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Relativistic Particle and Relativistic Fluids: Magnetic Moment and Spin-Orbit Interactions

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We consider relativistic charged-particle dynamics and relativistic magnetohydrodynamics using symplectic structures and actions given in terms of coadjoint orbits of the Poincaré group. The particle case is meant to clarify some points such as how minimal coupling (as defined in the text) leads to a gyromagnetic ratio $g = 2$ and to set the stage for fluid dynamics. The general group-theoretic framework is further explained and is then used to set up Abelian magnetohydrodynamics including spin effects. An interesting new physical effect is precession of spin density induced by gradients in pressure and energy density. The Euler equation (the dynamics of the velocity field) is also modified by gradients of the spin density.

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I. INTRODUCTION

The study of the relativistic point particle is a story as old as the theory of relativity itself. This long history might suggest that there would be very little new one can say on this matter. Nevertheless, over the years, new approaches and clarifications have been obtained [1]. In the quantum theory, a point particle is defined as a unitary irreducible representation (UIR) of the Poincaré group. Thus, even classically, a description which highlights this connection is interesting. The development of geometric quantization furnished the basic framework for carrying this out. The use of symplectic forms defined on coadjoint orbits of the group led to new Lagrangian descriptions incorporating spin and, in the case of charged particles, magnetic moments and spin-orbit couplings [2]. A Lagrangian derivation of the Bargmann–Michel–Telegdi equation for spin precession [3] was another result of such descriptions [1,2,4].

Our return to this old problem is motivated by the potential to generalize it to fluid mechanics. A charged fluid would obviously be described by magnetohydrodynamics. An action-based canonical approach to magnetohydrodynamics does exist, but we can go further and ask the following question: How do we incorporate the effects of magnetic moments and spin-orbit couplings in magnetohydrodynamics?

The description of fluids in terms of group theory has been developed over the last few years [5]; see also Ref. [6]. Fluids for which the constituents carry spin or internal (Abelian or non-Abelian) symmetries can be described using the Lorentz or internal symmetry groups. One particular advantage of this is the straightforward symmetry-based inclusion of anomalies, which has led to formulas for the chiral magnetic and chiral vorticity effects [7,8]. (The chiral magnetic effect, although not from a group-theory point of view, was first discussed in Ref. [9]. The effects of anomalies in fluids have been analyzed in Refs. [10–12].) However, even though spin is naturally included in this framework via the Lorentz group, the extension of this to the full Poincaré group had some subtleties and nuances related to the fact that individual particle positions have no meaning from a fluid point of view [13]. We sort out these issues in this paper, as they have not been fully clarified in previous work. Finally, in working out the fluid connection, we realized that some aspects of the role of the symplectic structure for the orbit of the Poincaré group for charged particles were also not entirely clear in the literature. This is another issue that is addressed in this paper.

To summarize, we will start, in Sec. II, by considering the symplectic structure for charged point particles defined purely group theoretically in terms of the Poincaré group. The equations of motion will be shown to lead naturally to magnetic moment and spin-orbit interactions (Sec. III). At the minimal level of gauging or introducing the electromagnetic field, the Lorentz force will fix the gyromagnetic ratio to be 2, just as it is for spinning particles in single-particle quantum mechanics. The Hamiltonian for proper time evolution, which is worked out in Sec. IV, explicitly shows this result. This is the $(3+1)$-dimensional analog (with its own complications) of the similar situation in $(2+1)$ dimensions [14–16]. Nonminimal coupling can be introduced to account for the anomalous magnetic moment, just as in $(2+1)$ dimensions [16]. (We should caution that the word "minimal" is used with somewhat different meanings here, in Ref. [2] and in Ref. [16]. Also, one might entertain variants of the Lorentz force. But the following conditional statement is valid: The Lorentz force in terms of the coordinate as it appears in the symplectic

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form, the latter being given purely by the group structure, leads to $g = 2$.) In the $3 + 1$ case, an arbitrary magnetic moment has been included in the proposed Lagrangians in Ref. [2]; however, the special role of $g = 2$ is not manifest, or, at least, not highlighted in this approach. We also show how a modification of the symplectic structure can accommodate $g \neq 2$. The proper time Hamiltonian should be zero as a constraint, for example, on states upon quantization. This is the single-particle wave equation. Writing this in terms of mutually commuting coordinates, which we do in Sec. V, again shows explicitly the magnetic moment and spin-orbit interactions. This is also very much a replay of the similar situation in $2 + 1$ dimensions.

Starting from the symplectic structure for the point particle and following Lagrange's method of obtaining fluid dynamics from particle dynamics, we obtain the required fluid action in Sec. VI. The variation of the particle coordinates, now viewed as a field, can still lead to the equations of motion, but incorporating nonzero pressure is awkward in this language. This is again related to the lack of a suitable fluid interpretation of particle positions alluded to before. For this reason, we switch to the Clebsch parametrization at this stage. The end result is an action for a relativistic fluid with spin, magnetic moment, spin-orbit interactions, etc. In Sec. VII, we examine standard Abelian magnetohydrodynamics in some detail, but now including spin. The Euler equation for the charged fluid now shows additional force terms involving the gradients of the spin density. This is not surprising by itself since it is related to the magnetic moment interaction. What is more interesting is the precession equation for the spin density. This has, in addition to the usual term proportional to the external field, terms involving gradients of the pressure and energy density.

A concluding short discussion summarizes the new results in this paper.

Before concluding this section, we also mention that there has been some recent work on the use of the symplectic form derived from the Poincaré group to discuss Dirac and Weyl particles [17]. While our focus has been on fluids, and hence not directly along these lines, clearly there is some resonance with our work.

