



# BPS degeneracies and superconformal index in diverse dimensions

Amer Iqbal<sup>1,2,\*</sup> and Cumrun Vafa<sup>3,†</sup>

<sup>1</sup>*Department of Physics, LUMS School of Science & Engineering, U-Block, D.H.A, Lahore 54792, Pakistan*

<sup>2</sup>*Department of Mathematics, LUMS School of Science & Engineering,  
U-Block, D.H.A, Lahore 54792, Pakistan*

<sup>3</sup>*Jefferson Physical Laboratory, Harvard University, Cambridge, Massachusetts 02138, USA*

(Received 3 June 2014; published 26 November 2014)

We present a unifying theme relating BPS partition functions and superconformal indices. In the case with complex supersymmetry central charges (as in  $\mathcal{N} = 2$  in  $d = 4$  and  $\mathcal{N} = (2, 2)$  in  $d = 2$ ), the known results can be reinterpreted as the statement that the BPS partition functions can be used to compute a specialization of the superconformal indices. We argue that in the case with real central charge in the supersymmetry algebra, as in  $\mathcal{N} = 1$  in  $d = 5$  (or the  $\mathcal{N} = 2$  in  $d = 3$ ), the BPS degeneracy captures the full superconformal index. Furthermore, we argue that refined topological strings, which capture five-dimensional (5d) BPS degeneracies of M theory on Calabi-Yau 3-folds, can be used to compute a 5d supersymmetric index including in the sectors with three-dimensional defects for a large class of 5d superconformal theories. Moreover, we provide evidence that distinct Calabi-Yau singularities which are expected to lead to the same Superconformal field theory yield the same index.

DOI: [10.1103/PhysRevD.90.105031](https://doi.org/10.1103/PhysRevD.90.105031)

PACS numbers: 12.60.Jv, 11.10.Kk

## I. INTRODUCTION

Supersymmetric BPS states have played an important role in many aspects of string theory. Their mass is typically protected by supersymmetry (SUSY) and provides a tool to analyze various limits of string theory. On the other hand, superconformal theories have also figured prominently in many developments of string theory. As we deform conformal theory away from the conformal point, (BPS) states arise in the resulting theory. It is natural to ask what is the relation between BPS states that appear and the properties of the superconformal theory they come from. In fact, there is evidence that the BPS spectrum away from the conformal point is faithful, and the superconformal theories are entirely captured by the BPS spectrum. In particular, we do not have a single example of two distinct superconformal theories which give the same BPS spectrum upon deformation. Of course, the arbitrary BPS spectrum does not give a consistent theory, and consistency conditions on what the allowed BPS states can be have been proposed as a way to classify conformal theories for  $\mathcal{N} = (2, 2)$  in  $d = 2$  [1] and  $\mathcal{N} = 2$  in  $d = 4$  [2,3]. If this is the case, it should be possible to recover all the data at the conformal fixed point solely from the BPS data. In particular, it is natural to ask if the superconformal partition functions such as supersymmetric indices [4–6] are reproducible from the BPS spectrum.

The most natural idea would be to treat BPS states as if they are the elementary building blocks of the theory and use them to compute the superconformal partition

functions. However, the story is not always so simple. For example, for theories with complex central charge, the BPS spectrum has different chambers separated by walls. Nevertheless, as we will review (and partially reinterpret), it is known that at least in the cases of  $d = 2$  with  $\mathcal{N} = (2, 2)$  [2] and  $d = 4$  with  $\mathcal{N} = 2$  [7] a specialization of the superconformal index can be recovered from the BPS spectrum in any chamber.

We will provide evidence that the situation is similar but more powerful in the case of theories in  $d = 3, 5$  dimensions with a Coulomb branch, with  $\mathcal{N} = 2, 1$  supersymmetries, respectively. Both of these cases involve a real central charge. In these cases, we propose that one can recover the full superconformal index solely from the BPS data in a Coulomb branch of the theory. In the case of  $d = 3$ , we reinterpret the computations already done as computing contributions from BPS states. The main new case involves the superconformal index in  $d = 5$ .

The basic class of examples we consider is obtained from M theory on Calabi-Yau 3-folds leading to  $\mathcal{N} = 2$  theories in  $d = 5$  dimensions. It is known that for these cases the topological string captures the BPS degeneracies (corresponding to M2-branes wrapping 2-cycles) [8,9]. In addition, one can introduce M5-branes wrapping Lagrangian submanifolds of Calabi-Yau. These lead to three-dimensional defects in the five-dimensional (5d) theory. Furthermore, it is known that open topological strings capture the open BPS state degeneracy for these sectors corresponding to M2-branes ending on M5-branes [10,11]. We will argue that superconformal index, i.e., the partition function on  $S^1 \times S^4$  where the three-dimensional (3d) defects wrap  $S^1 \times S^2$  where  $S^2 \subset S^4$ , can be computed simply by considering

\*amer.iqbal@lums.edu.pk

†vafa@physics.harvard.edu

$$\int \frac{dQ_i dU_j}{Q_i U_j} |Z_{\text{top}}(Q_i, U_j, \tilde{Q}_k; q_1, q_2)|^2,$$

$$\text{Tr}M = Z(S^1 \times S^{d-1}) \tag{1.2}$$

where  $Z_{\text{top}}$  is the refined open and closed topological string amplitudes;  $Q_i$  correspond to the Wilson line associated with normalizable Kähler moduli of Calabi–Yau;  $\tilde{Q}_k$  is the non-normalizable Kähler moduli, which correspond to mass parameters;  $U_j$  correspond to the Wilson lines for the Lagrangian branes; and  $(q_1, q_2)$  are the two coupling constants of the refined topological string. Here, complex conjugation sends<sup>1</sup>  $(Q_i, U_j, \tilde{Q}_k; q_1, q_2) \rightarrow (Q_i^{-1}, U_j^{-1}, \tilde{Q}_k^{-1}; q_1^{-1}, q_2^{-1})$ . Furthermore, this computation can be viewed as computing the scattering amplitudes of a string theory in four dimensions proposed recently [12].

A unifying theme seems to emerge about the connection of BPS states to the index, which can be summarized roughly as follows: We order the BPS states according to the phase of their BPS central charge. In the case of real central charge, this simply means dividing the BPS states to *CPT* conjugate pairs where one-half of the states are on right and the other on the left of the real line. In the case of complex central charge, this means organizing the states on a circle according to the phase of the central charge where *CPT* conjugate pairs are diametrically opposite. Whether it is real or complex central charge, we can consider a “partition function” of the BPS states where each BPS state  $i$  is represented by an operator  $\Phi_i$  and we take the product over all the BPS states. The operator acts on a different Hilbert space depending on the dimension and the theory in question: In the two-dimensional (2d) case, it involves the space of massive vacua; in the 3d and 5d cases, it is the space of flat connections on  $S^1$  for the corresponding Abelian gauge groups, and in the four-dimensional (4d) case, it is the Hilbert space of a  $U(1)$  Chern–Simons theory on the Seiberg–Witten curve.

In the complex central charge case,  $\Phi_i$  do not commute, and we have to order them according to the phase of the central charge in the SUSY algebra. In the real central charge case, they commute. Moreover, knowing the contribution for half the states is sufficient, because the *CPT* conjugate case can be obtained from them. Let

$$S = \prod_i \Phi_i$$

denote the (ordered) product over the BPS states for which the phase is on one side. The full partition over BPS states will take the form

$$M = SS^{-t}. \tag{1.1}$$

Then the statement is that

<sup>1</sup>As we will discuss later, for the defect sector, we can turn on monopole flux which would correspond to allowing  $U_j$  to be complex.

for suitably defined partition function  $Z$  of the theory on  $S^1 \times S^{d-1}$ . For  $d = 3, 5$ , this gives the full index, and for  $d = 2, 4$ , this gives a specialization of the index.

The intuitive idea for why such a picture holds may be that we can view operators at the conformal fixed point as being made of the composite of operators which create BPS states. In some cases where there is a weak coupling description of the theory, as in  $d = 3$  gauge theories, this picture can be fully justified.

The fact that we propose that the superconformal index in five dimensions can be computed only from the knowledge of BPS particles is surprising in the following sense: These theories also have BPS strings. If we go to the conformal point, we will have a system of interacting massless particles and tensionless strings. Upon going to the Coulomb branch, the particles pick up mass, and tensionless strings pick up tension. Moreover, the mass scales for both the interacting strings and the particles are the same [13]. What is surprising is that nevertheless the knowledge of only BPS particles is enough to recapture the full superconformal index in five dimensions. Perhaps this can be explained by the fact that  $S^1 \times S^4$  has no 2-cycles for the worldsheet of BPS strings to wrap around and the properties of the BPS strings are secretly encoded by the particle states, as far as the index is concerned.

The organization of this paper is as follows. In Sec. II, we discuss the relation between superconformal indices and BPS states in the two- and four-dimensional theories with complex central charges. In Sec. III, we discuss the three- and five-dimensional gauge theories with real central charges, superconformal indices, and their relation with BPS states including coupling to the 3d defects. In Sec. IV, we review the refinement of topological strings and how the refined amplitudes can be calculated. In Sec. V, we discuss some properties of the 5-branes webs which give rise to superconformal field theories. In Sec. VI, we give some examples of index computations for certain 5d theories coming from local CY 3-folds including in the presence of 3d defects. In Sec. VII, we present our conclusions.

## II. BPS STATES AND THEORIES WITH COMPLEX CENTRAL CHARGE IN $d = 2, 4$

In this section, we review (and partially reinterpret) what is known for the relation between BPS states and superconformal partition functions in the case of  $\mathcal{N} = (2, 2)$  theories in  $d = 2$  and  $\mathcal{N} = 2$  theories in  $d = 4$ .

### A. $\mathcal{N} = (2, 2)$ theories in $d = 2$

Consider an  $\mathcal{N} = (2, 2)$  conformal theory in  $d = 2$ . In this context, we can define a superconformal index (which is an elliptic genus) [14] given by the following trace in the Ramond sector:

$$Z(q, z) = \text{Tr}(-1)^F z^{J_L} q^{H_L} \bar{q}^{H_R},$$

where  $J_L$  is the left-moving  $U(1)_R$  charge and  $F = F_L - F_R$ ,  $F_{L,R}$  being the fermion numbers of the left and the right movers. Since the Ramond sector is supersymmetric, by the SUSY argument as in the Witten index, the above index only depends on  $q, z$  and is independent of moduli of conformal theory. It receives contributions from all the states which are ground states of the  $H_R$ , and it is an arbitrary eigenstate of  $H_L$ . Note that in the limit  $q \rightarrow 0$  this receives contribution only from the ground state  $H_L = H_R = 0$ . In this case,  $Z(0, z)$  simply computes the partition function of the ground states in the Ramond sector weighted by their  $R$  charge  $J_L$ .

We will consider a subset of  $\mathcal{N} = (2, 2)$  theories which admits deformations which flow in the IR to a trivial theory. For this to be possible, in particular  $J_L - J_R \in \mathbf{Z}$ , and the ground states have equal  $J_L, J_R$  charges. The index of such theories, which is also equal to the number of distinct vacua upon mass deformations, is  $N = Z(0, 1)$ .

For special values of  $z$ , the index simplifies and becomes  $q$  independent: Let

$$z = \exp(2\pi i k);$$

note that since  $J_L$  is not necessarily an integer, putting  $z = \exp(2\pi i k)$  is not the same as  $z = 1$ . Moreover, in this limit, the left-moving supercharges also commute with the elements in the trace, and the partition function is  $q$  independent and in particular can be evaluated by taking the  $q \rightarrow 0$  limit:

$$Z(q, \exp(2\pi i k)) = Z(0, \exp(2\pi i k)) = Z_k.$$

In particular, as shown in Ref. [15] using spectral flow,  $Z_k$  counts the index of the theory relative to  $(G_k^+, \bar{G}_0^+)$ , where  $(G, \bar{G})$  refer to (left, right)-moving supercharges in the Ramond sector.

This theory will have BPS kinks connecting the various vacua. The number of kinks depends on how we deform the superconformal theory to the massive ones, and there are domain walls in parameter space where the BPS degeneracies change [16]. Let  $m_{ij}$  be the number of kinks connecting the  $i$ th vacuum to the  $j$ th one, taking into account the  $(-1)^F$  acting on the lowest state of the multiplet. BPS kinks come with complex central charges. Order the vacua such that the phase of the corresponding central charges  $Z_{i,i+1}$  goes counterclockwise as we increase  $i$ . In this basis, let  $A$  be the upper triangular matrix given by  $A_{ij} = m_{ij}$  for each  $i < j$ . Consider the matrix

$$S = 1 - A,$$

and furthermore construct the matrix

$$M = SS^{-t} = (1 - A) \cdot \frac{1}{1 - A^t}, \tag{2.1}$$

where  $S^{-t}$  is the inverse transpose of  $S$ . Since  $A$  is upper triangular, we have

$$S^{-t} = 1 + A^t + A^{2t} + \dots + A^{(N-1)t}.$$

The wall crossing formula for the BPS states [16] implies that the eigenvalues of  $M$  do not depend on which chamber we compute it them (even though  $S$  does change). So the eigenvalues are purely a property of the conformal fixed point. Moreover, using  $tt^*$  equations [17], it was shown in Ref. [1] that<sup>2</sup>

$$\begin{aligned} \text{Tr} M^k &= \text{Tr}_{H=0} \exp(2\pi i k J_L) \\ &= Z_k. \end{aligned} \tag{2.2}$$

Moreover, this was used as a starting point of a program to classify  $\mathcal{N} = (2, 2)$  theories in  $d = 2$ . For a recent discussion of the meaning of this relation, see Ref. [18].

### 1. An example

As an example, consider the case of Landau-Ginzburg theory with superpotential  $W = \frac{1}{3}x^3$  for which a conformal fixed point is expected [19,20]. The spectrum of the  $R$  charges at the conformal point is  $\pm \frac{1}{6}$ . The chiral ring consists of  $\{1, x\}$ , and when the theory is deformed so that the superpotential becomes  $W = \frac{1}{3}x^3 - ax$ , we get two vacua for  $x_{\pm} = \pm \sqrt{a}$ . There is a single BPS kink connecting them; therefore,

$$S = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$



$M = SS^{-t}$  has two eigenvalues  $\exp(\pm \frac{2\pi i}{6})$  which agrees with the spectrum of the  $R$  charges of the theory at the conformal point.

It is interesting to note that Eq. (2.1) has the structure of the partition function of fermions and bosons. It is as if we are constructing composite operators from the fields creating the kinks. Moreover, consider the kink operators placed on a circle ordered by the phase of their central charge and the ones on the left semicircle are fermionic and the ones on the right half are bosonic. Then the  $\text{Tr} M$  can be viewed as the totality of operators we can make out of them which can be placed on a circle (i.e., start from one vacuum and end on the same vacuum). This structure will repeat, as we shall see in all the other dimensions where we connect BPS degeneracies with partition functions at superconformal points.

<sup>2</sup>Furthermore, it was shown how this can be refined to compute the  $Z(0, z)$  for arbitrary  $z$ .

**B. BPS states and  $\mathcal{N} = 2$  in  $d = 4$  dimensions**

The connection between degeneracies of BPS states for  $\mathcal{N} = 2$  theories in  $d = 4$  and certain partition functions at the superconformal point was found in Ref. [7]. We consider the theory in the background involving  $S^1 \times MC_q$  where  $MC_q$  is the Melvin cigar:  $MC_q$  is given by  $\mathbf{C} \times S^1$  where we rotate  $\mathbf{C}$  by  $q$  as we go around  $S^1$ . Moreover, as we go around the other  $S^1$ , we twist by  $t^{r-R}$  where  $r$  is the extra r-charge which is a symmetry at the conformal point and  $R$  is a Cartan in the  $SU(2)_R$ . The  $MC_q$  can be viewed topologically as  $\frac{1}{2}S^3$  with squashing parameter  $q$ . We will denote this by

$$MC_q = \frac{1}{2}S_q^3.$$

One considers the partition function on  $S^1 \times \frac{1}{2}S_q^3$  which can be represented in the operator formulation as (suppressing the irrelevant  $e^{-\beta H}$ )

$$Z(t, q) = \text{Tr}_{\frac{1}{2}S_q^3}(-1)^F t^{r-R}. \tag{2.3}$$

We now explain the relation of this partition function with the deformed theory. Each BPS state is characterized by a charge  $\gamma$  which belongs to the lattice of electric and magnetic charges. Note that this lattice has a canonical skew-symmetric product pairing the electric with the corresponding magnetic charges. Consider the quantum torus algebra given by introducing for each element  $\gamma$  of the lattice an operator  $U_\gamma$  satisfying<sup>3</sup>

$$U_\gamma U_\beta = q^{(\gamma, \beta)} U_\beta U_\gamma.$$

For each BPS state of charge  $\gamma$  and spin  $s$ , introduce the operator

$$\Phi(\gamma, s) = \prod_n (1 - q^{n+s+\frac{1}{2}} U_\gamma)^{(-1)^{2s}}. \tag{2.4}$$

Consider BPS states for which the central charges lie on the upper half-plane,

$$S = T \left( \prod_{\text{BPS-upper}} \Phi(\gamma_i, s_i) \right), \tag{2.5}$$

where  $T$  denote ordering the product in order of the increasing phase of the central charge as we go in the counterclockwise direction. Furthermore, consider the matrix

<sup>3</sup>When the  $\mathcal{N} = 2, d = 4$  theory is realized in terms of an M5-brane wrapping  $\Sigma \times S^1$  inside a CY 3-fold,  $S^1$  being the time direction, then BPS states are given by M2-branes bounding  $\gamma \in H_1(\Sigma, \mathbb{Z})$ , and  $U_\gamma$  is the holonomy of the gauge field coming from the B field on the M5-brane reduced along the cycles of  $\Sigma$  [7].

$$M = SS^{-t},$$

as in the 2d case, where the inverting of  $S$  means replacing  $U_\gamma \rightarrow U_\gamma^{-1}$  and  $q \rightarrow q^{-1}, s \rightarrow -s$ , and taking the inverse of the products. Furthermore, transposition means the order in the product continues in the order of increasing phase of central charge. It was found in Ref. [7] that

$$\text{Tr} M^k = Z(t = e^{2\pi i k}, q) = \text{Tr}_{\frac{1}{2}S_q^3}(-1)^F e^{2\pi i k r}. \tag{2.6}$$

The fact that this gives the same result in all chambers follows from the work of Kontsevich and Soibelman [21] and its refinement [22]. The similarity of the setup to the 2d case is striking and was explained in Ref. [7]. For alternative derivation, see Ref. [23].

