PHYSICAL REVIEW D **90,** 105024 (2014)

Nearly degenerate heavy sterile neutrinos in cascade decay: Mixing and oscillations

Daniel Boyanovsky*

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260, USA

(Received 15 September 2014; published 20 November 2014)

Some extensions beyond the Standard Model propose the existence of nearly degenerate heavy sterile neutrinos. If kinematically allowed these can be resonantly produced and decay in a cascade to common final states. The common decay channels lead to mixing of the heavy sterile neutrino states and interference effects. We implement nonperturbative methods to study the dynamics of the cascade decay to common final states, which features similarities but also noteworthy differences with the case of neutral meson mixing. We show that mixing and oscillations among the nearly degenerate sterile neutrinos can be detected as *quantum beats* in the distribution of final states produced from their decay. These oscillations would be a telltale signal of mixing between heavy sterile neutrinos. We study in detail the case of two nearly degenerate sterile neutrinos produced in the decay of pseudoscalar mesons and decaying into a purely leptonic "visible" channel: $\nu_h \to e^+e^-\nu_a$. Possible cosmological implications for the effective number of neutrinos $N_{\rm eff}$ are discussed.

DOI: 10.1103/PhysRevD.90.105024 PACS numbers: 11.10.-z, 11.15.Tk, 11.90.+t

I. INTRODUCTION

Many extensions of the Standard Model that propose explanations of neutrino masses via seesaw-type mechanisms [1–4] predict the existence of heavy "sterile" neutrinos, namely, $SU(2) \times U(1)$ singlets that mix very weakly with "active" neutrinos [5–13]. Heavy sterile neutrinos may play an important role in baryogenesis through leptogenesis [14–17] or via neutrino oscillations [18] motivating several models for leptogenesis which may also yield dark matter candidates [19,20]. Furthermore, heavy sterile neutrinos may contribute to the energy transport during type II supernovae explosions [21]; their decay may be a source of early reionization [22]; they have been argued to play an important role in the thermal history of the early Universe and to contribute to the cosmological neutrino background [23]. For a review of the role of sterile neutrinos in cosmology and astrophysics, see Refs. [20,24–26].

If the mass of the heavy sterile neutrino $m_h \lesssim M_{\pi,K}, M_\tau$, they can be produced as resonances in the decay of pseudoscalar mesons (or charged leptons) opening a window for current and future experimental searches. A comprehensive study of leptonic and semileptonic weak decays of heavy sterilelike neutrinos was carried out in Ref. [27] and extended in Ref. [28], and various experimental studies searching for heavy neutral leptons [29–43] provide constraints on the values of the mixing matrix elements between heavy sterile and active neutrinos for a wide range of masses with stringent bounds within the mass range $140\ \mathrm{MeV} \leq M_h \leq 500\ \mathrm{MeV}$ [40]. Recent bounds on the mixing matrix elements between active (light) and sterile (heavy) neutrinos [40,44,45] yield $|U_{eh}|^2; |U_{\mu h}|^2 \lesssim 10^{-7}$–$10^{-5}$ in the mass range $30\ \mathrm{MeV} \lesssim m_h \lesssim 300\ \mathrm{MeV}$. If heavy sterile neutrinos are Majorana, they can mediate lepton number violating transitions with $|\Delta l| = 2$ motivating further studies of their production and decay [46–48]. Furthermore, resonant production and mixing of *nearly degenerate* heavy sterile neutrinos may lead to enhanced CP violation and baryogenesis [15–20]. A thorough analysis of production and decay rates and cross sections of heavy neutral leptons in various mass regimes is available in Refs. [27,28,46,47,49–54], providing the theoretical backbone to current and proposed experimental searches.

A. Motivation and goals

The astrophysical, cosmological and phenomenological importance of heavy sterile neutrinos and their ubiquitous place in well-motivated extensions beyond the Standard Model motivates a series of recent proposals [49–54]. These make a compelling case for rekindling the search for heavy sterile neutrinos in various current and next generation experiments.

As pointed out in Refs. [15–20], extensions beyond the Standard Model that feature nearly degenerate heavy sterile neutrinos provide mechanisms for resonantly enhanced CP violation with important consequences for baryogenesis through leptogenesis. If these nearly degenerate heavy sterile neutrinos are produced resonantly, they may decay in a cascade into common channels leading to mixing [15–17,55]. Mixing and the ensuing time-dependent oscillation phenomena associated with the decay of (nearly) degenerate states into a common channel is a hallmark of the dynamics of neutral meson mixing such as $K^0\bar{K}^0$, $B^0\bar{B}^0$ [56–58].

*boyan@pitt.edu

The goal of this article is to explore in detail the mixing of two heavy but nearly degenerate sterile neutrinos as a consequence of a common decay channel, the concomitant *time-dependent oscillations* from their interference and the observational consequences in the distribution of the decay products.

Previous discussions of particle mixing focused either on the self-energy corrections featuring off-diagonal matrix elements because of common intermediate states [15–17] or effective Hamiltonian descriptions akin to the case of neutral meson mixing [55–58].

Our goal is complementary in that we study the complete time evolution from the decay of an initial unstable state into channels that include the nearly degenerate heavy sterile neutrinos, which in turn decay into the final states, and assess the impact of the interference between the nearly degenerate states upon the distribution of final states.

For this purpose, we implement a systematic quantum field theoretical generalization of the Wigner-Weisskopf approach [59,60] that includes the decay dynamics of the initial state and the time evolution of the final states. We consider the case of *two* nearly degenerate heavy sterile neutrinos produced from the decay of a pseudoscalar meson (or a heavy charged lepton) first within a general framework of cascade decay to common final states, and then consider the explicit case of a purely leptonic "visible" decay channel for the heavy sterile neutrinos as a potential observable in future experiments.

We find that while there are similarities with the case of neutral meson mixing ($K^0\bar{K}^0, B^0\bar{B}^0$), there are important differences primarily as a consequence of the production of the heavy steriles from the decay of a parent particle (here a pseudoscalar meson) and also from the decay of the nearly degenerate heavy neutrinos into the final states.

II. GENERAL FORMULATION

We generalize the framework described in Refs. [59,60] to describe the production, evolution and decay of two heavy sterile neutrinos.

Consider a total Hamiltonian $H = H_0 + H_I$ with H_0 the free field Hamiltonian and

$$H_I = H_{\mathcal{P}} + H_{\mathcal{D}} + H_{ct}, \tag{2.1}$$

where $H_{\mathcal{P}}$, $H_{\mathcal{D}}$ refer generically to the production ($\mathcal{P}$) and decay ($\mathcal{D}$) interaction vertices, and H_{ct} refers to local renormalization counterterms.

To be specific, and motivated by current and future neutrino experiments, we consider the case where sterile neutrinos are produced in the decay of a charged pseudoscalar meson $\Phi = \pi$, K into a charged lepton α and a neutrino i where $i = a$ refers to the "activelike" (light) and $i = h$ to the "sterilelike" heavy neutrinos *mass eigenstates*, with

$$H_{\mathcal{P}} = i\frac{F_\Phi}{2}\sum_{\alpha=e,\mu}\sum_i U_{\alpha i}\int d^3x[\bar{\Psi}_{l_\alpha}(\vec{x},t)\gamma^\mu(1-\gamma^5) \times \Psi_{\nu_i}(\vec{x},t)\partial_\mu\Phi(\vec{x},t)] + \text{H.c.}, \tag{2.2}$$

with

$$F_\pi = \sqrt{2}G_F V_{ud} f_\pi; \quad F_K = \sqrt{2}G_F V_{us} f_K, \tag{2.3}$$

where f_π, f_K are the corresponding decay constants and $U_{\alpha i}$ is the neutrino mixing matrix with $i = a, h$.

Specifically, the decay interaction vertex $H_{\mathcal{D}}$ is taken to be the usual Standard Model charged current and neutral current vertices, namely $H_{\mathcal{D}} = H_{CC} + H_{NC}$ written in the neutrino mass basis.

Although we consider these specific production and decay vertices for the main discussion in this article, the formulation is more general and applicable for any other production and decay interaction Hamiltonians beyond the Standard Model. To make the discussion general, we consider the case in which $H_{\mathcal{D}}$ describes the decay of ν_h into a multiparticle final state $\{X\}$ ($\nu_h \to \{X\}$).

Let us consider an initial state with one Φ meson of momentum $\vec{k}$ and the vacuum for the other fields, namely (to simplify, we use the same notation for the spatial Fourier transform of a field),

$$|\Psi(t=0)\rangle = |\Phi_{\vec{k}}\rangle. \tag{2.4}$$

Upon time evolution in the Schrödinger picture, this state evolves into $|\Psi(t)\rangle$ obeying

$$\frac{d}{dt}|\Psi(t)\rangle_S = -i(H_0 + H_I)|\Psi(t)\rangle_S. \tag{2.5}$$

When $M_\Phi > m_{L^\alpha} + m_{\nu_h}; m_{\nu_h} > m_X$, where m_X is the invariant mass of the multiparticle final state $\{X\}$, the interaction Hamiltonian (2.1) describes the cascade process depicted in Fig. 1.

We now pass to the interaction picture, wherein

$$H_I(t) = e^{iH_0 t} H_I e^{-iH_0 t}, \tag{2.6}$$

and the state obeys

$$i\frac{d}{dt}|\Psi(\vec{k},t)\rangle_I = H_I(t)|\Psi(\vec{k},t)\rangle_I. \tag{2.7}$$

Consider that at $t = 0$ the initial state is the single meson state of spatial momentum $\vec{k}$ given by (2.4); at any later time, the state $|\Psi(\vec{k},t)\rangle_I$ is expanded in the basis $|n\rangle$ of eigenstates of H_0, namely

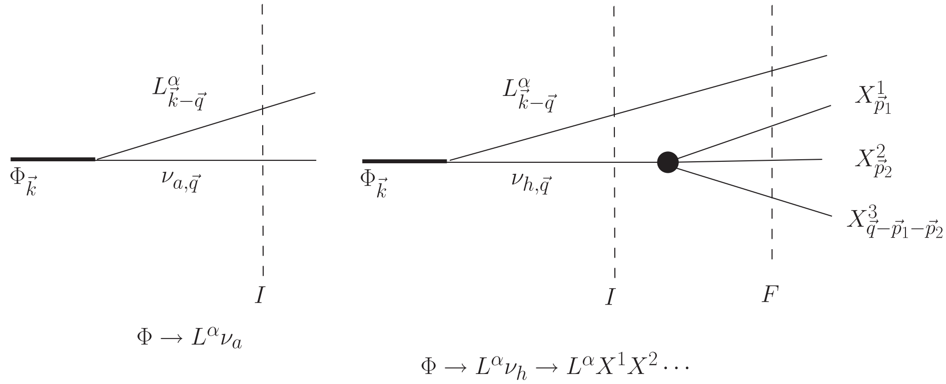

FIG. 1. Decay $\Phi \to L^{\alpha}\nu_a$ (left) and cascade decay $\Phi \to L^{\alpha}\nu_h \to L^{\alpha}\{X\}$ (right) where $\{X\} = X^1_{\vec{p}_1}X^2_{\vec{p}_2}X^3_{\vec{p}_3}\cdots$ is a multiparticle state with $\vec{p}_1+\vec{p}_2+\vec{p}_3+\cdots=\vec{q}$. The dashed lines depict the intermediate two-particle state (I) and the final multiparticle state (F).

$$|\Psi(\vec{k},t)\rangle_I = \sum_n A_n(t)|n\rangle. \tag{2.8}$$

Up to second order in the interaction, the cascade decay depicted in Fig. 1 is described by the following multiparticle state:

$$|\Psi(\vec{k},t)\rangle_I = A_\Phi(\vec{k},t)|\Phi_{\vec{k}}\rangle + \sum_{\alpha;\vec{q};i=a,h} A^{\alpha i}_I(\vec{k},\vec{q};t)|\nu_{i,\vec{q}};L^{\alpha}_{\vec{k}-\vec{q}}\rangle + \sum_{\alpha;\vec{q};\{X\};\{\vec{p}\}_X} A^{\alpha X}_F(\vec{k},\vec{q},\{\vec{p}\}_X;t)|L^{\alpha}_{\vec{k}-\vec{q}};\{X\}\rangle + \cdots. \tag{2.9}$$

For simplicity of notation we do not distinguish between neutrino and antineutrino; furthermore, the framework discussed below is general, independent of whether neutrinos are Dirac or Majorana.

In the last term in (2.9), the sum over $\{X\}$ is over all the decay channels of ν_h, and for each channel the sum over $\{\vec{p}\}_X$ is over the momenta $\vec{p}_1;\vec{p}_2\cdots$ of the multiparticle state $\{X\}$ constrained so that $\vec{p}_1+\vec{p}_2+\cdots=\vec{q}$ (see Fig. 1). There is also an implicit sum over helicity states of the fermionic fields. The coefficients A_Φ, A_I, A_F are the amplitudes of the initial, intermediate and final states respectively, $\alpha = e,\ \mu$ are the charged leptons (we are considering either π or K decay but τ decay can be considered along the same lines as described below), and each α represents a different decay channel for the pseudoscalar meson Φ. The processes that lead to the state (2.9) to second order in the interaction(s) are depicted in Fig. 1; the dots stand for higher order processes, and each vertex in Fig. 1 corresponds to one power of the couplings in H_I, either at the production or decay vertices.