II. SYMPLECTIC FORM

We begin by recalling the essence of the coadjoint orbit method. If g denotes a general element of a Lie group G, in a particular representation, the action is

$$S = i\sum_\alpha w_\alpha \int d\tau \mathrm{Tr}(h_\alpha g^{-1}\dot{g}), \qquad \dot{g} = \frac{dg}{d\tau}, \tag{1}$$

where h_α give a basis of the diagonal generators of the Lie algebra (the Cartan subalgebra) and w_α are a set of numbers. We are envisaging a matrix representation, say, the fundamental representation, with $\mathrm{Tr}(h_\alpha h_\beta) = \delta_{\alpha\beta}$. The basic theorem is that the quantization of this action leads to a Hilbert space which carries a unitary irreducible representation of G, this UIR being specified by the highest weight $(w_1, w_2, \ldots, w_r)$, r being the rank of the group, which is also the range of summation for α. The canonical 1-form associated to (1) is evidently

$$\mathcal{A} = i\sum_\alpha w_\alpha \mathrm{Tr}(h_\alpha g^{-1}dg). \tag{2}$$

Under transformations $g \to g\exp(-ih_\alpha\varphi_\alpha)$, we find $\mathcal{A} \to \mathcal{A} + df$, $f = \sum w_\alpha\varphi_\alpha$. Thus, the symplectic 2-form $\Omega = d\mathcal{A}$ is defined on G/T, T being the maximal torus. Further, the transformation $\mathcal{A} \to \mathcal{A} + df$ shows that in the quantum theory, where wave functions transform as e^{iS}, there will be restrictions or quantization conditions on w_α, these being the appropriate conditions for $(w_1, w_2, \ldots, w_r)$ to qualify as the highest weight of a UIR.

In extending this directly to the Poincaré group, first of all, there is a small technical difficulty due to the lack of a matrix representation with a well-defined trace. One can use infinite dimensional representations, but the notion of the trace has to be carefully defined. This can be done with a suitable regularization or a cutoff on integrals, but a simpler solution is to use a finite-dimensional representation of the de Sitter group, obtaining the Poincaré group as a contraction. Thus, we will use

$$\begin{aligned} P_\mu &= \gamma_\mu/r_0 \\ J_{\mu\nu} &= \gamma_{\mu\nu} = (i/4)[\gamma_\mu, \gamma_\nu] \end{aligned} \tag{3}$$

as the representation of the de Sitter algebra, γ_μ being the standard 4×4 Dirac matrices. The parameter r_0, which is the radius of curvature of the de Sitter space, can be taken to be very large to recover the Poincaré limit. A general group element is of the form

$$g = \exp(iP_\mu x^\mu)\Lambda, \tag{4}$$

where Λ is the Lorentz matrix constructed using the generators $J_{\mu\nu}$. The group has rank equal to 2, and, as the diagonal generators, we will choose P_0 and J_{12}. The corresponding weights will be mass and spin. Our philosophy here will not be to write down actions *a priori*, that will come later, but to start with Ω and work out a Hamiltonian for τ evolution. The trajectories should be invariant under reparametrizations of τ; thus, the Hamiltonian for τ evolution is the generator of a gauge symmetry. We must thus set $H \approx 0$ on states in the quantum theory. This will become the wave equation for single-particle states. This was the approach followed in Ref. [14] to obtain wave equations for anyons. In this approach, there is no fixed mass that arises from the constraint $H \approx 0$. Rather, we should use $\sqrt{p^2}$ in place of the mass. Finally, Λ can be parametrized in terms of the boost operator connecting the rest frame to an arbitrary frame of momentum p_μ. It is thus given by

$$\Lambda = B(p)R$$
$$B(p) = \frac{1}{\sqrt{2m(p_0+m)}}\begin{bmatrix} p_0+m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & p_0+m \end{bmatrix}, \quad (5)$$

where m is a shorthand notation for $\sqrt{p^2}$. R is a pure spatial rotation matrix generated by J_{12}, J_{23}, J_{31}.

A comment about the Lorentz transformation properties may be useful. Λ being an element of the Lorentz group, we may expect that additional Lorentz transformations $\tilde{\lambda}$ may be represented as left translations on Λ, as $\Lambda \to \tilde{\lambda}\Lambda$. But Λ is a dynamical variable made of p^μ and elements of R [as in (5)], and we may simply regard it as a matrix for which the transformation properties are to be determined from the transformation of the variables it depends on, namely, p^μ and R. Consider then a Lorentz transformation of p^μ as $p^\mu \to p'^\mu = \lambda^\mu{}_\nu p^\nu$. One can show that $B(p)$ transforms as $B(\lambda p) = \tilde{\lambda}B(p)R_w$, where $\tilde{\lambda}$ is the (Dirac) spinorial representative of $\lambda^\mu{}_\nu$ and R_w is the Wigner rotation corresponding to the parameters in λ. We then find that the combined transformation $p \to \lambda p$ and $R \to R_w^{-1}R$ is equivalent to $\Lambda \to \tilde{\lambda}\Lambda$ as expected, namely,

$$B(p)R \to B(\lambda p)R_W^{-1}R = \tilde{\lambda}(BR). \quad (6)$$

This relationship between the transformations of p^μ, R and Λ is what is obtained, as is well known, in Wigner's construction of the representation of the Poincaré group.