It is tempting to connect this to a more standard superconformal index. In fact, as noted in Refs. [7,24–26], if we consider the double space  $S_q^3$ , the partition function on this space gets related to a doubled version of BPS contributions given by

$$\hat{\Phi}(\gamma, s) = \frac{\prod_n (1 - q^{n+s+\frac{1}{2}} U_\gamma)^{(-1)^{2s}}}{\prod_n (1 - \hat{q}^{n+s+\frac{1}{2}} \hat{U}_\gamma)^{(-1)^{2s}}}$$

where  $\hat{q} = \exp(-2\pi i/\tau)$  with the parametrizations  $q = \exp(2\pi i\tau)$ , and  $\hat{U} = U^\tau$ . It can be checked that  $\hat{U}_\gamma$  satisfy

$$\hat{U}_\gamma \hat{U}_\beta = q^{(\gamma, \beta)} \hat{U}_\beta \hat{U}_\gamma.$$

Moreover,  $U_\gamma$  and  $\hat{U}_\beta$  commute. Then, it was proposed in Ref. [7] that if we consider

$$\hat{M} = \hat{S} \hat{S}^{-t},$$

where  $\hat{S}$  is constructed out of  $\hat{\Phi}$ , then

$$\text{Tr} \hat{M}^k = \text{Tr}_{S_q^3}(-1)^F e^{2\pi i k r}.$$

It is natural to compare this with the usual superconformal index. Given the relation between the superconformal index in 4d and the partition function on squashed  $S^3$  [27–29], it is natural to propose<sup>4</sup>

$$\text{Tr} \hat{M}^k = \text{Tr}(-1)^F e^{2\pi i k(r-R)} q^{J_{12}-J_{34}},$$

which can be viewed as a special limit of the  $N = 2$  superconformal index:

$$\text{Tr}(-1)^F t^{r-R} q^{J_{12}-R} p^{J_{34}-R},$$

with the specialization  $pq = 1, t = e^{2\pi i k}$ .

<sup>4</sup>Here, the  $J_{12} - J_{34}$  is naturally suggested by the relation with topological strings.

### III. BPS STATES AND GAUGE THEORIES WITH REAL CENTRAL CHARGES IN $d = 3, 5$

In this section, we review the computation of the superconformal index for gauge theories with  $\mathcal{N} = 2$  in  $d = 3$  and  $\mathcal{N} = 1$  in  $d = 5$ . In both cases, we argue that they can be written entirely in terms of BPS states of the corresponding theories in the Coulomb branches. This reinterpretation leads to our general proposal for relation between BPS states and the index for all superconformal theories in  $d = 3, 5$ .

#### A. Superconformal index in $d = 3, \mathcal{N} = 2$

Here, we review the basic statement for computation of the superconformal index for gauge theories on  $S^1 \times S^2$  [26,30–32].

Consider a 3d theory with gauge group  $G$  and some matter representations  $\mathcal{R}$ . Moreover, depending on what interactions are turned on, certain flavor symmetries can be introduced. The superconformal index can be viewed as computation of

$$I_3 = \text{Tr}(-1)^F q^{R-J} \prod_i z_i^{F_i},$$

where  $J$  is the rotation generator on  $S^2$ ,  $R$  denotes the  $R$  charge, and  $F_i$  are some flavor charges. The basic statement is that we can compute  $I_3$  simply by taking the contribution of all the fields in the UV to the index, where it can be taken to be a weakly coupled theory. Since the index does not change upon flow, this would give the superconformal index at the conformal point as well. If we have gauge group factors, we can turn on flat connections on  $S^1$ , which we denote by  $U_i$ , which need to be integrated over. This is equivalent to projecting to gauge invariant fields. Moreover, for each flavor charge, we introduce a fugacity  $z_i$  around the circle.

The contribution for each particle splits up formally to a square due to the  $CPT$  structure of each multiplet. Let  $\Phi_a(z_i, U_j, q)$  be the contribution of one of the particles. Let the spin of the particle be  $s$  and charges  $f_i$  under the flavor symmetries, and charge  $p_i$  under the gauge symmetries. Then<sup>5</sup>

$$\Phi_a(z_i, U_j, q) = \prod_n \left( 1 - q^{n+\delta_a+\frac{1}{2}} U_j^{p_j} z_i^{f_i} \right)^{(-1)^{2s}},$$

where  $\delta_a$  is the  $R$  charge of the field (and for free theory gets identified with  $s$ ). Including the  $CPT$  conjugate is the same as introducing  $\Phi_a^{-t} = 1/\Phi_a(z_i^{-1}, U_j^{-1}, q)$ . Let

<sup>5</sup>Here, we are turning off the fugacity associated with monopole number which can be viewed as complexification of  $U_j$  [26].

$$S = \prod_a \Phi_a.$$

Then the index can be written as<sup>6</sup>

$$I_3 = \text{Tr}M = \text{Tr}SS^{-t} = \int \frac{dU_j}{U_j} \prod_a \Phi_a(z_i, U_j, q) \cdot \frac{1}{\Phi_a(z_i^{-1}, U_j^{-1}, q)},$$

which has the same structure as what we had seen in  $d = 2, 4$ . Indeed, if we go to the Coulomb branch, the basic field becomes BPS states, and so this can also be viewed as computation using the BPS states,<sup>7</sup> in the same sense as we had seen in  $d = 2, 4$ . Note that at least formally this can be written in the form

$$I_3 = \int \frac{dU_j}{U_j} \left| \prod_a \Phi_a(z_i, U_j, q) \right|^2$$

using the fact that (not worrying about regions of convergence of  $q$ )

$$\Phi(z_i^{-1}, U_j^{-1}, q^{-1}) = \frac{1}{\Phi(z_i, U_j, q)}.$$

This computes the index at zero monopole number. One can also include the effect of the global symmetries associated with shifting the dual photon. This can be done most naturally by considering a generalized index [32] with fixed monopole numbers  $m_j$ . This is equivalent [26] to shifting the holonomies  $U_j \rightarrow U_j X_j$ , where  $X_j$  are viewed as real, and substituting  $X_j = q^{m_j/2}$  after taking  $|\dots|^2$ .

#### B. $\mathcal{N} = 1, d = 5$ and BPS states

The superconformal index in  $d = 5$  is defined [5] by the twisted partition function on  $S^1 \times S^4$ :

$$I_5 = \text{Tr}(-1)^F q_1^{J_{12}-R} q_2^{J_{34}-R} z_j^{f_j},$$

where  $J_{12}$  and  $J_{34}$  are the rotations of two planes in  $S^4$  and  $R$  denotes the Cartan of the  $SU(2)$   $R$  symmetry and  $f_i$  denote flavor symmetries. The fact that there are nontrivial  $\mathcal{N} = 1$  superconformal theories has been argued from many different viewpoints [13,33–36]. There are nontrivial superconformal field theories of which the existence is

<sup>6</sup>The integration is over the Cartan of  $U(1)^n$ . This is also true in the non-Abelian case where the extra measure factors can be viewed as arising from the contributions  $\Phi_a$  of the massive gauge particles of the non-Abelian group in the Coulomb branch.

<sup>7</sup>More precisely, what we mean by this is that if we ungauged the  $U(1)$ 's the BPS partition function of the global symmetries determines what are the BPS states. The index for the  $U(1)$ 's which we gauge is determined entirely in terms of by global symmetries.

signaled by the existence of massless particles and tensionless strings. Moreover, as argued in Ref. [33], many superconformal theories deform upon mass deformations to gauge theories. In turn, in the IR limit, the gauge theories become weakly coupled, and one can use this weakly coupled IR theory to compute the index. Since the index is independent of deformations, this can be used to recover the index at the conformal point. This idea has been considered in Ref. [37] where the superconformal index for some theories was computed using localization techniques. This includes that of  $SU(2)$  with up to  $N_f = 7$  fundamental matter. Moreover, the expected  $E_{N_f+1}$  symmetry of these theories was successfully tested. The basic structure of the answer can be recast, which we discuss in more detail in Sec. VI, as<sup>8</sup>

$$I_5 = \int \frac{dU_i}{U_i} |Z_{5d}^{\text{Nekrasov}}(U_i, z_j; q_1, q_2)|^2,$$

where  $Z_{5d}^{\text{Nekrasov}}$  denotes the Nekrasov partition function for the 4d theories coming from compactification of the theory on  $S^1$  and  $U_i$  denote the holonomy of the gauge group along  $S^1$  and  $z_j$  are the exponential of mass parameters and the instanton number (which is one of the flavor symmetries). Moreover, in the above formula, the  $|\dots|^2$  involves complex conjugating the  $U_i, z_j \rightarrow U_i^{-1}, z_j^{-1}$  but keeping  $q_{1,2}$  unchanged. Of course, this result was already anticipated by the computation of Pestun [38] relating the 4d Nekrasov partition function with the gauge theory partition function on  $S^4$ . This can be viewed as a special instance of that general argument where the argument is applied to the 4d theory obtained by compactification from five dimensions.

The question is what is the relation of this index with BPS states? Unlike the 3d case, where the basic fields can be viewed as BPS states in the Coulomb branch, in the 5d case, the gauge fields and matter fields are not the only BPS states. Indeed, this is consistent with the fact that  $I_5$  is considerably more complicated than the 3d case where the index is given by treating the basic fields as the only relevant ingredients for the computations. Indeed, there are infinitely many BPS states in this case. The question is whether  $I_5$  can be reinterpreted just in terms of BPS states, as was the case in  $d = 2, 3, 4$ .

As is well known, the partition function of refined topological strings on a CY which engineers the

corresponding gauge theory [39] is identical with Nekrasov's partition function. Therefore, we can interpret the above statement as

$$I_5 = \int \frac{dU_i}{U_i} |Z_{CY}^{\text{top}}(U_i, z_j; q_1, q_2)|^2.$$

On the other hand, it is known that topological strings capture BPS degeneracies [8] (see Ref. [9] for the refined version),

$$Z^{\text{top}} = \prod_{s_i, n_i, m_j} \prod_{m, n=1}^{\infty} (1 - q_1^{m+s_1+\frac{1}{2}} q_2^{n+s_2+\frac{1}{2}} U_i^{n_i} z_j^{m_j})^{(-1)^{2s} N_{s_1, s_2, n_i, m_j}},$$

where  $N_{s_1, s_2, n_i, m_j}$  is the BPS degeneracy with  $SO(4)$  spins  $s_i$  written in an orthogonal basis of Cartan, gauge charges  $n_i$  and flavor charge  $m_j$  [where in topological strings  $(n_i, m_j)$  translate to an element of  $H_2$  of CY where the M2-brane wraps to give rise to the BPS state]. Thus, we can view this  $Z^{\text{top}}$  as a partition function of BPS particles,

$$Z^{\text{top}} = \prod_{i \in \text{BPS}} \Phi_i = S,$$

with  $\Phi$  identified as the above, counting the BPS states as if they are the elementary building blocks of the theory, even though there is no weak coupling Lagrangian which describes them as fundamental fields. Nevertheless, they seem to behave as such. Moreover,  $S^{-t}$  is given by

$$\begin{aligned} S^{-t} &= Z^{\text{top}}(q_1^{-1}, q_2^{-1}, U_i^{-1}, z_j^{-1}) = \frac{1}{Z^{\text{top}}(q_1, q_2^{-1}, U_i^{-1}, z_j^{-1})} \\ &= Z^{\text{top}}(q_1, q_2, U_i^{-1}, z_j^{-1}). \end{aligned}$$

The proof of this is given in Sec. IV above Eq. (4.15) when we discuss the properties of the refined partition function.

Therefore, we can again write the index as

$$I_5 = \text{Tr} M = \text{Tr} S S^{-t} = \int \frac{dU_i}{U_i} |Z^{\text{top}}(q_1, q_2, U_i, z_j)|^2.$$

Thus, we have a unified picture in  $d = 2, 3, 4, 5$  on the relation between BPS states and supersymmetric partition functions.

### C. Inclusion of codimension-2 defects

In the context of topological strings, we can also consider M5-branes wrapping special Lagrangian submanifolds. These correspond to 3d defects in gauge theory, giving the analog of surface operators in the context of 4d gauge theory [40,41]. We will describe the detailed definition of them shortly. We can then ask how one may compute the index of the 5d theory in the presence

<sup>8</sup>Just as in the 3d case, we can consider a generalized 5d index, where we turn on  $U(1)$  instantons for the flavor symmetries on  $S^4$ . We can consider sectors with first Chern class of  $U(1)$  along the two planes of  $S^4$  be given by  $(m_1, m_2)$ . In this context, it is natural to expect that this formula gets changed simply by shifting  $(U_i, z_j) \rightarrow (U_i q_1^{m_1/2} q_2^{m_2/2}, z_j q_1^{m_1/2} q_2^{m_2/2})$ , coming from the shift of spins of the BPS states in that background. For simplicity of exposition, we restrict our attention in this paper to the sectors where  $m$ 's are zero.

of 3d defects. This fits nicely with the above formalism by simply combining the degrees of freedom of the BPS states involving M2-branes ending on M5-branes, which open topological string counts [10,11]:

$$I_{5,3} = \int \frac{dU_i dV_j}{U_i V_j} |Z_{\text{open,closed}}^{\text{top}}(q_1, q_2, U_i, V_j, z_k)|^2,$$

where  $U_i, V_j$  are the bulk and defect holonomies around  $S^1$ , respectively, and  $z_k$  are the flavor symmetries.  $z_k$  correspond to Kähler classes in the Calabi–Yau. This computes the index in the zero monopole number sector. To obtain the generalized index of Refs. [26,32] with fixed monopole numbers  $m_j$ , it suffices to take  $V_j$  to have a real piece  $V_j \rightarrow V_j X_j$  and substituting, *after* taking the  $|\dots|^2$ ,  $X_j = q_1^{m_j/2}$ , where we have taken the M5-brane to be in the 12-plane.<sup>9</sup>

Next, we discuss in more detail the connection between M5-branes wrapping Lagrangian submanifolds and gauge theoretic defects (see also Ref. [42]). M5-branes wrapped on special Lagrangian submanifolds and filling an  $\mathbb{R}^3 \subset \mathbb{R}^5$  in spacetime correspond to supersymmetric defects preserving half of the supersymmetries (i.e., leading to  $\mathcal{N} = 2$  supersymmetry in three dimensions). We will be mainly considering noncompact Calabi–Yau 3-folds which are toric. A distinguished class of special Lagrangian cycles in these cases [43,44] has the topology of  $\mathbb{R} \times T^2$  for which a cycle of  $T^2$  shrinks at each end. In the compact region of the toric 3-fold, where one cycle  $w_0$  of  $T^2$  shrinks it ends on the web of the toric diagram. With no loss of generality, let us call this the  $(1, 0)$  cycle of  $T^2$ . At infinity, a cycle  $w_\infty$  of the  $T^2$  shrinks ending on the “spectator brane.” Let us call this direction the  $w_\infty = (p, q)$ . The topology of this Lagrangian submanifold is the lens space  $L(q, p)$  which has fundamental group  $\mathbf{Z}_q$ . As discussed in Ref. [25], there is an  $\mathcal{N} = 2$  supersymmetric  $U(1)$  Chern–Simons gauge theory living on the noncompact three dimensions of the wrapped M5-brane, with level  $q$ . Furthermore, this theory has a flavor  $U(1)$  symmetry associated with the monopole number (corresponding to shifting the angular scalar dual to the photon). The  $p$  corresponds to the Chern–Simons level for this flavor symmetry. Furthermore, the position of the brane on the web is determined by the Fayet-Iliopoulos term  $\xi_0$  for the  $U(1)$  gauge symmetry. Such a defect can be defined for any 5d conformal theory arising from toric CY, where the lines of the web pass through the same point. As the web is resolved through breathing modes of the web, one per cycle, the spectator  $(q, p)$  line where the brane is suspended can intersect a number of edges in the diagram, and the brane can end on any of the lines (see Fig 1). To

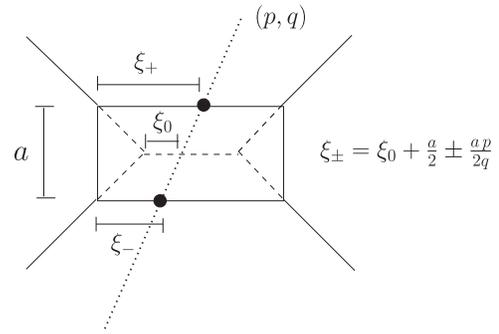


FIG. 1. The geometry of Lagrangian brane on local  $\mathbb{P}^1 \times \mathbb{P}^1$ . Here, we have chosen the spectator brane to be  $(p, q)$  with slope  $p/q$ . The Chern-Simons level on the brane is at  $k = q$ . The Lagrangian brane is suspended from the spectator brane at either of the two points (denoted by black dots). The Coulomb branch parameter is labeled by  $a$ . Moreover, the slope being  $p/q$  affects how the effective FI terms  $\xi_\pm = \xi_0 + \frac{a}{2} \pm \frac{ap}{2q}$  change with  $a$ .

make sure the amplitudes are invariant under resolutions, and they are defects associated to a superconformal theory, we need to sum over all such possible endings.

In case the toric geometry engineers an  $SU(N)$  gauge theory (corresponding to  $N$  parallel lines, the  $(p, q)$  spectator line will intersect the ladder of parallel lines at any of  $N$  points, and we will need to sum over all of them. This would correspond to breaking  $SU(N)$  to  $SU(N - 1) \times U(1)$  near the defect position. Moreover, as discussed in Ref. [41] in the analogous situation of surface defects in four dimensions, the surface defect generates a deficit angle  $0 \leq \alpha \leq 2\pi$  in the  $U(1) \subset U(1) \times SU(N - 1)$ , proportional to FI term  $\xi_0$ . We have

$$\xi_0 = \frac{\alpha}{2\pi g_{YM}^2}$$

corresponding to moving the end brane along the line for which the length is  $1/g_{YM}^2$ ; as the brane traverses the line, the deficit angle varies from 0 to  $2\pi$ . As we change the Coulomb branch parameters, the effective  $\xi_\pm$  depends not only on the Coulomb parameter  $a$  but also on the slope  $p/q$  (see Fig. 1).

In computing the index  $I$  in the presence of the defect, we choose a number of defect spectators with various slopes  $(p_i, q_i)$  and some fixed positions (corresponding to their FI term  $\xi_i$ ). We can also have more than one brane suspended from each. In the gauge theory setup, this will translate to more general patterns of breaking the gauge symmetry near the defect. We then integrate over the breathing modes of the loops (i.e., Wilson lines of the 5d gauge theory) and the Wilson lines associated to the gauge field on the brane, fixing the position of the suspended lines at infinity and the external lines of the web, which collectively play the role of mass parameters.

