Straightforward prescription for computing the interparticle potential energy related to *D*-dimensional electromagnetic models

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A simple expression for calculating the interparticle potential energy concerning *D*-dimensional electromagnetic models is obtained via Feynman path integral. This prescription converts the hard task of computing this potential into a trivial algebraic exercise. Since this method is equivalent to that based on the merging of quantum mechanics (to leading order, i.e., in the first Born approximation) with the nonrelativistic limit of quantum field theory, and keeping in mind that the latter relies basically on the computation of the nonrelativistic Feynman amplitude ($\mathcal{M}_{\mathrm{NR}}$), a trivial expression for calculating $\mathcal{M}_{\mathrm{NR}}$ is obtained from the alluded prescription as an added bonus. To test the efficacy and simplicity of the method, *D*-dimensional interparticle potential energy is found for a well-known extension of the standard model in which the massless electrodynamics $\mathrm{U}(1)_{\mathrm{QED}}$ is coupled to a hidden sector $\mathrm{U}(1)_{\mathrm{h}}$, as well as Lee-Wick electrodynamics.

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I. INTRODUCTION

Now and again new electromagnetic models appear in the literature. The reasons for studying these systems are many and of different kinds: (i) to control the ultraviolet (UV) divergences that are generally present in the electromagnetic models [1–23]; (ii) to obtain a system where a pointlike charge has a finite self-energy (Born-Infeld electrodynamics is a case in point [24–37]); (iii) to find a system that, besides having a pointlike charge with finite self-energy, also exhibits the feature of birefringence (the logarithmic electrodynamics is an example of this fact [38]); (iv) to analyze Lorentz-violating models [39–50]; and so on.

Nevertheless, as is well known, these electromagnetic models must reproduce the Coulomb potential energy in the nonrelativistic limit plus a correction to the aforementioned energy. Accordingly, it is of fundamental importance to have an easy method for finding the alluded potential so that its behavior at low energies can be analyzed promptly and efficiently.

There are, of course, many powerful methods in the literature for obtaining this potential in the nonrelativistic limit. Unfortunately, all these methods require excessive algebraic computations and, as a consequence, are time-consuming processes.

Our main aim here is to devise a method in which the above mentioned hurdles can be overcome, or at least reduced to a minimum. To accomplish this, we shall build out in Sec. II a prescription for getting the interparticle potential energy concerning *D*-dimensional electromagnetic models based on a Feynman path integral. The main ingredient of the method is a "propagator" in momentum space found by discarding all terms of the usual Feynman propagator in momentum space that are orthogonal to the external conserved currents, and making afterward $k^0 = 0$, where k^μ is the momentum of the exchanged particle.

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In Sec. III, a straightforward expression for calculating the nonrelativistic Feynman amplitude ($\mathcal{M}_{\rm NR}$)—the key point of the method for obtaining the D-dimensional potential based on the marriage between quantum mechanics and the nonrelativistic limit of quantum field theory—is found as a byproduct of our prescription.

To test our method, we shall compute in Secs. IV and V, respectively, the D-dimensional interparticle potential energy for (i) a well-known extension of the standard model in which the massless electrodynamics $\mathrm{U}(1)_{\rm QED}$ is coupled to a hidden sector $\mathrm{U}(1)_{\rm h}$ [51–55], and (ii) Lee-Wick electrodynamics [1,2].

Finally, in Sec. VI, we present our conclusions.

We use natural units throughout, and our Minkowski metric is $\mathrm{diag}(1,-1,\ldots,-1)$.

II. SIMPLE PRESCRIPTION FOR COMPUTING THE *D*-DIMENSIONAL INTERPARTICLE POTENTIAL ENERGY FOR ELECTROMAGENTIC MODELS

It is well known that the generating functional for the connected Feynman diagrams $W_D(J)$ is related to the generating functional for a free electromagnetic theory $Z_D(J)$ by $Z_D(J)=e^{iW_D(J)}$ [56,57], where

$$W_D(J)=-\frac{1}{2}\iint d^Dxd^DyJ^\mu(x)D_{\mu\nu}(x-y)J^\nu(y). \quad (1)$$

Here $J_\mu(x)$ and $D_{\mu\nu}(x-y)$ are, respectively, the external conserved current and the propagator.

Now, bearing in mind that

$$D_{\mu\nu}(x-y)=\int\frac{d^Dk}{(2\pi)^D}e^{ik(x-y)}D_{\mu\nu}(k),$$
$$J_\mu(k)=\int d^Dxe^{-ikx}J_\mu(x),$$

we promptly obtain

$$W_D(J)=-\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}J^\mu(k)^*D_{\mu\nu}(k)J^\nu(k),$$

which can be written as

$$W_D(J)=-\frac{1}{2}\int\frac{d^Dk}{(2\pi)^D}J^\mu(k)^*\mathcal{P}_{\mu\nu}(k)J^\nu(k), \quad (2)$$

where $\mathcal{P}_{\mu\nu}(k)$ is the "propagator" in momentum space obtained by neglecting all terms of the usual Feynman propagator in momentum space that are orthogonal to the external conserved currents.

From (2), we then get

$$W_D(J)=-\frac{1}{2}\int\frac{d^Dk}{(2\pi)^{D-1}}\Bigg[\delta(k^0)T\mathcal{P}_{\mu\nu}(k)\iint d^{D-1}\mathbf{x}\times d^{D-1}\mathbf{y}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}J^\mu(x)J^\nu(y)\Bigg], \quad (3)$$

where the time interval T is produced by the factor $\int dx^0$.