Returning to the canonical 1-form, it is given, in our case, by

$$\mathcal{A} = ir_0^2\sqrt{p^2}\,\mathrm{Tr}\left(\frac{\gamma_0}{r_0}g^{-1}dg\right) + is\,\mathrm{Tr}(J_{12}g^{-1}dg), \quad (7)$$

where s denotes the spin of the particle. Using $B\gamma_0B^{-1} = \gamma^\alpha p_\alpha/\sqrt{p^2}$ and taking $r_0 \to \infty$, we find, for the Poincaré group,

$$\mathcal{A} = -p_\mu dx^\mu + i\frac{s}{2}\mathrm{Tr}(\Sigma_3\Lambda^{-1}d\Lambda),$$
$$\Sigma_a = \begin{bmatrix} \sigma_a & 0 \\ 0 & \sigma_a \end{bmatrix}. \quad (8)$$

Upon using (5), this simplifies as

$$\mathcal{A} = -p_\mu dx^\mu + i\frac{s}{2}\mathrm{Tr}(\Sigma_3R^{-1}DR)$$
$$DR = dR + CR$$
$$C = -i\left(\frac{\Sigma_a}{2}\right)\frac{\epsilon_{abc}p_bdp_c}{m(p_0+m)},$$
$$R = \exp\left[i\left(\frac{\Sigma_a}{2}\right)\theta^a\right]. \quad (9)$$

We will see that R describes the spin degrees of freedom, with $2s$ quantized as an integer, so that the wave functions are appropriately single valued or double valued for fermions. The expression for $\mathcal{A}$ has the expected Lorentz invariance properties, even though this is not manifest when $\mathcal{A}$ is written in terms of p_μ and R as in (9).

Defining the unit vector N_a by

$$N_a\Sigma_a = R\Sigma_3R^{-1}, \quad (10)$$

the symplectic potential can be simplified as

$$\mathcal{A} = -p_\mu dx^\mu + s\frac{\epsilon_{abc}N_ap_bdp_c}{m(p_0+m)} + i\frac{s}{2}\mathrm{Tr}(\Sigma_3R^{-1}dR). \quad (11)$$

It is easy enough to see that the transformations $R \to [1 - i\vec{\Sigma}\cdot\vec{\epsilon}/2]R$ are generated by sN_a in the rest frame identifying $S_a = sN_a$ as the spin vector in the rest frame. We can also work out $\Omega^{(0)} = d\mathcal{A}$ as

$$\Omega^{(0)} = dx^\mu dp_\mu + \frac{S^{\mu\nu}dp_\mu dp_\nu}{2m^2} + s\frac{\epsilon_{abc}dN_ap_bdp_c}{m(p_0+m)}$$
$$- \frac{s}{2}(\vec{N}\cdot\vec{p})\frac{\epsilon_{abc}p_adp_bdp_c}{m^2(p_0+m)^2} - \frac{s}{2}\epsilon_{abc}N_adN_bdN_c, \quad (12)$$

where $S^{\mu\nu}$ is the canonical generator of the Lorentz transformation $\Lambda \to (1 - i\omega^{\mu\nu}J_{\mu\nu})\Lambda$ given by

$$S_{\mu\nu} = \frac{s}{2}\mathrm{Tr}(\Sigma_3\Lambda^{-1}J_{\mu\nu}\Lambda). \quad (13)$$

This can be explicitly written in terms of the spin vectors as

$$S_{0i} = -\epsilon_{ijk}\frac{p_jS_k}{m}$$
$$S_{ij} = \epsilon_{ijk}\left[\frac{p_0S_k}{m} - \frac{\vec{S}\cdot\vec{p}p_k}{m(p_0+m)}\right]. \quad (14)$$

Evidently

$$S^{\mu\nu}p_\nu = 0. \quad (15)$$

Since $N^2 = 1$, $\epsilon_{abc}N_adN_bdN_c$ is proportional to the volume (area) of the 2-sphere defined by N_a. The requirement that the integral of Ω on any closed 2-surface should be 2π times an integer shows that $2s$ must be an integer upon quantization.

We now turn to the introduction of the electromagnetic field. The usual minimal prescription amounts to adding $\int d\tau e A_\mu \dot{x}^\mu$ to the action. This is equivalent to adding $eA_\mu dx^\mu$ to $\mathcal{A}$ so that

$$\Omega = \Omega^{(0)} + \frac{e}{2} F_{\mu\nu} dx^\mu dx^\nu. \tag{16}$$

We will refer to this as the "minimal prescription." (As mentioned in the Introduction, there are some variations in the meaning attributed to the word minimal in the literature.)

III. EQUATIONS OF MOTION

Our next step is to consider the equations of motion, which will determine the Hamiltonian as the generator for τ evolution. We consider uniform fields $F_{\mu\nu}$ viewed as a good approximation to slowly varying fields. The minimal equations of motion will be taken to be

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu}{M} \tag{17}$$

$$\frac{dp^\mu}{d\tau} = -eF^{\mu\nu} \frac{dx_\nu}{d\tau} + \mathcal{O}(\partial F), \tag{18}$$

which produce the Lorentz force equation

$$M \frac{d^2 x^\mu}{d\tau^2} = -eF^{\mu\nu} \frac{dx_\nu}{d\tau} + \mathcal{O}(\partial F). \tag{19}$$

Thus, our definition of minimal coupling and Lorentz force amounts to the Ω given in (12) and (16) and the equations of motion given in (17) and (18).

It is useful to consider some arguments motivating our labeling of (12), (16), (17) and (18) as minimal coupling. The symplectic structure given in (12) and (16) corresponds to an action

$$S = \int \frac{dx^\mu}{d\tau} p_\mu + eA_\mu \frac{dx^\mu}{d\tau} + \cdots \tag{20}$$

Since x_μ only appears in the terms explicitly shown here, we obtain (18) as an exact equation of motion for any translationally invariant Hamiltonian.

