_2q'^2+A_4q'^4)+i(A_1q'+A_3q'^3+A_5q'^5)}{(1+A_2q'^2+A_4q'^4)-i(A_1q'+A_3q'^3+A_5q'^5)}}{(q'^2+1)(q'^2-q^2)}dq', \tag{22}$$

in which $m_\pi = 1$ is assumed. Taking into account the fact that the integrand is an even function in its argument, i.e., it is invariant under the transformation $q' \to -q'$, the latter expression can be transformed into the following form:

$$\Gamma_\pi(t) = P_n(t)\exp\frac{(q^2+1)}{2\pi i} \times \int_{-\infty}^\infty \frac{q'\ln\frac{(1+A_2q'^2+A_4q'^4)+i(A_1q'+A_3q'^3+A_5q'^5)}{(1+A_2q'^2+A_4q'^4)-i(A_1q'+A_3q'^3+A_5q'^5)}}{(q'^2+1)(q'^2-q^2)}dq', \tag{23}$$

where the integral is already suitable to be calculated by means of the theory of residua.

In order to carry out this program one has to identify all poles of the integrand and simultaneously calculate complex roots of the polynomial in the numerator and complex conjugate roots in the denominator under the logarithm, which generates branch points in the q plane.

Considering the case $q^2 < 0$, i.e., $q = i\sqrt{\frac{4-t}{4}} \equiv ib$, one finds the poles of the integrand in $q' = \pm i$ and $q' = \pm ib$.

Concerning the roots of the polynomials under the logarithm, it is clear that it is enough to investigate the roots of the numerator, as the roots of the denominator are complex conjugate to the roots of the numerator.

So, let us start with an investigation of the numerator $(1 + A_2q'^2 + A_4q'^4) + i(A_1q' + A_3q'^3 + A_5q'^5) = 0$.

In order to have an equation with real coefficients, one substitutes $q' = ix$.

Then, $1 - A_1x - A_2x^2 + A_3x^3 + A_4x^4 - A_5x^5 = 0$ or $-x^5 + \frac{A_4}{A_5}x^4 + \frac{A_3}{A_5}x^3 - \frac{A_2}{A_5}x^2 - \frac{A_1}{A_5}x + \frac{1}{A_5} = 0$.

Solutions of the latter equation are straightforward, where one finds roots of the numerator and the denominator under the logarithm of integrand $\phi(q', q)$ to be

$$\begin{aligned} q_1 &= -i1.45626,\\ q_2 &= -3.67646 + i0.40771,\\ q_3 &= -1.17396 + i1.27308,\\ q_4 &= 3.67646 + i0.40771,\\ q_5 &= 1.17396 + i1.27308, \end{aligned} \tag{24}$$

and

$$\begin{aligned} q_1^* &= -q_1,\\ q_2^* &= -q_4,\\ q_3^* &= -q_5,\\ q_4^* &= -q_2,\\ q_5^* &= -q_3, \end{aligned} \tag{25}$$

respectively.

Then the integral in (23) takes the form

$$I = \int_{-\infty}^{\infty} \frac{q'\ln\frac{(q'-q_1)(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)}dq', \tag{26}$$

with all of the singularities of its integrand to be explicitly presented in Fig. 2.

For an explicit calculation of the latter integral (26), it is convenient to split it into a sum of two integrals

$$I = \int_{-\infty}^{\infty}\frac{q'\ln\frac{(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)}dq' + \int_{-\infty}^{\infty}\frac{q'\ln\frac{(q'-q_1)}{(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)}dq' = I_1 + I_2,$$

according to singularities to be placed in the upper or lower half-plane, respectively.

Let us start to calculate the first integral I_1 by the theory of residua,

$$\oint \frac{q' \ln \frac{(q'-q_2)(q'-q_3)(q'-q_4)(q'-q_5)}{(q'-q_1^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n, \quad (27)$$

where the contour of integration is closed in the upper half-plane (see Fig. 2).

As the integral on the half-circle is 0, then

$$I_1 = \int_{-\infty}^{\infty} \phi_1(q')dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n - \left[-\int_{1^*} + \int_2 + \int_3 + \int_4 + \int_5 \right], \quad (28)$$

where the integrals on the right-hand side represent the contributions of the cuts generated by the branch points $q_1^*, q_2, q_3, q_4, q_5$ in Fig. 2.