Simple algebraic manipulations, on the other hand, reduce (3) to the form

$$W_D(J)=-T\int\frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}}\mathcal{P}_{\mu\nu}(\mathbf{k})\Delta^{\mu\nu}(\mathbf{k}), \quad (4)$$

where $\mathcal{P}_{\mu\nu}(\mathbf{k})\equiv\mathcal{P}_{\mu\nu}(k)|_{k^0=0}$, and

$$\Delta^{\mu\nu}(\mathbf{k})=\iint d^{D-1}\mathbf{x}d^{d-1}\mathbf{y}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\frac{J^\mu(x)J^\nu(y)}{2}. \quad (5)$$

Now, in the specific case of two charges Q_1 and Q_2 located, respectively, at $\mathbf{a_1}$ and $\mathbf{a_2}$, the current assumes the form

$$J^\mu(x)=\eta^{\mu o}[Q_1\delta^{D-1}(\mathbf{x}-\mathbf{a_1})+Q_2\delta^{D-1}(\mathbf{x}-\mathbf{a_2})]. \quad (6)$$

Therefore,

$$\Delta^{\mu\nu}(\mathbf{k})=Q_1Q_2e^{i\mathbf{k}\cdot\mathbf{r}}\eta^{\mu0}\eta^{\nu0}, \quad (7)$$

where $\mathbf{r}=\mathbf{a_2}-\mathbf{a_1}$, and

$$W_D(J)=-T\frac{Q_1Q_2}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{P}_{00}(\mathbf{k}). \quad (8)$$

On the other hand,

$$Z_D(J)=<0|e^{-iH_DT}|0>=e^{-iE_DT}, \quad (9)$$

which implies that

$$E_D=-\frac{W_D(J)}{T}. \quad (10)$$

As a consequence, the D-dimensional potential energy can be computed through the straightforward expression

$$E_D(r)=\frac{Q_1Q_2}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{P}_{00}(\mathbf{k}). \quad (11)$$

We remark that this expression, *mutatis mutandis*, can be easily extended to scalar and tensorial models.

We call attention to the fact that Zee [57] has utilized Eq. (1) to show that the electromagnetic force between like charges is repulsive.

III. FINDING $\mathcal{M}_{\rm NR}$ FROM OUR PRESCRIPTION

Obviously, the calculations made via Feynman diagrams must coincide in the nonrelativistic limit with those coming from quantum mechanics, where the interaction between particles is described by a potential energy E_D. Our goal here is to compare the cross sections for the interaction of two particles obtained, respectively, using first quantum mechanics and afterward the nonrelativistic limit of quantum field theory, in order to find an expression for computing the potential energy E_D experienced by them.

We begin by recalling a very known expression from quantum mechanics, i.e., the formula for the elastic scattering cross section (to leading order, i.e., in the first Born approximation) for a particle of mass m in the potential E_D:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \tag{12}$$

where

$$f(\theta) = -\frac{m}{2\pi}\int d^{D-1}\mathbf{r}e^{-i\mathbf{k}'\cdot\mathbf{r}}E_D(r)e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{13}$$

Here, θ is the scattering angle, $\mathbf{k}$ the incident moment, and $\mathbf{k}'$ the outgoing moment ($|\mathbf{k}| = |\mathbf{k}'|$, since we are considering an elastic process).

Let us then make a comparison of this result with that obtained through the relativistic formalism. For the sake of clarity, we shall consider the scattering of a nonrelativistic particle of momentum $\mathbf{k}$ and mass m, with $|\mathbf{k}| \ll m$, off a heavy target A, with mass $M_A \gg m$. We can imagine, for instance, an electron being scattered by an atom. Since $|\mathbf{k}| \ll m \ll M_A$, the recoil of the atom can be neglected. In this discussion we shall consider only elastic scattering. Let us assume also that the incident particle and the particle A interact via the exchange of a massless or massive boson.

Now, assuming that M_A is much larger than the electron energy, the elastic cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_A^2}|\mathcal{M}|^2, \tag{14}$$

where $\mathcal{M}$ is the Feynman amplitude for the process at hand. Denoting, in turn, the Feynman amplitude in the nonrelativistic limit by $\mathcal{M}_{\rm NR}$, we get [58]

$$\mathcal{M} = (2m)(2M_A)\mathcal{M}_{\rm NR}, \tag{15}$$

which allows us to rewrite (14) in the form

$$\frac{d\sigma}{d\Omega} = \left(\frac{m}{2\pi}\right)^2|\mathcal{M}_{\rm NR}|^2. \tag{16}$$

From (12) and (16), we find

$$\mathcal{M}_{\rm NR} = \int d^{D-1}\mathbf{r}E_D e^{i\mathbf{q}\cdot\mathbf{r}}, \tag{17}$$

where $\mathbf{q} = \mathbf{k}' - \mathbf{k}$ is the exchanged momentum. Note that the phase of (17) was chosen in such a way that a repulsive force (positive potential) corresponds to a positive $\mathcal{M}_{\rm NR}$.

Accordingly,

$$E_D = \frac{1}{(2\pi)^{D-1}}\int \mathcal{M}_{\rm NR}e^{i\mathbf{q}\cdot\mathbf{r}}d^{D-1}\mathbf{q}. \tag{18}$$

It is worth noting that in the case of two identical fermions, the matrix element $\mathcal{M}_{\rm NR}$ that appears in (18) is just that part of the covariant matrix element which corresponds to direct scattering, since the use of antisymmetric wave functions in nonrelativistic wave mechanics automatically takes care of the contributions due to exchange scattering [59].

Now, since to leading order the methods developed in Secs. II and III are equivalent, we come to the conclusion that

$$\mathcal{M}_{\rm NR} = Q_1 Q_2 \mathcal{P}_{00}. \tag{19}$$

Therefore, our prescription yields an added bonus: a trivial expression for computing $\mathcal{M}_{\rm NR}$.