In our approach, the Lorentz boost transformation $B(p)$ takes us from the rest frame to one which is moving with 4-velocity $p^\mu/\sqrt{p^2}$, as is clear from the explicit formula for $B(p)$. On the other hand, the 4-velocity in terms of x^μ is

$$u^\mu = \frac{dx^\mu}{d\tau} \frac{1}{\sqrt{\frac{dx^\alpha}{d\tau}\frac{dx_\alpha}{d\tau}}}. \tag{21}$$

(We write in this way to make it independent of the parameter τ.) Thus, *a priori*, there are two velocities which enter the description. Since a moving particle (say, defined by nonzero u_i) can be brought to rest by a suitable boost defined by $B(p)$, we should expect these two velocities to be the same, namely, u^μ [as defined in (21)] to be $p^\mu/\sqrt{p^2}$. This implies that in general

$$\frac{dx^\mu}{d\tau} = c \frac{p^\mu}{M}. \tag{22}$$

If c is a constant of motion, we can write $\tau \to \frac{\tau}{c}$ so that the equations of motion will produce the Lorentz force equation (19). This means that, since we have reparametrization invariance for τ (once we set its evolution operator to zero), we may, without loss of generality, set $c = 1$. (The freedom of such a function c was noted in Ref. [16], where the authors also noted that it amounted to just reparametrization of τ.)

In short, if the symplectic structure is entirely defined by the Poincaré group, boost transformations [with the velocity parameter occurring in $B(p)$] should implement the transformation from the rest frame to the comoving frame of the particle. Incorporating this feature and requiring that c is a constant of motion (this will essentially restrict the choice of Hamiltonian) will produce the equations of motion (17), (18) which in turn lead to the Lorentz force equation. This motivates our qualification of (12), (16), (17) and (18) as minimal.

So what is nonminimal? Our expression for $\Omega^{(0)}$ is given by Poincaré symmetry. The gauging is done by adding the term $eA_\mu dx^\mu$ in $\mathcal{A}$, to the canonical 1-form; this is basically singled out as the leading coupling for slowly varying fields by gauge invariance. Going beyond what we termed minimal would include additional terms in $\Omega^{(0)}$, possible changes to the equations of motion themselves, etc. For example, one could envision corrections to the equations of motion involving powers of F, even when we ignore gradients of the fields. (We may, however, note that corrections to the equations of motion with higher powers of the field strength must involve powers of $F_{\mu\nu}/p^2$ or $F_{\mu\nu}/M^2$ for dimensional reasons. Thus, they are significant only at high field strengths, of magnitudes needed for pair production. Single-particle dynamics may not be adequate for such field strengths anyway.)

IV. HAMILTONIAN

We are now ready to obtain the Hamiltonian for τ evolution. This can be done in terms of the τ evolution vector field given by (17) and (18), namely,

$$V \rfloor \Omega = -dH$$
$$V = \frac{p^\mu}{M} \frac{\partial}{\partial x^\mu} - eF^{\mu\nu} \frac{p^\nu}{M} \frac{\partial}{\partial p^\mu} + \cdots, \tag{23}$$

where the ellipsis refers to the spin part. Another way to calculate the Hamiltonian is by computing the Poisson brackets for the dynamical variables and identifying the Hamiltonian that produces the equations of motion (17), (18). These two ways of identifying the Hamiltonian are equivalent, of course, but since the Poisson brackets are interesting in their own right and we can derive them to all orders in F (up to ∂F corrections), we will follow the latter.

Either way, we need Hamiltonian vector fields V_f, corresponding to a function f, defined by

$$V_f \rfloor \Omega = -df, \tag{24}$$

where $V \rfloor \Omega$ denotes the interior contraction of V with Ω given by

$$\begin{aligned} V\rfloor\Omega &= V^\mu \Omega_{\mu\nu} d\xi^\nu \\ \Omega &= \frac{1}{2}\Omega_{\mu\nu} d\xi^\mu d\xi^\nu, \\ V &= V^\mu \frac{\partial}{\partial \xi^\mu}, \end{aligned} \tag{25}$$

ξ^μ are phase space coordinates given by x^μ, p_μ and the two coordinates on the two-sphere are defined by N_a. First define a vector v^μ, $v^0 = 0$ and v^i by its action on N_a as

$$v^i \rfloor dN_a = \frac{N_i p_a - \delta_{ia} \vec{N}\cdot\vec{p}}{m(p_0 + m)}, \qquad m = \sqrt{p^2}. \tag{26}$$

Although tedious, it is then straightforward to verify that the vector fields corresponding to p_α and x^μ are

$$\begin{aligned} (V_p)_\alpha &= -(M^{-1})_\alpha{}^\mu \left[\frac{\partial}{\partial x^\mu} + eF_{\mu\nu} Q^\nu\right] \\ (V_x)^\mu &= \left[\delta^\mu{}_\nu - \frac{S^{\mu\alpha}(M^{-1})_\alpha{}^\lambda (eF_{\lambda\nu})}{m^2}\right] Q^\nu \\ &\quad - \frac{S^{\mu\alpha}(M^{-1})_\alpha{}^\lambda}{m^2}\frac{\partial}{\partial x^\lambda}, \end{aligned} \tag{27}$$

where

$$M_\mu{}^\alpha = \delta^\alpha_\mu + \frac{eF_{\mu\nu}S^{\nu\alpha}}{m^2}, \qquad Q^\mu = \frac{\partial}{\partial p_\mu} + v^\mu. \tag{28}$$

One can also verify that

$$Q^\mu S^{\alpha\beta} = \frac{S^{\mu\alpha}p^\beta - S^{\mu\beta}p^\alpha}{m^2}. \tag{29}$$