In what follows, we distinguish the labels for the heavy sterile neutrinos as $h = 1, 2$, which *should not* be confused with the activelike neutrinos simply labeled as $\underline{a}$ without further specification.

Unitary time evolution with the initial condition $A_\Phi(\vec{k},0) = 1$ implies

$$|A_\Phi(\vec{k},t)|^2 + \sum_{\alpha;\vec{q};i=a,h}|A^{\alpha i}_I(\vec{k},\vec{q};t)|^2 + \sum_{\alpha;\vec{q};\{X\};\{\vec{p}\}_X}|A^{\alpha X}_F(\vec{k},\vec{q},\{\vec{p}\}_X;t)|^2 + \cdots = 1, \tag{2.10}$$

which has been explicitly confirmed in general in Ref. [59] and in particular for the case of single sterile neutrinos in Ref. [60].

We introduce the following notation:

$$E_\Phi \equiv E_\Phi(k);\quad E^i_I \equiv E_\alpha(|\vec{k}-\vec{q}|)+E_i(q);\quad i=a,h, \tag{2.11}$$

$$E^X_F \equiv E_\alpha(|\vec{k}-\vec{q}|)+E^X; \qquad E^X \equiv E_{X_1}(p_1)+E_{X_2}(p_2)+\cdots, \tag{2.12}$$

$$\langle \nu_{i,\vec{q}};L^{\alpha}_{\vec{k}-\vec{q}}|H_I(t)|\Phi_{\vec{k}}\rangle \equiv M^{\alpha i}_{\mathcal{P}}(\vec{k},\vec{q})e^{-i(E_\Phi-E^i_I)t}, \tag{2.13}$$

$$\langle L^{\alpha}_{\vec{k}-\vec{q}};\{X\}|H_I(t)|\nu_{h,\vec{q}};L^{\alpha}_{\vec{k}-\vec{q}}\rangle \equiv M^{hX}_{\mathcal{D}}(\vec{k},\vec{q},\vec{p})e^{-i(E^h_I-E^X_F)t}, \tag{2.14}$$

where $E_\Phi(k),\ E_i(q),\ E_\alpha(|\vec{k}-\vec{q}|)$ are the single-particle energies for the quanta of the respective fields and E^X is the energy of the multiparticle state with the set of momenta $\{\vec{p}\}_X$. The matrix elements $M_{\mathcal{P}}, M_{\mathcal{D}}$ refer to production ($\mathcal{P}$) and decay ($\mathcal{D}$) vertices.

For example, for the specific production vertex described by (2.2), we find

$$M^{\alpha i}_{\mathcal{P}}(\vec{k},\vec{q};s,s') = U_{\alpha i}F_\Phi\frac{\bar{\mathcal{U}}_{\alpha,s}(\vec{k}-\vec{q})\gamma^\mu(1-\gamma^5)\mathcal{V}_{i,s'}(\vec{q})k_\mu}{\sqrt{32VE_\Phi(k)E_\alpha(|\vec{k}-\vec{q}|)E_i(q)}}; \qquad i=a,h, \tag{2.15}$$

where $\bar{\mathcal{U}}_{\alpha,s}(\vec{k}-\vec{q}),\ \mathcal{V}_{i,s'}(\vec{q})$ are the Dirac spinors for the charged lepton α and neutrino $i = a,\ h$, and the labels $s,\ s'$

refer to helicity states and will be suppressed in what follows. If neutrinos are Majorana, it follows that

$$\mathcal{V}_{i,s'}(\vec{q}) \to \mathcal{U}^{c}_{i,s'}(-\vec{q}). \tag{2.16}$$

The counterterm in the interaction Hamiltonian H_{ct} yields the matrix elements

$$\langle \nu_{h,\vec{q}} | H_{ct} | \nu_{h',\vec{q}} \rangle = \delta\mathcal{E}_{hh'} = \delta\mathcal{E}^{*}_{h'h} \tag{2.17}$$

and renormalizes the masses by subtracting the Hermitian parts of the self-energies as discussed in detail below. The second equality in (2.17) is a consequence of Hermiticity of the interaction Hamiltonian.

To simplify notation, we suppress the momentum arguments of the amplitudes, energies and matrix elements; they are displayed explicitly in the expansion (2.9) and the definitions (2.11), (2.12), (2.13), and (2.14) respectively.

The time evolution of the amplitudes A_Φ, $A^{\alpha i}_I$, and $A^{\alpha X}_F$ is obtained from the Schrödinger equation (2.7) by projecting onto the Fock states; namely, with the interaction picture state written as (2.8), it follows that

$$\begin{aligned}\dot{A}_m(t) &= -i\sum_n \langle m|H_I(t)|n\rangle A_n(t)\\ &= -i\sum_n \mathcal{M}_{mn} e^{i(E_m-E_n)t} A_n(t),\end{aligned} \tag{2.18}$$

where we have used that the matrix elements are of the form

$$\langle m|H_I(t)|n\rangle = e^{i(E_m-E_n)t}\mathcal{M}_{mn}; \quad \mathcal{M}_{mn} = \langle m|H_I(0)|n\rangle, \tag{2.19}$$

and the relevant matrix elements are given by Eqs. (2.13) and (2.14).

Using Eq. (2.18), we obtain the following equations:

$$\begin{aligned}\dot{A}_\Phi(t) &= -i\sum_{\alpha,\vec{q},a} M^{\alpha a *}_{\mathcal{P}} e^{i(E_\Phi - E^a_I)t} A^{\alpha a}_I(t)\\ &\quad - i\sum_{\alpha,\vec{q},h=1,2} M^{\alpha h *}_{\mathcal{P}} e^{i(E_\Phi - E^h_I)t} A^{\alpha h}_I(t); \quad A_\Phi(0) = 1,\end{aligned} \tag{2.20}$$

$$\dot{A}^{\alpha a}_I(t) = -ie^{-i(E_\Phi - E^a_I)t} M^{\alpha a}_{\mathcal{P}} A_\Phi(t); \quad A^{\alpha a}_I(0) = 0 \quad \text{(active)}, \tag{2.21}$$

$$\begin{aligned}\dot{A}^{\alpha h}_I(t) &= -ie^{-i(E_\Phi - E^h_I)t} M^{\alpha h}_{\mathcal{P}} A_\Phi(t)\\ &\quad - i\sum_{h'=1,2} \delta\mathcal{E}_{hh'} e^{i(E_h - E_{h'})t} A^{\alpha h'}_I(t)\\ &\quad - i\sum_{\{X\};\{\vec{p}\}_X} M^{hX*}_{\mathcal{D}} e^{-i(E^X_F - E^h_I)t} A^{\alpha X}_F(t);\\ A^{\alpha h}_I(0) &= 0, \quad h = 1,2 \quad \text{(sterile)},\end{aligned} \tag{2.22}$$

$$\dot{A}^{\alpha X}_F(t) = -i\sum_{h=1,2} M^{hX}_{\mathcal{D}} e^{i(E^X_F - E^h_I)t} A^{\alpha h}_I(t); \quad A^{\alpha X}_F(0) = 0. \tag{2.23}$$

The higher order terms in the expansion of the quantum state represented by the dots in (2.9) lead to higher order terms in the hierarchy of equations. The label α in A_I, A_F refers to the fact that the (charged) lepton α is entangled with the intermediate neutrino and final state, and the kinematics of the production and decay depend on its mass.

In Ref. [59] it is shown that truncating the hierarchy at the order displayed above and solving the coupled set of equations provides a nonperturbative real time resummation of Dyson-type self-energy diagrams with self-energy corrections up to second order in the interactions. In Appendix A we provide a similar analysis for the case of mixing considered here and establish a correspondence with the self-energy treatment in Refs. [15–17].

The three terms on the right-hand side in Eq. (2.22) have a clear interpretation: the first term describes the buildup of the amplitude from the decay of the parent meson, the second term is the counterterm [see Eq. (2.17)] and the third term describes the decay of the heavy steriles into the final states.

The solution of the set of equations (2.20)–(2.23) proceeds from the bottom up. The solution of (2.23) is

$$\begin{aligned}A^{\alpha X}_F(t) &= -i\int_0^t \{M^{1X}_{\mathcal{D}} e^{i(E^X_F - E^1_I)t'} A^{\alpha 1}_I(t')\\ &\quad + M^{2X}_{\mathcal{D}} e^{i(E^X_F - E^2_I)t'} A^{\alpha 2}_I(t')\} dt'.\end{aligned} \tag{2.24}$$

Introducing this solution into Eqs. (2.22), we obtain

$$\begin{aligned}\dot{A}^{\alpha 1}_I(t) &= -ie^{-i(E_\Phi - E^1_I)t} M^{\alpha 1}_{\mathcal{P}} A_\Phi(t) - i\delta\mathcal{E}_{11} A^{\alpha 1}_I(t)\\ &\quad - i\delta\mathcal{E}_{12} e^{i(E_1 - E_2)t} A^{\alpha 2}_I(t)\\ &\quad - \sum_{\{X\};\{\vec{p}\}_X} \int_0^t \{|M^{1X}_{\mathcal{D}}|^2 e^{-i(E^X_F - E^1_I)(t-t')} A^{\alpha 1}_I(t')\\ &\quad + M^{1X*}_{\mathcal{D}} M^{2X}_{\mathcal{D}} e^{i(E_1 - E_2)t} e^{-i(E^X_F - E^2_I)(t-t')} A^{\alpha 2}_I(t')\} dt',\end{aligned} \tag{2.25}$$

$$\begin{aligned}\dot{A}^{\alpha 2}_I(t) &= -ie^{-i(E_\Phi - E^2_I)t} M^{\alpha 2}_{\mathcal{P}} A_\Phi(t) - i\delta\mathcal{E}_{22} A^{\alpha 2}_I(t)\\ &\quad - i\delta\mathcal{E}_{21} e^{i(E_2 - E_1)t} A^{\alpha 1}_I(t)\\ &\quad - \sum_{\{X\};\{\vec{p}\}_X} \int_0^t \{|M^{2X}_{\mathcal{D}}|^2 e^{-i(E^X_F - E^2_I)(t-t')} A^{\alpha 2}_I(t')\\ &\quad + M^{2X*}_{\mathcal{D}} M^{1X}_{\mathcal{D}} e^{i(E_2 - E_1)t} e^{-i(E^X_F - E^1_I)(t-t')} A^{\alpha 1}_I(t')\} dt'.\end{aligned} \tag{2.26}$$

A. The Wigner-Weisskopf approximation

In solving the hierarchy of coupled equations from the bottom up, we encounter linear integrodifferential equations for the coefficients of the general form [see (2.25) and (2.26)]

$$\dot{A}(t) + \int_0^t \sum_{\vec{p}} |M|^2 e^{i(E_I - E_F)(t-t')} A(t') dt' = I(t), \tag{2.27}$$

where $I(t)$ is an inhomogeneity. These types of equations can be solved in terms of Laplace transforms (as befits an initial value problem). In Ref. [59] it is shown that the solution of the hierarchy of equations via Laplace transform yields a real time nonperturbative resummation of Dyson-type self-energy diagrams and a similar proof for the case of mixing is provided in Appendix A. An alternative but equivalent method relies on that the matrix elements M are typically of $\mathcal{O}(g)$ where g refers to a generic coupling in H_I [59]. Therefore, in perturbation theory the amplitudes evolve *slowly* in time since $\dot{A} \propto g^2 A$, suggesting an expansion in *derivatives*. This is implemented as follows [59,60]; consider

$$W_0(t,t') = \sum_{\vec{p}} |M|^2 \int_0^{t'} dt'' e^{-i(E_I - E_F)(t-t'')}, \tag{2.28}$$

which has the properties

$$\frac{d}{dt'} W_0(t,t') = \sum_{\vec{p}} |M|^2 e^{-i(E_I - E_F)(t-t')} \sim \mathcal{O}(g^2);$$
$$W_0(t,0) = 0 \tag{2.29}$$

and is the kernel of the integral term in (2.27). An integration by parts in (2.27) yields

$$\int_0^t dt' \frac{d}{dt'} W_0(t,t') A(t') = W_0(t,t) A(t) - \int_0^t dt' \dot{A}(t') W_0(t,t'). \tag{2.30}$$

From the amplitude equations it follows that $\dot{A} \propto g^2 A$ and $W_0 \propto g^2$; therefore, the second term on the right-hand side in (2.30) is $\propto g^4$ and can be neglected to leading order $\mathcal{O}(g^2)$, which is consistent with the order at which the hierarchy is truncated. This procedure can be repeated systematically, producing higher order derivatives, which are in turn higher order in g^2 providing a systematic quantum field theoretical generalization of the Wigner-Weisskopf method ubiquitous in the treatment of neutral meson mixing [56–58].

The Wigner-Weisskopf approximation is the leading order in the coupling(s) and consists in keeping the first term in (2.30) and taking the long time limit,

$$W_0(t,t) \to \sum_{\vec{p}} |M|^2 \int_0^{t\to\infty} e^{i(E_I - E_F + i\epsilon)(t-t'')} dt'' = i \sum_{\vec{p}} \frac{|M|^2}{(E_I - E_F + i\epsilon)}, \tag{2.31}$$

where $\epsilon \to 0^+$ is a convergence factor for the long time limit.