We point out that formula (18) was used to justify the Coulomb law from QED in the early days of this theory. In the seminal book by Sakurai [59], for instance, the Coulomb potential is found from (18). It is also obtained using the same formula in the excellent and up-to-date book by Maggiore [58]. Gupta and Radford [60], on the other hand, derived the Coulomb potential from the scattering operator using the techniques of standard field theory. It is important to remember that prior to Möller's work, G. Breit had already worked out all the correction terms to the Coulomb potential using essentially classical arguments and applied them to the He atom. In [61–63], we can find interesting results deduced by the mentioned author in the beginning of the quantum electrodynamics era.

IV. *D*-DIMENSIONAL POTENTIAL FOR AN EXTENSION OF THE STANDARD MODEL IN WHICH THE MASSLESS ELECTROMAGNETIC $\mathrm{U}(1)_{\rm QED}$ IS COUPLED TO A HIDDEN-SECTOR $\mathrm{U}(1)_{\rm h}$

Most standard extensions of the standard model, in particular the ones based on string theory, often involve a hidden sector—a set of degrees of freedom very weakly coupled to standard-model particles. On the other hand, an interesting model for studying the massive hidden-sector photons is based on the assumption that the low-energy dynamics contains, in addition to the familiar massless electromagnetic $\mathrm{U}(1)_{\mathrm{QED}}$, another hidden-sector $\mathrm{U}(1)_{\mathrm{h}}$ under which all standard-model particles have zero charge. These two U(1) gauge groups at low energies can be described by the renormalizable Lagrangian [51–55]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}B_{\mu\nu}^2 - \frac{1}{2}\chi F^{\mu\nu}B_{\mu\nu} + \frac{1}{2}m^2 B^\mu B_\mu + J^\mu_{(\mathrm{A})}A_\mu + J^\mu_{(\mathrm{B})}B_\mu, \tag{20}$$

where $F_{\mu\nu}$ is the field strength tensor for the usual electromagnetism $\mathrm{U}(1)_{\mathrm{QED}}$ gauge field A_μ, $B_{\mu\nu}$ is the field strength for the hidden-sector $\mathrm{U}(1)_{\mathrm{h}}$ field B_μ, and $J^\mu_{(\mathrm{A})}$ and $J^\mu_{(\mathrm{B})}$ are the respective currents. The first two terms are the standard kinetic terms for the photon and hidden-sector photon fields, in that order, while the third term, a so-called kinetic mixing, corresponds to a nondiagonal kinetic term. For the sake of positiveness of the energy, the χ parameter must be such that $|\chi| < 1$. It is worth noting that kinetic mixing arises generally both in field theoretic [64,65] as well as in string theoretic [66–69] setups, and typical predicted values for this parameter range between 10^{-16} and 10^{-4}. The second-to-last term in the above Lagrangian accounts for a possible mass of the paraphoton. The purpose of analyzing this model here is to investigate the impact of paraphotons $[\mathrm{U}(1)_{\mathrm{h}}]$ on physical observables. To accomplish this task, we shall work out the D-dimensional interparticle potential energy between the charges using the prescription previously mentioned.

To begin with, we shall write the first four terms in Lagrangian (20) in terms of the usual vector projection operators $\theta_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$ and $\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}$. In the Lorentz gauge ($\mathcal{L}_{\mathrm{gf}} = -\frac{1}{2\lambda}(\partial_\mu A^\mu)^2$), the mentioned terms assume the following form in momentum space:

$$\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3,$$

where

$$\begin{aligned}
\mathcal{L}_1 &= \frac{1}{2}A_\mu K_1^{\mu\nu}A_\nu \\
&= \frac{1}{2}A_\mu\left[-k^2\theta^{\mu\nu} - \frac{k^2}{\lambda}\omega^{\mu\nu}\right]A_\nu, \\
\mathcal{L}_2 &= \frac{1}{2}B_\mu K_3^{\mu\nu}B_\nu \\
&= \frac{1}{2}B_\mu[(m^2-k^2)\theta^{\mu\nu} + m^2\omega^{\mu\nu}]B_\nu, \\
\mathcal{L}_3 &= A_\mu K_2^{\mu\nu}B_\nu \\
&= A_\mu[-\chi k^2\theta^{\mu\nu}]B_\nu.
\end{aligned}$$

Now we gather together A_μ and B_ν in a doublet

$$\Theta_\mu = \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix},$$

which greatly facilitates the computation of the propagators. As a result, (20) can be rewritten in the form

$$\mathcal{L} = \frac{1}{2}\tilde{\Theta}_\mu K^{\mu\nu}\Theta_\nu + \tilde{J}^\mu\Theta_\mu,$$

where

$$K^{\mu\nu} = \begin{bmatrix} K_1^{\mu\nu} & K_2^{\mu\nu} \\ K_2^{\mu\nu} & K_3^{\mu\nu} \end{bmatrix}$$

and

$$J^\mu = \begin{pmatrix} J^\mu_{(\mathrm{A})} \\ J^\mu_{(\mathrm{B})} \end{pmatrix}.$$

After lengthy algebraic calculations, we are able to read off the operator $K^{-1}_{\mu\nu}$:

$$K^{-1}_{\mu\nu} = \begin{bmatrix} a_{\mu\nu} \equiv \langle A_\mu A_\nu\rangle & b_{\mu\nu} \equiv \langle A_\mu B_\nu\rangle \\ b_{\mu\nu} \equiv \langle A_\mu B_\nu\rangle & c_{\mu\nu} \equiv \langle B_\mu B_\nu\rangle \end{bmatrix},$$

where

$$\begin{aligned}
a_{\mu\nu} &= \frac{k^2 - m^2}{k^4\chi^2 + k^2(m^2 - k^2)}\theta_{\mu\nu} - \frac{\lambda}{k^2}\omega_{\mu\nu}, \\
b_{\mu\nu} &= -\frac{\chi}{m^2 - k^2(1-\chi^2)}\theta_{\mu\nu}, \\
c_{\mu\nu} &= \frac{\theta_{\mu\nu}}{m^2 - k^2(1-\chi^2)} + \frac{\omega_{\mu\nu}}{m^2}.
\end{aligned}$$