<sup>9</sup>As noted earlier, we can do a similar generalization of the index for the 5d sector, which for simplicity we will not consider in this paper.

**IV. TOPOLOGICAL STRINGS AND BPS STATES**

The  $N = 2$  topological strings propagating on a CY 3-fold  $X$  have been intensely studied in recent years from both mathematical and physical viewpoints. They not only provide an exactly solvable sector of the full string theory but also provide very useful insight into the spacetime physics. In this section, we will summarize the relation between topological strings on  $X$  and BPS states which arise in the M-theory compactification on  $X$ .

Consider a Calabi–Yau 3-fold  $X$ , and let  $\omega = \sum_{a=1}^{h^{1,1}(X)} t_a \omega_a$  be the Kähler class. The classes  $\{\omega_1, \omega_2, \dots, \omega_{h^{1,1}}\}$  span  $H^2(X, \mathbb{Z})$ , and we denote with  $D_a$  the 4-cycle dual to  $\omega_a$ . The genus  $g$  A-model topological string amplitude on the Calabi–Yau 3-fold  $X$  is then given by [45]

$$\begin{aligned}
 F_0(\omega) &= \frac{c_{abc} t_a t_b t_c}{6} + \sum_{\beta \in H_2(X, \mathbb{Z})} N_\beta^0 e^{-\int_\beta \omega} \\
 F_1(\omega) &= -\frac{1}{24} \sum_{a=1}^{h^{1,1}} t_a \int_X c_2(X) \wedge \omega_a + \sum_{\beta \in H_2(X, \mathbb{Z})} N_\beta^1 e^{-\int_\beta \omega} \\
 F_{g \geq 2}(\omega) &= (-1)^g \left( \int_{\mathcal{M}_g} \lambda_{g-1}^3 \right) \frac{\chi(X)}{2} + \sum_{\beta \in H_2(X, \mathbb{Z})} N_\beta^g e^{-\int_\beta \omega},
 \end{aligned} \tag{4.1}$$

where  $c_{abc} = \int_X \omega_a \wedge \omega_b \wedge \omega_c$  are the triple intersection number  $D_a \cdot D_b \cdot D_c$  of the divisors  $D_a$  dual to  $\omega_a$ ,  $N_\beta^g$  are the genus  $g$  Gromov–Witten invariants, the  $\lambda_{g-1}$  is the  $(g - 1)$ th Chern class of the Hodge bundle over the moduli space of genus  $g$  curves,  $\mathcal{M}_g$ , and

$$\int_{\mathcal{M}_g} \lambda_{g-1}^3 = \frac{|B_{2g}||B_{2g-2}|}{(2g)(2g-2)(2g-2)!}. \tag{4.2}$$

In the above equation,  $B_{2g}$  are the Bernoulli numbers,  $\sum_{n=0}^\infty B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}$ .

The topological string partition function is given by

$$Z(\omega, g_s) = \exp\left(\sum_{g=0}^\infty g_s^{2g-2} F_g(\omega)\right), \tag{4.3}$$

where  $g_s$  is the topological string coupling constant. In Ref. [8], topological strings on a CY 3-fold  $X$  were studied from a spacetime point of view, and it was shown that the topological string partition function captures the degeneracy of BPS particles in the 5d theory coming from M theory on  $X$ . We present a short summary of their argument linking the BPS states in five dimensions with topological strings. Consider M-theory compactification on CY 3-fold  $X$  which gives a 5d theory. The massive BPS particles will form a representation of the little group in 5d

$SO(4) = SU(2)_L \times SU(2)_R$ . These BPS particles in five dimensions arise from M2-branes wrapping a holomorphic curve in  $X$  and have mass equal to the area of the curve. These BPS particles are electrically charged under the  $h^{1,1}(X)$  Abelian gauge fields  $A^{(a)}$  coming from the 3-form  $C$ ,

$$C = \sum_{a=1}^{h^{1,1}(X)} A^{(a)} \wedge \omega_a. \tag{4.4}$$

As has been mentioned before, the 5d theory also has states which are magnetically charged under  $A^{(a)}$ . These magnetically charged states are not point particles but are strings coming from M5-branes wrapping the 4-cycles in  $X$ . The M2-brane wrapping a holomorphic curve in the class  $\beta$  gives rise to a set of BPS particles in five dimensions with mass equal to  $\int_\beta \omega$  and certain  $SU(2)_L \times SU(2)_R$  spin content. Let us denote by  $N_\beta^{j_L, j_R}$  the number of particles with spin  $(j_L, j_R)$  and charge  $\beta$  (which determines the mass), and let

$$n_\beta^{j_L} = \sum_{j_R} (-1)^{2j_R} (2j_R + 1) N_\beta^{j_L, j_R}. \tag{4.5}$$

The integers  $n_\beta^{j_L}$  are invariant under complex structure deformations of  $X$  and are the BPS degeneracies captured by the topological strings. In terms of  $n_\beta^{j_L}$ , the topological string partition function can be written as ( $q = e^{ig_s}$ )

$$\begin{aligned}
 Z(\omega, g_s) &= Z_0(\omega, g_s) \prod_{\beta \in H_2(X, \mathbb{Z})} \prod_{j_L} \prod_{k_L = -j_L}^{+j_L} \prod_{m=1}^\infty \\
 &\times \left(1 - q^{2k_L + m} e^{-\int_\beta \omega}\right)^{m(-1)^{2j_L} n_\beta^{j_L}} \\
 Z_0(\omega, g_s) &= \frac{\exp\left(\frac{c_{ijk} t_i t_j t_k}{6g_s^2} - \frac{1}{24} \sum_{a=1}^{h^{1,1}} t_a \int_X c_2(X) \wedge \omega_a\right)}{\exp\left(-\frac{\zeta(3)}{g_s^2} + \sum_{g=2}^\infty g_s^{2g-2} (-1)^g \int_{\mathcal{M}_g} \lambda_{g-1}^3\right)^{\frac{\chi(X)}{2}}}.
 \end{aligned}$$

In  $Z_0(\omega, g_s)$  above, the numerator is the classical contribution from genus zero worldsheet with three punctures (the cubic term) and genus one worldsheet with one puncture. The denominator is the contribution coming from constant maps and can also be written as

$$\begin{aligned}
 &-\frac{\zeta(3)}{g_s^2} + \sum_{g=2}^\infty g_s^{2g-2} (-1)^g \int_{\mathcal{M}_g} \lambda_{g-1}^3 \\
 &= -\sum_{n=1}^\infty n \log(1 - q^n) = \log M(q),
 \end{aligned} \tag{4.6}$$

where  $M(q) = \prod_{n=1}^\infty (1 - q^n)^{-n}$  is the generating function of the number of plane partitions known as the MacMahon

function. Thus, the full topological string partition function is given by

$$Z(\omega, g_s) = e^{-\frac{c_{ijk}t^i t^j t^k}{6g_s^2} - \frac{1}{24} \sum_{a=1}^{h^{1,1}} t_a \int_X c_2(X) \wedge \omega_a} M(q)^{\frac{\chi(X)}{2}} \times \prod_{\beta, j_L, k_L, m} (1 - q^{2k_L+m} e^{-\int_{\beta} \omega})^{m(-1)^{2j_L} n_{\beta}^{j_L}}.$$

There also exists a refinement of the above topological string partition function. Gopakumar-Vafa invariant  $n_{\beta}^{j_L}$  is

$$Z(\omega, t, q) = e^{-\frac{c_{ijk}t^i t^j t^k}{6\epsilon_1 \epsilon_2} - \frac{1}{24} \sum_{a=1}^{h^{1,1}} t_a \int_X c_2(X) \wedge \omega_a} (M(t, q)M(q, t))^{\frac{\chi(X)}{4}} \times \prod_{\beta \in H_2(X, \mathbb{Z})} \prod_{j_L, j_R} \prod_{k_L = -j_L}^{+j_L} \prod_{k_R = -j_R}^{+j_R} \prod_{m_1, m_2=1}^{\infty} (1 - t^{k_L+k_R+m_1-\frac{1}{2}} q^{k_L-k_R+m_2-\frac{1}{2}} Q^{\beta})^{M_{\beta}^{j_L, j_R}} M_{\beta}^{j_L, j_R} = (-1)^{2(j_L+j_R)} N_{\beta}^{j_L, j_R}, \tag{4.7}$$

where  $M(t, q)$  is the refined MacMahon function,

$$M(t, q) = \prod_{i, j=1}^{\infty} (1 - q^i t^{j-1})^{-1}, \tag{4.8}$$

and  $q = e^{i\epsilon_1}, t = e^{-i\epsilon_2}$ . The usual topological string partition function is recovered in the limit  $\epsilon_1 = -\epsilon_2 = g_s$ . Notice that we have kept the classical contribution and the constant map contribution. Equation (4.7) can also be written as

$$Z(\omega, t, q) = e^{-\frac{c_{ijk}t^i t^j t^k}{6\epsilon_1 \epsilon_2} - \frac{1}{24} \sum_{a=1}^{h^{1,1}} t_a \int_X c_2(X) \wedge \omega_a} (M(t, q)M(q, t))^{\frac{\chi(X)}{4}} \times \text{PE}[F(\omega, t, q)] F(\omega, t, q) = \sum_{\beta \in H_2(X, \mathbb{Z})} \sum_{j_L, j_R} e^{-\int_{\beta} \omega} \times \frac{(-1)^{2(j_L+j_R)} N_{\beta}^{j_L, j_R} \text{Tr}_{j_R}(\frac{q}{t})^{j_R, 3} \text{Tr}_{j_L}(qt)^{j_L, 3}}{(q^{1/2} - q^{-1/2})(t^{1/2} - t^{-1/2})}, \tag{4.9}$$

where  $\text{PE}[f(x_1, x_2, \dots)]$  is the Plethystic exponential of  $f(x)$  defined as

$$\text{PE}[f(x_1, x_2, \dots)] = \exp\left(\sum_{n=1}^{\infty} \frac{f(x_1^n, x_2^n, \dots)}{n}\right). \tag{4.10}$$

<sup>10</sup>In the previous sections, the coupling constants of the refined topological strings were denoted by  $q_1$  and  $q_2$ . From now on, we will denote them by  $t$  and  $q$  which are more familiar in the context of calculations involving the refined topological vertex.

an index over the Hilbert space of states coming from  $\beta$  and the index structure is needed since complex structure deformations can change  $N_{\beta}^{j_L, j_R}$  but do not change  $n_{\beta}^{j_L}$ . This is the story for the generic CY 3-fold. For a local CY 3-fold (noncompact toric CY 3-folds), the story is much more interesting. The local CY 3-folds enjoy extra R symmetry, and therefore  $N_{\beta}^{j_L, j_R}$  are also invariants. The refinement of the topological string partition function captures these full BPS degeneracies<sup>10</sup>:

In general for a local CY 3-fold,  $\chi(X)$  is not well defined; however, if we only consider compact homologies in its definition, we get it equal to twice the number of 4-cycles. This is the value we will use in writing the factors of the MacMahon function in the refined partition functions.

Before we discuss how the refined topological string partition function can be calculated for local CY 3-folds, let us discuss an important property of the partition function which has been mentioned before and which will be of importance later. We would like to see how the refined partition function transforms under complex conjugation. In the later calculations of the index, as have been discussed earlier, the Kähler parameters will be taken to be pure imaginary, and some of them will be integrated over. Keeping this in mind, the complex conjugation acts as follows on the variables  $(\omega, t, q)$ :

$$(\omega, t, q) \mapsto (-\omega, t^{-1}, q^{-1}). \tag{4.11}$$

Now it is easy to see from Eq. (4.9) that<sup>11</sup>

$$F(-\omega, t^{-1}, q^{-1}) = F(-\omega, t, q). \tag{4.12}$$

The MacMahon function, which is part of the closed topological string partition function, behaves in a nontrivial way under the complex conjugation,

<sup>11</sup>As long as for each  $\beta$  we have the full spin content corresponding to  $(j_L, j_R)$ . This is indeed the case for the class  $\beta$  if the corresponding moduli space of D-brane  $\mathcal{M}_{\beta}$  is compact. A counterexample to this is the case of  $\mathcal{O}(-2) \oplus \mathcal{O}(0) \mapsto \mathbb{P}^1$ . In this case, the moduli space of the  $\mathbb{P}^1$  is  $\mathbb{C}$ , and the corresponding  $F(T, t, q) = e^{-T} \frac{\sqrt{t}}{(q^{1/2} - q^{-1/2})(t^{1/2} - t^{-1/2})}$ .

$$\begin{aligned}
 M(t^{-1}, q^{-1}) &= \prod_{i,j=1}^{\infty} (1 - q^{-i} t^{-j+1})^{-1} \\
 &= \exp\left(\sum_{i,j=1}^{\infty} \sum_{n=1}^{\infty} \frac{q^{-ni} t^{-n(j-1)}}{n}\right) \\
 &= \exp\left(\sum_{n=1}^{\infty} \frac{q^{-n}}{n(1-q^{-n})(1-t^{-n})}\right) \\
 &= \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n(1-q^n)(1-t^n)}\right) \quad (4.13)
 \end{aligned}$$

$$= \prod_{i,j=1}^{\infty} (1 - q^{i-1} t^j)^{-1} = M(q, t). \quad (4.14)$$

Thus, from Eqs. (4.12) and (4.14), it follows that

$$\overline{Z(\omega, t, q)} = Z(-\omega, t^{-1}, q^{-1}) = Z(-\omega, t, q), \quad (4.15)$$

and therefore

$$\begin{aligned}
 |Z(\omega, t, q)|^2 &= (M(t, q)M(q, t))^{\frac{\chi(X)}{2}} \text{PE}[F(\omega, t, q) \\
 &\quad + F(-\omega, t, q)], \quad (4.16)
 \end{aligned}$$

where the classical piece cancelled because it was odd in  $\omega$ .

Now, we will briefly discuss the open string case which will be of use when we consider the 5d index with a 3d defect. In the A-model topological string, one can consider a worldsheet with boundaries as long as proper boundary conditions are enforced which preserve the A-model supersymmetry. The boundary conditions in this case require the boundary of the worldsheet to end on a Lagrangian submanifold of the target space. These Lagrangian submanifolds on which the worldsheet can have boundaries are the Lagrangian branes of the theory. For the local CY 3-folds, we are considering these Lagrangian branes are noncompact and have the topology of  $S^1 \times \mathbb{R}^2$ . The partition function of the A model in the presence of branes was studied in Ref. [10] from a spacetime viewpoint, and it was shown that in this case, just as in the case of closed strings, the partition function captures certain BPS degeneracies. The spacetime picture arises if we consider type IIA compactification and consider a D4-brane wrapped on the Lagrangian cycle. In this, D2-branes can wrap holomorphic curves in  $X$  and end on the D4-brane. The open topological string partition function captures the degeneracies of BPS states arising from D2-branes ending on the D4-brane. If we denote the Lagrangian brane by  $\mathcal{L}$ , then the D4-brane wraps  $\mathcal{L} \times \mathbb{R}^2$  where  $\mathbb{R}^2$  is part of the spacetime  $\mathbb{R}^4$ . The theory on the  $\mathbb{R}^2$  has a  $U(1)_s$  rotation and a  $U(1)_r$  R symmetry. We combine these two  $U(1)$ 's and define  $S_L = S + R$  and  $S_R = S - R$ . In addition to these quantum numbers, the D2-brane couples to the gauge field on the D4-brane, and we can introduce a holonomy factor  $\text{Tr}_R U$  where  $U$  is the holonomy of the

gauge field on the D4-brane around the nontrivial  $S^1$  of  $\mathcal{L}$ . If we denote by  $N_{R,\beta}^{s_L, s_R}$  the number of particles with charge  $\beta$  and  $U(1)_L \times U(1)_R$  quantum numbers  $s_L, s_R$  in the representation  $R$ , then the open topological string partition function is given by

$$Z_{\text{open}}(\omega, t, q, U) = \text{PE}[F_{\text{open}}(\omega, t, q, U)] \quad (4.17)$$

$$F_{\text{open}}(\omega, q) = \sum_{R, \beta, s_L} e^{-\int_{\beta} \omega} (-1)^{2s_L} n_{R, \beta}^{s_L} \frac{q^{s_L}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \text{Tr}_R U, \quad (4.18)$$

where

$$n_{R, \beta}^{s_L} = \sum_{s_R} N_{\beta, R}^{s_L, s_R} (-1)^{2s_L + 2s_R}. \quad (4.19)$$

The  $s_L + s_R$  is the fermion number, and the above index is invariant under complex structure deformations.

A refinement of the above partition function also exists and is given by Ref. [11] (see also Ref. [46]):

$$Z_{\text{open}}(\omega, t, q) = \text{PE}[F(\omega, t, q)] \quad (4.20)$$

$$\begin{aligned}
 F(\omega, t, q) &= \sum_{\beta, R, s_L, s_R} e^{-\int_{\beta} \omega} (-1)^{2s_L + 2s_R} N_{R, \beta}^{s_L, s_R} \\
 &\quad \times \frac{q^{s_L} t^{s_R}}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} \text{Tr}_R U. \quad (4.21)
 \end{aligned}$$

The action of complex conjugation on the open string partition function is different than in the case of the closed string partition function that we discussed above. The action of complex conjugation on the open string variables is given by

$$(\omega, t, q, U) \mapsto (-\omega, t^{-1}, q^{-1}, U^{-1}). \quad (4.22)$$

With this action, Eq. (4.21) gives

$$\begin{aligned}
 F(-\omega, t^{-1}, q^{-1}) &= -F(-\omega, t, q, U^{-1}), \\
 Z_{\text{open}}(-\omega, t^{-1}, q^{-1}, U^{-1}) &= \frac{1}{Z_{\text{open}}(-\omega, t, q, U^{-1})}. \quad (4.23)
 \end{aligned}$$

Thus, for the open string case

$$|Z_{\text{open}}(\omega, t, q, U)|^2 = \frac{Z_{\text{open}}(\omega, t, q, U)}{Z_{\text{open}}(-\omega, t, q, U^{-1})}. \quad (4.24)$$

### A. An example

Consider the case of  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1$  with a Lagrangian brane on the  $\mathbb{P}^1$ . In this case, the open string partition function is given by

$$Z_{\text{open}}(Q, t, q, z) = \prod_{m=1}^{\infty} (1 - q^{m-\frac{1}{2}}z)^{-1} (1 - q^{m-\frac{1}{2}}Qz^{-1})^{-1}.$$

Using

$$\begin{aligned} \prod_{m=1}^{\infty} (1 - q^{-m+\frac{1}{2}}z)^{-1} &= \exp\left(\sum_{n=0}^{\infty} \frac{z^n}{n} \frac{q^{-n/2}}{1 - q^{-n}}\right) \\ &= \exp\left(-\sum_{n=0}^{\infty} \frac{z^n}{n} \frac{q^{n/2}}{1 - q^n}\right) \\ &= \prod_{m=1}^{\infty} (1 - q^{m-\frac{1}{2}}z), \end{aligned} \tag{4.25}$$

it is easy to see that

$$Z_{\text{open}}(Q^{-1}, t^{-1}, q^{-1}, z^{-1}) = \frac{1}{Z_{\text{open}}(Q^{-1}, t, q, z^{-1})}.$$

### V. FIVE-DIMENSIONAL SUPERCONFORMAL THEORIES FROM TORIC CALABI-YAU 3-FOLDS

In this section, we briefly recall the class of 5d superconformal theories for which our methods yield the corresponding index. See Ref. [12] and references therein for more detail.