A more detailed analysis of the long time limit presented in Refs. [59,61] allows us to extract the contribution from wave function renormalization; we will not pursue this contribution here, as it is not directly relevant to the time evolution and oscillations which are the focus of this study.

In Ref. [59] it is shown explicitly that this approximation is indeed equivalent to the exact solution via Laplace transform in the weak coupling and long time limit, where the Laplace transform is dominated by a narrow Breit-Wigner resonance in the Dyson-resummed propagator. The generalization of this equivalence to the case of mixing is discussed in Appendix A.

In the Wigner-Weisskopf approximation up to second order in H_I, we obtain

$$\dot{A}_I^{\alpha 1}(t) + i\Sigma_{11} A_I^{\alpha 1}(t) + i\Sigma_{12} A_I^{\alpha 2}(t) e^{i(E_1 - E_2)t} = -i e^{-i(E_\Phi - E_I^1)t} M_{\mathcal{P}}^{\alpha 1} A_\Phi(t), \tag{2.32}$$

$$\dot{A}_I^{\alpha 2}(t) + i\Sigma_{22} A_I^{\alpha 2}(t) + i\Sigma_{21} A_I^{\alpha 1}(t) e^{i(E_2 - E_1)t} = -i e^{-i(E_\Phi - E_I^2)t} M_{\mathcal{P}}^{\alpha 1} A_\Phi(t). \tag{2.33}$$

The oscillatory factors $e^{\pm i(E_1 - E_2)t}$ in (2.32) and (2.33) can be absorbed by defining

$$\mathcal{A}^h(t) \equiv e^{-iE_h t} A_I^{\alpha h}(t); \quad h = 1,2 \tag{2.34}$$

leading to the following matrix equations for these amplitudes

$$\frac{d}{dt}\begin{pmatrix} \mathcal{A}^1(t) \\ \mathcal{A}^2(t) \end{pmatrix} + i\mathbb{H}\begin{pmatrix} \mathcal{A}^1(t) \\ \mathcal{A}^2(t) \end{pmatrix} = -i e^{i(E_\alpha - E_\Phi)t} A_\Phi(t) \begin{pmatrix} M_{\mathcal{P}}^{\alpha 1} \\ M_{\mathcal{P}}^{\alpha 2} \end{pmatrix}, \tag{2.35}$$

where the "effective Hamiltonian" is

$$\mathbb{H} \equiv \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} = \begin{pmatrix} E_1 + \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & E_2 + \Sigma_{22} \end{pmatrix}. \tag{2.36}$$

The right-hand side of (2.35) describes the production from Φ decay.

The matrix elements are given by

$$\Sigma_{11} = \sum_{\{X\};\{\vec{p}\}_X} \frac{|M_{\mathcal{D}}^{1X}|^2}{E_1 - E^X + i\epsilon} + \delta\mathcal{E}_{11} \equiv \Delta E_{11} + \delta\mathcal{E}_{11} - i\frac{\Gamma_{11}}{2}, \tag{2.37}$$

$$\Sigma_{22} = \sum_{\{X\};\{\vec{p}\}_X} \frac{|M^{2X}_{\mathcal{D}}|^2}{E_2 - E^X + i\epsilon} + \delta\mathcal{E}_{22} \equiv \Delta E_{22} + \delta\mathcal{E}_{22} - i\frac{\Gamma_{22}}{2}, \tag{2.38}$$

$$\Sigma_{12} = \sum_{\{X\};\{\vec{p}\}_X} \frac{M^{1X*}_{\mathcal{D}} M^{2X}_{\mathcal{D}}}{E_2 - E^X + i\epsilon} + \delta\mathcal{E}_{12} \equiv \Delta E_{12} + \delta\mathcal{E}_{12} - i\frac{\Gamma_{12}}{2}, \tag{2.39}$$

$$\Sigma_{21} = \sum_{\{X\};\{\vec{p}\}_X} \frac{M^{1X}_{\mathcal{D}} M^{2X*}_{\mathcal{D}}}{E_1 - E^X + i\epsilon} + \delta\mathcal{E}_{21} \equiv \Delta E_{21} + \delta\mathcal{E}_{21} - i\frac{\Gamma_{21}}{2}, \tag{2.40}$$

where

$$\Delta E_{ij} = \sum_{\{X\};\{\vec{p}\}_X} \mathcal{P}\left(\frac{M^{iX*}_{\mathcal{D}} M^{jX}_{\mathcal{D}}}{E_j - E^X}\right) \tag{2.41}$$

and

$$\Gamma_{ij} = 2\pi \sum_{\{X\};\{\vec{p}\}_X} M^{iX*}_{\mathcal{D}} M^{jX}_{\mathcal{D}} \delta(E_j - E^X), \tag{2.42}$$

where we used $E^h_I - E^X_F = E_h - E^X$ from Eqs. (2.11) and (2.12). In the expressions above, the sum over $\{X\}$ refers to the sum over all decay channels and $\{\vec{p}\}_X$ refers to the sum over the momenta for a fixed channel.

The off-diagonal matrix elements Σ_{12}, Σ_{21} can be understood from the fact that the interaction Hamiltonian has nonvanishing matrix elements between the two sterile neutrinos and the *same final state*. For the case of a three-body common decay channel, the self-energy that mixes ν_h, $\nu_{h'}$ is depicted in Fig. 2; the imaginary part of this self-energy yields the widths Γ_{ij} in Eq. (2.42).

As discussed in Refs. [59,60], the quantum field theoretical Wigner-Weisskopf approximation is equivalent to a Dyson resummation of Feynman diagrams and a Breit-Wigner approximation (complex pole) of the Dyson-resummed propagator. This equivalence is discussed in Appendix A and is confirmed by the results of Ref. [15] where mixing has been studied in terms of self-energy corrections obtained from Feynman diagrams and compared to the effective Hamiltonian description within a different context. In particular, the renormalization corrections and decay widths are exactly those obtained from a Breit-Wigner approximation to the full propagator with self-energy corrections obtained from Feynman diagrams [15].

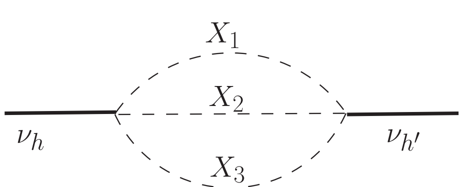

FIG. 2. Self-energy that mixes ν_h, $\nu_{h'}$ for the case of a common three-body decay channel $\nu_h; \nu_{h'} \to X_1 X_2 X_3$.

We emphasize that if the heavy sterile neutrinos are not exactly degenerate, namely, if $E_1 \neq E_2$ then $\Delta E_{ij} \neq (\Delta E_{ji})^*$. As a consequence of the Hermiticity of the counterterm Hamiltonian, it follows that $\delta\mathcal{E}_{ij} = (\delta\mathcal{E}_{ji})^*$; therefore, the counterterms *cannot* completely cancel the real part of the self-energy corrections ΔE_{ij}.

It is convenient to introduce the following quantities:

$$\bar{E} = \frac{1}{2}(E_1 + E_2); \quad \Delta = \frac{1}{2}(E_1 - E_2), \tag{2.43}$$

$$\bar{\Sigma} = \frac{1}{2}(\Sigma_{11} + \Sigma_{22}); \quad \sigma = \frac{1}{2}(\Sigma_{11} - \Sigma_{22}), \tag{2.44}$$

in terms of which the complex eigenvalues of $\mathbb{H}$ are

$$\lambda^{\pm} = (\bar{E} + \bar{\Sigma}) \pm [(\Delta + \sigma)^2 + \Sigma_{12}\Sigma_{21}]^{\frac{1}{2}} \equiv E^{\pm} - i\frac{\Gamma^{\pm}}{2}, \tag{2.45}$$

where $E^{\pm}$ and $\Gamma^{\pm}$ are real corresponding to the energy and decay width of the propagating modes.

Consider now the eigenvalue problem

$$\mathbb{H}\begin{pmatrix} \alpha^{\pm}_1 \\ \alpha^{\pm}_2 \end{pmatrix} = \lambda^{\pm}\begin{pmatrix} \alpha^{\pm}_1 \\ \alpha^{\pm}_2 \end{pmatrix} \tag{2.46}$$

and the matrices

$$\mathcal{U}^{-1} = \begin{pmatrix} \alpha^{+}_1 & \alpha^{-}_1 \\ \alpha^{+}_2 & \alpha^{-}_2 \end{pmatrix}; \quad \mathcal{U} = \frac{1}{(\alpha^{+}_1\alpha^{-}_2 - \alpha^{+}_2\alpha^{-}_1)}\begin{pmatrix} \alpha^{-}_2 & -\alpha^{-}_1 \\ -\alpha^{+}_2 & \alpha^{+}_1 \end{pmatrix} \tag{2.47}$$

from which it follows that

$$\mathcal{U}\mathbb{H}\mathcal{U}^{-1} = \begin{pmatrix} \lambda^{+} & 0 \\ 0 & \lambda^{-} \end{pmatrix}. \tag{2.48}$$

Therefore, defining

$$\begin{pmatrix} \mathcal{A}^1(t) \\ \mathcal{A}^2(t) \end{pmatrix} = \mathcal{U}^{-1}\begin{pmatrix} V^{+}(t) \\ V^{-}(t) \end{pmatrix} \tag{2.49}$$

and right multiplying (2.35) by $\mathcal{U}$ and using (2.48), we find

$$\frac{d}{dt}\begin{pmatrix} V^{+}(t) \\ V^{-}(t) \end{pmatrix} + i\begin{pmatrix} \lambda^{+} & 0 \\ 0 & \lambda^{-} \end{pmatrix}\begin{pmatrix} V^{+}(t) \\ V^{-}(t) \end{pmatrix} = -ie^{i(E_\alpha - E_\Phi)t} A_\Phi(t)\begin{pmatrix} \widetilde{M}^{\alpha+}_{\mathcal{P}} \\ \widetilde{M}^{\alpha-}_{\mathcal{P}} \end{pmatrix}; \quad V^{\pm}(0) = 0 \tag{2.50}$$

with

$$\begin{pmatrix} \widetilde{M}_{\mathcal{P}}^{\alpha+} \\ \widetilde{M}_{\mathcal{P}}^{\alpha-} \end{pmatrix} = \mathcal{U}\begin{pmatrix} M_{\mathcal{P}}^{\alpha 1} \\ M_{\mathcal{P}}^{\alpha 2} \end{pmatrix}. \tag{2.51}$$

The solutions are

$$V^{\pm}(t) = -i\widetilde{M}_{\mathcal{P}}^{\alpha\pm} e^{-i\lambda^{\pm} t} \int_0^t e^{i(\lambda^{\pm}+E_\alpha-E_\Phi)t'} A_\Phi(t')dt', \tag{2.52}$$

and from the relation (2.49) we obtain

$$\mathcal{A}^1(t) = \alpha_1^+ V^+(t) + \alpha_1^- V^-(t);$$
$$\mathcal{A}^2(t) = \alpha_2^+ V^+(t) + \alpha_2^- V^-(t). \tag{2.53}$$

Full expressions for the products $\alpha_j^{\pm}\widetilde{M}_{\mathcal{P}}^{\alpha\pm}$ are given in Appendix B where it is recognized that these products are independent of the normalization of the eigenvectors of $\mathbb{H}$.

The solution of (2.21) is

$$A_I^{\alpha a}(t) = -iM_{\mathcal{P}}^{\alpha a*} \int_0^t e^{-i(E_\Phi - E_I^a)t'} A_\Phi(t')dt'. \tag{2.54}$$

We now insert the solutions (2.53) and (2.54) into the evolution equation for $A_\Phi(t)$ (2.20); using the definitions (2.12) and (2.34) we find

$$\begin{aligned} \dot{A}_\Phi(t) = &-\sum_{\alpha;\vec{q}} \int_0^t \{\overline{M}_{\mathcal{P}}^{\alpha+}\widetilde{M}_{\mathcal{P}}^{\alpha+} e^{-i(\lambda^+ + E_\alpha - E_\Phi)(t-t')} \\ &+ \overline{M}_{\mathcal{P}}^{\alpha-}\widetilde{M}_{\mathcal{P}}^{\alpha-} e^{-i(\lambda^- + E_\alpha - E_\Phi)(t-t')}\} A_\Phi(t')dt' \\ &- \sum_{\alpha;\vec{q};a} \int_0^t \{|M_{\mathcal{P}}^{\alpha a}|^2 e^{-i(E_a + E_\alpha - E_\Phi)(t-t')} A_\Phi(t')\}dt', \end{aligned} \tag{2.55}$$

where

$$\bar{M}_{\mathcal{P}}^{\alpha\pm} = \alpha_1^{\pm} M_{\mathcal{P}}^{\alpha 1*} + \alpha_2^{\pm} M_{\mathcal{P}}^{\alpha 2*}. \tag{2.56}$$

The first line in Eq. (2.55) is the contribution from the intermediate heavy sterile states, and the second line is the contribution from the active neutrinos.