From Sec. II, we come to the conclusion that

$$E_D = \frac{1}{2T}\iint d^D x d^D y J^\mu(x)K^{-1}_{\mu\nu}(x-y)J^\nu(y). \tag{21}$$

Consequently,

$$E_D = E_I + 2E_{II} + E_{III},$$

where

$$E_I = \frac{1}{2T}\iint d^Dxd^DyJ^\mu_{(\mathrm{A})}(x)a_{\mu\nu}(x-y)J^\nu_{(\mathrm{A})}(y),$$
$$E_{II} = \frac{1}{2T}\iint d^Dxd^DyJ^\mu_{(\mathrm{A})}(x)b_{\mu\nu}(x-y)J^\nu_{(\mathrm{B})(y)},$$
$$E_{III} = \frac{1}{2T}\iint d^Dxd^DyJ^\mu_{(\mathrm{B})}(x)c_{\mu\nu}(x-y)J^\nu_{(\mathrm{B})}(y).$$

It is straightforward to show, on the other hand, that the above expressions can be rewritten as

$$E_I = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}}\mathcal{P}^{(a)}_{\mu\nu}(\mathbf{k})\Delta^{\mu\nu}_{(a)}(\mathbf{k}),$$
$$E_{II} = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}}\mathcal{P}^{(b)}_{\mu\nu}(\mathbf{k})\Delta^{\mu\nu}_{(b)}(\mathbf{k}),$$
$$E_{III} = \int \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1}}\mathcal{P}^{(c)}_{\mu\nu}(\mathbf{k})\Delta^{\mu\nu}_{(c)}(\mathbf{k}),$$

where $\mathcal{P}^{(a)}_{\mu\nu}(\mathbf{k})$, $\mathcal{P}^{(b)}_{\mu\nu}(\mathbf{k})$, and $\mathcal{P}^{(c)}_{\mu\nu}(\mathbf{k})$ are the "propagators" in momentum space found by getting rid of all terms of the corresponding Feynman propagators

$$a_{\mu\nu}(k), b_{\mu\nu}(k), c_{\mu\nu}(k)$$

in this order that are orthogonal to the conserved currents, and afterward making $k^0 = 0$ in all of them. The $\Delta^{\mu\nu}_{(i)}$, $i = a, b, c$, in turn, are defined as follows:

$$\Delta^{\mu\nu}_{(a)}(\mathbf{k}) = \iint d^{D-1}\mathbf{x}d^{D-1}\mathbf{y}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\frac{J^\mu_{(\mathrm{A})}(x)J^\nu_{(\mathrm{A})}(y)}{2},$$
$$\Delta^{\mu\nu}_{(b)}(\mathbf{k}) = \iint d^{D-1}\mathbf{x}d^{D-1}\mathbf{y}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\frac{J^\mu_{(\mathrm{A})}(x)J^\nu_{(\mathrm{B})}(y)}{2},$$
$$\Delta^{\mu\nu}_{(c)}(\mathbf{k}) = \iint d^{D-1}\mathbf{x}d^{D-1}\mathbf{y}e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}\frac{J^\mu_{(\mathrm{B})}(x)J^\nu_{(\mathrm{B})}(y)}{2}.$$

If the currents are expressed as

$$J^\mu_{(\mathrm{A})}(x) = \eta^{\mu 0}[\sigma_1\delta^{D-1}(\mathbf{x}-\mathbf{a_1}) + \sigma_2\delta^{D-1}(\mathbf{x}-\mathbf{a_2})],$$
$$J^\mu_{(\mathrm{B})}(y) = \eta^{\mu 0}[\rho_1\delta^{D-1}(\mathbf{y}-\mathbf{a_1}) + \rho_2\delta^{D-1}(\mathbf{y}-\mathbf{a_2})],$$

we get immediately

$$E_I(r) = \frac{\sigma_1\sigma_2}{(2\pi^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{P}^{(a)}_{00}(\mathbf{k})$$
$$= \frac{\sigma_1\sigma_2}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\left[\frac{1}{\mathbf{k}^2} + \frac{\chi^2}{1-\chi^2}\frac{1}{\mathbf{k}^2+M^2}\right],$$
$$E_{II}(r) = \frac{\sigma_1\rho_2+\sigma_2\rho_1}{2}\frac{1}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{P}^{(b)}_{00}(\mathbf{k})$$
$$= \frac{\chi}{\chi^2-1}\frac{\sigma_1\rho_2+\sigma_2\rho_1}{2(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{\mathbf{k}^2+M^2},$$
$$E_{III}(r) = \frac{\rho_1\rho_2}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{P}^{(c)}_{00}(\mathbf{k})$$
$$= \frac{\rho_1\rho_2}{1-\chi^2}\frac{1}{(2\pi)^{D-1}}\int d^{D-1}\mathbf{k}e^{i\mathbf{k}\cdot\mathbf{r}}\frac{1}{\mathbf{k}^2+M^2}.$$

Here $M^2 \equiv \frac{m^2}{1-\chi^2}$.