We consider M theory on toric Calabi-Yau 3-folds, or equivalently type IIB string theory with a web of  $(p, q)$  5-branes. Let  $x^0, x^1, \dots, x^9$  be the coordinates of the ten-dimensional spacetime. The  $(p, q)$  5-branes fill the  $\mathbb{R}^{1,4}$  part of the spacetime given by  $x^0, x^1, \dots, x^4$  and extend as a web of piecewise straight lines in the plane given by  $x^5$  and  $x^6$ . The generic  $(p, q)$  5-brane web can be viewed as a trivalent graph in  $\mathbb{R}^2$  depicting each 5-brane as a line segment in  $\mathbb{R}^2$  (filling the  $\mathbb{R}^{1,4}$  spacetime) where the slope of each  $(p, q)$  line is given by the  $q/p$ . The generic graph is trivalent with  $\sum_i (p_i, q_i) = 0$  on each vertex. An example of such a web is shown in Fig. 2.

In the limit where the web becomes singular, consisting of lines all passing through the same point, we get a superconformal theory in five dimensions. An example is shown in Fig. 3 where the singular web gives a superconformal theory with  $SU(2)$  global symmetry.

The resolutions of the web, fixing the external line, correspond to going to the Coulomb branch of the 5d gauge

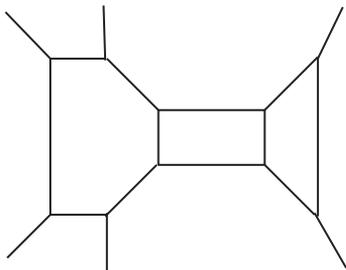


FIG. 2. A generic  $(p, q)$  5-brane web.



FIG. 3. The singular limit of the web gives a superconformal theory. In this case, the theory has  $SU(2)$  global symmetry at the superconformal point. In the M-theory compactification, this corresponds to a 4-cycle ( $\mathbb{P}^1 \times \mathbb{P}^1$ ) shrinking to a point.

theory. Some of these theories correspond to gauge theories upon resolutions [39]. However, most of them do not have a direct gauge theory interpretation. Our method for computing the index applies equally well to all of them.

Moving the external lines corresponds to changing the mass parameters of the theory. The data of the conformal theory are thus captured by a collection of external lines characterized by  $\mathbf{w}_i = (p_i, q_i)$  5-branes, with the condition that

$$\sum_i \mathbf{w}_i = 0.$$

Moreover, for each  $\vec{w}_i$ , one can introduce a mass parameter  $m_i$  corresponding to moving the external lines parallel to itself. They add up to zero, and there is in addition a two parameter redundancy due to shifting the origin of the  $\mathbb{R}^2$ , so the number of mass parameters is 3 less than the number of external lines. It was proposed in Ref. [12] that this data can be identified with the states of a 4d string on  $T^*T^2$ . Moreover, the scattering amplitudes of the resulting string states are identified with the superconformal index  $I_5$  of the resulting theory in five dimensions:

$$\left\langle \prod_i \Phi_{\mathbf{w}_i}(m_i) \right\rangle = I_5 \cdot \delta\left(\sum m_i\right) \delta\left(\sum \mathbf{w}_i\right).$$

In addition, we can select a number of spectator branes from which the Lagrangian branes can be suspended, giving rise to defects of the 5d theory. The slope of the spectator branes determines the type of defect we introduce. Its position is a mass parameter associated to the FI term on the defect. These correspond to degrees of freedom of the unwound string in the proposal of Ref. [12].

#### A. Loop variables and Kähler parameters

In calculating the index, we need to integrate over the loop variables associated with the 4-cycles in the geometry. Each loop variable corresponds to a  $U(1)$  coming from the 4-cycle as discussed in Sec. IV. Since the partition function depends on the Kähler parameters, we need to determine how the Kähler parameters depend on the loop variables. This relation can be easily determined either from the web diagram or from the geometry.

Let us first show how we can determine the dependence of the Kähler parameters on the loop variables using the

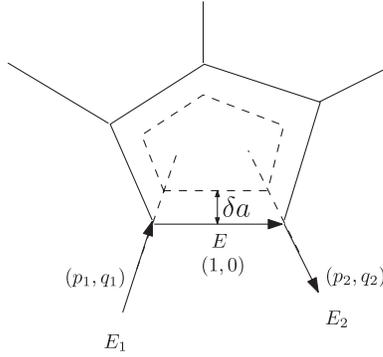


FIG. 4. A local deformation of 5-brane web.

web diagram. Consider an edge  $E$  which is one of the edges forming the loop (4-cycle) in the web diagram. Let  $E_1$  and  $E_2$  be the two edges connected with  $E$  but not part of the loop as shown in Fig. 4 where we have used  $SL(2, \mathbb{Z})$  transformation to convert the edge  $E$  to a horizontal line.

From Fig. 4, it is clear that as the 4-cycle size changes the size of the edge  $E$  also changes with it. The relation between the deformation of the 4-cycle given by the change in the loop variable  $\delta a$  and the change in the size of the edge  $E$   $\delta t_E$  depends on the slope of the connected edges  $E_1$  and  $E_2$  and is given by

$$\delta t_E = \delta a \left| \frac{p_1}{q_1} - \frac{p_2}{q_2} \right|. \quad (5.1)$$

If we define  $Q_e = e^{it_E}$ , the loop variable  $U = e^{ia}$ , and let  $w_1, w_2$ , and  $w_e$  be the winding vectors associated with  $E_1, E_2$ , and  $E$ , then the  $SL(2, \mathbb{Z})$  invariant version of the relation between the edge variable  $Q_e$  and the loop variable  $U$  is given by

$$Q_e = Q_0 U^n, \quad n = \left| \frac{w_1 \wedge w_2}{(w_1 \wedge w_e)(w_2 \wedge w_e)} \right|, \quad (5.2)$$

where  $Q_0$  is the value of  $Q_e$  for  $a = 0$  and is determined by the position of the external legs.

If the geometry has many 4-cycles, then it may become difficult to determine the dependence of the Kähler parameters on the loop variables using the web diagram although the basic idea still is same. A more geometric way of obtaining the relation follows from the fact that holomorphic curves in the geometry give rise to BPS particles in the 5d theory which are electrically charged under the  $U(1)^g$  gauge group (assuming there are  $g$  4-cycles in the geometry). The scaling relation between the Kähler parameter of a curve  $C$  and the loop variables is just given by the electric charge of the corresponding state:

$$Q_C = Q_{C,0} e^{i(d_1 a_1 + d_2 a_2 + \dots + d_g a_g)}, \quad (5.3)$$

where  $\{a_1, a_2, \dots, a_g\}$  are the loop variables corresponding to the  $g$  4-cycles and  $d_i$  is electric charge of the state coming

from  $C$  under the  $U(1)_i$  [the  $U(1)$  coming from the  $i$ th 4-cycle]. The electric charge of the curve  $C$  is a purely geometric quantity given by the intersection of the curve  $C$  with the 4-cycle. If we denote the 4-cycles in the geometry by  $D_1, D_2, \dots, D_g$ , then

$$\begin{aligned} d_i(C) &= D_i \cdot C \\ &= -K_{D_i} \cdot C, \end{aligned} \quad (5.4)$$

where  $-K_{D_i}$  is the anticanonical class of the divisor  $D_i$ . We will use Eqs. (5.3) and (5.4) to determine the relation between the Kähler parameters and the loop variables when calculating the index in Sec. VI.

## VI. COMPUTATION OF THE 5D INDEX THROUGH TOPOLOGICAL STRING

In this section, we will calculate the index for certain 5d theories using the refined topological string partition function. The refined partition function will be calculated using the refined topological vertex. We will give a short introduction to the refined vertex formalism.

### A. Refined vertex formalism

The topological vertex, which was derived using large  $N$  transition from Chern–Simons theory, can be used to calculate the topological string partition function for a toric CY 3-fold [47]. A refinement of the topological vertex was found in Ref. [48] and allows the calculation of refined topological string partition function for a large class of toric CY 3-folds.<sup>12</sup> The refined topological vertex is given by

$$\begin{aligned} C_{\lambda\mu\nu}(t, q) &= f_{\mu^t}(q, t) q^{\frac{\|\mu\|^2}{2}} \tilde{Z}_\nu(t, q) \sum_{\eta} \left( \frac{q}{t} \right)^{\frac{|\eta| + |\lambda| - |\mu|}{2}} \\ &\quad \times s_{\lambda^t/\eta}(t^{-\rho} q^{-\nu}) s_{\mu/\eta}(t^{-\nu^t} q^{-\rho}), \end{aligned} \quad (6.1)$$

where  $s_{\lambda/\eta}(\mathbf{x})$  is the skew-Schur function and the following table summarizes other quantities:

---


$$\begin{aligned} \lambda &= \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)} > 0\}, \lambda^t = \{\lambda_1^t \geq \lambda_2^t \geq \dots \geq \lambda_i^t = \#\{a|\lambda_a \geq i\}\} \\ |\lambda| &= \sum_{a=1}^{\ell(\lambda)} \lambda_a, \|\lambda\|^2 = \sum_{a=1}^{\ell(\lambda)} (\lambda_a)^2 \\ f_\lambda(t, q) &= (-1)^{|\lambda|} t^{\frac{\|\lambda^t\|^2}{2}} q^{-\frac{\|\lambda\|^2}{2}}, \\ \tilde{Z}_\lambda(t, q) &= \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (1 - q^{\lambda_i - j} t^{\lambda_j^t - i + 1})^{-1} \\ \rho &= \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \dots\}, t^{-\rho} q^{-\lambda} = \{t^{\frac{1}{2}} q^{-\lambda_1}, t^{\frac{3}{2}} q^{-\lambda_2}, t^{\frac{5}{2}} q^{-\lambda_3}, \dots\} \\ s_{\lambda/\mu}(\mathbf{x}) &= \sum_{\eta} N_{\mu\eta}^\lambda s_\eta(\mathbf{x}), N_{\mu\eta}^\lambda = \text{Littlewood–Richardson coefficients} \end{aligned}$$


---

<sup>12</sup>See Ref. [49] for an earlier attempt at refining the topological vertex by replacing Schur polynomials with Macdonald polynomials.

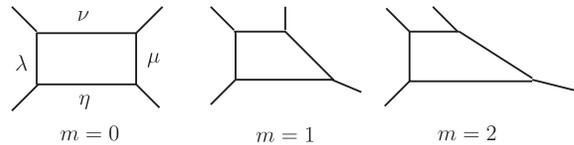
Given any web diagram corresponding to a toric Calabi–Yau 3-fold, we give orientation to the edge and associate to each internal edge  $e_\alpha$  a partition  $\lambda^{(\alpha)}$ . To each external edge, we associate the trivial partition, i.e., the empty set. Since in the web diagram three edges meet at each vertex, we have a set of three partitions for each vertex. If an edge is oriented such that it is going out from the vertex, the corresponding partition is changed to its transpose. We use these three partitions, say,  $\lambda, \mu, \nu$ , associated to the incoming edges of the vertex, to associate with the vertex the refined topological vertex  $C_{\lambda\mu\nu}(t, q)$ . The ordering of the three partitions in writing the refined vertex is taken to be anticlockwise as we go around the vertex, and this should be the same for all vertices in the web diagram. To each edge  $e_\alpha$  of the web diagram, we had associated a partition  $\lambda^{(\alpha)}$ , and we now associate a factor of  $e^{-|\lambda^{(\alpha)}|(t_\alpha+i\pi)}(f_{\lambda^{(\alpha)}}(t, q))^{p_\alpha}$  where  $t_\alpha$  is the length of this edge  $e_\alpha$  and  $p_\alpha$  is an integer which is determined by the local geometry of the  $\mathbb{P}^1$  associated to the edge  $e_\alpha$  in the CY 3-fold. In the neighborhood of a  $\mathbb{P}^1$  in a CY 3-fold, the geometry looks like  $\mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$  with  $m_1 + m_2 = -2$ , the integer  $p = (m_2 - m_1)/2$ . Another important constraint that needs to be considered in the case of the refined topological vertex, but not for the usual topological vertex, is that at each vertex we need to assign one edge as the preferred edge, and all preferred edges in the web diagram should be parallel to each other. This constraint

comes from the construction of the refined topological vertex in terms of plane partitions and restricts the class of toric CY 3-folds to which the refined vertex can be applied to those geometries which are fibrations over a  $\mathbb{P}^1$  or a chain of  $\mathbb{P}^1$ 's. In writing the refined vertex for a vertex, of the web diagram, the partition associated with the preferred edge is always the last partition in the refined vertex, and the two refined vertex factors which appear for two vertices connected by a preferred edge should have  $(t, q)$  parameters switched between them. With these constraints in place, the refined topological string partition function is given by taking the product over all vertices of the corresponding refined vertex factors and taking a product over all edges of the corresponding edge factors and summing over all partitions:

$$Z_{\text{refined}}(t_\alpha, t, q) := \sum_{\text{all partitions } \alpha} \prod_{\alpha} (e^{-|\lambda^{(\alpha)}|(t_\alpha+i\pi)}(f_{\lambda^{(\alpha)}}(t, q))^{p_\alpha}) \times \prod_{\text{vertices}} C_{\lambda^{(\alpha)}\lambda^{(\beta)}\lambda^{(\gamma)}}.$$

**1. An example**

Consider local  $\mathbb{F}_m$  which is the total space of the canonical bundle on the Hirzebruch surface  $\mathbb{F}_m$ . The web diagram of this geometry is shown in the figure below. We take the two horizontal lines to be the preferred edges:



Edge factor :  $(-1)^{|\lambda|+|\mu|+|\nu|+|\eta|} e^{-t_f(|\lambda|+|\mu|)+t_b|\nu|+(t_b+mt_f)|\eta|} \times$  (6.2)

$$f_\nu(t, q)^{-m+1} f_\eta(q, t)^{m+1} f_\lambda(t, q) f_\mu(q, t)$$

Vertex factors :  $C_{\lambda^t\emptyset\nu}(t, q) C_{\emptyset\lambda\eta^t}(t, q) C_{\mu^t\emptyset\eta}(q, t) C_{\emptyset\mu\nu^t}(q, t)$

The refined partition function is then given by

$$Z_{\text{local}\mathbb{F}_m} = \sum_{\lambda\mu\nu^{(1)}\nu^{(2)}} \text{edge factor} \times \text{vertex factor}. \tag{6.3}$$

After some simplification and using the identity  $\sum_\lambda s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) = \prod_{i,j}(1-x_i y_j)^{-1}$ , we get  $(Q_b = e^{-t_b}, Q_f = e^{-t_f})$

$$Z_{\text{local}\mathbb{F}_m} = \sum_{\nu\eta} ((-1)^m Q_b)^{(\nu+|\eta|)} Q_f^{m|\eta|} (f_{\nu'}(q, t))^m (f_\eta(q, t))^m q^{\|\eta'\|^2} t^{\|\nu'\|^2} \tilde{Z}_\nu(t, q) \tilde{Z}_{\eta'}(t, q) \tilde{Z}_\eta(q, t) \tilde{Z}_{\nu'}(q, t) \times \prod_{i,j=1}^\infty [(1 - Q_f t^{i-\eta_j} q^{j-1-\nu_i})(1 - Q_f q^{i-\nu_j} t^{j-1-\eta_i})]^{-1}. \tag{6.4}$$

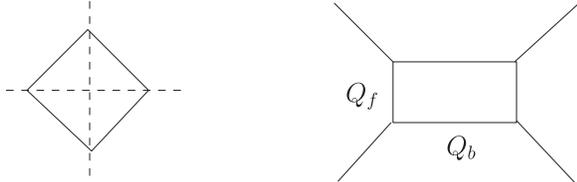


FIG. 5. The Newton polygon (a) and web diagram (b) of local  $\mathbb{P}^1 \times \mathbb{P}^1$ . The Newton polygon has a unique triangulation; therefore, this geometry has only one phase.

**B. Example 1: Local  $\mathbb{P}^1 \times \mathbb{P}^1$**

Let us begin with a very interesting example of local  $\mathbb{P}^1 \times \mathbb{P}^1$ . This CY 3-fold gives rise to  $N_f = 0$   $SU(2)$  gauge theory, and we will be able to compare the answer we get from topological strings with the gauge theoretic calculation of Ref. [37].

The web diagram corresponding to this CY 3-fold (which is dual to the Newton polygon encoding the toric data of this CY 3-fold) is shown below in Fig. 5.

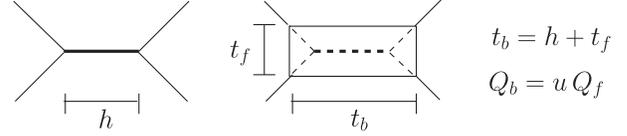


FIG. 6. The parameter  $h$  is determined by the position of the external legs and is fixed. In the 5d gauge theory,  $h$  is proportional to the inverse of the tree level gauge coupling and has dimensions of mass.