Implementing the Wigner-Weisskopf approximation and taking the long time limit (no convergence factor is needed in the first sum because $\lambda^{\pm}$ feature a negative imaginary part arising from the decay of the intermediate state) this evolution equation simplifies to

$$\dot{A}_\Phi(t) + i\mathcal{E}_\Phi A_\Phi(t) = 0; \quad A_\Phi(0) = 1, \tag{2.57}$$

where

$$\begin{aligned} \mathcal{E}_\Phi \equiv \Delta E_\Phi - i\frac{\Gamma_\Phi}{2} = &\sum_{\alpha;\vec{q};a} \frac{|M_{\mathcal{P}}^{\alpha a}|^2}{E_\Phi - E_a - E_\alpha + i\epsilon} \\ &+ \sum_{\alpha;\vec{q}} \left\{ \frac{\bar{M}_{\mathcal{P}}^{\alpha+}\widetilde{M}_{\mathcal{P}}^{\alpha+}}{(E_\Phi - E_\alpha - E^+ + \frac{i}{2}\Gamma^+)} + \frac{\bar{M}_{\mathcal{P}}^{\alpha-}\widetilde{M}_{\mathcal{P}}^{\alpha-}}{(E_\Phi - E_\alpha - E^- + \frac{i}{2}\Gamma^-)} \right\} \end{aligned} \tag{2.58}$$

with ΔE_Φ and Γ_Φ real, leading to

$$A_\Phi(t) = e^{-i\Delta E_\Phi t} e^{-\frac{\Gamma_\Phi}{2}t}. \tag{2.59}$$

ΔE_Φ will be absorbed into a renormalization of the single meson energy, namely $E_\Phi + \Delta E_\Phi \to E_\Phi$ (from now on, E_Φ denotes the renormalized single-particle energy), and Γ_Φ is the *total decay width* of the parent meson.

It only remains to introduce the result (2.59) into (2.52) to obtain the time evolution of all the amplitudes. We find

$$V^{\pm}(t) = \widetilde{M}_{\mathcal{P}}^{\alpha\pm} \frac{[e^{-i(E_\Phi - E_\alpha - i\frac{\Gamma_\Phi}{2})t} - e^{-i(E^{\pm} - i\frac{\Gamma^{\pm}}{2})t}]}{[E_\Phi - E_\alpha - E^{\pm} - \frac{i}{2}(\Gamma_\Phi - \Gamma^{\pm})]}. \tag{2.60}$$

Using the definition (2.34) and inserting the results (2.53) and (2.60) into (2.23), we find for the final state amplitude

$$\begin{aligned} A_F^X(t) = &\frac{(\alpha_1^+\widetilde{M}_{\mathcal{P}}^{\alpha+}M_{\mathcal{D}}^{1X} + \alpha_2^+\widetilde{M}_{\mathcal{P}}^{\alpha+}M_{\mathcal{D}}^{2X})}{[E_\Phi - E_\alpha - E^+ - \frac{i}{2}(\Gamma_\Phi - \Gamma^+)]} \\ &\times \left\{ \frac{[e^{-i(E_\Phi - E_F^X - i\frac{\Gamma_\Phi}{2})t} - 1]}{[E_\Phi - E_F^X - i\frac{\Gamma_\Phi}{2}]} - \frac{[e^{-i(E^+ - E^X - i\frac{\Gamma^+}{2})t} - 1]}{[E^+ - E^X - i\frac{\Gamma^+}{2}]} \right\} \\ &+ \frac{(\alpha_1^-\widetilde{M}_{\mathcal{P}}^{\alpha-}M_{\mathcal{D}}^{1X} + \alpha_2^-\widetilde{M}_{\mathcal{P}}^{\alpha-}M_{\mathcal{D}}^{2X})}{[E_\Phi - E_\alpha - E^- - \frac{i}{2}(\Gamma_\Phi - \Gamma^-)]} \\ &\times \left\{ \frac{[e^{-i(E_\Phi - E_F^X - i\frac{\Gamma_\Phi}{2})t} - 1]}{[E_\Phi - E_F^X - i\frac{\Gamma_\Phi}{2}]} - \frac{[e^{-i(E^- - E^X - i\frac{\Gamma^-}{2})t} - 1]}{[E^- - E^X - i\frac{\Gamma^-}{2}]} \right\}. \end{aligned} \tag{2.61}$$

In the probability of detecting the final state $|A_F^X(t)|^2$, the interference between the terms with $e^{-i(E^{\pm} - i\frac{\Gamma^{\pm}}{2})t}$ leads to oscillations. These will be studied in Sec. III below.

Going back to the Schrödinger picture with $|\Psi(t)\rangle_S = e^{-iH_0 t}|\Psi(t)\rangle_I$ we obtain

$$|\Psi(\vec{k},t)\rangle_S = e^{-iE_\Phi t}e^{-\frac{\Gamma_\Phi}{2}t}|\Phi_{\vec{k}}\rangle + \sum_{\alpha;\vec{q};a} e^{-iE_I^\alpha t} A_I^{\alpha a}(\vec{k},\vec{q};t)|\nu_{a,\vec{q}};L^\alpha_{\vec{k}-\vec{q}}\rangle + \sum_{\alpha;\vec{q};h} e^{-iE_\alpha t}\mathcal{A}^h(\vec{k},\vec{q};t)|\nu_{h,\vec{q}};L^\alpha_{\vec{k}-\vec{q}}\rangle + \sum_{\alpha;\vec{q};\{X\};\{\vec{p}\}_X} e^{-iE_F^X t} A_F^{\alpha X}(\vec{k},\vec{q},\{\vec{p}\}_X;t)|L^\alpha_{\vec{k}-\vec{q}};\{X\}\rangle + \cdots. \tag{2.62}$$

Using the result (2.53) and the definition (2.34) we note that we can write in the second term in (2.62)

$$\sum_{h=1,2}\mathcal{A}^h(t)|\nu_h\rangle = V^+(t)|\nu^+\rangle + V^-(t)|\nu^-\rangle; \quad |\nu^\pm\rangle = \alpha_1^\pm|\nu_1\rangle + \alpha_2^\pm|\nu_2\rangle. \tag{2.63}$$

Namely, the states $|\nu^\pm\rangle$ are coherent superpositions of the mass eigenstates of the unperturbed Hamiltonian. In particular, under the assumption that $\Gamma_\Phi \gg \Gamma^\pm$, it follows that for time scales $1/\Gamma_\Phi \ll t \lesssim 1/\Gamma_\pm$,

$$V^\pm(t) = C_\pm e^{-iE^\pm t}e^{-\Gamma^\pm t}, \tag{2.64}$$

where the normalization constants

$$C_\pm = -\frac{\widetilde{M}^{\alpha\pm}_{\mathcal{P}}}{[E_\Phi - E_\alpha - E^\pm - \frac{i}{2}(\Gamma_\Phi - \Gamma^\pm)]} \tag{2.65}$$

reflect the Lorentzian distribution from the decay of the parent meson. However, because $\mathbb{H}$ is non-Hermitian, the states $|\nu^\pm\rangle$ are not orthogonal, namely $\langle\nu^+|\nu^-\rangle = (\alpha_1^+)^*(\alpha_1^-) + (\alpha_2^+)^*(\alpha_2^-) \neq 0$.

B. Comparison to neutral meson mixing

The evolution equations for the amplitudes of the intermediate state (2.35) in terms of an effective Hamiltonian (2.36) are similar to the case of neutral meson mixing but with noteworthy differences:

(i) The inhomogeneity on the right-hand side of (2.35) describes the *production* of the intermediate state from the decay of the initial state. In the description of neutral meson mixing, the production stage is not included but the initial state is assumed to be a linear superposition of the unperturbed neutral mesons (K^0, $\bar{K}^0$, B^0, $\bar{B}^0$ etc.), and the equivalent of Eq. (2.35) is homogeneous. Since the amplitude $A_\Phi(t) \to 0$ for $t \gg 1/\Gamma_\Phi$, the production contribution vanishes and Eq. (2.35) becomes homogeneous describing an initial value problem for the amplitudes for time scales $t \gg 1/\Gamma_\Phi$; therefore, one would conclude that for $t \gg 1/\Gamma_\Phi$ the two cases are similar. However, it is clear from the expressions (2.60) that in this limit, the amplitudes for the heavy sterile neutrinos *are not determined from arbitrary initial conditions*, but are determined by the Lorentzian distribution function that results from the decay of the parent particle. This is manifest in the prefactors $C_\pm$ in (2.64) which are given by (2.65) as a direct consequence of production of sterile neutrinos from the decay process; in other words, these coefficients are a manifestation of the "memory" of the initial state and of the decay dynamics of the parent meson.

The probability for finding a particular mode $\pm$ after the decay of the parent meson for $t \gg 1/\Gamma_\Phi$ is

$$\frac{|\widetilde{M}^{\alpha\pm}_{\mathcal{P}}|^2 e^{-\Gamma^\pm t}}{[E_\Phi - E_\alpha - E^\pm]^2 + [(\Gamma_\Phi - \Gamma^\pm)/2]^2}. \tag{2.66}$$

Namely, the exponential decay factor multiplies a Lorentzian probability distribution of decay products. The difference in the decay widths in the denominator has a simple interpretation: Γ_Φ describes the rate at which the sterile neutrinos are *produced*, whereas $\Gamma^\pm$ are the rate at which they *decay* into the final states so that the effective production rate is $\Gamma_\Phi - \Gamma^\pm$.

(ii) Unlike the neutral meson case under the assumption of CPT symmetry, the diagonal entries in the matrices (2.41) and (2.42) are *not* the same. This is because the sterile neutrinos in the intermediate state are *not exactly degenerate.* As a consequence of this nondegeneracy, it also follows that $\Delta E_{ij} \neq \Delta E^*_{ji}$; $\Gamma_{ij} \neq \Gamma^*_{ji}$, unlike the case of neutral meson mixing. Therefore, as mentioned above, the counterterms $\delta\mathcal{E}_{ij}$ obeying the Hermiticity condition cannot completely cancel the self-energy corrections ΔE_{ij}. In the case of neutral meson mixing, the unperturbed (bare) masses of the meson and antimeson are the same; hence, the denominators in ΔE_{12}, ΔE_{21} are the same and ΔE_{ij} is Hermitian [56]. Indeed the original derivation in [56] manifestly uses that the meson and antimeson have the same (unperturbed) energy (mass). Allowing for different energies and following the derivation in [56], the results for ΔE_{ij} obtained above follow directly.

(iii) The time-dependent prefactors $V^\pm(t)$ are given by (2.60); the first term $\propto e^{-i(E_\Phi - E_\alpha - i\frac{\Gamma_\Phi}{2})t}$ is a direct consequence of the *production* of sterile neutrinos via the decay of the pseudoscalar meson and can be traced to the right-hand side of Eq. (2.35). *If* $\Gamma_\Phi \gg \Gamma^\pm$ and for $t \gg 1/\Gamma_\Phi$, it follows that $V^\pm(t) \propto e^{-i(E^\pm - i\frac{\Gamma^\pm}{2})t}$ which is the usual time evolution obtained from the effective Hamiltonian in the Wigner-Weisskopf approximation for neutral meson mixing [56–58]. This is in agreement with the results of Ref. [59] wherein it was observed that if the decay rate of the parent particle is much larger than that of the intermediate resonant state, the time evolution proceeds sequentially: the decay of the parent particle leads to the formation of the intermediate state on a time scale much shorter than the lifetime of the intermediate resonant state; its amplitude grows initially from the production dynamics and decays on a longer time scale.

III. OSCILLATIONS IN THE DETECTION OF DECAY PRODUCTS

Oscillations in the decay products are observationally relevant on macroscopic scales when the heavy sterile neutrinos are nearly degenerate, namely when $E_1 + E_2 \gg |E_1 - E_2|$. In this limit there are two important cases to consider.

(i) $|E_1 - E_2| \gg \Sigma_{ij}$: Since $\Sigma_{ij} \propto g^2$ where g is a typical weak coupling in the interaction Hamiltonian, we find up to second order in couplings

$$\lambda^+ = E_1 + \Sigma_{11} + \mathcal{O}(g^4); \quad \lambda^- = E_2 + \Sigma_{22} + \mathcal{O}(g^4). \tag{3.1}$$

The counterterms can be chosen to cancel the real parts of the self-energy so that $E_{1,2}$ are the fully renormalized (real) energies and to leading order in the couplings for this case we find

$$\lambda^+ = E_1 - \frac{i}{2}\Gamma_{11}; \quad \lambda^- = E_2 - \frac{i}{2}\Gamma_{22}. \tag{3.2}$$

(ii) $|E_1 - E_2| \lesssim \Sigma_{ij}$: In this case the full expression for $\lambda^\pm$ are given by (2.45) and we can set $E_{1,2} \to \bar{E} = (E_1 + E_2)/2$ to leading order in the self-energies (2.37)–(2.40). Neglecting terms of $\mathcal{O}(g^2|\Delta|/\bar{E} \lesssim g^4)$, we find

$$\lambda^+ - \lambda^- = 2[(\Delta + \sigma)^2 + \Sigma_{12}\Sigma_{21}]^{\frac{1}{2}} \propto \mathcal{O}(g^2), \tag{3.3}$$

where

$$\Delta E_{ij} = \sum_{\{X\};\{\vec{p}\}_X} \mathcal{P}\left(\frac{M_{\mathcal{D}}^{iX*} M_{\mathcal{D}}^{jX}}{\bar{E} - E^X}\right) \tag{3.4}$$

and

$$\Gamma_{ij} = 2\pi \sum_{\{X\};\{\vec{p}\}_X} M_{\mathcal{D}}^{iX*} M_{\mathcal{D}}^{jX} \delta(\bar{E} - E^X), \tag{3.5}$$

with the corollary that $(\Delta E_{ij})^* = \Delta E_{ji}$; $(\Gamma_{ij})^* = \Gamma_{ji}$. Because the counterterms obey the Hermiticity conditions [see (2.17)] in this case we implement the "on-shell" renormalization scheme following [15] and request that

$$\Delta E_{ij} + \delta\mathcal{E}_{ij} = 0, \tag{3.6}$$

where ΔE_{ij} are given by (3.4).