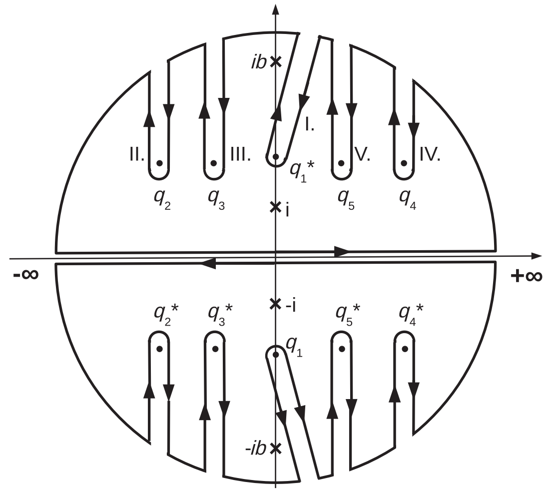

FIG. 2. Poles (cross) and branch points (circle) of the integrands $\phi_1(q')$ and $\phi_2(q')$, with contours of integrations in the upper and lower half-planes, respectively.

The residua at the poles $q' = i, q' = ib$ are straightforward to calculate and they are

$$\mathrm{Res}\phi_1(i,q) = -\frac{1}{2(q^2+1)} \ln \frac{(i-q_2)(i-q_3)(i-q_4)(i-q_5)}{(i-q_1^*)}, \quad (29)$$

$$\mathrm{Res}\phi_1(ib,q) = \frac{1}{2(q^2+1)} \ln \frac{(q-q_2)(q-q_3)(q-q_4)(q-q_5)}{(q-q_1^*)}, \quad (30)$$

as $ib = q$.

Now we move on to the contributions of the cuts. Let us start with the contribution of the cut to be generated by the branch point q_1^*:

$$\begin{aligned}\int_{1^*} &= \int_{\infty}^{q_1^*} \frac{q' \ln_+(q'-q_1^*)}{(q'^2+1)(q'^2+b^2)} dq' + \int_{q_1^*}^{\infty} \frac{q' \ln_-(q'-q_1^*)}{(q'^2+1)(q'^2+b^2)} dq' \\ &= \int_{q_1^*}^{\infty} \frac{q'}{(q'^2+1)(q'^2+b^2)} [\ln_-(q'-q_1^*) - \ln_+(q'-q_1^*)] dq' \\ &= -2\pi i \int_{q_1^*}^{\infty} \frac{q'}{(q'^2+1)(q'^2+b^2)} dq' \\ &= -\frac{\pi i}{(b^2-1)} \ln \frac{(q_1^{*2}+b^2)}{(q_1^{*2}+1)} \equiv \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q_1^{*2}-q^2)}{(q_1^{*2}+1)}. \end{aligned} \quad (31)$$

Similarly,

$$\int_j = -\frac{\pi i}{(b^2-1)} \ln \frac{(q_j^2+b^2)}{(q_j^2+1)} \equiv \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q_j^2-q^2)}{(q_j^2+1)}; \qquad j = 2,3,4,5. \quad (32)$$

Then the sum of all of these partial results according to (28) gives the final result for I_1 in the form

$$I_1 = \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q+q_1^*)}{(q+q_2)(q+q_3)(q+q_4)(q+q_5)} \frac{(i+q_2)(i+q_3)(i+q_4)(i+q_5)}{(i+q_1^*)}. \quad (33)$$

Similarly, one can also calculate the second integral I_2 by means of the theory of residua:

$$\oint \frac{q' \ln \frac{(q'-q_1)}{(q'-q_2^*)(q'-q_3^*)(q'-q_4^*)(q'-q_5^*)}}{(q'+i)(q'-i)(q'+ib)(q'-ib)} dq' = 2\pi i \sum_{n=1}^{2} \mathrm{Res}_n, \tag{34}$$

where the contour of integration is closed in the lower half-plane (see Fig. 2).

As the integral on the half-circle is 0, then

$$I_2 = \int_{-\infty}^{\infty} \phi_2(q')dq' = -2\pi i \sum_{n=1}^{2} \mathrm{Res}_n + \left[+ \int_1 - \int_{2^*} - \int_{3^*} - \int_{4^*} - \int_{5^*} \right]. \tag{35}$$

The residua at the poles $q' = -i, q' = -ib$ take the form

$$\mathrm{Res}\phi_2(-i,q) = -\frac{1}{2(q^2+1)} \ln \frac{(-i-q_1)}{(-i-q_2^*)(-i-q_3^*)(-i-q_4^*)(-i-q_5^*)}, \tag{36}$$

$$\mathrm{Res}\phi_2(-ib,q) = \frac{1}{2(q^2+1)} \ln \frac{(-q-q_1)}{(-q-q_2^*)(-q-q_3^*)(-q-q_4^*)(-q-q_5^*)}, \tag{37}$$

as $ib = q$.