In the above figure,  $Q_b$  and  $Q_f$  are related to the Kähler parameters  $t_b$  and  $t_f$  corresponding to the base  $\mathbb{P}^1$ , which we will denote by  $B$ , and the fiber  $\mathbb{P}^1$ , which we will denote by  $F$ , respectively, as

$$Q_b = e^{-t_b}, \quad Q_f = e^{-t_f}.$$

The refined partition function of this geometry was calculated above and is given by taking  $m = 0$  in Eq. (6.4),

$$\begin{aligned} Z_{\text{local}\mathbb{P}^1 \times \mathbb{P}^1}(Q_b, Q_f, t, q) &= (M(t, q)M(q, t))^{\frac{1}{2}} Z(Q_b, Q_f, t, q) \\ Z(Q_b, Q_f, t, q) &:= \sum_{\nu_1 \nu_2} Q_b^{|\nu_1| + |\nu_2|} q^{\|\nu_2\|^2} t^{\|\nu_1\|^2} \tilde{Z}_{\nu_1}(t, q) \tilde{Z}_{\nu_2}(q, t) \tilde{Z}_{\nu_1'}(q, t) \tilde{Z}_{\nu_2'}(t, q) \\ &\quad \times \prod_{i,j=1}^{\infty} [(1 - Q_f t^{i-1-\nu_{2,j}} q^{j-\nu_{1,i}})(1 - Q_f q^{i-1-\nu_{1,j}} t^{j-\nu_{2,i}})]^{-1}. \end{aligned} \tag{6.5}$$

The refined topological vertex calculation gives the last factor in Eq. (6.5). The first factor involving the refined MacMahon function  $M(t, q)$  has been added in accordance with Eq. (4.7) while taking  $\chi(X) = 2$ , as discussed in Sec. V, since there is only one 4-cycle. We have ignored the classical contribution in writing the refined partition function since it cancels when we take the absolute value square of the refined partition function as discussed in Sec. V.

The index for this geometry is given by

$$I = \int da |Z_{\text{local}\mathbb{P}^1 \times \mathbb{P}^1}(Q_b, Q_f, t, q)|^2,$$

where  $a$  is the loop variable (breathing mode) for the 4-cycle in the geometry. In Sec. VI, we discussed the general relation between the Kähler parameters and the loop variable. In this case, we see that the  $Q_f$  is related to the loop variable as

$$Q_f = e^{2ia}. \tag{6.6}$$

$Q_b$  also depends on the loop variable and therefore on  $Q_f$ . This dependence can be easily determined using the web diagram. Consider the web diagram shown in Fig. 5. If the external legs are fixed, then the two parameters  $Q_b$  and  $Q_f$  are not independent anymore; instead, the choice of the external legs determines a parameter  $u = e^{-h}$  such that  $\frac{Q_b}{Q_f} = u$  as shown in Fig. 6.

Thus, the index is given by

$$I_{\text{local}\mathbb{P}^1 \times \mathbb{P}^1} = \int da |Z_{\text{local}\mathbb{P}^1 \times \mathbb{P}^1}(ue^{2ia}, e^{2ia}, t, q)|^2.$$

We can now use Eq. (6.5) to determine the above index to obtain<sup>13</sup>

<sup>13</sup>The index can also be written as a infinite product

$$I = \prod_{a,b,c} (1 - x^a y^b u^c)^{C(a,b,c)}, \tag{6.7}$$

where  $C(a, b, c) \in \mathbb{Z}$ . It would be interesting to see if  $C(a, b, c)$  have a direct physical meaning.

$$\begin{aligned}
 I_{\text{local}\mathbb{P}^1 \times \mathbb{P}^1} := & 1 + \chi_3(u)x^2 + \chi_2(y)(1 + \chi_3(u))x^3 + (\chi_3(y)[1 + \chi_3(u)] + 1 + \chi_5(u))x^4 \\
 & + (\chi_4(y)[1 + \chi_3(u)] + \chi_2(y)[1 + \chi_3(u) + \chi_5(u)])x^5 + (\chi_5(y)[1 + \chi_3(u)] \\
 & + \chi_3(y)[1 + \chi_3(u) + \chi_5(u) + \chi_3(u)\chi_3(u)] + \chi_3(u) + \chi_7(u) - 1)x^6 + (\chi_6(y)[1 + \chi_3(u)] \\
 & + \chi_4(y)[2 + 4\chi_3(u) + 2\chi_5(u)] + \chi_2(y)[1 + 3\chi_3(u) + 2\chi_5(u) + \chi_7(u)])x^7 + (\chi_7(y)[1 + \chi_3(u)] \\
 & + \chi_5(y)[4 + 5\chi_3(u) + 3\chi_5(u)] + \chi_3(y)[2 + 7\chi_3(u) + 3\chi_5(u) + 2\chi_7(u)] + 3 + 2\chi_3(u) + 2\chi_5(u) + \chi_9(u))x^8 \\
 & + (\chi_8(y)[1 + \chi_3(u)] + \chi_6(y)[3\chi_5(u) + 7\chi_3(u) + 4] + \chi_4(y)[3\chi_7(u) + 6\chi_5(u) + 10\chi_3(u) + 6] \\
 & + \chi_2(y)[\chi_9(u) + 2\chi_7(u) + 4\chi_5(u) + 7\chi_3(u) + 4])x^9 + (\chi_9(y)[1 + \chi_3(u)] + \chi_7(y)[4\chi_5(u) + 8\chi_3(u) + 6] \\
 & + \chi_5(y)[4\chi_7(u) + 9\chi_5(u) + 16\chi_3(u) + 7] + \chi_3(y)[2\chi_9(u) + 4\chi_7(u) + 10\chi_5(u) + 11\chi_3(u) + 10] \\
 & + \chi_{11}(u) + 3\chi_7(u) + 3\chi_5(u) + 7\chi_3(u) + 1)x^{10} + \dots, \tag{6.8}
 \end{aligned}$$

where  $x = \sqrt{\frac{q}{t}}$  and  $y = \sqrt{qt}$ .

Equation (6.8) agrees with the result of Ref. [37] for the case of  $SU(2)$  gauge theory with  $N_f = 0$ .

To understand the relation between the gauge theoretic calculation [37] and topological string result that we just derived, we will look carefully at the various factors which arise in the calculation of the index.

In Ref. [37], the index for  $SU(2)$  gauge theory with  $N_f = 0$  was calculated using equivariant localization and was given by

$$I = \int da 2\sin^2(a) \underbrace{\text{PE}[f_{\text{vec}}(a, x, y)]}_{\text{perturbative contribution}} \underbrace{|Z_{\text{Nekrasov}}(a, q, x, y)|^2}_{\text{instanton contribution}}, \tag{6.9}$$

where  $a$  is the parameter on the Coulomb branch,  $q$  is the instanton counting parameter, and  $x$  and  $y$  are related to the

equivariant parameters  $q$  and  $t$  for the  $U(1) \times U(1)$  action of  $\mathbb{C}^2$ ,

$$\begin{aligned}
 (z_1, z_2) \in \mathbb{C}^2 & \mapsto (qz_1, t^{-1}z_2), \\
 x = \sqrt{\frac{q}{t}}, \quad y = \sqrt{qt}. \tag{6.10}
 \end{aligned}$$

The perturbative contribution after subtracting the Haar measure is given by  $f_{\text{vec}}(a, x, y)$ ,

$$f_{\text{vec}}(a, x, y) = -\frac{x(y + \frac{1}{y})}{(1 - xy)(1 - \frac{x}{y})} (e^{2ia} + 1 + e^{-2ia}). \tag{6.11}$$

Now, we can identify different pieces of the integrand in Eq. (6.9) with different contributions to the topological string partition function. The topological string partition function of local  $\mathbb{P}^1 \times \mathbb{P}^1$  can be written as

$$Z_{\text{local}\mathbb{P}^1 \times \mathbb{P}^1}(Q_b, Q_f, t, q) = (M(t, q)M(q, t))^{\frac{1}{2}} Z_0(Q_f, t, q) Z'(Q_b, Q_f, t, q) \tag{6.12}$$

$$\begin{aligned}
 Z_0(Q_f, t, q) &= \prod_{i,j=1}^{\infty} [(1 - Q_f q^i t^{j-1})(1 - Q_f q^{i-1} t^j)]^{-1} \\
 Z'(Q_b, Q_f, t, q) &= \sum_{\nu_1 \nu_2} Q_b^{|\nu_1| + |\nu_2|} q^{|\nu_2|^2} t^{|\nu_1|^2} \tilde{Z}_{\nu_1}(t, q) \tilde{Z}_{\nu_2}(q, t) \tilde{Z}_{\nu_1'}(q, t) \tilde{Z}_{\nu_2'}(t, q) \\
 &\times \prod_{i,j=1}^{\infty} \frac{(1 - Q_f t^{i-1} q^j)(1 - Q_f q^{i-1} t^j)}{(1 - Q_f t^{i-1-\nu_{2,j}} q^{j-\nu_{1,i}})(1 - Q_f q^{i-1-\nu_{1,j}} t^{j-\nu_{2,i}})}. \tag{6.13}
 \end{aligned}$$

In Eq. (6.12),  $Z_0(Q_f, t, q)$  is the contribution to the partition function coming from branes wrapping the fiber curve  $F$  only, and  $Z'(Q_b, Q_f, t, q)$  is the contribution to the partition function coming from branes wrapping the base curve  $B$  at least once and wrapping the fiber curve arbitrary number of times. The contribution  $Z'(Q_b, Q_f, t, q)$  is such that

$$\lim_{Q_b \rightarrow 0} Z'(Q_b, Q_f, t, q) = 1. \tag{6.14}$$

Thus, in the limit  $Q_b \mapsto 0$ , the only contribution to the partition function comes from branes wrapping the fiber curve  $F$  and the D0-branes (the constant map contribution).  $Z_0(Q_f, t, q)$  gives the perturbative part of the 4d gauge theory partition function in the limit

$$Q_f = e^{2ia\beta}, \quad q = e^{\beta\epsilon_1} t = e^{-i\beta\epsilon_2}, \quad \beta \mapsto 0. \quad (6.15)$$

The index is expressed in terms of the variables  $x$  and  $y$  which couple to the  $SU(2)_R$  and  $SU(2)_L$  spins. To see the

relation between the integrand of the index and the topological string partition function, let us express partition function  $|M(t, q)Z_0(Q_f, t, q)|^2$  in terms of the variables  $x = \sqrt{\frac{q}{t}}$  and  $y = \sqrt{qt}$ :

$$M(t, q) = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^{-1} = \prod_{i,j=1}^{\infty} (1 - x^{i+j} y^{i-j}) \quad (6.16)$$

$$\begin{aligned} \overline{M(t, q)} &= M(t^{-1}, q^{-1}) = M(q, t) = \prod_{(i,j) \neq (1,1)}^{\infty} (1 - x^{i+j-2} y^{i-j}) \\ |M(t, q)|^2 &= M(t, q)M(q, t) = \prod_{i,j=1}^{\infty} (1 - x^{i+j} y^{i-j}) \prod_{(i,j) \neq (1,1)}^{\infty} (1 - x^{i+j-2} y^{i-j}) \\ &= \prod_{i,j=1}^{\infty} (1 - x^{i+j-1} y^{i-j+1})(1 - x^{i+j-1} y^{i-j-1}) = \left| \prod_{i,j=1}^{\infty} (1 - x^{i+j-1} y^{i-j+1}) \right|^2. \end{aligned} \quad (6.17)$$

Similarly,

$$\begin{aligned} Z_0(Q_f, t, q) &= \left[ \prod_{i,j=1}^{\infty} (1 - Q_f q^i t^{j-1})(1 - Q_f t^j q^{i-1}) \right]^{-1} = \prod_{i,j=1}^{\infty} (1 - Q_f x^{i+j} y^{i-j})(1 - Q_f x^{i+j-2} y^{i-j}) \\ &= (1 - Q_f) \prod_{i,j=1}^{\infty} (1 - Q_f x^{i+j} y^{i-j}) \prod_{(i,j) \neq (1,1)} (1 - Q_f x^{i+j-2} y^{i-j}) \\ &= (1 - Q_f) \prod_{i,j=1}^{\infty} (1 - Q_f x^{i+j-1} y^{i-j+1}) \prod_{i,j=1}^{\infty} (1 - Q_f x^{i+j-1} y^{i-j-1}) \end{aligned} \quad (6.18)$$

$$\begin{aligned} \overline{Z_0(Q_f, t, q)} &= \prod_{i,j=1}^{\infty} (1 - Q_f^{-1} x^{-i-j} y^{-i+j})(1 - Q_f^{-1} x^{-i-j+2} y^{-i+j}) \\ &= \prod_{i,j=1}^{\infty} (1 - Q_f^{-1} x^{i+j-2} y^{i-j})(1 - Q_f^{-1} x^{i+j} y^{i-j}) \\ &= (1 - Q_f^{-1}) \prod_{i,j=1}^{\infty} (1 - Q_f^{-1} x^{i+j-2} y^{i-j}) \prod_{(i,j) \neq (1,1)} (1 - Q_f^{-1} x^{i+j} y^{i-j}) \\ &= (1 - Q_f^{-1}) \prod_{i,j=1}^{\infty} (1 - Q_f^{-1} x^{i+j-1} y^{i-j+1}) \prod_{i,j=1}^{\infty} (1 - Q_f^{-1} x^{i+j-1} y^{i-j-1}). \end{aligned} \quad (6.19)$$

Using Eq. (6.11), it is easy to see that

$$\begin{aligned} 2\sin^2(a)\text{PE}[f_{\text{vec}}] &= \frac{1}{2} (1 - e^{2ia})(1 - e^{-2ia}) \prod_{i,j=1}^{\infty} [(1 - x^{i+j-1} y^{i-j+1})(1 - x^{i+j-1} y^{i-j-1})(1 - e^{2ia} x^{i+j-1} y^{i-j+1}) \\ &\quad \times (1 - e^{2ia} x^{i+j-1} y^{i-j-1})(1 - e^{-2ia} x^{i+j-1} y^{i-j+1})(1 - e^{-2ia} x^{i+j-1} y^{i-j-1})] \\ &= \frac{1}{2} \left| (1 - e^{2ia}) \prod_{i,j=1}^{\infty} (1 - x^{i+j-1} y^{i-j+1})(1 - e^{2ia} x^{i+j-1} y^{i-j+1})(1 - e^{2ia} x^{i+j-1} y^{i-j-1}) \right|^2. \end{aligned} \quad (6.20)$$

Comparing Eqs. (6.16), (6.18), (6.19), and (6.20), we see that

$$2\sin^2(\alpha)\text{PE}[f_{\text{vec}}] = \frac{1}{2}|M(t, q)Z_0(Q_f, t, q)|^2. \quad (6.21)$$

Thus, the perturbative part of the integrand in Eq. (6.9) is exactly given by the part of the topological string partition function which gets contributions from the D0-branes and D2-branes wrapping the fiber curve. The instanton part of the integrand in Eq. (6.9) is precisely the Nekrasov instanton partition function. It is known that Nekrasov's instanton partition function for  $SU(2)$  with  $N_f = 0$  is precisely equal to the part of the topological string partition function which includes contributions from the base curve, i.e.,  $Z'(Q_b, Q_f, t, q)$  given by Eq. (6.13).

### C. Example 2: Blowup of local $\mathbb{P}^1 \times \mathbb{P}^1$

The blowup of local  $\mathbb{P}^1 \times \mathbb{P}^1$  is another interesting example that we will work out in this section. The corresponding gauge theory is  $\mathcal{N} = 2 SU(2)$  with  $N_f = 1$ . The Newton polygon and the web diagram of this geometry are shown in Fig. 7 below.

The  $H_2(X, \mathbb{Z})$  is spanned by  $\{B, F, E\}$  where  $B$  and  $F$  are the base and the fiber curves and  $E$  is the exceptional curve coming from the blowup. The intersection numbers are given by

$$\begin{aligned} B \cdot B = 0, \quad F \cdot F = 0, \quad B \cdot F = +1, \\ B \cdot E = F \cdot E = 0, \quad E \cdot E = -1. \end{aligned} \quad (6.22)$$

The anticanonical class is given by

$$-K_X = 2(B + F) - E, \quad (6.23)$$

and using Eq. (6.22), we get

$$-K_X \cdot B = +2, \quad -K_X \cdot F = +2, \quad -K_X \cdot E = +1. \quad (6.24)$$

As discussed before, the intersection number of the curves with the anticanonical class (the degree of the curve) determines the electric charges of the state coming from the M2-brane wrapping the curve and determines the relation between the loop variables and the Kähler parameters. In this case, we get<sup>14</sup>

$$Q_b = ue^{2ia}, \quad Q_f = e^{2ia}, \quad Q = \tilde{u}e^{-ia}. \quad (6.25)$$

The refined partition function of this geometry is given by

<sup>14</sup>As before, we have chosen the position of the external line of the web diagram such that  $t_f \mapsto 0$  as  $a \mapsto 0$ .

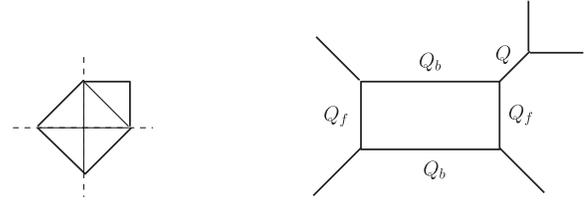


FIG. 7. Toric and web diagram of local  $\mathbb{P}^1 \times \mathbb{P}^1$  blown up at one point.

$$\begin{aligned} Z_X(Q_b, Q_f, Q, t, q) &= (M(t, q)M(q, t))^{\frac{1}{2}}Z(Q_b, Q_f, t, q) \\ &= \sum_{\nu_1 \nu_2} Q_b^{|\nu_1|+|\nu_2|} q^{\|\nu_2\|^2} t^{\|\nu_1\|^2} \tilde{Z}_{\nu_1}(t, q) \tilde{Z}_{\nu_2}(q, t) \\ &\quad \times \tilde{Z}_{\nu_1'}(q, t) \tilde{Z}_{\nu_2'}(t, q) \\ &\quad \times \prod_{i,j=1}^{\infty} \frac{(1 - Qt^{i-\frac{1}{2}-\nu_{1,j}} q^{j-\frac{1}{2}})(1 - QQ_f t^{i-\frac{1}{2}-\nu_{1,j}} q^{j-\frac{1}{2}})}{(1 - Q_f t^{i-1-\nu_{2,j}} q^{j-\nu_{1,i}})(1 - Q_f q^{i-1-\nu_{1,j}} t^{j-\nu_{2,i}})}. \end{aligned} \quad (6.26)$$

Using the refined partition function and Eq. (6.25), the index of this geometry is given by

$$\begin{aligned} I_X(u, \tilde{u}, t, q) &= \int da |Z_X(ue^{2ia}, e^{2ia}, \tilde{u}e^{ia}, t, q)|^2 \\ &= 1 + \left(2 + \tilde{u} + \frac{1}{\tilde{u}}\right)x^2 \\ &\quad + \left(3y + \frac{3}{y} + \tilde{u}y + \frac{\tilde{u}}{y} + \frac{1}{y\tilde{u}} + \frac{y}{\tilde{u}}\right)x^3 + \dots \end{aligned} \quad (6.27)$$

This agrees with the result of Ref. [37].