The probability of finding a particular final state X at time t is given by $|A_F^{\alpha X}(t)|^2$. Consider the visible decay of the heavy sterile neutrinos to the *common decay channel* $\{X\} = e^+e^-\nu_a$, namely $\nu_{h1,h2} \to e^+e^-\nu_a$ where ν_a is an active neutrino. The number of e^+e^- *pairs* in this state is given by (suppressing the appropriate quantum numbers)

$$\langle\Psi(t)|b_e^\dagger b_e d_e^\dagger d_e|\Psi(t)\rangle = |A_F^{\alpha e^+e^-\nu_a}(t)|^2, \tag{3.7}$$

and the total number of e^+e^- pairs in this particular decay channel is

$$N_{e^+e^-}(t) = \sum_{\{\vec{p}\}_X} |A_F^{\alpha e^+e^-\nu_a}(t)|^2. \tag{3.8}$$

The amplitude $A_F^{\alpha X}(t)$ (2.61) clearly indicates that $|A_F^{\alpha X}(t)|^2$ features oscillatory contributions from the interference between the terms with $e^{\pm iE^\pm t}$. These interference terms will be manifest over macroscopic distances if the real part of the eigenvalues $E^\pm$ are nearly degenerate. Since the self-energies are perturbative, from the expressions (2.45) it is clear that near degeneracy of $E^{1,2}$ implies near degeneracy of $E^\pm$. It is convenient to define

$$\bar{\mathcal{E}} = \frac{1}{2}(E^+ + E^-); \quad \delta = \frac{1}{2}(E^+ - E^-);$$
$$\bar{\Gamma} = \frac{1}{2}(\Gamma^+ + \Gamma^-) = \frac{1}{2}(\Gamma_{11} + \Gamma_{22}), \tag{3.9}$$

with $\delta \ll \bar{\mathcal{E}}$. Writing $A_F^{\alpha e^+e^-\nu_a}(t) \equiv A^+(t) + A^-(t)$ and assuming that the matrix elements are smooth functions of the energy so that to leading order we can evaluate them at the average energy $\bar{\mathcal{E}}$ thereby neglecting terms of $\mathcal{O}(\delta/\bar{\mathcal{E}})$, we find

$$(A^+(t))^* A^-(t) = \tau^+\tau^- \frac{4\pi^2\delta(\bar{\mathcal{E}} + E_\alpha - E_\Phi)\delta(\bar{\mathcal{E}} - E^X)}{[\Gamma_\Phi + \bar{\Gamma} + 2i\delta][\bar{\Gamma} - i2\delta]} \times [1 - e^{2i\delta t} e^{-\bar{\Gamma} t}], \tag{3.10}$$

where

$$\tau^\pm = (\alpha_1^\pm \widetilde{M}_{\mathcal{P}}^{\alpha\pm} M_{\mathcal{D}}^{1X} + \alpha_2^\pm \widetilde{M}_{\mathcal{P}}^{\alpha\pm} M_{\mathcal{D}}^{2X}). \tag{3.11}$$

The details of the calculation are given in Appendix C.

Integration over the final state phase space p_X and over $\bar{\mathcal{E}}$ yields the overall energy-momentum conservation and fixes the average $\bar{\mathcal{E}} = E_\Phi - E_\alpha$.

These oscillations in the probability of decay products are akin to "quantum beats" in the photodetection probability of radiative decays in multilevel atomic systems [62], and a similar phenomenon has been discussed in Ref. [63] within a different context.

In the nearly degenerate case, after imposing the on-shell renormalization condition [15] (3.6), we find

$$\Sigma_{ij} = -i\frac{\Gamma_{ij}}{2} \tag{3.12}$$

and

$$\lambda^{\pm} \equiv E^{\pm} - \frac{i}{2}\Gamma^{\pm} = \bar{E} - \frac{i}{2}(\Gamma_{11} + \Gamma_{22}) \pm \left[\left(\Delta - \frac{i}{2}(\Gamma_{11} - \Gamma_{22})\right)^2 - \frac{1}{4}|\Gamma_{12}|^2\right]^{1/2}. \tag{3.13}$$

These are general results in the nearly degenerate case.

IV. AN EXAMPLE: THE VISIBLE DECAY CHANNEL $\nu_h \to e^+e^-\nu_a$

We now study the specific example of two nearly degenerate heavy sterile neutrinos with a common purely leptonic visible decay channel: $\nu_h, \nu_{h'} \to e^+e^-\nu_a$ via a charged and/or neutral current vertex, with a an activelike neutrino [27].

In the Fermi limit the self-energy diagram that describes this common decay channel is shown in Fig. 3.

Adapting the results from Ref. [27] and neglecting corrections of order $|E^1 - E^2|/(E^1 + E^2) \ll 1$, we find

$$\Gamma_{ij}(\nu_h \to e^+e^-\nu_a) = \frac{G_F^2 \bar{M}_h{}^6}{192\pi^3 \bar{E}}\mathcal{H}\left(\frac{m_e^2}{\bar{M}_h^2}\right)U_{eh_i}U^*_{eh_j};$$

$$\bar{M}_h = \frac{1}{2}(m_{h_1} + m_{h_2}), \tag{4.1}$$

where [27]

$$\mathcal{H}(x) = (1-4x^2)^{\frac{1}{2}}(1 - 14x - 2x^2 - 12x^3) + 24x^2(1-x^2)\ln\frac{1+(1-4x^2)^{\frac{1}{2}}}{1-(1-4x^2)^{\frac{1}{2}}}, \tag{4.2}$$

and to leading order we have replaced $\bar{\mathcal{E}} \to \bar{E}$. Neglecting m_e it follows that[1]

$$\Gamma_{ij}(\nu_h \to e^+e^-\nu_a) \simeq 3.5 \times \left[\frac{\bar{M}_h}{100\ \mathrm{MeV}}\right]^5 \left[\frac{U_{eh_i}U^*_{eh_j}}{10^{-5}}\right]\left(\frac{\bar{M}_h}{\bar{E}}\right)\ \mathrm{s}^{-1}, \tag{4.3}$$

with an extra factor of 2 if ν_h is a Majorana neutrino.

Imposing the on-shell renormalization condition (3.6), we find the effective Hamiltonian

$$\mathbb{H} = \begin{pmatrix} E_1 - \frac{i}{2}\Gamma_{11}(\nu_h \to e^+e^-\nu_a) & -\frac{i}{2}\Gamma_{12}(\nu_h \to e^+e^-\nu_a) \\ -\frac{i}{2}\Gamma^*_{12}(\nu_h \to e^+e^-\nu_a) & E_2 - \frac{i}{2}\Gamma_{22}(\nu_h \to e^+e^-\nu_a) \end{pmatrix}. \tag{4.4}$$

[1]The factor $\bar{E}/\bar{M}_h$ is the average Lorentz factor.

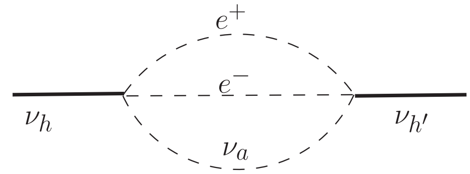

FIG. 3. Self-energy with $e^+e^-\nu_a$ in the intermediate state mixing ν_h, $\nu_{h'}$ in the Fermi limit.

Because the decay width is suppressed by the small neutrino mixing matrix elements, the decay vertices are expected to be displaced far from the production vertices and the space-time evolution of the sterile neutrinos becomes important. In Ref. [60] these aspects were studied within the context of a single sterile neutrino but the results are straightforwardly adapted to the present study. To address the space-time evolution, a wave packet description is necessary and it is discussed in detail in Ref. [60] for the case of a single sterile neutrino in a cascade decay. Consider now the nearly degenerate sterile neutrinos propagating as wave packets with nearly equal group velocities

$$v_g = \bar{p^*}/\bar{E}, \tag{4.5}$$

where we have approximated $\bar{\mathcal{E}} \simeq \bar{E}$ to leading order in weak coupling and $\bar{p^*}$ is the value of the momentum determined by energy-momentum conservation at the production vertex for a sterile neutrino of average energy $\bar{E} = E_\Phi - E_\alpha$. For pseudoscalar meson decaying at rest,

$$v_g = \frac{[\lambda(1,\delta_\alpha,\bar{\delta_h})]^{\frac{1}{2}}}{(1+\bar{\delta_h}-\delta_\alpha)}, \tag{4.6}$$

where

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz;$$

$$\delta_\alpha = \frac{m^2_{L^\alpha}}{M^2_\Phi}; \quad \bar{\delta_h} = \frac{\bar{M}_h{}^2}{M^2_\Phi}. \tag{4.7}$$

Consider a detector a distance L_d from the production vertex and fiducial length ΔL_d, so that $|E^+ - E^-|\Delta L_d/v_g \ll 1$, $\bar{\Gamma}\Delta L_d/v_g \ll 1$, then the oscillatory contribution to the number of events detected within the fiducial length simplifies; namely, the last term in (3.10) becomes (see Ref. [60] for details)

$$\frac{[1 - e^{2i\delta t}e^{-\bar{\Gamma}t}]}{[\bar{\Gamma} - i2\delta]} \to e^{2i\delta L_d/v_g}e^{-\bar{\Gamma}L_d/v_g}\Delta L_d. \tag{4.8}$$

Namely, after the phase space integrations, the number of e^+e^- pairs detected within the distance ΔL_d a distance L_d away from the production region is

$$N_{e^+e^-}(t)|_{\rm osc} = \mathcal{N} e^{2i\delta L_d/v_g} e^{-\bar{\Gamma} L_d/v_g} \Delta L_d;$$
$$\delta = \frac{1}{2}(E^+ - E^-); \quad \bar{\Gamma} = \frac{1}{2}(\Gamma^+ + \Gamma^-), \tag{4.9}$$

where $\mathcal{N}$ is the normalization factor arising from the phase space integrations and

$$2\delta = \mathrm{Re}[(E_1 - E_2 - i(\Gamma_{11} - \Gamma_{22}))^2 - |\Gamma_{12}|^2]^{1/2};$$
$$\bar{\Gamma} = \frac{1}{2}(\Gamma_{11} + \Gamma_{22}), \tag{4.10}$$

with Γ_{ij} given by Eq. (4.1). This result for $\bar{\Gamma}$ follows from that in (4.9) and (3.13).

A. Coherence aspects

The oscillatory behavior arising from the interference terms between the two nearly degenerate eigenstates bears many similarities with the case of oscillation and mixing of active neutrinos, but with noteworthy differences.

Oscillations in the decay products will be observed provided that $|E^+ - E^-| \gtrsim \bar{\Gamma}$; otherwise, the interference term damps out before any oscillation can occur. Furthermore, the result for the interference term (3.10) has been obtained under the assumption that the difference in energies of correct eigenstates $|E^+ - E^-|$ *cannot* be discriminated by the measurement. This is manifest in the derivation of Eqs. (C11) and (C9) in Appendix C leading to the result (3.10) which is obtained as a distribution integrated over a density of states (detector) that is insensitive to the energy difference. If the detector (final density of states) *can* discriminate between the energy eigenstates with a resolution smaller than the widths, the narrow width approximation to each Lorentzian yields a product $\propto \delta(\mathcal{E} - \delta)\delta(\mathcal{E} + \delta)$ which vanishes and the interference and quantum beats will be suppressed as the measurement is effectively projecting on a particular energy eigenstate. This is similar to the case of active neutrino oscillations when the neutrino mass eigenstates are produced in the decay of a parent meson whose decay width determines the energy resolution as analyzed in Refs. [53,64] and discussed further in Ref. [65].

The analysis leading to the result (4.9) made use of a wave packet description of the space-time evolution. The two different eigenstates with $E^{\pm}$ feature slightly different group velocities which result in that the corresponding wave packets slowly drift away from each other. Coherence leading to oscillatory interference is maintained provided these wave packets have a substantial overlap, which requires that $|v_g^+ - v_g^-| L_d \ll \sigma$ where σ is the width of the individual wave packets. This is similar to the case of oscillations of active neutrinos and has been analyzed in detail in Ref. [65] to which the reader is referred for further discussion. A detailed analysis of possible decoherence effects requires a firm assessment of the energies and energy differences as well as an estimate of the width of the wave packets, which is ultimately determined by characteristic localization length scale of the parent particle and determined by the experimental setup.