Carrying out the previous integrals, we arrive at the conclusion that for $D > 3$,

$$E_D = \frac{\sigma_1\sigma_2}{(2\pi)^{\frac{D-1}{2}}}\frac{2^{\frac{D-5}{2}}\Gamma(\frac{D-3}{2})}{r^{D-3}} + \frac{1}{(2\pi)^{\frac{D-1}{2}}}\left[\frac{\sigma_1\sigma_2\chi^2}{1-\chi^2} + \frac{\chi(\sigma_1\rho_2+\sigma_2\rho_1)}{\chi^2-1} + \frac{\rho_1\rho_2}{1-\chi^2}\right]\left(\frac{M}{r}\right)^{\frac{D-3}{2}}K_{\frac{D-3}{2}}(Mr), \quad (22)$$

whereas for $D = 3$,

$$E_3 = -\frac{\sigma_1\sigma_2}{2\pi}\ln\frac{r}{r_0} + \frac{1}{2\pi}\left[\frac{\sigma_1\sigma_2\chi^2}{1-\chi^2} + \frac{\chi(\sigma_1\rho_2+\sigma_2\rho_1)}{\chi^2-1} + \frac{\rho_1\rho_2}{1-\chi^2}\right]K_0(Mr), \quad (23)$$

where Γ is the gamma function, K_ν is the modified Bessel function of the second order of the order ν, and r_0 is an infrared regulator.

We remark that we could have reshuffled the A_μ and B_μ fields by diagonalizing the matrix $\mathcal{K}$ in the kinetic term of (20) [65],

$$\mathcal{K} = \begin{pmatrix} 1 & \chi \\ \chi & 1 \end{pmatrix},$$

by means of an orthogonal transformation. By doing that, we could rescale the new vector fields by the factors $\sqrt{1\pm\chi}$, and as a result we would end up with a basis of new fields with canonical kinetic terms. However, this would yield new mass-mixing terms amongst the vector fields of the new field basis, and nondiagonal propagators would appear anyhow, mixing the reshuffled fields. That is the reason why we have opted to work with the A_μ and B_μ fields with a mixing kinetic term as given in the

Lagrangian (20). Indeed, this can be easily seen if we follow the steps described below.

First of all, we rewrite (20) in the form

$$\mathcal{L} = \frac{1}{2}\tilde{\Theta}_\mu \mathcal{K}\Box\theta^{\mu\nu}\Theta_\nu + \frac{1}{2}\tilde{\Theta}_\mu M^2\Theta^\mu + \tilde{\Theta}_\mu J^\mu,$$

where

$$M^2 = \begin{pmatrix} 0 & 0 \\ 0 & m^2 \end{pmatrix}.$$

Now, if R is the $SO(2)$ matrix that diagonalizes $\mathcal{K}$, then

$$R\mathcal{K}\tilde{R} = \mathcal{K}_d,$$

where

$$\mathcal{K}_d = \begin{pmatrix} 1+|\chi| & 0 \\ 0 & 1-|\chi| \end{pmatrix}, \qquad 0 < |\chi| < 1.$$

On the other hand, by considering $0 < \chi < 1$, we promptly obtain

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

with $\theta = \frac{\pi}{4}$, which allows us to bring the Lagrangian into the form

$$\mathcal{L} = \frac{1}{2}\tilde{\Lambda}_\mu \mathcal{K}_d\Box\theta^{\mu\nu}\Lambda_\nu + \frac{1}{2}\tilde{\Lambda}_\mu \tilde{M^2}\Lambda^\mu + \tilde{\Lambda}_\mu J^\mu_\Lambda,$$

with $R\Theta_\mu = \Lambda_\mu, \tilde{M^2} = RM^2\tilde{R}, J_\Lambda = RJ$.

In order to absorb the $\mathcal{K}_d^{\frac{1}{2}}$ matrix, we define the Λ_μ field as

$$\mathcal{K}_d^{\frac{1}{2}}\Lambda_\mu = \Sigma_\mu,$$

so that

$$\mathcal{L} = \frac{1}{2}\tilde{\Sigma}_\mu\Box\Theta^{\mu\nu}\Sigma_\nu + \frac{1}{2}\tilde{\Sigma}_\mu \mu^2\Sigma^\mu + \tilde{\Sigma}_\mu J^\mu_\Sigma,$$

with

$$J_\Sigma \equiv \mathcal{K}_d^{-\frac{1}{2}} J_\Lambda, \qquad \mu^2 \equiv \mathcal{K}_d^{-\frac{1}{2}}\tilde{M}^2\mathcal{K}_d^{-\frac{1}{2}},$$

where

$$\mu^2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{m^2}{1-\chi^2} \end{pmatrix}.$$

At this point, the parameter χ has been moved into new (symmetric) mass matrix, μ^2. We can then conclude by understanding that we could work with the more physical basis of fields—namely, the one given by Σ_μ—for which we have the canonical kinetic term and a diagonal mass matrix. It is worth noting that with this field parametrization, the χ parameter moves from the kinetic term into the mass spectrum and the coupling to the external fields. For the sake of our computations to attain the interparticle potential, both field bases, Θ_μ and Σ_μ, are perfectly equivalent. The propagators, in both bases, exhibit exactly the same poles: 0 and $\frac{m^2}{1-\chi^2}$, which are the physical masses in the spectrum.

Let us then return to the subject we were discussing before this digression. Keeping in mind that $K_\nu(r) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-r}}{\sqrt{r}}(1+\mathcal{O}(\frac{1}{r}))$ for $r \longrightarrow \infty$, we clearly see that (22) and (23) reproduce asymptotically the Coulomb potential energy.