The contribution of the cut to be generated by the branch point q_1 is

$$\begin{aligned} \int_1 &= \int_{\infty}^{q_1} \frac{q' \ln_+(q'-q_1)}{(q'^2+1)(q'^2+b^2)} dq' + \int_{q_1}^{\infty} \frac{q' \ln_-(q'-q_1)}{(q'^2+1)(q'^2+b^2)} dq' \\ &= \int_{q_1}^{\infty} \frac{q'}{(q'^2+1)(q'^2+b^2)} [\ln_-(q'-q_1) - \ln_+(q'-q_1)] dq' \\ &= -2\pi i \int_{q_1}^{\infty} \frac{q'}{(q'^2+1)(q'^2+b^2)} dq' \\ &= -\frac{\pi i}{(b^2-1)} \ln \frac{(q_1^2+b^2)}{(q_1^2+1)} \equiv \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q_1^2-q^2)}{(q_1^2+1)}. \end{aligned} \tag{38}$$

Similarly,

$$\int_{j^*} = -\frac{\pi i}{(b^2-1)} \ln \frac{(q_j^{*2}+b^2)}{(q_j^{*2}+1)} \equiv \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q_j^{*2}-q^2)}{(q_j^{*2}+1)}; \qquad j = 2,3,4,5. \tag{39}$$

Then the sum of all of these partial results according to (35) gives the comprehensive result for I_2 in the form

$$\begin{aligned} I_2 = \frac{1}{2} \frac{2\pi i}{(q^2+1)} \Bigg[&\ln \frac{(q+q_1)}{(q+q_2^*)(q+q_3^*)(q+q_4^*)(q+q_5^*)} \frac{(i+q_2^*)(i+q_3^*)(i+q_4^*)(i+q_5^*)}{(i+q_1)} \\ &+ \ln \frac{q_1^2-q^2}{q_1^2+1} - \ln \frac{q_2^{*2}-q^2}{q_2^{*2}+1} - \ln \frac{q_3^{*2}-q^2}{q_3^{*2}+1} - \ln \frac{q_4^{*2}-q^2}{q_4^{*2}+1} - \ln \frac{q_5^{*2}-q^2}{q_5^{*2}+1} \Bigg], \end{aligned} \tag{40}$$

from which by using the relations (25) one finally gets

$$I_2 = \frac{1}{2} \frac{2\pi i}{(q^2+1)} \ln \frac{(q+q_1^*)}{(q+q_2)(q+q_3)(q+q_4)(q+q_5)} \frac{(i+q_2)(i+q_3)(i+q_4)(i+q_5)}{(i+q_1^*)}. \tag{41}$$

The sum of (41) with (33) represents the total integral,

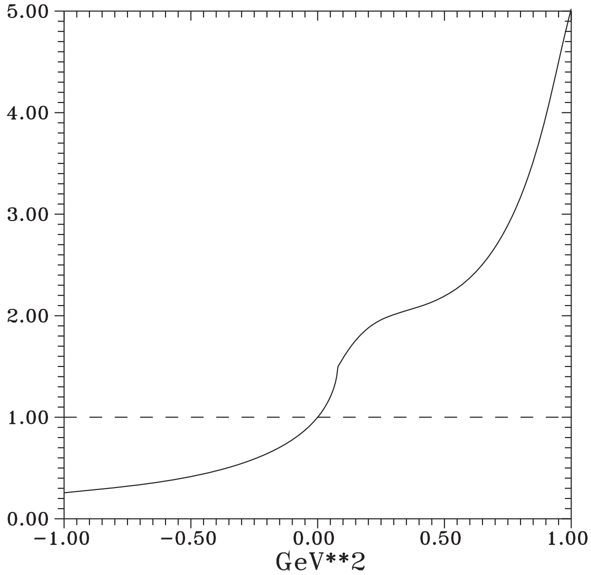

FIG. 3. Behavior of the pion scalar form factor in the region $-1\ \mathrm{GeV}^2 < t < 1\ \mathrm{GeV}^2$.

$$I = \frac{2\pi i}{(q^2+1)} \ln \frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)(q+q_5)} \times \frac{(i+q_2)(i+q_3)(i+q_4)(i+q_5)}{(i-q_1)}. \tag{42}$$

If the latter is substituted into the pion scalar FF phase representation (23), one obtains an explicit form for the pion scalar FF $\Gamma_\pi(t)$:

$$\Gamma_\pi(t) = P_n(t) \frac{(q-q_1)}{(q+q_2)(q+q_3)(q+q_4)(q+q_5)} \times \frac{(i+q_2)(i+q_3)(i+q_4)(i+q_5)}{(i-q_1)}, \tag{43}$$

with one zero and four poles whose behavior is graphically presented in Fig. 3.