The Poisson bracket of two functions f and g is given by

$$\{f, g\} = -(V_f \rfloor V_g \rfloor \Omega) = (V_f \rfloor dg). \tag{30}$$

Using the vector fields (27), the basic bracket relations (PB) we need are thus given by

$$\{x^\mu, x^\nu\} = -\frac{K^{\mu\nu}}{m^2} \tag{31}$$

$$\{x^\mu, p_\nu\} = \delta^\mu{}_\nu - \frac{K^{\mu\alpha}(eF_{\alpha\nu})}{m^2} \tag{32}$$

$$\begin{aligned} \{p_\mu, p_\nu\} &= -\left[eF_{\mu\nu} - \frac{(eF_{\mu\alpha})K^{\alpha\beta}(eF_{\beta\nu})}{m^2}\right] \\ &= -eF_{\mu\lambda}\{x^\lambda, p_\nu\} \end{aligned} \tag{33}$$

$$\{x^\mu, S^{\alpha\beta}\} = \frac{(p^\alpha K^{\beta\mu} - p^\beta K^{\alpha\mu})}{m^2} \tag{34}$$

$$\begin{aligned} \{p_\mu, S^{\alpha\beta}\} &= \frac{p^\alpha(eF_{\mu\nu})K^{\nu\beta} - p^\beta(eF_{\mu\nu})K^{\nu\alpha}}{m^2} \\ &= -eF_{\mu\lambda}\{x^\lambda, S^{\alpha\beta}\} \end{aligned} \tag{35}$$

$$K^{\mu\nu} = S^{\mu\alpha}(M^{-1})^\nu_\alpha = S^{\mu\nu} - S^{\mu\alpha}\frac{eF_{\alpha\beta}}{m^2}S^{\beta\nu} + \cdots \tag{36}$$

Since $S^{\alpha\beta}p_\beta = 0$ and H is to be made out of invariants, we consider a Hamiltonian of the form $H = H(p^2, \sigma)$, where $\sigma = eF_{\alpha\beta}S^{\alpha\beta}$. The PB relations (31)–(36) show that

$$\frac{dx^\mu}{d\tau} = 2p^\mu \frac{\partial H}{\partial p^2} + \frac{K^{\mu\alpha}eF_{\alpha\beta}p^\beta}{m^2} 2\left(\frac{\partial H}{\partial \alpha} - \frac{\partial H}{\partial p^2}\right) \tag{37}$$

$$\frac{dp^\mu}{d\tau} = -eF^{\mu\nu}\frac{dx_\nu}{d\tau}. \tag{38}$$

Equation (38) is exact for any Hamiltonian as mentioned earlier. Equation (37) agrees with (17) if the Hamiltonian is of the form

$$H = \frac{p^2}{2M} + \frac{1}{2M}\left(\frac{g}{2}\right) eF_{\alpha\beta}S^{\alpha\beta} + \text{constant} + \mathcal{O}(\partial F) \tag{39}$$

and further if we choose $g = 2$. Thus, starting with the symplectic structure Ω, which determines the PB relations among the dynamical variables, and requiring that the Lorentz force equation be satisfied fixes $g = 2$ in (39). This is the essence of our statement in the Introduction that, with the minimality conditions we have stated, the Lorentz force equations imply $g = 2$. There is no surprise here; after all, it is well known that the minimal gauging of single-particle wave equations for spinning particles does lead to $g = 2$.

We now turn to the possibility of the anomalous magnetic moment. We want to allow for the possibility of $g \neq 2$, while obtaining equations of motion consistent with the Lorentz force equation (19). A more general set of equations that allows this is

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu + B_\mu}{M} \tag{40}$$

$$\frac{d(p^\mu + B_\mu)}{d\tau} = -eF^{\mu\nu}\frac{dx_\nu}{d\tau} + \mathcal{O}(\partial F), \tag{41}$$

where $B = B(p, S, eF)$ and $p_\mu + B_\mu = f p_\mu$ for some function f, but f is not necessarily a constant of motion. The important point is that this more general set of equations is not compatible with the minimal symplectic structure given by the Poincaré group. In fact, Eq. (41) implies that the symplectic structure has to be modified by

$$\Omega \to \Omega + dx^\mu dB_\mu, \tag{42}$$

where B_μ is to be determined so that (40) is satisfied. We will determine B_μ as a series in powers of F. If we use $\Omega + \delta\Omega$ as the symplectic form, the modified PBs are given by

$$\{f, g\} = \{f, g\}_{(0)} + (V_f^{(0)} \rfloor V_g^{(0)} \rfloor \delta\Omega) + \cdots, \tag{43}$$

where $V^{(0)}$ are the Hamiltonian vector fields given by Ω. With $\delta\Omega = dx^\mu B_\mu$, we find

$$\{x^\mu, x^\nu\} = -\frac{K^{\mu\nu}}{m^2} + \frac{1}{m^2}[S^{\mu\lambda}(Q^\nu B_\lambda) - S^{\nu\lambda}(Q^\mu B_\lambda)] + \cdots \tag{44}$$

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu - \frac{K^{\mu\alpha}(eF_{\alpha\nu})}{m^2} - (Q^\mu B_\nu) + \cdots \tag{45}$$

Using the same Hamiltonian as before, namely, Eq. (39), we find that the new equations of motion become

$$\frac{dx^\mu}{d\tau} = \frac{p^\mu + B^\mu}{M} + \left(\frac{g}{2} - 1\right)\frac{K^{\mu\alpha} eF_{\alpha\beta} p^\beta}{Mm^2} - (Q^\mu B_\nu)\frac{p^\nu}{M} - \frac{B^\mu}{M} + \cdots \tag{46}$$

$$\frac{d(p_\mu + B_\mu)}{d\tau} = -eF^{\mu\nu}\frac{dx^\nu}{d\tau} \tag{47}$$

The choice

$$B_\mu = \frac{(g-2)}{2}\frac{(S^{\alpha\beta} eF_{\alpha\beta})}{2m^2} p^\mu + \mathcal{O}(F^2) \tag{48}$$

eliminates all the unwanted terms on the right-hand side of (46), giving us (40). As mentioned earlier, given the modified Ω in (42), we obtain (41) as an exact equation of motion. We then choose B_μ such that $\frac{dx^\mu}{d\tau} = \frac{p^\mu + B^\mu}{M}$ is also obtained as an exact equation, even though B^μ has to be determined in a series expansion. Thus, the Lorentz force is still an exact result, up to terms involving gradients of the fields.