Since the model we are analyzing was originally built up in a four-dimensional spacetime, for the sake of completeness we display below the expression for the potential energy for $D = 4$:

$$E_4 = \frac{\sigma_1\sigma_2}{4\pi}\frac{1}{r} + \frac{1}{4\pi}\left[\frac{\sigma_1\sigma_2\chi^2}{1-\chi^2} + \frac{\chi(\sigma_1\rho_2+\sigma_2\rho_1)}{\chi^2-1} + \frac{\rho_1\rho_2}{1-\chi^2}\right]\frac{e^{-Mr}}{r}. \tag{24}$$

Suppose now that we consider the model (20), with $J^\mu_{(\mathrm{A})} = J^\mu_{(\mathrm{B})} = 0$, in the limit of a very heavy B field, i.e., $m \gg m_\gamma$, where m_γ is the photon mass. If we are bound to energies $\ll m$, we can integrate over B_μ in order to obtain an effective model for the A_μ field. This can be done via the path-integral formulation of the generating functional related to the alluded model. Shifting the B_μ field through the expression

$$B_\mu = \hat{B}_\mu - \frac{1}{\Box + m^2}\eta_{\mu\nu}\partial_\alpha F^{\alpha\nu}, \tag{25}$$

and performing afterward the Gaussian integration over this field, we arrive at the following effective Lagrangian for A_μ:

$$\mathcal{L}_{\mathrm{eff}} = -\frac{1}{4}F^2_{\mu\nu} + \frac{\chi^2}{4}F_{\mu\nu}\frac{\Box}{\Box + m^2}F^{\mu\nu}. \tag{26}$$

We are now ready to compute the D-dimensional interparticle potential energy through the prescription developed in Sec. II.

In the Lorentz gauge, the propagator in momentum space concerning (26) reads

$$D_{\mu\nu}(k) = \left[-\frac{1}{k^2} + \frac{\chi^2}{\chi^2-1}\frac{1}{k^2-M^2}\right]\theta_{\mu\nu} - \frac{\lambda}{k^2}\omega_{\mu\nu},$$

where $M^2 \equiv \frac{m^2}{1-\chi^2}$.

Thus,

$$\mathcal{P}_{00}(\mathbf{k}) = \frac{1}{\mathbf{k}^2} - \frac{\chi^2}{\chi^2-1}\frac{1}{\mathbf{k}^2+M^2}. \tag{27}$$

As a result, the D-dimensional interparticle potential energy for the interaction of two static charges σ_1 and σ_2 is given for $D > 3$ by

$$E_D(r) = \frac{\sigma_1\sigma_2}{(2\pi)^{\frac{D-1}{2}}}\left[\frac{2^{\frac{D-5}{2}}\Gamma(\frac{D-3}{2})}{r^{D-3}} - \frac{\chi^2}{\chi^2-1}\left(\frac{M}{r}\right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(Mr)\right], \tag{28}$$

while for $D = 3$,

$$E_3(r) = -\frac{\sigma_1\sigma_2}{2\pi}\left[ln\frac{r}{r_0} + \frac{\chi^2}{\chi^2-1}K_0(Mr)\right]. \tag{29}$$

Both (28) and (29) can obviously be obtained from (22) and (23) by making $\rho_1 = \rho_2 = 0$ in the aforementioned equations. On the other hand, if we make $\rho_1 = \rho_2 = 0$ in (24), the resulting equation reproduces that found in Ref. [70].

V. E_D FOR LEE-WICK ELECTRODYNAMICS

D-dimensional higher-derivative models are the object of intensive research currently. The motivation for studying these systems lies in the fact that higher derivatives have been used frequently as a powerful mechanism to tame the wild UV divergences that are commonly found in relevant physical models. For instance, in the early 1970s, Lee and Wick (LW) claimed that they have found a finite version of QED [1,2]; nevertheless, the model they have put forward is affected by a severe problem: the presence of degrees of freedom associated with a nonpositive norm on the Hilbert space. To remedy this difficulty, these authors adopted *ad hoc* modifications for the analytic continuation of the amplitudes [2]. In summary, we may say that LW work consists essentially in the introduction of Pauli-Villars, wrong-sign propagator fields as physical degrees of freedom, which leads to amplitudes that are better behaved in the UV and render the logarithmically divergent QED finite. Recently, the LW theories have enjoyed a revival of interest owed to the introduction of non-Abelian LW gauge theories by Grinstein, O'Connell, and Wise [71,72]. Their model, usually referred to as the LW standard model, is naturally free of quadratic divergences, thus providing an alternative way to the solution of the hierarchy problem. Although the LW model is nonunitary in the framework of the usual quantum field theory, this does not imply that that it must be rejected. Indeed, higher-order systems can be utilized as effective field models at familiar scales [73]. It is worth noting that, in this sense, many interesting studies related to LW electrodynamics have recently been done [3–23]. Accordingly, owing to the great interest this electrodynamics has aroused in the literature, we shall study the role played by higher derivatives in this model via the analysis of its D-dimensional interparticle potential energy.

LW theory is defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4m^2}F_{\mu\nu}\Box F^{\mu\nu},$$

where m (> 0) is a parameter with mass dimension. The propagator related to this theory in the Lorentz gauge can in turn be written in momentum space as

$$D_{\mu\nu} = \frac{m^2}{k^2(k^2-m^2)}\theta_{\mu\nu} - \frac{\lambda}{k^2}\omega_{\mu\nu}. \tag{30}$$

Consequently,

$$\mathcal{P}_{00}(\mathbf{k}) = \frac{1}{\mathbf{k}^2} - \frac{1}{\mathbf{k}^2+m^2}. \tag{31}$$

As a result, the potential energy for the interaction of two pointlike charges Q_1 and Q_2 located, respectively, at $\mathbf{a_1}$ and $\mathbf{a_2}$ can be computed through the expression

$$E_D(r) = \frac{Q_1Q_2}{(2\pi)^{D-1}}\left[\int\frac{d^{D-1}\mathbf{k}}{\mathbf{k}^2}e^{i\mathbf{k}\cdot\mathbf{r}} - \int\frac{d^{D-1}\mathbf{k}}{\mathbf{k}^2+m^2}e^{i\mathbf{k}\cdot\mathbf{r}}\right].$$

It follows, then, that for $D \neq 3$,

$$E_D(r) = \frac{Q_1Q_2}{(2\pi)^{\frac{D-1}{2}}}\left[\frac{2^{\frac{D-5}{2}}\Gamma(\frac{D-3}{2})}{r^{D-3}} - \left(\frac{m}{r}\right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(mr)\right], \tag{32}$$

whereas for $D = 3$,

$$E_3(r) = -\frac{Q_1Q_2}{2\pi}\left[\ln\frac{r}{r_0} + K_0(mr)\right]. \tag{33}$$

Both (32) and (33) agree with Ref. [3].