The $-q_3$ pole of $\Gamma_\pi(t)$ on the second Riemann sheet in the t variable clearly corresponds to the $f_0(500)$ meson, as its mass and width are determined to be $m_{f_0(500)} = (388 \pm 23)$ MeV and $\Gamma_{f_0(500)} = (602 \pm 67)$ MeV, respectively. Investigating additional poles of $\Gamma_\pi(t)$ in (43), one finds that the $-q_2$ pole must correspond to the $f_0(980)$ meson, with $m_{f_0(980)} = (1066 \pm 142)$ MeV and $\Gamma_{f_0(980)} = (220 \pm 194)$ MeV. The errors correspond to the transferred errors of the coefficients of (18).

Finally one can only say that, if more precise data on the S-wave isoscalar $\pi\pi$ scattering phase shift is available, more precise parameters of mesons under consideration can be found in the framework of our fully solvable mathematical scheme.

V. CONCLUSIONS

In this paper we have elaborated for the pion scalar FF in the elastic region on the fully solvable mathematical scheme, in the framework of which we have demonstrated another confirmation of the existence of the $f_0(500)$ scalar meson resonance, whereby we did not pretend to make a determination of its worldwide parameters. In order to find an explicit form of the pion scalar FF, we have used the dispersion relation with one substraction, whereby the substraction constant has been taken to be the normalization of the pion scalar FF at $t = 0$, well known from χPT. A combination of this dispersion relation with the elastic unitarity condition led to the phase representation of the pion scalar FF, whereby the corresponding phase under the integral has been identified with S-wave isoscalar $\pi\pi$ phase shift $\delta_0^0(t)$. For the arctan of the latter, starting from fully general considerations, a sophisticated parametrization in the absolute valued pion c.m. three-momentum variable q has been found. By a comparison of such a parametrization with existing experimental information on δ_0^0 and the calculation of the integral in the phase representation, one finds the pion scalar FF in the form of a rational function with one zero and four poles, whereby two of them conjugated according to the imaginary axis of the q plane clearly correspond to the $f_0(500)$ scalar meson resonance. The latter alone can be considered to be another proof of the $f_0(500)$ scalar meson's existence.

ACKNOWLEDGMENTS

The support of the Slovak Grant Agency for Sciences VEGA under Grant No. 2/0158/13 and of the Slovak Research and Development Agency under Contract No. APVV-0463-12 is acknowledged.

[1] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).
[2] I. Caprini, G. Colangelo, and H. Leutwyler, Phys. Rev. Lett. **96**, 132001 (2006).
[3] R. Garcia-Martin, R. Kaminski, J. R. Pelaez, and J. Ruiz de Elvira, Phys. Rev. Lett. **107**, 072001 (2011).
[4] S. Dubnicka, πN Newsletter **11**, 167 (1995).
[5] J. Gasser and U.-G. Meissner, Nucl. Phys. **B357**, 90 (1991).
[6] M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Phys. Lett. **78B**, 443 (1978).
[7] M. B. Voloshin and V. I. Zakharov, Phys. Rev. Lett. **45**, 688 (1980).

[8] B. R. Martin, D. Morgan, and G. Shaw, *Pion-Pion Interactions in Particle Physics* (Academic Press, New York, 1976).
[9] B. Hyams *et al.*, Nucl. Phys. **B64**, 134 (1973).
[10] S. Protopopescu, M. Alston-Garnjost, A. Barbaro-Galtieri, S. Flatté, J. Friedman, T. Lasinski, G. Lynch, M. Rabin, and F. Solmitz, Phys. Rev. D **7**, 1279 (1973).
[11] P. Estabrooks and A. D. Martin, Nucl. Phys. **B79**, 301 (1974).
[12] R. Kaminski, L. Leśniak, and K. Rybicki, Eur. Phys. J. direct C **4**, 1 (2002).
[13] G. Grayer *et al.*, Nucl. Phys. **B75**, 189 (1974).
[14] J. Gunter *et al.* (E852 Collaboration), Phys. Rev. D **64**, 072003 (2001).
[15] S. Pislak *et al.* (BNL-E865 Collaboration), Phys. Rev. Lett. **87**, 221801 (2001).
[16] J. R. Batley *et al.* (NA48/2 Collaboration), Eur. Phys. J. C **70**, 635 (2010).
[17] I. Caprini, Phys. Rev. D **77**, 114019 (2008).
[18] G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. **B603**, 125 (2001).