A comment about canonical transformations would be appropriate at this stage. We note that once the equations of motion are specified the vector field generating this flow gives the Hamiltonian. There is no further freedom in H. In our case, the Lorentz force equations are expressed in terms of x^μ, which have the direct physical interpretation as the position, so we have chosen to keep these variables, as the starting point, in writing down the τ evolution vector field and, from there, the Hamiltonian. Certainly, further canonical transformations can be carried out on this version of τ evolution and the Hamiltonian, if desired. This would give a transformed set of equations and a corresponding transformed Hamiltonian. More explicitly, let X be a vector field generating a canonical transformation. Then the τ evolution vector field $V \to V + [X, V]$, where the second term is the Lie bracket. This also gives $(V + [X, V])\rfloor\Omega = -dH - d\{f_X, H\}$, where f_X is the corresponding function in phase space. Thus, $H \to H + \{f_X, H\}$ as expected. (Up to unitary transformations on Ψ, the wave equation $H\Psi = 0$, which is discussed in the next section, remains the same.)

V. WAVE EQUATION

Since τ is a gauge parameter, the generator of τ evolution, namely, H, must be set to zero. Upon quantization, the wave equation is thus given by $H\Psi = 0$. Taking the constant in (39) to be $-\mu^2/2M$, this becomes

$$\left[p^2 + \frac{eg}{2}F_{\alpha\beta}S^{\alpha\beta} - \mu^2\right]\Psi = 0. \tag{49}$$

This shows that the mass appearing in solutions of the wave equation is μ. This would also be the mass which appears in the classical equations of motion obtained starting from the wave equation and taking a classical limit. However, the mass parameter we used for the classical equations of motion was M. There is no real contradiction here by virtue of reparametrization invariance. Going back to (17), (18) or (19), we see that M can be replaced by μ by redefining $\tau \to \tau M/\mu$. Therefore, without loss of generality we can take $\mu = M$, and the wave equation can be written as

$$\left[p^2 + \frac{eg}{2}F_{\alpha\beta}S^{\alpha\beta} - M^2\right]\Psi = 0. \tag{50}$$

The orbit of the Poincaré group, and the corresponding Ω, should be specified by the values for the set of mutually commuting observables, which are the mass and the spin. But we relaxed the condition to retain unconstrained p_μ in Ω by using $\sqrt{p^2}$ in place of the mass, as in (7). The requirement (50) is thus the reinstatement of the definition of mass.

As is evident from (44), in the quantum theory, x^μ does not commute with x^ν. Thus, to write (50) as a differential equation, we must first transform to a mutually commuting set of coordinates. This is equivalent to the choice of Darboux coordinates in the classical theory. We will show how this can be done for the simpler case of $g = 2$, where $B_\mu = 0$. In this case, it is easily verified that the required transformation is

$$x^\mu = q^\mu - C^\mu(k) - \frac{e}{2} F_{\lambda\alpha}\left(\frac{\partial C^\mu}{\partial k_\alpha} q^\lambda + \frac{\partial C^\alpha}{\partial k_\mu} C^\lambda\right) + \cdots$$

$$p_\mu = k_\mu - \frac{e}{2} F_{\mu\alpha} q^\alpha + eF_{\mu\alpha} C^\alpha(k) - \frac{1}{4} q^\lambda e F_{\lambda\alpha}\left(\frac{\partial C^\beta}{\partial k_\alpha} - \frac{\partial C^\alpha}{\partial k_\beta}\right) eF_{\beta\mu} + \cdots, \qquad (51)$$

where q^μ, k_μ are standard canonical coordinates,

$$[q^\mu, q^\nu] = 0, \qquad [q^\mu, k_\nu] = \delta^\mu_\nu, \qquad [k_\mu, k_\nu] = 0. \qquad (52)$$

Further, C^α in (51) is given by

$$C^\alpha = S_a \frac{\epsilon_{abc} k_b}{\sqrt{k^2}(k_0 + \sqrt{k^2})} \delta^\alpha_c. \qquad (53)$$

This corresponds to the 1-form C in (9) written in terms of the commuting momenta k. One can verify that these definitions (51) reproduce the Poisson brackets (31) to (35) to the lowest nontrivial order. In evaluating Poisson brackets with (51), it should be kept in mind that

$$[S_a, S_b] = -\epsilon_{abc} S_c. \qquad (54)$$

The symplectic structure (12), specifically the term $-(s/2)\epsilon_{abc} N_a dN_b dN_c$, gives these PB relations for S_a.