A more detailed analysis can be carried out in this case to identify different pieces of the gauge theoretic calculation and the topological string calculation. The gauge theory calculation of Ref. [37] gives the index to be

$$I = \int da 2\sin^2(a) \underbrace{\text{PE}[f_{\text{vec}}(a, x, y) + f_{\text{matter}}(a, m, x, y)]}_{\text{perturbative contribution}}$$

In the previous example, we have already shown that part of the above perturbative contribution that depends on the Haar measure and  $f_{\text{vec}}(a, x, y)$  comes from the fiber curve and the D0-brane contribution (the constant map configurations in the worldsheet terms). The new contribution to the perturbative part here is the term that depends on  $f_{\text{matter}}(a, m, x, y)$  where  $(x = \sqrt{\frac{q}{t}}, y = \sqrt{qt})$ ,

$$\begin{aligned}
 f_{\text{matter}}(a, x, y, m) &= \frac{x}{(1-xy)(1-\frac{x}{y})} (e^{-ia-im} + e^{ia-im} + e^{-ia+im} + e^{ia+im}) \\
 &= -\frac{1}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(t^{\frac{1}{2}} - t^{-\frac{1}{2}})} \\
 &\quad \times (e^{-ia-im} + e^{ia-im} + e^{-ia+im} + e^{ia+im}). \quad (6.28)
 \end{aligned}$$

It is easy to see that the contribution of this term to the perturbative part obtained through the plethystic exponential is precisely equal to the contribution of the curve  $E$  and  $F + E$  to the partition function and its complex conjugate. These are the only holomorphic curves that do not involve the curve  $B$  (which would be the instanton contribution). Since the curves  $E$  and  $F + E$  are locally both  $(-1, -1)$  curves, they are rigid and have  $N_E^{j_L, j_R} = N_{E+F}^{j_L, j_R} = \delta_{j_L, 0} \delta_{j_R, 0}$ , and therefore from Eq. (4.7) the contribution to the partition function from these curves is given by

$$\tilde{Z}(Q, Q_f, t, q) = \prod_{i,j} (1 - Q q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}) (1 - Q Q_f q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}). \quad (6.29)$$

From Eq. (6.28), the definition of the plethystic exponential and the above equation it follows that

$$\begin{aligned}
 \text{PE}[f_{\text{matter}}(a, m, x, y)] &= \prod_{i,j=1}^{\infty} (1 - e^{-ia-im} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}) (1 - e^{ia-im} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}) \\
 &\quad \times (1 - e^{-ia+im} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}) \times (1 - e^{ia+im} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}) \\
 &= \tilde{Z}(Q, Q_f, t, q) \tilde{Z}(Q^{-1}, Q_f^{-1}, t, q) = |\tilde{Z}(Q, Q_f, t, q)|^2,
 \end{aligned}$$

where

$$Q = e^{-ia-im}, \quad Q Q_f = e^{ia-im}. \quad (6.30)$$

### D. Example 3: Local $\mathbb{F}_1$

Let us consider the CY 3-fold which is the total space of canonical bundle on the Hirzebruch surface  $\mathbb{F}_1$ . M theory on this CY 3-fold gives rise to pure  $SU(2)$  gauge theory which differs from the  $SU(2)$  gauge theory coming from local  $\mathbb{P}^1 \times \mathbb{P}^1$ , discussed in the last section, due to the  $\mathbb{Z}_2$  valued theta angle [34].

$\mathbb{F}_1$  is a nontrivial  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$ , and we will denote the base  $\mathbb{P}^1$  by  $B$  and the fiber  $\mathbb{P}^1$  by  $F$  with corresponding Kähler parameters  $t_b$  and  $t_f$ , respectively, such that  $B \cdot B = -1$ . As usual, we define  $Q_b = e^{-t_b}$  and  $Q_f = e^{-t_f}$ . The Newton polygon and the web diagram of local  $\mathbb{F}_1$  are shown in Fig. 8.

The refined partition function for this geometry was calculated in Ref. [48] and is given by (see Appendix A of Ref. [48] for notation and other details)

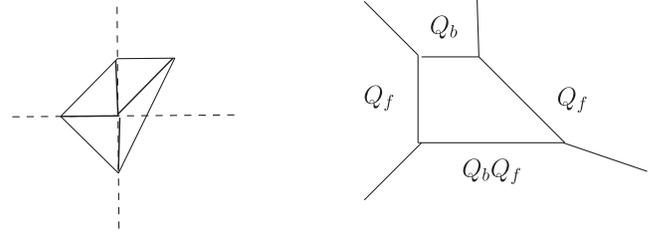


FIG. 8. The Newton polygon (a) and the web diagram (b) of local  $\mathbb{F}_1$ . In this case, there are two distinct triangulations of the Newton polygon corresponding to two different phases.

$$\begin{aligned}
 Z_{\text{local}\mathbb{F}_1}(Q_b, Q_f, t, q) &:= (M(t, q)M(q, t))^{\frac{1}{2}} Z(Q_b, Q_f, t, q) \\
 Z(Q_b, Q_f, t, q) &= \sum_{\nu_1, \nu_2} Q_b^{|\nu_1|+|\nu_2|} Q_f^{|\nu_2|} (-1)^{(|\nu_1|+|\nu_2|)} \left(\frac{q}{t}\right)^{\frac{\|\nu_1\|^2 + \|\nu_2\|^2}{2}} t^{\frac{\kappa(\nu_1) - \kappa(\nu_2)}{2}} \\
 &\quad \times q^{\|\nu_2\|^2} t^{\|\nu_1\|^2} \tilde{Z}_{\nu_1}(t, q) \tilde{Z}_{\nu_1'}(q, t) \tilde{Z}_{\nu_2}(q, t) \tilde{Z}_{\nu_2'}(t, q) \\
 &\quad \times \prod_{i,j=1}^{\infty} [(1 - Q_f t^{i-\nu_{2,j}} q^{j-1-\nu_{1,i}}) (1 - Q_f q^{i-\nu_{1,j}} t^{j-1-\nu_{2,i}})]^{-1}. \quad (6.31)
 \end{aligned}$$

The index for this geometry is therefore given by

$$I_{\text{local}\mathbb{F}_1} = \int da |Z_{\text{local}\mathbb{F}_1}(Q_b, Q_f, t, q)|^2, \quad (6.32)$$

where  $a$  is the loop variable corresponding to the only 4-cycle in the geometry. To calculate the index, we have to determine the dependence of the Kähler parameters  $t_b$  and  $t_f$  on the loop variable. It is easy to see from the general result given in Sec. VI that

$$Q_f = e^{2ia}. \quad (6.33)$$

The geometry of the web determines the relation between  $t_f$  and  $t_b$ . If we fix the external legs of the web, we can change the size of the 4-cycle by changing  $t_f$ ; if we take  $t_f = 0$ , then the web diagram is shown in Fig. 9, and the parameter  $h$  is determined by the position of the external legs. The index will be a function of this parameter  $h$  (along with  $x$  and  $y$ ). The relation between the  $Q_b$  and  $Q_f$  can be easily determined from the web diagram and is given by

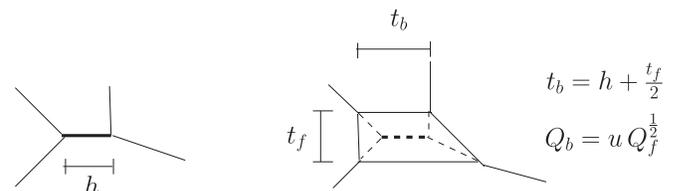


FIG. 9. The web corresponding to local  $\mathbb{F}_1$ .

$$Q_b = uQ_f^{\frac{1}{3}}. \tag{6.34}$$

Thus, the index is given by

$$\begin{aligned} I_{\text{local}\mathbb{F}_1} &= \int da |Z_{\text{local}\mathbb{F}_1}(ue^{ia}, e^{2ia}, t, q)|^2 \\ &= \int \frac{1}{2} \frac{dQ_f}{2\pi i Q_f} |Z_{\text{local}\mathbb{F}_1}(uQ_f^{\frac{1}{3}}, Q_f, t, q)|^2. \end{aligned} \tag{6.35}$$

Using Eqs. (6.31) and (6.35), we get

$$\begin{aligned} I_{\text{local}\mathbb{F}_1} &= 1 + x^2 + 2\left(y + \frac{1}{y}\right)x^3 + \left(3 + 2y^2 + \frac{2}{y^2}\right)x^4 \\ &\quad + \left(2y^3 + 3y + \frac{3}{y} + \frac{2}{y^3}\right)x^5 \\ &\quad + \left(u^2 + \frac{1}{u^2} + 5 + 2y^4 + 5y^2 + \frac{5}{y^2} + \frac{2}{y^4}\right)x^6 \\ &\quad + \left(2y^5 + 6y^3 + u^2y + 10y + \frac{y}{u^2} + \frac{u^2}{y} + \frac{1}{u^2y}\right)x^7 \\ &\quad + \frac{10}{y} + \frac{6}{y^3} + \frac{2}{y^5} x^7 + \dots \end{aligned}$$

**1. Flop invariance of the index**

Recall that the Newton polygon of the local  $\mathbb{F}_1$  has two distinct triangulations. These two triangulations correspond to two different geometries which are related with each other by a flop transition. Here, we will show that the index we have computed above is invariant under the flop transition.

In Fig. 10, the two triangulations and the corresponding web diagrams are shown. The neighborhood of the base curve  $B$  of the local  $\mathbb{F}_1$  is  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1$  which can undergo a flop, giving the resulting geometry which is

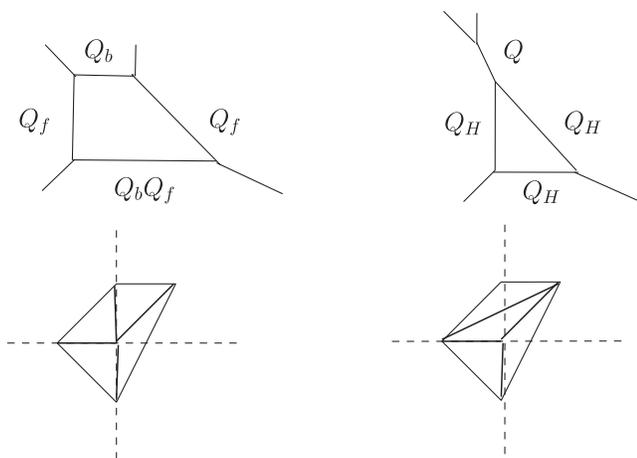


FIG. 10. The two triangulations of toric diagram of local  $\mathbb{F}_1$  and corresponding webs.

local  $\mathbb{P}^2$  together with the flopped curve. We will denote by  $X$  the local  $\mathbb{F}_1$  geometry and will denote by  $Y$  the geometry obtained by the flop from  $X$ .

The relation between the Kähler parameters on the two sides of the flop is given by

$$Q_H = Q_b Q_f, \quad Q = Q_b^{-1}. \tag{6.36}$$

The refined partition function of  $Y$  can be determined from the partition function of  $X$  by carefully following the flop and is given by [50]

$$Z_Y(Q_H, Q, t, q) = Z_{\text{local}\mathbb{F}_1}(Q^{-1}, Q_H Q, t, q). \tag{6.37}$$

In the corresponding web diagram Fig. 10(b), if we fix the external legs, then the size of the  $\mathbb{P}^1$ , which came from the flop, changes with the size of the  $\mathbb{P}^2$ , and the relation between them, as shown in Fig. 11, is given by

$$Q = uQ_H^{-\frac{1}{3}}, \quad u = e^{-h}, \tag{6.38}$$

where  $h$  is the size of the flopped curve when  $t_H = 0$ .

The index of  $Y$  is then given by

$$I_Y(u, t, q) := \int da |Z_Y(Q_H, uQ_H^{-\frac{1}{3}}, t, q)|^2, \tag{6.39}$$

where  $a$  is the loop variable corresponding to the 4-cycle  $\mathbb{P}^2$ . The relation between the loop variable  $a$  and the Kähler parameter  $t_H$  is given by

$$t_H = 3a. \tag{6.40}$$

Equation (6.39) becomes

$$I_Y(u, t, q) = \int \frac{dz}{2\pi iz} |Z_Y(z^3, uz^{-1}, t, q)|^2, \tag{6.41}$$

where  $z = e^{ia}$ . Using the relation between the partition function of  $Y$  and that of  $X$ , Eq. (6.37) then gives

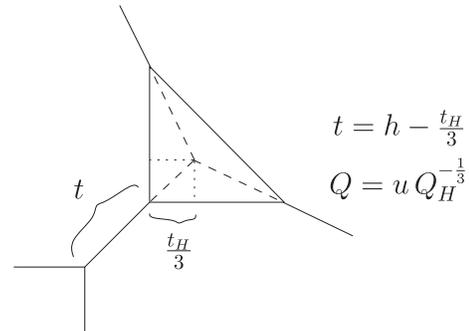


FIG. 11. Local  $\mathbb{P}^2$  blown up at one point.

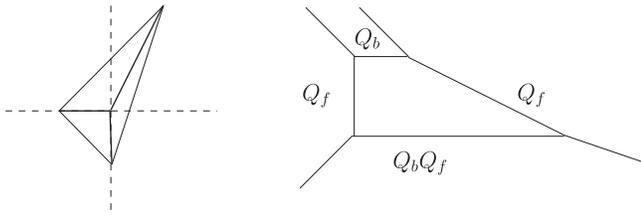


FIG. 12. The Newton polygon (a) and the web diagram (b) of local  $\mathbb{F}_2$ .

$$I_Y(u, t, q) = \int \frac{dz}{2\pi iz} |Z_X(u^{-1}z, uz^2, t, q)|^2. \quad (6.42)$$

Changing the integration variable  $z \mapsto \frac{z}{\sqrt{u}}$ , we get

$$\begin{aligned} I_Y(u, t, q) &= \int \frac{dz}{2\pi iz} |Z_X(u^{-\frac{3}{2}}z, z^2, t, q)|^2, \\ &= \int da |Z_X(u^{-\frac{3}{2}}e^{ia}, e^{2ia}, t, q)|^2. \end{aligned} \quad (6.43)$$

Comparing the above with Eq. (6.35), we see that

---


$$\begin{aligned} Z_{\text{local}\mathbb{F}_2}(Q_b, Q_f, t, q) &= (M(t, q)M(q, t))^{\frac{1}{2}} Z(Q_b, Q_f, t, q) \\ Z(Q_b, Q_f, t, q) &= \sum_{\nu_1, \nu_2} Q_b^{|\nu_1|+|\nu_2|} Q_f^{2|\nu_2|} \left(\frac{q}{t}\right)^{\|\nu_1\|^2 + \|\nu_2\|^2} t^{\kappa(\nu_1) - \kappa(\nu_2)} q^{\|\nu_2'\|^2} t^{\|\nu_1'\|^2} \tilde{Z}_{\nu_1}(t, q) \tilde{Z}_{\nu_1'}(q, t) \tilde{Z}_{\nu_2}(q, t) \tilde{Z}_{\nu_2'}(t, q) \\ &\quad \times \prod_{i,j=1}^{\infty} [(1 - Q_f t^{i-\nu_{2,j}} q^{j-1-\nu_{1,i}})(1 - Q_f q^{i-\nu_{1,j}} t^{j-1-\nu_{2,i}})]^{-1}. \end{aligned} \quad (6.45)$$

The index is given by

$$I_{\text{local}\mathbb{F}_2} = \int da |Z_{\text{local}\mathbb{F}_2}(Q_b, Q_f, t, q)|^2. \quad (6.46)$$

The relation between the loop variable  $a$  and the fiber parameter  $t_f$  is the same as before:

$$t_f = 2a. \quad (6.47)$$

Just as before, we fix the external legs of the web so that we have one parameter  $h$  as shown in Fig. 13. The relation between the  $Q_b$  and  $Q_f$  in this case is given by<sup>15</sup>

$$Q_b = e^{-h} = u. \quad (6.48)$$

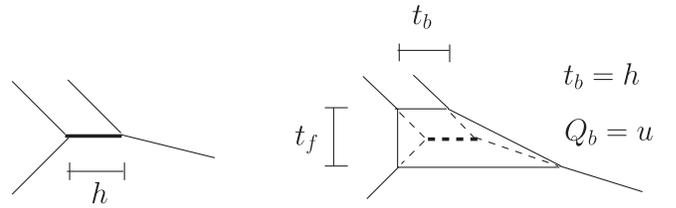


FIG. 13. In this case, since there are two parallel legs, the area of the base curve does not change with the area of the fiber curve.

$$I_Y(u, t, q) = I_X(u^{-2/3}, t, q). \quad (6.44)$$

### E. Example 4: Local $\mathbb{F}_2$

The Hirzebruch surface  $\mathbb{F}_2$  is also a  $\mathbb{P}^1$  bundle over  $\mathbb{P}^1$ . The total space of the canonical bundle on  $\mathbb{F}_2$  gives a local CY 3-fold. As before, the Kähler parameters corresponding to the base  $B$  and the fiber  $F$  will be called  $t_b$  and  $t_f$ , and we define  $Q_b := e^{-t_b}$  and  $Q_f := e^{-t_f}$ . The Newton polygon and the web diagram of local  $\mathbb{F}_2$  are shown in Fig. 12.

The refined partition function for this geometry is given by

---

Thus, the index becomes

$$I_{\text{local}\mathbb{F}_2} = \int da |Z_{\text{local}\mathbb{F}_2}(u, e^{2ia}, t, q)|^2. \quad (6.49)$$

Using Eq. (6.45), we get

$$\begin{aligned} I_{\text{local}\mathbb{F}_2} &= 1 + x^2 + 2\left(y + \frac{1}{y}\right)x^3 + \left(3 + 2y^2 + \frac{2}{y^2}\right)x^4 \\ &\quad + \left(2y^3 + 3y + \frac{3}{y} + \frac{2}{y^3} - \left(u + \frac{1}{y}\right)\left(y + \frac{1}{y}\right)\right)x^5 \\ &\quad \times \left(2y^4 + 5y^2 + \frac{5}{y^2} + \frac{2}{y^4} + 4\right. \\ &\quad \left. - \left(u + \frac{1}{u}\right)\left(3 + y^2 + \frac{1}{y^2}\right)\right)x^6 \\ &\quad + \left(2y^5 + 6y^3 + 9y + \frac{9}{y} + \frac{6}{y^3} + \frac{2}{y^5}\right. \\ &\quad \left. - \left(u + \frac{1}{u}\right)\left(y^3 + 3y + \frac{3}{y} + \frac{1}{y^3}\right)\right)x^7 + \dots \end{aligned}$$

<sup>15</sup>In general for local  $\mathbb{F}_m$ , the relation is  $Q_b = uQ_f^{1-\frac{m}{2}}$ .