V. CONCLUSIONS, POSSIBLE COSMOLOGICAL IMPLICATIONS AND FURTHER QUESTIONS

Motivated by their astrophysical, cosmological and phenomenological relevance, their important place in compelling extensions beyond the Standard Model and recent proposals to search for heavy neutral leptons, we have studied the production, propagation and decay of nearly degenerate heavy sterile neutrinos with common decay channels.

We have implemented a nonperturbative field theoretical systematic generalization of the Wigner-Weisskopf theory ubiquitous in the study of neutral meson mixing, here extended to include both the production and the decay into the full dynamics for the general case of sterile neutrinos with a common decay channel. Mixing between them is a consequence of a common set of intermediate states which lead to off-diagonal terms in the self-energies. Within the Wigner-Weisskopf description, mixing is manifest in off-diagonal terms in the effective Hamiltonian that describes the time evolution of the *amplitudes* for the sterile neutrino states.

Our study has focused on heavy sterile neutrinos produced by pseudoscalar meson decay, as this is one important avenue for possible study in current and future neutrino experiments, however the method may be straightforwardly generalized to alternative production reactions.

While the dynamical evolution features similarities with the cases of neutral meson mixing, there are noteworthy differences primarily as a consequence of including the dynamics of the production and decay in the treatment.

Although the framework is general, we considered the case of a visible leptonic common decay channel $\nu_h, \nu_{h'} \to e^+e^-\nu_a$ (a is an active neutrino), as an explicit example of experimental relevance and obtained the (nearly degenerate) complex energies. Interference between the "mass eigenstates" are manifest in damped oscillations in the e^+e^- distribution function akin to the quantum beat phenomenon in the radiative decay of multilevel atoms.

In combination with a wave packet description, we obtained the oscillatory contribution to the number of e^+e^- pairs within a detector of length ΔL_d placed at a distance L_d from the production region.

These oscillations in the decay products would be a telltale signature of mixing between heavy neutral leptons.

A. Possible cosmological implications

The decay width of the propagating modes [see Eq. (4.3)] suggests that sterile neutrinos in the mass range $M_h \sim 100$ MeV and with $|U_{eh}|^2 \lesssim 10^{-7}$–$10^{-5}$ feature a lifetime ranging from a few seconds to a few minutes depending on the strength of the mixing matrix elements. If these sterile neutrinos are produced from pion decay shortly after the QCD (hadronization) transition (at $\simeq 10\mu s$; $T \simeq 150$ MeV), they may decay into $e^+e^-\nu_a$ several minutes after the freeze-out of active neutrinos. In this case the active neutrinos from decay are "injected" into the cosmic neutrino background with a nonequilibrium distribution function and cannot thermalize after neutrino freeze-out. These extra nonthermal neutrinos would not contribute to big bang nucleosynthesis (BBN) as they are produced well after the time scale for BBN, but *may* modify the effective number of relativistic neutrinos, $N_{\rm eff}$, with a nonequilibrium distribution function.

While the results obtained here yield insights into this possibility, the formulation introduced in this article is not directly applicable to the cosmological case which requires the time evolution of a density matrix instead of an initial single-particle state. Furthermore the production of sterile neutrinos must be studied within the quantum kinetics from pion decay in the thermal medium and freeze-out of the distribution function when the pion abundance becomes suppressed as the temperature decreases during the cosmological expansion. This study will be reported elsewhere [66].

B. Further questions

In this study we focused on understanding the interference effects between the nearly degenerate sterile neutrinos and their manifestation in the decay products within a general framework.

We did not consider specifically either CP-violating transitions, or $|\Delta l| = 2$ transitions in the case of Majorana neutrinos. As pointed out in Refs. [15,16,19,20], CP violation may be resonantly enhanced in the case of nearly degenerate heavy sterile neutrinos; furthermore, lepton violating transitions are suppressed in the case of small (Majorana) neutrino masses but may be enhanced by heavy sterile neutrinos in intermediate states. Of particular interest would be possible oscillations in $|\Delta l| = 2$ transitions. Furthermore, a complete assessment of the probability of detection requires us to consider specific cases for the production reaction as well as the decay interaction vertex. These determine the explicit form of the matrix elements, the coefficients $\alpha^{\pm}_{1,2}$ in the superposition (2.63), the normalization of the Lorentzian distribution in the coefficients $C_\pm$ in (2.65) and ultimately the overall normalization factor $\mathcal{N}$ in the final expression (4.9). All of these aspects merit further study, which will be reported elsewhere.

ACKNOWLEDGMENTS

The author acknowledges partial support from Grant No. NSF-PHY-1202227.

APPENDIX A: EQUIVALENCE WITH DYSON-RESUMMED PROPAGATORS

In the Schrödinger picture the full quantum state is

$$|\Psi(\vec{k},t)\rangle_S = C_\Phi(\vec{k},t)|\Phi_{\vec{k}}\rangle + \sum_{\alpha;\vec{q};i=a,h} C_I^{\alpha i}(\vec{k},\vec{q};t)|\nu_{i,\vec{q}};L^\alpha_{\vec{k}-\vec{q}}\rangle + \sum_{\alpha;\vec{q};\{X\};\{\vec{p}\}_X} C_F^{\alpha X}(\vec{k},\vec{q},\{\vec{p}\}_X;t)|L^\alpha_{\vec{k}-\vec{q}};\{X\}\rangle + \cdots, \tag{A1}$$

where the coefficients in this expression and those of (2.9) are related by

$$C_\Phi(\vec{k},t) = e^{-iE_\Phi t}A_\Phi(\vec{k},t);$$
$$C_I^{\alpha i}(\vec{k},\vec{q};t) = e^{-iE_I^i t}A_I^{\alpha i}(\vec{k},\vec{q};t)$$
$$C_F^{\alpha X}(\vec{k},\vec{q},\{\vec{p}\}_X;t) = e^{-iE_F^X t}A_F^{\alpha X}(\vec{k},\vec{q},\{\vec{p}\}_X;t). \tag{A2}$$

The state (A1) obeys the Schrödinger equation

$$i\frac{d}{dt}|\Psi(\vec{k},t)\rangle_S = -i(H_0+H_I)|\Psi(\vec{k},t)\rangle_S. \tag{A3}$$

The equations for the coefficients are obtained by projection in a similar fashion as in Sec. II, with the same notation as in Sec. II [see Eqs. (2.11)–(2.14)], and neglecting the momenta arguments in the coefficients, we obtain

$$\dot{C}_\Phi(t) = -iE_\phi C_\Phi(t) - i\sum_{\alpha,\vec{q},a} M_{\mathcal{P}}^{\alpha a *}C_I^{\alpha a}(t) - i\sum_{\alpha,\vec{q},h=1,2} M_{\mathcal{P}}^{\alpha h *}C_I^{\alpha h}(t); \quad C_\Phi(0)=1, \tag{A4}$$

$$\dot{C}_I^{\alpha a}(t) = -iE_I^a C_I^{\alpha a}(t) - iM_{\mathcal{P}}^{\alpha a}C_\Phi(t); \quad C_I^{\alpha a}(0)=0, \tag{A5}$$

$$\dot{C}_I^{\alpha h}(t) = -iE_I^h C_I^{\alpha h}(t) - iM_{\mathcal{P}}^{\alpha h}C_\Phi(t) - i\sum_{h'=1,2}\delta\mathcal{E}_{hh'}C_I^{\alpha h'}(t) - i\sum_{\{X\};\{\vec{p}\}_X} M_{\mathcal{D}}^{hX*}C_F^{\alpha X}(t); \quad C_I^{\alpha h}(0)=0, \quad h=1,2, \tag{A6}$$

$$\dot{C}_F^{\alpha X}(t) = -iE_F^X C_F^{\alpha X}(t) - i\sum_{h=1,2} M_{\mathcal{D}}^{hX}C_I^{\alpha h}(t); \quad C_F^{\alpha X}(0)=0. \tag{A7}$$

This hierarchy of coupled differential equations becomes a set of coupled algebraic equations by Laplace transform. Defining

$$\widetilde{C}(s) = \int_0^\infty e^{-st}C(t)dt \tag{A8}$$

for all the coefficients, we find beginning from the bottom up

$$\widetilde{C}_F^{\alpha X}(s) = -i\frac{[M_{\mathcal{D}}^{1X}\widetilde{C}_I^{\alpha 1}(s) + M_{\mathcal{D}}^{2X}\widetilde{C}_I^{\alpha 2}(s)]}{s + iE_F^X}. \quad \text{(A9)}$$

Introducing this solution into the Laplace transform of Eqs. (A6), we find

$$\begin{bmatrix} s + iE_I^1 + i\widetilde{\Sigma}_{11}(s) & i\widetilde{\Sigma}_{12}(s) \\ i\widetilde{\Sigma}_{21}(s) & s + iE_I^1 + i\widetilde{\Sigma}_{22}(s) \end{bmatrix}\begin{pmatrix} \widetilde{C}_I^{\alpha 1}(s) \\ \widetilde{C}_I^{\alpha 2}(s) \end{pmatrix} = -i\widetilde{C}_\Phi(s)\begin{pmatrix} M_{\mathcal{P}}^{\alpha 1} \\ M_{\mathcal{P}}^{\alpha 2} \end{pmatrix}, \quad \text{(A10)}$$

where

$$i\widetilde{\Sigma}_{ij}(s) = \sum_{\{X\};\{\vec{p}\}_X} \frac{M_{\mathcal{D}}^{iX*}M_{\mathcal{D}}^{jX}}{s + iE_F^X} + i\delta\mathcal{E}_{ij}. \quad \text{(A11)}$$

The first term in $\widetilde{\Sigma}_{ij}(s)$ corresponds to the intermediate states $\{X\}$. Figure 2 shows the self-energy for the case of a common three-body decay channel.

The solution of the set of Eqs. (A10) is given by

$$\begin{pmatrix} \widetilde{C}_I^{\alpha 1}(s) \\ \widetilde{C}_I^{\alpha 2}(s) \end{pmatrix} = -i\widetilde{\mathbb{G}}(s)\begin{pmatrix} M_{\mathcal{P}}^{\alpha 1} \\ M_{\mathcal{P}}^{\alpha 2} \end{pmatrix}\widetilde{C}_\Phi(s), \quad \text{(A12)}$$

where

$$\widetilde{\mathbb{G}}(s) = \frac{1}{D(s)}\begin{bmatrix} s + iE_I^1 + i\widetilde{\Sigma}_{22}(s) & -i\widetilde{\Sigma}_{12}(s) \\ -i\widetilde{\Sigma}_{21}(s) & s + iE_I^1 + i\widetilde{\Sigma}_{11}(s) \end{bmatrix} \quad \text{(A13)}$$

with

$$D(s) = [(s + iE_I^1 + i\widetilde{\Sigma}_{11}(s))(s + iE_I^2 + i\widetilde{\Sigma}_{22}(s)) - \widetilde{\Sigma}_{12}(s)\widetilde{\Sigma}_{21}(s)]. \quad \text{(A14)}$$

For the amplitudes corresponding to the active neutrinos, we find for their Laplace transform

$$\widetilde{C}_I^{\alpha a}(s) = -i\frac{M_{\mathcal{P}}^{\alpha a}}{s + iE_I^a}\widetilde{C}_\Phi(s). \quad \text{(A15)}$$

Introducing (A12) and (A15) into the Laplace transform of (A4), we find

$$\widetilde{C}_\Phi(s) = \frac{1}{s + iE_\Phi + i\Sigma_\Phi(s)}, \quad \text{(A16)}$$

where

$$\Sigma_\Phi(s) = \Sigma_\Phi^{(a)}(s) + \Sigma_\Phi^{(s)}(s) \quad \text{(A17)}$$

with

$$\Sigma_\Phi^{(a)}(s) = -i\sum_{\alpha,\vec{q},a}\frac{|M_{\mathcal{P}}^{\alpha a}|^2}{s + iE_I^a}, \quad \text{(A18)}$$

and

$$\Sigma_\Phi^{(s)}(s) = -i\sum_{\alpha,\vec{q}}(M_{\mathcal{P}}^{\alpha 1*}, M_{\mathcal{P}}^{\alpha 2*})\widetilde{\mathbb{G}}(s)\begin{pmatrix} M_{\mathcal{P}}^{\alpha 1} \\ M_{\mathcal{P}}^{\alpha 2} \end{pmatrix} \quad \text{(A19)}$$

are the contributions to the Φ self-energy from the active (a) and sterile (s) neutrinos. This latter contribution highlights the nature of the resonant heavy neutrino states because $\mathbb{G}(s)$ includes the self-energy corrections in the mixed heavy neutrino propagator.