Taking into account that $K_\nu(r) \sim \sqrt{\frac{\pi}{2}}\frac{e^{-r}}{\sqrt{r}}(1+\mathcal{O}(\frac{1}{r}))$ for $r \longrightarrow \infty$, it is straightforward to see that (32) and (33) agree asymptotically with the Coulomb potential energy.

We analyze in the following the small-distance behavior of the potential energy, considering the two possible situations: $D \neq 3$ and $D = 3$.

A. $D \neq 3$

Here we have to take into account whether $\nu \notin N$ or $\nu \in N$.

In the first case, i.e., $\nu \notin N$, for $z \to 0$,

$$K_\nu(z) \sim \frac{1}{2}\left[\Gamma(\nu)\left(\frac{2}{z}\right)^\nu\left(1+\frac{z^2}{4(1-\nu)}+\frac{z^4}{32(1-\nu)}\times\frac{1}{(2-\nu)}+\cdots\right)+\Gamma(-\nu)\left(\frac{z}{2}\right)^\nu\left(1+\frac{z^2}{4(\nu+1)}+\frac{z^4}{32(\nu+1)(\nu+2)}+\cdots\right)\right],$$

which implies that $D=2$ or $3<D<5$. Therefore, if D is an even number, there exist only two models that are finite at $r=0$: LW electrodynamics in two or four dimensions. These systems are renormalizable, and their potential energies at $r=0$ are equal to

$$E_2(r=0) = -\frac{Q_1Q_2}{2m},\, E_4(r=0) = \frac{Q_1Q_2m}{4\pi}.$$

On the other hand, if $\nu \in N$,

$$K_\nu(z) = (-1)^{\nu-1}\ln\left(\frac{z}{2}\right)\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{(\frac{z}{2})^{2k}}{k!(k+\nu)!} + \frac{1}{2}\left(\frac{2}{z}\right)^\nu\sum_{k=0}^{\nu-1}\frac{(-1)^k(\nu-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k} + \frac{(-1)^\nu}{2}\left(\frac{z}{2}\right)^\nu\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(k+\nu+1)}{k!(k+\nu)!}\times\left(\frac{z}{2}\right)^{2k},$$

where $\psi(z) \equiv \frac{d}{dz}\ln\Gamma(z)$ is the psi function. Unfortunately, if D is an odd number, the corresponding system is singular at $r=0$.

B. $D = 3$

In this case, the potential energy is finite at $r=0$ and equal to

$$E_3(r=0) = \frac{Q_1Q_2}{2\pi}\ln(mr_0).$$

C. Summary

In short, we may say that for $D=2$, $D=3$, and $D=4$, the higher derivatives present in the model are able to tame the wild divergences of the LW electrodynamics at the origin; unluckily, for $D>4$, these higher derivatives are unable to control the mentioned divergences.

VI. FINAL REMARKS

We have developed a trivial prescription for computing the potential energy for D-dimensional electromagnetic models. The essential part of the method consists in finding the "propagator" $\mathcal{P}_{\mu\nu}(\mathbf{k})$, which is a quite straightforward calculation. The potential energy can then be easily computed via the expression

$$E_D(r) = \frac{Q_1Q_2}{(2\pi)^{D-1}}\int d^{D-1}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k}\mathcal{P}_{00}(\mathbf{k}).$$

This prescription can also be utilized to find E_D for the interaction of two static scalar charges σ_1 and σ_2, as well as for the interaction of two static masses M_1 and M_2. In the former case,

$$E_D(r) = \frac{\sigma_1\sigma_2}{(2\pi)^{D-1}}\int d^{D-1}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k}\mathcal{P}(\mathbf{k}),$$

while in the latter one,

$$E_D(r) = \frac{M_1M_2}{(2\pi)^{D-1}}\kappa_D\int d^{D-1}e^{i\mathbf{k}\cdot\mathbf{r}}\mathbf{k}\mathcal{P}_{00,00}(\mathbf{k}),$$

where κ_D is the D-dimensional Einstein constant [74].

We have also shown how to generalize the method at hand for the case of a doublet of currents. In essence, all we have to do in this case is to compute $\mathcal{P}^{(i)}_{\mu\nu}(\mathbf{k})$, $i=1,2,3,4$.

On the other hand, we have obtained a trivial expression for calculating the Feynman amplitude in the nonrelativistic limit ($\mathcal{M}_{\rm NR}$), as a side effect of our prescription. If we appeal to orthodox methods to make this calculation, we are frequently faced with time-consuming work, particularly in processes mediated by gravitons. We illustrate the efficacy and simplicity of the prescription through two examples: LW electrodynamics and higher-derivative gravity.