**F. Example 5: Local  $\mathbb{P}^2$**

The local  $\mathbb{P}^2$  is the total space of  $\mathcal{O}(-3) \mapsto \mathbb{P}^2$ . As discussed in Sec. IV, the refined topological vertex alone cannot be used to calculate its partition function since there is no set of edges which covers the vertices and has edges parallel to each other. However, some recent developments have made it possible to calculate the refined partition function for any local toric CY 3-fold [50–52]. We use the form of the partition function given in Ref. [50]. The web diagram of the local  $\mathbb{P}^2$  is shown in Fig. 14.

The refined partition function of local  $\mathbb{P}^2$  is given by [50,51]

$$Z_{\text{local}\mathbb{P}^2}(Q, t, q) = \sum_{\lambda\mu\nu} (-Q)^{|\lambda|+|\mu|+|\nu|} q^{\frac{3|\nu'|^2}{2}} t^{-\frac{|\nu|}{2}} \times \tilde{Z}_\nu(q, t) \tilde{Z}_{\nu'}(t, q) s_\lambda(q^{-\rho} t^{-\nu}) \times s_\mu(q^{-\rho} t^{-\nu}) \left(\frac{q}{t}\right)^{\frac{|\lambda|+|\mu|}{2}} N_{\lambda\mu}^\eta R_\eta, \quad (6.50)$$

where

$$R_\eta = \sum_\sigma U_{\eta\sigma} t^{\frac{|\sigma t|^2}{2}} q^{-\frac{|\sigma \nu|^2}{2}} P_\sigma(t^{-\rho}; q, t).$$

$N_{\lambda\mu}^\eta$  are the Littlewood–Richardson coefficients, and  $U_{\eta\sigma}$  is the matrix which takes Macdonald polynomials to Schur polynomials,  $s_\eta(\mathbf{x}) = \sum_\sigma U_{\eta\sigma} P_\sigma(\mathbf{x}; q, t)$ . The matrix elements  $U_{\eta\sigma}$  are rational functions of  $q$  and  $t$ , for example,

$$U_{(1)(1)} = 1 \quad U_{(2)(2)} = 1, \\ U_{(2)(11)} = \frac{t-q}{1-tq}, \quad U_{(11)(2)} = 0, \quad U_{(11)(11)} = 1.$$

The above partition function can also be written as

$$Z(Q, t, q) = \sum_\nu (-Q)^{|\nu|} [q^{\frac{3|\nu'|^2}{2}} t^{-\frac{|\nu|}{2}} \tilde{Z}_\nu(q, t) \tilde{Z}_{\nu'}(t, q)] \times Z_\nu(Q, t, q), \quad (6.51)$$

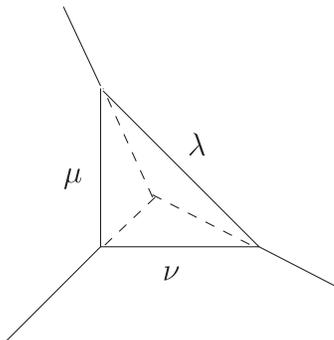


FIG. 14. The web diagram of local  $\mathbb{P}^2$ .

where

$$Z_\nu(Q, t, q) = \sum_{\lambda\mu\eta} (-Q)^{|\lambda|+|\mu|} s_\lambda(q^{-\rho} t^{-\nu}) s_\nu(q^{-\rho} t^{-\nu}) \times \left(\frac{q}{t}\right)^{\frac{|\lambda|+|\mu|}{2}} N_{\lambda\mu}^\eta R_\eta. \quad (6.52)$$

In Eq. (6.51), the factor in the square bracket has expansion in positive powers of  $q$  and  $t^{-1}$  (i.e., positive powers of  $x$ ), but the factor in the second line,  $Z_\nu$ , has expansion in positive powers of  $q^{-1}$  and  $t$  (i.e., negative powers of  $x$ ). This is the generic case for partition functions calculated using the refined topological vertex and can be understood from Eq. (4.9). Since the variables  $x$  and  $y$  couple to the  $SU(2)_R$  and  $SU(2)_L$ , as long as we have full  $(j_L, j_R)$  spin content, negative power of  $x$  cannot be avoided. However, in some cases we can sum over a class of curves and get a product representation which allows expansion in positive powers of  $x$  at the expense of introducing  $Q$  and  $Q^{-1}$ . This is how the index as expansion in positive powers of  $x$  was determined in the last three examples. Therefore, what is required here is some way of summing up the contribution from the curves labelled by  $\lambda$  and  $\mu$  in Fig. 14,  $Z_\nu(Q, t, q)$ , to obtain a product representation which can then be expanded in positive powers of  $x$ .

The index is then given by

$$I_{\text{local}\mathbb{P}^2} = \int da |Z_{\text{local}\mathbb{P}^2}(Q, t, q)|^2. \quad (6.53)$$

The relation between the loop variable and  $Q$  is the same as we derived in showing the flop invariance of the index in Sec. VID 1,

$$Q = e^{3ia}. \quad (6.54)$$

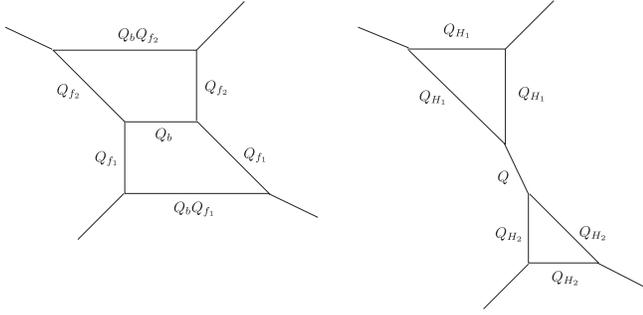
Thus, the index becomes

$$I_{\text{local}\mathbb{P}^2} = \int da |Z_{\text{local}\mathbb{P}^2}(e^{3ia}, t, q)|^2. \quad (6.55)$$

**G. Example 6: Flop invariance of the index**

Here, we present another example which shows that the index is invariant under flop transition. The web diagram of the geometry we will discuss is shown in Fig. 15. The geometry consists of two 4-cycles  $D_1$  and  $D_2$ , both a Hirzebruch surface  $\mathbb{F}_1$ , intersecting along a  $\mathbb{P}^1$  which is the base of the fibration for both divisors. In the neighborhood of the base curve, the geometry looks like  $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \mapsto \mathbb{P}^1$ , and therefore the base curve can flop as shown in Fig. 15.

We will call the geometry before the flop (the one with two  $\mathbb{F}_1$  divisors)  $X$  and the geometry after the flop (the one with two  $\mathbb{P}^2$  divisors)  $Y$ . The relation between the Kähler parameters of geometry  $X$  and  $Y$  is given by


 FIG. 15. Local Calabi-Yau 3-fold with two  $\mathbb{F}_1$  divisors.

$$Q_{H_1} = Q_b Q_{f_1}, \quad Q_{H_2} = Q_b Q_{f_2}, \quad Q = Q_b^{-1}. \quad (6.56)$$

The relation between the partition functions follows from the above relation between the Kähler parameters,

$$Z_Y(Q_{H_1}, Q_{H_2}, Q, t, q) = Z_X(Q^{-1}, Q_{H_1} Q, Q_{H_2} Q, t, q). \quad (6.57)$$

The geometry  $X$  has two 4-cycles and hence two loop variables  $a_1$  and  $a_2$ . As discussed in Sec. V, the relation between the Kähler parameters and the loop variables can be determined using the intersection between curves and the 4-cycles, i.e., intersection of the curve and the anticanonical class of the 4-cycle. Let us denote the 4-cycles in  $X$  by  $D_1$  and  $D_2$ ; then, the anticanonical class is given by

$$-K_{D_1} = 2B + 3F_1, \quad -K_{D_2} = 2B + 3F_2. \quad (6.58)$$

Then, the charge vector of the curve  $C = nB = m_1 F_1 + m_2 F_2$  is given by

$$\begin{aligned} \vec{d} &= (-K_{D_1} \cdot C, -K_{D_2} \cdot C) \\ &= (n + 2m_1 - m_1, n - m_1 + 2m_2), \end{aligned} \quad (6.59)$$

where we have used the following intersection numbers in calculating Eq. (6.59):

$$\begin{aligned} B \cdot B &= -1, & B \cdot F_1 &= +1, & B \cdot F_2 &= +1, \\ F_1 \cdot F_2 &= -1. \end{aligned} \quad (6.60)$$

Thus, the Kähler parameter corresponding to  $C$  in terms of the loop variables is given by

$$Q_C = Q_{C,0} e^{i\vec{d}\cdot\vec{a}} = e^{i(n+2m_1-m_2)a_1 + i(n-m_1+2m_2)a_2}. \quad (6.61)$$

Thus, for geometry  $X$ ,

$$Q_{f_1} = e^{i(2a_1-a_2)}, \quad Q_{f_2} = e^{i(2a_2-a_1)}, \quad Q_b = u e^{i(a_1+a_2)}. \quad (6.62)$$

In the case of geometry  $Y$ , the two 4-cycles will be denoted by  $P_1$  and  $P_2$  with corresponding loop variables  $b_1$  and  $b_2$ , respectively. Both these divisors are  $\mathbb{P}^2$ , and the anticanonical class of these divisors is given by

$$-K_{P_1} = 3H_1 - E, \quad -K_{P_2} = 3H_2 - E, \quad (6.63)$$

where  $H_1$  is the hyperplane class of  $P_1$ ,  $H_2$  is the hyperplane class of  $P_2$ , and  $E$  is the curve connecting the two which comes from the flop of the curve  $B$ . The intersection numbers of these curves are

$$\begin{aligned} H_1 \cdot H_1 &= 1, & H_2 \cdot H_2 &= 1, & H_1 \cdot H_2 &= 0, \\ H_1 \cdot E &= H_2 \cdot E = 0. \end{aligned} \quad (6.64)$$

Using the above intersection numbers, we can easily determine the charge vector of the curve  $C = nH_1 + mH_2 - kE$ ,

$$\vec{d} = (-K_{P_1} \cdot C, -K_{P_2} \cdot C) = (3n - k, 3m - k); \quad (6.65)$$

thus, the Kähler parameter of  $C$  scales with loop variables as

$$Q_C = Q_{C,0} e^{i\vec{d}\cdot\vec{b}} = Q_{C,0} e^{i(3n-k)b_1 + i(3m-k)b_2}. \quad (6.66)$$

Thus, for geometry  $Y$ ,

$$Q_{H_1} = e^{3ib_1}, \quad Q_{H_2} = e^{3ib_2}, \quad Q = Q_E = \tilde{u} e^{-i(b_1+b_2)}. \quad (6.67)$$

Now that we have the relation between the Kähler parameters and the loop variables, we can discuss the index of the two geometries. The index for geometry  $X$  and  $Y$  is given by

$$\begin{aligned} I_X(u, t, q) &= \int da_1 da_2 |Z_X(Q_b, Q_{f_1}, Q_{f_2})|^2 \\ &= \int da_1 da_2 |Z_X(u e^{i(a_1+a_2)}, \\ &\quad e^{i(2a_1-a_2)}, e^{i(2a_2-a_1)})|^2 \\ I_Y(\tilde{u}, t, q) &= \int db_1 db_2 |Z_Y(Q_{H_1}, Q_{H_2}, Q)|^2 \\ &= \int db_1 db_2 |Z_Y(e^{3ib_1}, e^{3ib_2}, \tilde{u} e^{-i(b_1+b_2)})|^2. \end{aligned} \quad (6.68)$$

Now, using Eq. (6.57), we get

$$\begin{aligned}
 I_Y(\tilde{u}, t, q) &= \int db_1 db_2 |Z_X(\tilde{u}^{-1} e^{i(b_1+b_2)}, \tilde{u} e^{i(2b_1-b_2)} \tilde{u} e^{i(2b_2-b_1)})|^2 \\
 &= \int \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} |Z_X(\tilde{u}^{-1} z_1 z_2, \tilde{u} z_1^2 z_2^{-1}, \tilde{u} z_2^2 z_1^{-1})|^2.
 \end{aligned}
 \tag{6.69}$$

Let  $z_1 \mapsto \tilde{u}^{-1} z_1, z_2 \mapsto \tilde{u}^{-1} z_2$ ; then, we get

$$\begin{aligned}
 I_Y(\tilde{u}, t, q) &= \int \frac{dz_1}{2\pi i z_1} \frac{dz_2}{2\pi i z_2} |Z_X(\tilde{u}^{-3} z_1 z_2, z_1^2 z_2^{-1}, z_2^2 z_1^{-1})|^2 \\
 &= I_X(\tilde{u}^{-3}, t, q),
 \end{aligned}
 \tag{6.70}$$

which proves the flop invariance of the index.

### H. Computation of the index with 3d defects

As discussed in Sec. III, 3d defects in the 5d theory can be engineered using Lagrangian branes. In this section, we consider some examples in which there is a single Lagrangian brane in the geometry.

#### 1. Lagrangian brane on $\mathbb{C}^3$

Let us begin by considering the simplest of the brane configurations, a Lagrangian brane on  $\mathbb{C}^3$ . The geometry is shown in Fig. 16 below.

The partition function of the brane is given by

$$Z_{\text{Brane}}(Q, U, t, q) = \sum_{\alpha} (-Q)^{|\alpha|} \text{Tr}_{\alpha} U S_{\mathcal{H}}(q^{-\rho}), \tag{6.71}$$

where  $U$  is the holonomy on the brane and  $-\log Q$  is the area of the disk ending on the brane. Since we are considering a single brane,  $U = e^{i\theta}$ , and the sum over the partitions is restricted to partitions of type  $\{(k) | k = 0, 1, 2, \dots\}$ . The partition function is then given by

$$\begin{aligned}
 Z_{\text{Brane}} &= \sum_{k=0}^{\infty} (-QU)^k s_{(k)}(q^{-\rho}) \\
 &= \sum_{k=0}^{\infty} (-QU)^k q^{k/2} \prod_{i=1}^k (1 - q^i)^{-1} \\
 &= \prod_{n=1}^{\infty} (1 - QUq^{n-\frac{1}{2}}).
 \end{aligned}
 \tag{6.72}$$

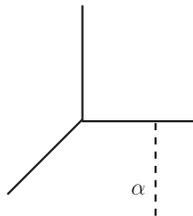


FIG. 16. A Lagrangian brane on  $\mathbb{C}^3$  with boundary condition given by  $\alpha$ .

Define  $z = QUq^{-\frac{1}{2}}$ . Then,

$$|Z_{\text{Brane}}|^2 = \prod_{r=0}^{\infty} \left( \frac{1 - zq^{r+1}}{1 - \bar{z}q^r} \right);$$

this is precisely the result given in Ref. [26] [Eq. (3.6)] if we take  $z = q^{\frac{m}{2}} \zeta$  for the generalized index [32] where  $m$  is the monopole charge.

#### 2. Lagrangian brane on local $\mathbb{P}^1 \times \mathbb{P}^1$

Here, we will consider a single Lagrangian brane on  $\mathbb{P}^1 \times \mathbb{P}^1$ . The brane configuration is shown in Fig. 17 below.

In the limit  $a \mapsto 0$ , the 4-cycle collapses to the curve  $B$ , and the position of the Lagrangian brane on the  $B$  is determined by  $\xi_0$ . When we deform away from this point, there are two possibilities for the Lagrangian brane. Either it ends on the upper horizontal line or the lower one as shown in Fig. 18. We will consider both possibilities in calculating the index.

The partition function of the brane depends on  $(Q_1, Q_b, Q_f)$ , which in turn depend on the parameters of the geometry for a given fixed position of the external legs, and the holonomy on the brane. For the brane attached to the lower horizontal leg (see Fig. 18),

$$Q_b = uQ_f, \quad Q_1 = ue^{\xi_0} Q_f^{\frac{p+1}{2}}. \tag{6.73}$$

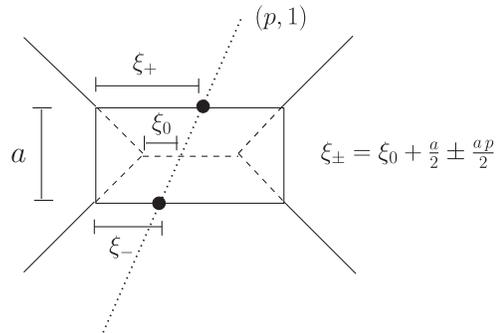


FIG. 17. The geometry of the Lagrangian brane on local  $\mathbb{P}^1 \times \mathbb{P}^1$ .

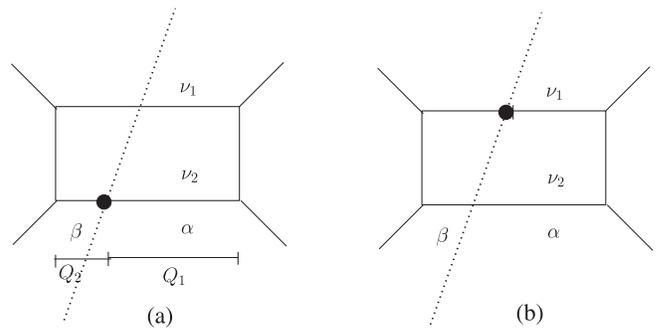


FIG. 18. The two choices for the Lagrangian brane on local  $\mathbb{P}^1 \times \mathbb{P}^1$ .