The time evolution is obtained from the anti-Laplace transform, namely, for all the amplitudes

$$C(t) = \int_{\mathcal{C}} e^{st}\widetilde{C}(s)\frac{ds}{2\pi i}, \quad \text{(A20)}$$

where $\mathcal{C}$ is the Bromwich contour running parallel to the imaginary axis in the complex s plane to the right of all the singularities of $\widetilde{C}(s)$. Decaying states are described by complex poles in $\widetilde{C}(s)$ with a negative real part; therefore, along the Bromwich contour $s = i\omega + \epsilon$ with $-\infty \le \omega \le \infty$, $\epsilon \to 0^+$ and

$$C(t) = \int_{-\infty}^{\infty} e^{i\omega t}\widetilde{C}(s = i\omega + \epsilon)\frac{d\omega}{2\pi}. \quad \text{(A21)}$$

In perturbation theory, $\widetilde{C}_\Phi(s = i\omega + \epsilon)$ features a complex pole near $\omega \sim -E_\Phi$. Writing to leading order

$$\Sigma_\Phi(s = -iE_\phi + \epsilon) = \Delta E_\Phi - i\frac{\Gamma_\Phi}{2}, \quad \text{(A22)}$$

it follows that $\widetilde{C}_\Phi(s = i\omega + \epsilon)$ near this pole is of the Breit-Wigner form[2]

$$\widetilde{C}_\Phi(s = i\omega + \epsilon) \simeq -\frac{i}{\omega + E_\Phi^R - i\frac{\Gamma_\Phi}{2}}; \qquad E_\Phi^R = E_\Phi + \Delta E_\Phi \quad \text{(A23)}$$

and

$$C_\Phi(t) = e^{-iE_\Phi^R t}e^{-\frac{\Gamma_\Phi}{2}t}. \quad \text{(A24)}$$

From the convolution theorem for Laplace transforms, we find

[2]Again we neglect wave function renormalization.

$$\begin{pmatrix} C_I^{\alpha 1}(t) \\ C_I^{\alpha 2}(t) \end{pmatrix} = -i \int_0^t \mathbb{G}(t-t') \begin{pmatrix} M_{\mathcal{P}}^{\alpha 1} \\ M_{\mathcal{P}}^{\alpha 2} \end{pmatrix} C_\Phi(t') dt', \quad \text{(A25)}$$

where

$$\mathbb{G}(t) = \int_{-\infty}^{\infty} \widetilde{G}(\omega) e^{i\omega t} \frac{d\omega}{2\pi}; \quad \widetilde{G}(\omega) \equiv \widetilde{G}(s = i\omega + \epsilon). \quad \text{(A26)}$$

To simplify notation, we define

$$\mathcal{E}_{11}(\omega) \equiv E_1 + E_\alpha + \widetilde{\Sigma}_{11}(\omega); \quad \mathcal{E}_{22}(\omega) \equiv E_1 + E_\alpha + \widetilde{\Sigma}_{22}(\omega), \quad \text{(A27)}$$

$$\mathcal{E}_{12}(\omega) \equiv \widetilde{\Sigma}_{12}(\omega); \quad \mathcal{E}_{21}(\omega) \equiv \widetilde{\Sigma}_{21}(\omega), \quad \text{(A28)}$$

with

$$\widetilde{\Sigma}_{ij}(\omega) \equiv \widetilde{\Sigma}_{ij}(s = i\omega + \epsilon). \quad \text{(A29)}$$

It follows that the analytic continuation

$$\widetilde{\mathbb{G}}(\omega) = -\frac{1}{[\omega - \omega^+(\omega)][\omega - \omega^-(\omega)]} \times \begin{bmatrix} \omega + \mathcal{E}_{22}(\omega) & -\mathcal{E}_{12}(\omega) \\ -\mathcal{E}_{21}(\omega) & \omega + \mathcal{E}_{11}(\omega) \end{bmatrix}, \quad \text{(A30)}$$

where

$$\omega^\pm(\omega) = -\frac{1}{2}\{(\mathcal{E}_{11}(\omega) + \mathcal{E}_{22}(\omega)) \pm [(\mathcal{E}_{11}(\omega) - \mathcal{E}_{22}(\omega))^2 + 4\mathcal{E}_{12}(\omega)\mathcal{E}_{21}(\omega)]^{1/2}\}. \quad \text{(A31)}$$

The propagator $\widetilde{G}(\omega)$ features (simple) complex poles at

$$\omega = \omega^\pm(\omega). \quad \text{(A32)}$$

These self-consistent conditions can be solved perturbatively. Again there are two cases:

(i) $|E_1 - E_2| \gg \widetilde{\Sigma}_{ij}(E_{1,2})$ for which we find that

$$\omega^+ = -\mathcal{E}_{11}(E_I^1) = -[\lambda^+ + E_\alpha]; \quad \omega^- = -\mathcal{E}_{22}(E_I^2) = -[\lambda^- + E_\alpha], \quad \text{(A33)}$$

where $\lambda^\pm$ is given by (3.2).

(ii) $|E_1 - E_2| \lesssim \widetilde{\Sigma}_{ij}(\bar{E})$: In this case we can set $\omega = \bar{E} + E_\alpha$ in the arguments of the self-energies to leading order $\mathcal{O}(g^2)$, and again we find

$$\omega^+ = -[\lambda^+ + E_\alpha]; \quad \omega^- = -[\lambda^- + E_\alpha], \quad \text{(A34)}$$

where in this case $\lambda^\pm$ are given by (2.45) with $E_{1,2} \to \bar{E}$ in the arguments of the self-energies.

In both cases, straightforward contour integration finally yields

$$\mathbb{G}(t) = \frac{e^{-iE_\alpha t}}{\lambda^+ - \lambda^-}\left\{ e^{-i\lambda^+ t} \begin{bmatrix} \lambda^+ - E_{22} & E_{12} \\ E_{21} & \omega + \lambda^+ - E_{11} \end{bmatrix} + e^{-i\lambda^- t} \begin{bmatrix} \lambda^- - E_{22} & E_{12} \\ E_{21} & \lambda^- - E_{11} \end{bmatrix} \right\}, \quad \text{(A35)}$$

where the E_{ij} are the same as in (2.36) with $E_{1,2} \to \bar{E}$ in the self-energies. With the result (A24), it is now straightforward to find the coefficients $C_I^{\alpha 1}(t)$, $C_I^{\alpha 2}(t)$ from Eq. (A25). Using the results of Appendix B we confirm the Wigner-Weisskopf result (2.53) with (2.60) to leading order, thereby establishing that the Wigner-Weisskopf approximation is indeed equivalent to the Dyson resummation of the propagators in terms of the self-energy. This is a nonperturbative result that generalizes the simpler case analyzed in Ref. [59] and establishes the relation to the field theoretical propagator approach studied in Ref. [15].

APPENDIX B: USEFUL IDENTITIES

The eigenvalue equation (2.46)

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} \alpha_1^\pm \\ \alpha_2^\pm \end{pmatrix} = \lambda^\pm \begin{pmatrix} \alpha_1^\pm \\ \alpha_2^\pm \end{pmatrix}, \quad \text{(B1)}$$

we obtain

$$\lambda^\pm = \frac{1}{2}\{(H_{11} + H_{22}) \pm [(H_{11} - H_{22})^2 + 4H_{12}H_{21}]^{1/2}\}, \quad \text{(B2)}$$

from which it follows that

$$\lambda^- - H_{11} = -(\lambda^+ - H_{22}) \quad \text{(B3)}$$

and

$$\frac{\lambda^+ - H_{11}}{\lambda^- - H_{11}} = \frac{\lambda^- - H_{22}}{\lambda^+ - H_{22}}, \quad \text{(B4)}$$

along with the ratios

$$\frac{\alpha_1^-}{\alpha_2^-} = \frac{\lambda^- - H_{22}}{H_{21}}; \quad \frac{\alpha_2^+}{\alpha_1^+} = \frac{\lambda^+ - H_{11}}{H_{12}}. \quad \text{(B5)}$$

With these results, after straightforward algebra we find the following identities:

$$\alpha_1^+ \widetilde{M}_{\mathcal{P}}^{\alpha +} = \frac{(\lambda^+ - H_{22})M_{\mathcal{P}}^{\alpha 1} + H_{12}M_{\mathcal{P}}^{\alpha 2}}{\lambda^+ - \lambda^-}, \quad \text{(B6)}$$

$$\alpha_1^- \widetilde{M}_{\mathcal{P}}^{\alpha -} = -\frac{(\lambda^- - H_{22})M_{\mathcal{P}}^{\alpha 1} + H_{12}M_{\mathcal{P}}^{\alpha 2}}{\lambda^+ - \lambda^-}, \quad \text{(B7)}$$

$$\alpha_2^+ \widetilde{M}_{\mathcal{P}}^{\alpha +} = \frac{(\lambda^+ - H_{11}) M_{\mathcal{P}}^{\alpha 2} + H_{21} M_{\mathcal{P}}^{\alpha 1}}{\lambda^+ - \lambda^-}, \tag{B8}$$

$$\alpha_2^- \widetilde{M}_{\mathcal{P}}^{\alpha -} = -\frac{(\lambda^- - H_{11}) M_{\mathcal{P}}^{\alpha 2} + H_{21} M_{\mathcal{P}}^{\alpha 1}}{\lambda^+ - \lambda^-}. \tag{B9}$$

The important aspect is that these products are independent of the normalization of $\alpha_{1,2}^{\pm}$.

APPENDIX C: DERIVATION OF EQ. (3.10)

$A_X^F(t)$ given by (2.61) can be written in obvious notation as $A^+(t) + A^-(t)$ corresponding to the first and second lines in (2.61). The interference terms are

$$(A^+(t))^* A^-(t) + \text{c.c.} \tag{C1}$$

To simplify notation, we introduce the auxiliary quantities

$$\tau^{\pm} = (\alpha_1^{\pm} \widetilde{M}_{\mathcal{P}}^{\alpha\pm} M_{\mathcal{D}}^{1X} + \alpha_2^{\pm} \widetilde{M}_{\mathcal{P}}^{\alpha\pm} M_{\mathcal{D}}^{2X}) \tag{C2}$$

and

$$\mathcal{E} = \bar{\mathcal{E}} + E_\alpha - E_\Phi; \quad \eta = E_F^X - E_\Phi; \quad \Delta_\gamma^{\pm} = \Gamma_\Phi - \Gamma^{\pm}, \tag{C3}$$

where $\bar{\mathcal{E}}$, δ have been defined in Eq. (3.9). The interference term $(A^+(t))^* A^-(t)$ is given by

$$(A^+(t))^* A^-(t) = \frac{\tau^+ \tau^-}{(\mathcal{E} + \delta - i\frac{\Delta_\gamma^+}{2})(\mathcal{E} - \delta + i\frac{\Delta_\gamma^-}{2})} \times \{(1) + (2) + (3) + (4)\}, \tag{C4}$$

where

$$(1) = \frac{(e^{-i\eta t} e^{-\frac{\Gamma_\Phi}{2}t} - 1)}{(\eta - i\frac{\Gamma_\Phi}{2})} \frac{(e^{i\eta t} e^{-\frac{\Gamma_\Phi}{2}t} - 1)}{(\eta + i\frac{\Gamma_\Phi}{2})}, \tag{C5}$$

$$(2) = \frac{(e^{-i(\eta - \mathcal{E})t} e^{i\delta t} e^{-\frac{\Gamma^+}{2}t} - 1)}{(\eta - \mathcal{E} - \delta - i\frac{\Gamma^+}{2})} \frac{(e^{i(\eta - \mathcal{E})t} e^{i\delta t} e^{-\frac{\Gamma^-}{2}t} - 1)}{(\eta - \mathcal{E} + \delta + i\frac{\Gamma^-}{2})}, \tag{C6}$$

$$(3) = -\frac{(e^{-i\eta t} e^{-\frac{\Gamma_\Phi}{2}t} - 1)}{(\eta - i\frac{\Gamma_\Phi}{2})} \frac{(e^{i(\eta - \mathcal{E})t} e^{i\delta t} e^{-\frac{\Gamma^-}{2}t} - 1)}{(\eta - \mathcal{E} + \delta + i\frac{\Gamma^-}{2})}, \tag{C7}$$

$$(4) = -\frac{(e^{-i(\eta - \mathcal{E})t} e^{i\delta t} e^{-\frac{\Gamma^+}{2}t} - 1)}{(\eta - \mathcal{E} - \delta - i\frac{\Gamma^+}{2})} \frac{(e^{i\eta t} e^{-\frac{\Gamma_\Phi}{2}t} - 1)}{(\eta + i\frac{\Gamma_\Phi}{2})}. \tag{C8}$$

Out of these four contributions, it is only contribution (2) that survives at long time with an oscillatory behavior on long time scales, (1) does not feature oscillatory interference and (3),(4) feature rapidly varying phases $e^{\pm i\mathcal{E}t}$ but not interference terms and decay on time scales $1/\Gamma_\Phi$.