The Darboux coordinates (51) show that, in going to the quantum theory, we can represent p_μ in terms of derivatives with respect to q^μ as

$$p_\mu = -i \frac{\partial}{\partial q^\mu} + eA_\mu + eF_{\mu\alpha} C^\alpha + \cdots, \qquad (55)$$

where $A_\mu = -\frac{1}{2} F_{\mu\alpha} q^\alpha$ is the vector potential for us, since we are ignoring derivatives of $F_{\mu\alpha}$. The wave equation (50) thus takes the form

$$\left[-(\partial_\mu + ieA_\mu + ieF_{\mu\alpha} C^\alpha + \cdots)^2 + \frac{e}{2} g S^{\alpha\beta} F_{\alpha\beta} - M^2\right]\Psi = 0. \qquad (56)$$

One point worth emphasizing here is that, while the term $(eg/2) S^{\alpha\beta} F_{\alpha\beta}$ gives the correct magnetic moment interaction, the spin-orbit interaction part of this term is twice what is needed. The extra term $-ie(\partial_\mu F_{\mu\alpha} C^\alpha + F_{\mu\alpha} C^\alpha \partial_\mu)\Psi$ compensates for this and leads to the correct spin-orbit interaction in the wave equation. More specifically, if we introduce the nonrelativistic wave function $\Psi_{\rm NR}$ by writing $\Psi = e^{iMq^0}\Psi_{\rm NR}$, (56) can be simplified in the nonrelativistic limit as

$$\begin{aligned} H\Psi_{\rm NR} &\equiv -i \frac{\partial}{\partial q^0} \Psi_{\rm NR} \\ &\approx \left[-\frac{(\nabla + ie\vec{A})^2}{2M} - eA_0 - eF_{0i} C^i - \frac{eg}{4M}(2S^{0i} F_{0i} + S^{ij} F_{ij})\right]\Psi_{\rm NR} \\ &\approx \left[-\frac{(\nabla + ie\vec{A})^2}{2M} - eA_0 + \frac{e}{2M^2} \vec{S}\cdot(\vec{k}\times\vec{E}) - \frac{e}{M}\vec{S}\cdot\vec{B}\right]\Psi_{\rm NR}, \end{aligned} \qquad (57)$$

where we have used the nonrelativistic limit of $S^{\mu\nu}$ in (14) and C^i in (53). Equation (57) shows the correct magnetic moment and the correct spin-orbit interaction for $g = 2$.

VI. FLUIDS

As the first step in generalizing these considerations to fluids, we consider the action for a number of particles, each described by $\mathcal{A}$ in (11). For the point we want to make, it suffices to consider spinless particles. The action is thus given by

$$S = \sum_\alpha^N \int [p^{(\alpha)}_\mu \dot{x}^{\mu(\alpha)} - f(n^\alpha)], \qquad (58)$$

where $n^2 = p^2$. The particles are labeled by the index α. We are interested in a continuum approximation where N is very large and the index α becomes almost a continuous variable. Lagrange's key observation was that the initial positions of particles may be used to label them, so that $\sum_\alpha^N \to \int d^3\alpha$. Further, the transformation between the present positions $x_i(t)$ and the initial positions is a diffeomorphism, so we can replace $d^3\alpha$ by d^3xJ, J being the Jacobian $|\partial\alpha/\partial x|$. The action can then be expressed as a spacetime integral. The Jacobian J can be absorbed into the definition of p_μ. We then find

$$S = \int d^4x \left[p_\mu(x)\left(\frac{\partial x^\mu}{\partial\tau}\right) - f(n)\right]. \qquad (59)$$

The 4-velocity $u^\mu(x) = \dot{x}^\mu$ is the flow velocity of a stream of particles. The Hamiltonian will depend on p_μ.

We know that such a simple generalization *at the level of the action* will not suffice. Lagrange's derivation of hydrodynamics was done at the level of the equations of motion and a derivation based on the action came much later, with the replacement of the coarse-grained particle velocity in terms of the Clebsch variables. Nevertheless, let us go a little further with (59) and work out the equations of motion by variation with respect to x^μ. For this, we write (59) in terms of $d^3\alpha$ again and find

$$\begin{aligned}\delta S &= \int d\tau d^3\alpha p^{(\alpha)}_\mu \frac{\partial}{\partial\tau}(\delta x^\mu)\\ &= \int d\tau d^3\alpha p^{(\alpha)}_\mu u^\lambda \frac{\partial}{\partial x^\lambda}(\delta x^\mu)\\ &= \int d^4x J p^{(\alpha)}_\mu u^\lambda \frac{\partial}{\partial x^\lambda}(\delta x^\mu)\\ &= -\int d^4x \left[\frac{\partial}{\partial x^\lambda}(u^\lambda p_\mu)\right]\delta x^\mu, \end{aligned} \tag{60}$$

where we used the fact that $\partial/\partial\tau$ acting on a function g of x^μ is $u^\lambda(\partial g/\partial x^\lambda)$, and, further, $p_\mu = J p^{(\alpha)}_\mu$. The equation of motion is thus

$$\frac{\partial}{\partial x^\lambda}(u^\lambda p_\mu) = 0. \tag{61}$$

The energy-momentum tensor corresponding to (59) is given by

$$T_{\mu\nu} = u_\mu p_\nu - \eta_{\mu\nu}(nf' - f), \tag{62}$$

where $f' = (\partial f/\partial n)$. The equation of motion (61) only coincides with the conservation of $T_{\mu\nu}$ provided $f(n) = n$, so that the pressure $P = nf' - f = 0$. In this case $p_\mu = nu_\mu$. Thus, (61) only describes the pressureless flow of a large number of particles. There is no surprise here, since, beyond using a continuous set to label the particles, we have not included interparticle interactions which is needed for nonzero pressure. We can modify $f(n)$ to allow for nonzero pressure, but then individual particle positions lose meaning as independent dynamical variables. There are two points to be made here. First the use of individual positions as dynamical variables to be varied to get equations of motion will not work, even nonrelativistically. Second, having four positions x^μ is appropriate for particles because we would eliminate one of them by the constraint of $H \approx 0$ eventually. We do not have such a constraint in terms of the fluid variables. Thus, we need three variables for the fluid velocity for which the variation in the action can lead to the correct equations of motion. For this, it is useful to recall that, even for the nonrelativistic case, an action for fluid dynamics requires the Clebsch parametrization of the velocities, given by