The partition function of the geometry when the Lagrangian brane is on the upper horizontal leg is given by

$$Z_{\text{Brane}}^p(Q'_1, Q_b, Q_f), \quad (6.74)$$

where

$$\begin{aligned} Z_{\text{Brane}} &:= \sum_{\nu_1 \nu_2} (-Q_b)^{|\nu_1|+|\nu_2|} (-Q_1)^{|\alpha|} (-Q_2)^{|\beta|} (f_{\nu_1 \otimes \beta}(t, q))^{p-1} (f_{\nu_1' \otimes \alpha}(q, t))^p f_{\nu_2'}(t, q) \\ &\times P_{\nu_1 \otimes \alpha}(t^\rho; q, t) P_{\nu_1' \otimes \beta}(q^\rho; t, q) P_{\nu_2}(q^\rho; t, q) P_{\nu_2'}(t^\rho; q, t) \text{Tr}_\alpha U \text{Tr}_\beta U^{-1} \\ &\times \prod_{i,j} [(1 - Q_f t^{-i+1+\nu_{2,j}} q^{-j+(\nu_1 \otimes \alpha)_{1,i}}) (1 - Q_f q^{-i+1+(\nu_1 \otimes \beta')_{1,j}} t^{-j+\nu_{2,i}})]^{-1}, \end{aligned}$$

where  $p$  is the framing of the brane. Since we are considering a single brane, the sum over  $\alpha$  and  $\beta$  is restricted to partitions of type  $\{(n)|n = 0, 1, \dots\}$ . It is clear from Fig. 18 that

$$Q_1 Q_2 = Q_b; \quad (6.76)$$

therefore,

$$\begin{aligned} Z_{\text{Brane}} &:= \sum_{\nu_1 \nu_2} (-Q_b)^{|\nu_1|+|\nu_2|+|\beta|} (-Q_1)^{|\alpha|-|\beta|} (f_{\nu_1 \otimes \beta}(t, q))^{p-1} (f_{\nu_1' \otimes \alpha}(q, t))^p f_{\nu_2'}(t, q) \\ &\times P_{\nu_1 \otimes \alpha}(t^\rho; q, t) P_{\nu_1' \otimes \beta}(q^\rho; t, q) P_{\nu_2}(q^\rho; t, q) P_{\nu_2'}(t^\rho; q, t) \text{Tr}_\alpha U \text{Tr}_\beta U^{-1} \\ &\times \prod_{i,j} [(1 - Q_f t^{-i+1+\nu_{2,j}} q^{-j+(\nu_1 \otimes \alpha)_{1,i}}) (1 - Q_f q^{-i+1+(\nu_1 \otimes \beta')_{1,j}} t^{-j+\nu_{2,i}})]^{-1}. \end{aligned}$$

Taking into account contributions of order  $Q_b$  and  $Q_1$ , we get

$$Z_{\text{Brane}}^p(Q_1, Q_b, Q_f) = Z_0(\tilde{Z} - Q_b(Z_1 + Z_2 + Z_3) + \dots), \quad (6.77)$$

where

$$\begin{aligned} Z_0 &= \left[ \prod_{i,j} [(1 - Q_f t^i q^{j-1}) (1 - Q_f q^i t^{j-1})]^{-1} \right] \\ \tilde{Z} &:= \sum_{\alpha} (-Q_1 U)^{|\alpha|} (f_{\alpha}(q, t))^p P_{\alpha}(t^\rho; q, t) \prod_{(i,j) \in \alpha} (1 - Q_f q^{-i} t^j)^{-1} \\ &= \left( 1 + \frac{Q_1 U Q_f^{-1} q^{\frac{p}{2}+1} t^{-\frac{p}{2}-3}}{(1-t)(1-qt^{-1}Q_f^{-1})} - \frac{Q_1^2 U^2 Q_f^{-2} q^{p+3} t^{-2p-3}}{(1-t)(1-tq)(1-qt^{-1}Q_f^{-1})(1-qt^{-2}Q_f^{-1})} + \dots \right), \end{aligned} \quad (6.78)$$

and

$$\begin{aligned} Z_1 &= -\frac{q}{(1-q)(1-t)(1-qt^{-1}Q_f)} \sum_{\alpha} (-Q_1 U)^{|\alpha|} (f_{\alpha}(q, t))^p P_{\alpha}(t^\rho; q, t) \prod_{i,j} \frac{1 - Q_f t^{-i+1} q^{-j}}{1 - Q_f t^{-i+1} q^{-j+(\square \otimes \alpha)_i}} \\ &= \frac{qt^{-1}}{(1-q)(1-t)(1-Q_f)(1-qt^{-1}Q_f)} \left[ 1 - Q_1 U \frac{q^{\frac{p}{2}} t^{-\frac{p+1}{2}}}{(1-t)(1-Q_f)(1-qQ_f)(1-t^{-1}Q_f)} \right. \\ &\quad \left. - Q_1^2 U^2 \frac{q^{p+1} t^{-2p}}{(1-t)(1-qt)(1-Q_f)(1-t^{-1}Q_f)(1-qQ_f)^2(1-q^2Q_f)} + \dots \right] \end{aligned}$$

$$\begin{aligned}
 Z_2 &= \frac{1}{(1-q)(1-t^{-1})(1-Q_f)} \sum_{\alpha} (-Q_1 U)^{|\alpha|} (f_{\alpha}(q, t))^p P_{\alpha}(t^p; q, t) \prod_{i,j} \frac{1 - Q_f t^{-i+1} q^{-j}}{1 - Q_f t^{-i+1+\square_j} q^{-j+\alpha_i}} \\
 &= \frac{qt^{-1}}{(1-q)(1-t^{-1})(1-Q_f)} \left[ -\frac{Q_f^{-1}}{(1-qt^{-1}Q_f^{-1})} - Q_1 U \frac{q^{\frac{p-1}{2}} t^{-\frac{p-1}{2}}}{(1-t^{-1})(1-Q_f)} \right. \\
 &\quad \left. - Q_1^2 U^2 \frac{q^p t^{-2p+1}}{(1-t^{-1})(1-qt)(1-Q_f)(1-qQ_f)} + \dots \right] \\
 Z_3 &= -\frac{Q_1^{-1} U^{-1} \left(\sqrt{\frac{t}{q}}\right)^{p-1} \sqrt{q}}{(1-q)(1-\frac{q}{t}Q_f)} \sum_{\alpha} (-Q_1 U)^{|\alpha|} (f_{\alpha}(q, t))^p P_{\alpha}(t^p; q, t) \prod_{i=1}^{\alpha} (1 - Q_f q^{\alpha-i})^{-1} \\
 &= -\frac{Q_1^{-1} U^{-1} \left(\sqrt{\frac{t}{q}}\right)^{p-1} \sqrt{q}}{(1-q)(1-\frac{q}{t}Q_f)} \left[ 1 - Q_1 U \frac{q^{\frac{p}{2}} t^{-\frac{p+1}{2}}}{(1-Q_f)} - Q_1^2 U^2 \frac{q^{p+1} t^{-2p}}{(1-t^{-1})(1-qt)(1-Q_f)(1-qQ_f)} + \dots \right]. \tag{6.79}
 \end{aligned}$$

In the above equations,  $P_{\alpha}(\mathbf{x}; q, t)$  are Macdonald polynomials, and since  $\alpha$  only takes the values  $\{(m) | m = 0, 1, \dots\}$ , we give below the explicit expression for  $P_{(m)}(t^p; q, t)$  as a function of  $x$  and  $y$  which we will need later:

$$\begin{aligned}
 P_{(m)}(t^p; q, t) &= (-1)^m t^{m/2} q^{m(m-1)/2} \prod_{j=1}^m (1 - tq^{m-j})^{-1} \\
 &= (-1)^{m-1} \frac{y^{\frac{m^2}{2}-1} x^{\frac{m^2}{2}+1-m}}{(1-xy^{-1}) \prod_{j=1}^{m-1} (1 - y^{m-j+1} x^{m-j-1})} \\
 P_{(m)}(q^p; t, q) &= -\frac{x^{1-\frac{m}{2}} y^{1-\frac{m}{2}}}{(1-xy) \prod_{j=1}^{m-1} (1 - x^{m-j-1} y^{j-m-1})}. \tag{6.80}
 \end{aligned}$$

The partition function of the geometry with the Lagrangian brane has a closed string factor which is the partition function of the geometry without the brane and an open string factor,

$$Z_{\text{Brane}} = Z_{\text{closed}} \times Z_{\text{open}}. \tag{6.81}$$

In our case, there are two different possibilities for the brane to end when the loop variable is deformed. We denote the open string partition function of the brane on the lower leg by  $Z_{\text{open}}$  and the open string partition function of the brane on the upper leg by  $\tilde{Z}_{\text{open}}$ . The two are related as follows:

$$\tilde{Z}_{\text{open}} = Z_{\text{open}}(e^{-\xi_0} Q_f^{\frac{p+1}{2}} U^{-1}, uQ_f, Q_f), \tag{6.82}$$

where the open string partition function of the brane on the lower leg is

$$Z_{\text{open}}(ue^{\xi_0} Q_f^{\frac{p+1}{2}} U, uQ_f, Q_f).$$

The index of the defect theory is given by

$$\begin{aligned}
 I_p &= \int \frac{1}{2} \frac{dQ_f}{2\pi i Q_f} \frac{dU}{2\pi i U} |Z_{\text{open}} \tilde{Z}_{\text{open}}|^2 \\
 &= \int \frac{1}{2} \frac{dQ_f}{2\pi i Q_f} \frac{dU}{2\pi i U} |Z_{\text{open}}(ue^{\xi_0} Q_f^{\frac{p+1}{2}} U, uQ_f, Q_f) \\
 &\quad \times Z_{\text{open}}(e^{-\xi_0} Q_f^{\frac{p+1}{2}} U^{-1}, uQ_f, Q_f)|^2, \tag{6.83}
 \end{aligned}$$

where under complex conjugation

$$(e^{-\xi_0}, u, Q_f, U) \mapsto (e^{-\xi_0}, u^{-1}, Q_f^{-1}, U^{-1}). \tag{6.84}$$

For  $p = 1$ , the above index up to order  $x^2$  is ( $v = e^{\xi_0}$ )

$$\begin{aligned}
 I_{p=1} &= \left[ 1 + \frac{x}{u^3 y} + \left( \frac{2}{u^3} + \frac{4}{u^2} - \frac{1}{uv^2} - \frac{v^2}{u} + \frac{1}{u^6 y^2} + \frac{4}{u^3 y^2} \right. \right. \\
 &\quad \left. \left. - \frac{10y^2}{u^4(1-y^2)^2} - \frac{12}{u^5(1-y^2)} - \frac{2}{u(1-y^2)} \right. \right. \\
 &\quad \left. \left. + \frac{2y^2}{u(1-y^2)} \right) x^2 + \dots \right].
 \end{aligned}$$

To compute the index in the monopole sector  $m$ , we simply substitute  $v = x^{m/2}$ .

## VII. CONCLUSION

We have seen in this paper that the BPS states which arise in the IR flow of superconformal theories upon deformations are a powerful tool in computing superconformal indices at the conformal point. This is particularly so in  $d = 3, 5$ , where we have proposed how one may recover the full index in terms of the BPS partition functions. We have given an indirect argument for our proposal based on reduction of the theory to four dimensions and Pestun's result for partition functions on  $S^4$ . We have checked that it works in all the known examples. It should be possible<sup>16</sup> to derive these results by compactifying M theory on toric 3-folds times  $S^1 \times S^4$  and applying localization ideas to the full string theory similar to the derivation of Ooguri-Strominger-Vafa conjecture in Ref. [55].

It is natural to ask whether we can compute the partition function of supersymmetric theories on  $S^5$  and  $S^3$  using BPS data. Indeed, there is a natural proposal for this [56], which shows how this may be done using topological

<sup>16</sup>We conjecture that this should make sense in the full M-theory context, at least for a noncompact Calabi–Yau 3-fold, i.e., that one could embed the constructions of Refs. [38,54] and similar extensions in other dimensions in the full string theory.

strings. Moreover, this leads to computation of the index of six-dimensional superconformal theories with  $\mathcal{N} = (1, 0)$  and  $\mathcal{N} = (2, 0)$  supersymmetry on  $S^1 \times S^5$  [56]. We thus see that BPS states, as captured by topological strings, are powerful enough to capture the partition function and the superconformal index of a large number of theories in diverse dimensions.

Our work gives further motivation for a reformulation of supersymmetric theories entirely in terms of their BPS data in the IR, in diverse dimensions with varying amounts of supersymmetry. It would be very important to see if one can fully reconstruct the superconformal theories solely from their BPS data.

## ACKNOWLEDGMENTS

We would like to thank M. Aganagic, S. Cecotti, C. Cordova, T. Dimofte, A. Gadde, S. Gukov, J. Heckman, Y. Imamura, K. Intriligator, D. Jafferis, G. Lockhart, S. Minwalla, V. Pestun, L. Rastelli, and N. Seiberg for useful discussions. C. V. would also like to thank the Simons Center for Geometry and Physics where he attended the 10th Simons Workshop on math and physics. The work of A. I. is supported in part by the Higher Education Commission Grant No. HEC-2052. The work of C. V. is supported in part by NSF Grant No. PHY-0244821.

- 
- [1] S. Cecotti and C. Vafa, *Commun. Math. Phys.* **158**, 569 (1993).
  - [2] S. Cecotti and C. Vafa, [arXiv:1103.5832](#).
  - [3] M. Alim, S. Cecotti, C. Cordova, S. Espahbodi, A. Rastogi, and C. Vafa, [arXiv:1112.3984](#).
  - [4] J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, *Commun. Math. Phys.* **275**, 209 (2007).
  - [5] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and S. Raju, *J. High Energy Phys.* **02** (2008) 064.
  - [6] F. A. Dolan and H. Osborn, *Ann. Phys. (Amsterdam)* **307**, 41 (2003).
  - [7] S. Cecotti, A. Neitzke, and C. Vafa, [arXiv:1006.3435](#).
  - [8] R. Gopakumar and C. Vafa, [arXiv:hep-th/9809187](#); [arXiv:hep-th/9812127](#).
  - [9] T. J. Hollowood, A. Iqbal, and C. Vafa, *J. High Energy Phys.* **03** (2008) 069.
  - [10] H. Ooguri and C. Vafa, *Nucl. Phys.* **B577**, 419 (2000).
  - [11] S. Gukov, A. S. Schwarz, and C. Vafa, *Lett. Math. Phys.* **74**, 53 (2005).
  - [12] C. Vafa, [arXiv:1209.2425](#).
  - [13] E. Witten, *Nucl. Phys.* **B471**, 195 (1996).
  - [14] E. Witten, *Int. J. Mod. Phys. A* **09**, 4783 (1994).
  - [15] C. Vafa, *Mod. Phys. Lett. A* **04**, 1169 (1989).
  - [16] S. Cecotti, P. Fendley, K. A. Intriligator, and C. Vafa, *Nucl. Phys.* **B386**, 405 (1992).
  - [17] S. Cecotti and C. Vafa, *Nucl. Phys.* **B367**, 359 (1991).
  - [18] S. Cecotti and C. Vafa, [arXiv:1002.3638](#).
  - [19] E. J. Martinec, *Phys. Lett. B* **217**, 431 (1989).
  - [20] C. Vafa and N. P. Warner, *Phys. Lett. B* **218**, 51 (1989).
  - [21] M. Kontsevich and Y. Soibelman, [arXiv:0811.2435](#).
  - [22] T. Dimofte, S. Gukov, and Y. Soibelman, *Lett. Math. Phys.* **95**, 1 (2011).
  - [23] D. Gaiotto, G. W. Moore, and A. Neitzke, [arXiv:1103.2598](#).
  - [24] S. Pasquetti, *J. High Energy Phys.* **04** (2012) 120.
  - [25] S. Cecotti, C. Cordova, and C. Vafa, [arXiv:1110.2115](#).
  - [26] T. Dimofte, D. Gaiotto, and S. Gukov, *Adv. Theor. Math. Phys.* **17**, 975 (2013).
  - [27] Y. Imamura, *J. High Energy Phys.* **09** (2011) 133.
  - [28] A. Gadde and W. Yan, *J. High Energy Phys.* **12** (2012) 003.
  - [29] F. A. H. Dolan, V. P. Spiridonov, and G. S. Vartanov, *Phys. Lett. B* **704**, 234 (2011).
  - [30] S. Kim, *Nucl. Phys.* **B821**, 241 (2009); **B864**, 884 (2012).
  - [31] Y. Imamura and S. Yokoyama, *J. High Energy Phys.* **04** (2011) 007.
  - [32] A. Kapustin and B. Willett, [arXiv:1106.2484](#).
  - [33] N. Seiberg, *Phys. Lett. B* **388**, 753 (1996).
  - [34] M. R. Douglas, S. H. Katz, and C. Vafa, *Nucl. Phys.* **B497**, 155 (1997).
  - [35] D. R. Morrison and N. Seiberg, *Nucl. Phys.* **B483**, 229 (1997).
  - [36] K. A. Intriligator, D. R. Morrison, and N. Seiberg, *Nucl. Phys.* **B497**, 56 (1997).

- [37] H.-C. Kim, S.-S. Kim, and K. Lee, *J. High Energy Phys.* **10** (2012) 142.
- [38] V. Pestun, *Commun. Math. Phys.* **313**, 71 (2012).
- [39] S. Katz, P. Mayr, and C. Vafa, *Adv. Theor. Math. Phys.* **1**, 53 (1998).
- [40] S. Gukov and E. Witten, [arXiv:hep-th/0612073](https://arxiv.org/abs/hep-th/0612073).
- [41] S. Gukov and E. Witten, *Adv. Theor. Math. Phys.* **14**, 87 (2010).
- [42] T. Dimofte, S. Gukov, and L. Hollands, *Lett. Math. Phys.* **98**, 225 (2011).
- [43] M. Aganagic and C. Vafa, [arXiv:hep-th/0012041](https://arxiv.org/abs/hep-th/0012041).
- [44] M. Aganagic, A. Klemm, and C. Vafa, *Z. Naturforsch. A* **57**, 1 (2002).
- [45] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, *Commun. Math. Phys.* **165**, 311 (1994).
- [46] M. Aganagic and S. Shakirov, [arXiv:1105.5117](https://arxiv.org/abs/1105.5117).
- [47] M. Aganagic, A. Klemm, M. Marino, and C. Vafa, *Commun. Math. Phys.* **254**, 425 (2005).
- [48] A. Iqbal, C. Kozcaz, and C. Vafa, *J. High Energy Phys.* **10** (2009) 069.
- [49] H. Awata and H. Kanno, *J. High Energy Phys.* **05** (2005) 039.
- [50] A. Iqbal and C. Kozcaz, [arXiv:1210.3016](https://arxiv.org/abs/1210.3016).
- [51] M. Aganagic and S. Shakirov, [arXiv:1210.2733](https://arxiv.org/abs/1210.2733).
- [52] A. Okounkov and N. Nekrasov (unpublished).
- [53] H. Awata, H. Fuji, H. Kanno, M. Manabe, and Y. Yamada, *Adv. Theor. Math. Phys.* **16**, 725 (2012).
- [54] T. T. Dumitrescu, G. Festuccia, and N. Seiberg, *J. High Energy Phys.* **08** (2012) 141.
- [55] C. Beasley, D. Gaiotto, M. Guica, L. Huang, A. Strominger, and X. Yin, [arXiv:hep-th/0608021](https://arxiv.org/abs/hep-th/0608021).
- [56] G. Lockhart and C. Vafa (unpublished).