In the first case, $\mathcal{M}_{\rm NR}$ for the interaction of two electrons (each one with charge $-e$ $(e>0)$) is given by [see Eq. (31)]

$$\mathcal{M}_{\rm NR} = e^2\left[\frac{1}{\mathbf{k}^2}-\frac{1}{\mathbf{k}^2+m^2}\right].$$

As far as higher-derivative gravity is concerned, whose appropriate Lagrangian for computing the propagator is

$$\mathcal{L} = \sqrt{(-1)^{D-1}g}\left(\frac{2R}{\kappa^2}+\frac{\alpha}{2}R^2+\frac{\beta}{2}R^2_{\mu\nu}\right),$$

where $\kappa^2 = 4\pi\kappa_D$, it can be shown that [74]

$$\mathcal{P}_{00,00}(\mathbf{k}) = \frac{3-D}{D-1}\frac{1}{\mathbf{k}^2} + \frac{D-2}{D-1}\frac{1}{\mathbf{k}^2+m_2^2} - \frac{1}{(D-1)(D-2)}\frac{1}{\mathbf{k}^2+m_0^2},$$

where $m_2^2 \equiv -\frac{4}{\beta\kappa^2}$, $m_0^2 \equiv \frac{4(D-2)}{\kappa^2[4\alpha(D-1)+D\beta]}$.

Accordingly, for the interaction of two masses M_1 and M_2, we promptly find that

$$\mathcal{M}_{\rm NR} = M_1 M_2 \kappa_D \left[-\frac{D-3}{D-2}\frac{1}{\mathbf{k}^2} + \frac{D-2}{D-1}\frac{1}{\mathbf{k}^2+m_2^2} - \frac{1}{(D-1)(D-2)}\frac{1}{\mathbf{k}^2+m_0^2}\right].$$

We point out that it is very hard work to calculate, via traditional methods, the nonrelativistic Feynman amplitude related to this specific example.

Summarizing, we may say that in comparison with the conventional methods, our prescription presents the following advantages:

(i) The computations are easier to perform in the context of our method.

(ii) Our prescription provides, as an added bonus, the nonrelativistic Feynman amplitude in a straightforward way.

(iii) Our method can be used for finding the interparticle potential energy for models with a doublet of currents. In addition, if one of the two fields of the system is a very heavy one, we can find the effective model concerning the remaining field by simply reducing to zero the charges related to the current concerning the heavy field. In other words, there is no need of shifting the heavy field and performing afterward the Gaussian integration over this field in order to arrive at the effective field model.

To conclude, we comment on a recent proposed experiment to search for extra hidden-sector $U(1)$ gauge bosons with small kinetic mixing χ with ordinary photons [75]. The setup consists in putting a sensitive magnetometer inside a superconducting shielding, which is in turn placed inside a strong magnetic field. The authors of the proposed experiment argued that photo-hidden-sector photon-photon oscillations would allow the magnetic field to leak into the shielding volume and register on the magnetometer. For this purpose, they considered the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{4}B_{\mu\nu}^2 - \frac{\chi}{2}F^{\mu\nu}B_{\mu\nu} + \frac{1}{2}m^2 B^\mu B_\mu + \frac{{M_{\rm Lon}}^2}{2}A^\mu A_\mu, \tag{34}$$

where $M_{\rm Lon}$ is the London mass—i.e., the mass the photon acquires inside the superconductor. Since in vacuum $M_{\rm Lon}=0$, (34) reduces to (20) in the absence of currents. Owing to the special relevance of this experiment, it is important to analyze the stability of the above scenario.

Following the same steps that have led to (26), we get

$$\mathcal{L}_{\rm eff} = \frac{1}{2}A_\mu\left[\frac{k^4(1-\chi^2) - k^2(M_{\rm Lon}^2+m^2) + M_{\rm Lon}^2 m^2}{m^2-k^2}\theta^{\mu\nu} + \frac{M_{\rm Lon}^2\lambda - k^2}{\lambda}\omega^{\mu\nu}\right]A_\nu. \tag{35}$$

Therefore, the propagator associated with (35) is given by

$$D_{\mu\nu}(k) = \frac{1}{(1-\chi^2)}\frac{1}{(M_2^2-M_1^2)}\left[\frac{M_1^2-m^2}{k^2-M_1^2} - \frac{M_2^2-m^2}{k^2-M_2^2}\right]\theta_{\mu\nu} + \frac{\lambda}{\lambda M_{\rm Lon}^2 - k^2}\omega_{\mu\nu}, \tag{36}$$

where

$$M_1^2 = \frac{(m^2+M_{\rm Lon}^2)[1+\sqrt{1-\frac{4m^2M_{\rm Lon}^2(1-\chi^2)}{(m^2+M_{\rm Lon}^2)^2}}]}{2(1-\chi^2)},$$

$$M_2^2 = \frac{(m^2+M_{\rm Lon}^2)[1-\sqrt{1-\frac{4m^2M_{\rm Lon}^2(1-\chi^2)}{(m^2+M_{\rm Lon}^2)^2}}]}{2(1-\chi^2)}.$$

Consequently,

$$\mathcal{P}_{00}(\mathbf{k}) = \frac{1}{(1-\chi^2)}\frac{1}{(M_2^2-M_1^2)}\left[\frac{M_2^2-m^2}{\mathbf{k}^2+M_2^2} - \frac{M_1^2-m^2}{\mathbf{k}^2+M_1^2}\right].$$

Thus, the potential energy for $D=4$ reads

$$E_4(r) = \frac{\sigma_1\sigma_2}{4\pi(1-\chi^2)(M_2^2-M_1^2)}\left[(M_2^2-m^2)\frac{e^{-M_2 r}}{r} - (M_1^2-m^2)\frac{e^{-M_1 r}}{r}\right]. \tag{37}$$

Now, if we make $\rho_1=\rho_2=0$ in (24), we get

$$E_4(r) = \frac{\sigma_1\sigma_2}{4\pi}\left[\frac{1}{r} + \frac{\chi^2}{1-\chi^2}\frac{e^{-Mr}}{r}\right]. \tag{38}$$

Thence, for $M_{\rm Lon}=0$, (37) reduces to (38).

Last but not least, we call attention to the interesting fact that inside a superconductor box, the model at hand describes precisely a screening phase [See (37)].

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