The denominators in (2) feature resonances at $E^{\pm} \simeq E^X$ precisely when the heavy sterile neutrinos (the correct eigenstates) can decay into the common channel. In the narrow width limit, these resonant denominators become energy-conserving delta functions. We can obtain the coefficient functions of these delta functions by integrating in the complex η plane and extracting the residues at the complex poles. In the nearly degenerate limit, $\bar{\mathcal{E}} \gg \delta$, and Eq. (C6) is understood as a distribution that is integrated over a density of states that is insensitive to the energy difference δ and we find

$$(2) = \frac{2\pi}{\bar{\Gamma}} \frac{[1 - e^{2i\delta t} e^{-\bar{\Gamma} t}]}{[1 - i\frac{2\delta}{\bar{\Gamma}}]} \delta(\eta - \mathcal{E}). \tag{C9}$$

Similarly, in the narrow-width limit and in the nearly degenerate case, the product

$$\frac{1}{(\mathcal{E} + \delta - i\frac{\Delta_\gamma^+}{2})(\mathcal{E} - \delta + i\frac{\Delta_\gamma^-}{2})} \propto \delta(\mathcal{E}). \tag{C10}$$

To find the proportionality factor, we integrate in the complex $\mathcal{E}$ plane extracting the residues at the complex poles and find this product (as a distribution integrated over smooth density of states that is insensitive to the energy difference δ) with the result

$$\frac{1}{(\mathcal{E} + \delta - i\frac{\Delta_\gamma^+}{2})(\mathcal{E} - \delta + i\frac{\Delta_\gamma^-}{2})} = \frac{2\pi\delta(\mathcal{E})}{\Gamma_\Phi + \bar{\Gamma} + 2i\delta}. \tag{C11}$$

This result can be easily understood as follows: in the narrow-width limit,

$$\frac{1}{(\mathcal{E} + \delta - i\frac{\Delta_\gamma^+}{2})(\mathcal{E} - \delta + i\frac{\Delta_\gamma^-}{2})} = \frac{1}{-2\delta + i(\Gamma_\Phi + \bar{\Gamma})} \left[i\pi((\delta(\mathcal{E} + \delta) + (\delta(\mathcal{E} - \delta)) + \mathcal{P}\left(\frac{1}{(\mathcal{E} + \delta)} - \frac{1}{(\mathcal{E} - \delta)}\right) \right]. \tag{C12}$$

Upon integrating over a density of final states that is insensitive to the energy difference δ, the result (C11) above follows. The same analysis applies to the result (C9).

Therefore, the final result for the interference term is

$$(A^+(t))^* A^-(t) = \tau^+ \tau^- \frac{2\pi\delta(\bar{\mathcal{E}} + E_\alpha - E_\Phi)}{[\Gamma_\Phi + \bar{\Gamma} + 2i\delta]} \frac{2\pi\delta(\bar{\mathcal{E}} - E^X)}{[\bar{\Gamma} - 2i\delta]} \times [1 - e^{2i\delta t} e^{-\bar{\Gamma} t}]. \tag{C13}$$

[1] P. Minkowski, Phys. Lett. **67B**, 421 (1977).
[2] M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by D. Freedman *et al.* (North-Holland, Amsterdam, 1979).
[3] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).
[4] T. Yanagida, Prog. Theor. Phys. **64**, 1103 (1980).
[5] C. W. Kim and A. Pevsner, *Neutrinos in Physics and Astrophysics* (Harwood Academic, Chur, Switzerland, 1993).
[6] S. M. Bilenky, C. Giunti, and W. Grimus, Prog. Part. Nucl. Phys. **43**, 1 (1999).
[7] R. N. Mohapatra and P. B. Pal, *Massive Neutrinos in Physics and Astrophysics* (World Scientific, Singapore, 2004).
[8] M. Fukugita and T. Yanagida, *Physics of Neutrinos and Applications to Astrophysics* (Springer-Verlag, Berlin, 2003).
[9] W. Grimus, Lect. Notes Phys. **629**, 169 (2004).
[10] R. N. Mohapatra *et al.*, Rep. Prog. Phys. **70**, 1757 (2007).
[11] A. de Gouvea, Mod. Phys. Lett. A **19**, 2799 (2004).
[12] S. M. Bilenky, arXiv:1105.2306.
[13] C. Giunti and C. W. Kim, *Fundamentals of Neutrino Physics and Astrophysics* (Oxford University Press, Oxford, 2007).
[14] M. Fukugita and T. Yanagida, Phys. Lett. B **174**, 45 (1986).
[15] A. Pilaftsis, Nucl. Phys. **B504**, 61 (1997); Int. J. Mod. Phys. A **14**, 1811 (1999); *Proc. Sci.*, CORFU98 (1998) 075 [arXiv:hep-ph/9810211]; arXiv:hep-ph/9704227.
[16] A. Pilaftsis, Phys. Rev. D **56**, 5431 (1997); O. Kittel and A. Pilaftsis, Nucl. Phys. **B856**, 682 (2012); S. Bray, J. S. Lee, and A. Pilaftsis, Nucl. Phys. **B786**, 95 (2007); A. Pilaftsis and T. E. J. Underwood, Nucl. Phys. **B692**, 303 (2004).
[17] P. S. Bhupal Dev, P. Millington, A. Pilaftsis, and D. Teresi, Nucl. Phys. **B886**, 569 (2014); P. S. Bhupal Dev, A. Pilaftsis, and U.-k. Yang, Phys. Rev. Lett. **112**, 081801 (2014); A. Das, P. S. Bhupal Dev, and N. Okada, Phys. Lett. B **735**, 364 (2014).
[18] E. K. Akhmedov, V. Rubakov, and A. Y. Smirnov, Phys. Rev. Lett. **81**, 1359 (1998).
[19] T. Asaka and M. Shaposhnikov, Phys. Lett. B **620**, 17 (2005); M. Shaposhnikov, J. High Energy Phys. 08 (2008) 008; T. Asaka and H. Ishida, Phys. Lett. B **692**, 105 (2010); T. Asaka, S. Blanchet, and M. Shaposhnikov, Phys. Lett. B **631**, 151 (2005); M. Shaposhnikov, J. High Energy Phys. 08 (2008) 008; T. Asaka and H. Ishida, Phys. Lett. B **692**, 105 (2010); L. Canetti and M. Shaposhnikov, J. Cosmol. Astropart. Phys. 09 (2010) 001; L. Canetti, M. Drewes, T. Frossard, and M. Shaposhnikov, Phys. Rev. D **87**, 093006 (2013); L. Canetti, M. Drewes, and M. Shaposhnikov, Phys. Rev. Lett. **110**, 061801 (2013); L. Canetti, M. Drewes, and B. Garbrecht, arXiv:1404.7114 [Phys. Rev. D (to be published)].
[20] See the recent reviews: L. Canetti, M. Drewes, and M. Shaposhnikov, New J. Phys. **14**, 095012 (2012); M. Drewes, Int. J. Mod. Phys. E **22**, 1330019 (2013); A. Boyarsky, O. Ruchayskiy, and M. Shaposhnikov, Annu. Rev. Nucl. Part. Sci. **59**, 191 (2009).
[21] G. M. Fuller, A. Kusenko, and K. Petraki, Phys. Lett. B **670**, 281 (2009).
[22] S. H. Hansen and Z. Haiman, Astrophys. J. **600**, 26 (2004).
[23] G. M. Fuller, C. T. Kishimoto, and A. Kusenko, arXiv:1110.6479.
[24] A. Kusenko, Phys. Rep. **481**, 1 (2009); arXiv:hep-ph/0703116.
[25] D. Gorbunov and M. Shaposhnikov, J. High Energy Phys. 10 (2007) 015.
[26] A. D. Dolgov, Surveys High Energy Phys. **17**, 91 (2002); Phys. Rep. **370**, 333 (2002).
[27] R. E. Shrock, Phys. Lett. **96B**, 159 (1980); Phys. Rev. D **24**, 1232 (1981).
[28] M. Gronau, C. N. Leung, and J. L. Rosner, Phys. Rev. D **29**, 2539 (1984).
[29] F. P. Calaprice, D. F. Schreiber, M. B. Schneider, M. Green, and R. E. Pollock, Phys. Lett. **106B**, 175 (1981).
[30] R. Minehart, K. Ziock, R. Marshall, W. Stephens, M. Daum, B. Jost, and P. Kettle, Phys. Rev. Lett. **52**, 804 (1984).
[31] M. Daum, P.-R. Kettle, B. Jost, R. Marshall, R. Minehart, W. Stephens, and K. Ziock, Phys. Rev. D **36**, 2624 (1987); M. Daum, R. Frosch, W. Hajdas, M. Janousch, P.-R. Kettle, St. Ritt, and Z. G. Zhao, Phys. Lett. B **361**, 179 (1995); M. Daum, M. Janousch, P.-R. Kettle, J. Koglin, D. Počanić, J. Schottmüller, C. Wigger, and Z. Zhao, Phys. Rev. Lett. **85**, 1815 (2000).
[32] R. Bilger *et al.*, Phys. Lett. B **363**, 41 (1995).
[33] P. Astier *et al.*, Phys. Lett. B **527**, 23 (2002); **506**, 27 (2001).
[34] R. Abela, M. Daum, G. H. Eaton, R. Frosch, B. Jost, P.-R. Kettle, and E. Steiner, Phys. Lett. **105B**, 263 (1981).
[35] Y. Asano *et al.*, Phys. Lett. **104B**, 84 (1981).
[36] R. S. Hayano *et al.*, Phys. Rev. Lett. **49**, 1305 (1982).
[37] F. Bergsma *et al.*, Phys. Lett. **128B**, 361 (1983); J. Dorenbosch *et al.*, Phys. Lett. **166B**, 473 (1986).
[38] S. Baranov *et al.* Phys. Lett. B **302**, 336 (1993).
[39] A. Vaitaitis *et al.*, Phys. Rev. Lett. **83**, 4943 (1999).
[40] G. Bernardi *et al.*, Phys. Lett. **166B**, 479 (1986); Phys. Lett. B **203**, 332 (1988).
[41] D. I. Britton *et al.*, Phys. Rev. D **46**, R885 (1992).
[42] C. Hagner, M. Altmann, F. Feilitzsch, L. Oberauer, Y. Declais, and E. Kajfasz, Phys. Rev. D **52**, 1343 (1995).
[43] H. O. Back *et al.*, JETP Lett. **78**, 261 (2003).
[44] M. Gronau, Phys. Rev. D **28**, 2762 (1983).
[45] A. Kusenko, S. Pascoli, and D. Semikoz, J. High Energy Phys. 11 (2005) 028.
[46] A. Atre, T. Han, S. Pascoli, and B. Zhang, J. High Energy Phys. 05 (2009) 030.
[47] C. Dib and C. S. Kim, Phys. Rev. D **89**, 077301 (2014); G. Cvetic, C. Dib, and C. S. Kim, J. High Energy Phys. 06 (2012) 149; C. Dib, J. C. Helo, M. Hirsch, S. Kovalenko, and I. Schmidt, Phys. Rev. D **85**, 011301 (2012).
[48] G. Lopez Castro and N. Quintero, Phys. Rev. D **87**, 077901 (2013); arXiv:1212.0037; Phys. Rev. D **85**, 076006 (2012); **86**, 079904(E) (2012); D. Delepine, G. Lopez Castro, and N. Quintero, Phys. Rev. D **84**, 096011 (2011); **86**, 079905 (E) (2012).
[49] D. Gorbunov and M. Shaposhnikov, J. High Energy Phys. 10 (2007) 015.
[50] T. Asaka, S. Eijima, and A. Watanabe, J. High Energy Phys. 03 (2013) 125.
[51] W. Bonivento *et al.*, arXiv:1310.1762.

[52] S. N. Gninenko, D. S. Gorbunov, and M. E. Shaposhnikov, Adv. High Energy Phys. **2012**, 718259 (2012).
[53] L. Lello and D. Boyanovsky, Phys. Rev. D **87**, 073017 (2013).
[54] M. Masip and P. Masjuan, Phys. Rev. D **83**, 091301 (2011); M. Masip, P. Masjuan, and D. Meloni, J. High Energy Phys. 01 (2013) 106.
[55] J. Liu and G. Segre, Phys. Rev. D **49**, 1342 (1994).
[56] P. K. Kabir, *The CP Puzzle* (Academic Press, London, 1968).
[57] I. I. Bigi and A. I. Sanda, *CP Violation* (Cambridge University Press, Cambridge, England, 2009); Q. Wang and A. I. Sanda, Phys. Rev. D **55**, 3131 (1997).
[58] L. Lavoura, Phys. Rev. D **62**, 056002 (2000); L. Lavoura and J. P. Silva, Phys. Rev. D **60**, 056003 (1999); G. C. Branco, L. Lavoura, and J. P. Silva, *CP Violation* (Oxford University Press, New York, 1999); A. Amorim, M. G. Santos, and J. P. Silva, Phys. Rev. D **59**, 056001 (1999).
[59] D. Boyanovsky and L. Lello, New J. Phys. **16**, 063050 (2014).
[60] D. Boyanovsky, Nucl. Phys. **B888**, 248 (2014).
[61] D. Boyanovsky and R. Holman, J. High Energy Phys. 05 (2011) 47.
[62] P. Meystr and M. Sargent III, *Elements of Quantum Optics* (Springer, New York, 1999).
[63] A. Merle, Phys. Rev. C **80**, 054616 (2009).
[64] D. Boyanovsky, Phys. Rev. D **84**, 065001 (2011).
[65] E. Kh. Akhmedov and J. Kopp, J. High Energy Phys. 04 (2010) 008.
[66] L. Lello and D. Boyanovsky (to be published).