$$v_i = \nabla_i\theta + \alpha\nabla_i\beta. \tag{63}$$

For constructing an action for fluids, it is necessary to fix a value for the helicity C (which is $C = \frac{1}{8\pi}\int \epsilon_{ijk} v_i \partial_j v_k$) for which the value is superselected in the sense that local field transformations do not change it. For vanishing C, the velocity in three dimensions can be brought to the form (63) with three independent fields θ, α, β. (A similar form exists for other values of C as well.) For a general discussion of the Clebsch variables, see Ref. [6].

The action for nonrelativistic fluids is given by

$$\begin{aligned} S &= \int d^4x \left[j^0(\partial_0\theta + \alpha\partial_0\beta) - \frac{1}{2}j^0(\nabla\theta + \alpha\nabla\beta)^2 - V(j^0)\right]\\ &\quad - \int d^4x j^0, \end{aligned} \tag{64}$$

where $j^0 = \rho$ is the density. Usually the last term can be omitted as it does not contribute to the equations of motion, $\int d^3x\rho$ being fixed. Introducing an auxiliary field j_i, we can rewrite this as

$$\begin{aligned} S &= \int d^4x \Big[j^0(\partial_0\theta + \alpha\partial_0\beta) - j_i(\nabla_i\theta + \alpha\nabla_i\beta)\\ &\quad - j^0 + \frac{j_i j_i}{2j^0} - V(j^0)\Big]. \end{aligned} \tag{65}$$

The elimination of j_i evidently leads back to (64). We notice that this action is the approximation, for $(j^0)^2 \gg j_i j_i$, of the action

$$\begin{aligned} S &= \int d^4x[j^\mu(\partial_\mu\theta + \alpha\partial_\mu\beta) - f(n)]\\ f(n) &= n + V(n),\\ n^2 &= \eta_{\mu\nu}j^\mu j^\nu = (j^0)^2 - j_i j_i. \end{aligned} \tag{66}$$

Clearly we can take (66) as the relativistic action which reproduces the nonrelativistic one in the appropriate limit. The point of this argument is that what takes the place of $\dot{x}^\mu$ is the Clebsch parametrization $\partial_\mu\theta + \alpha\partial_\mu\beta$. The distinction between the use of the Clebsch variables and $\dot{x}^\mu$ is not important if we were just to transform the equations of motion, as Lagrange did; it is relevant only when we want to construct an action.

With this understanding of the replacement of $\dot{x}_\mu$ by the Clebsch parametrization, we can now easily generalize the point-particle action to fluids. Going back to (8), we can write the action

$$\begin{aligned} S &= \int d^4x \Big[j^\mu(\partial_\mu\theta + \alpha\partial_\mu\beta) - \frac{i}{4}j^\mu_{(s)}\mathrm{Tr}(\Sigma_3\Lambda^{-1}\partial_\mu\Lambda)\\ &\quad - f(n, n_{(s)})\Big], \end{aligned} \tag{67}$$

where Λ is a function on spacetime, depending on all x^μ in general. It is again given explicitly by $\Lambda = BR$, with

$$B = \frac{1}{\sqrt{2n_{(s)}(j^0_{(s)} + n_{(s)})}}\begin{bmatrix} j^0_{(s)} + n_{(s)} & \vec{\sigma}\cdot\vec{j}_{(s)} \\ \vec{\sigma}\cdot\vec{j}_{(s)} & j^0_{(s)} + n_{(s)} \end{bmatrix}. \tag{68}$$

There are two currents in (67), j^μ for mass and $j^\mu_{(s)}$ for spin. As before, $n^2 = \eta_{\mu\nu} j^\mu j^\nu$, $n^2_{(s)} = \eta_{\mu\nu} j^\mu_{(s)} j^\nu_{(s)}$. For point particles, these two currents were proportional to each other; rather the corresponding momenta were the same. However, for fluids, we can consider independent transport of mass and spin, so the general situation is to have separate currents.

So far we have considered only the mass and spin of the fluid. The inclusion of additional quantum numbers is straightforward. We consider the full symmetry group, which is of the form of the Poincaré group times the internal symmetry group G. The latter could be the full symmetry group of the standard model, for example, including the gauged subgroup $U(1)_Y \times SU(2)_L \times SU(3)_c$, as well as the relevant chiral symmetries, the difference between the baryon and lepton numbers $(B - L)$, etc. The generalization of the action (67) is then

$$S = \int d^4x \left[j^\mu (\partial_\mu \theta + \alpha \partial_\mu \beta) - \frac{i}{2} j^\mu_{(s)} \mathrm{Tr}(\Sigma_3 \Lambda^{-1} \partial_\mu \Lambda) + i \sum_a j^\mu_{(a)} \mathrm{Tr}(h_a g^{-1} D_\mu g) - f(\{n\}) \right] + S(A), \tag{69}$$

where we have, in principle, separate currents $j^\mu, j^\mu_{(s)}, j^\mu_{(a)}$ for all the diagonal generators of the symmetry group, i.e., as many currents as the rank of the group. We use θ, α, β to describe mass transport; Λ for spin transport; and $g \in G$ for the transport of internal symmetries corresponding to a group G. (We use a current of mass dimension 3, so that j^μ is like the mass current divided by a mass parameter.) The covariant derivative D_μ indicates gauging with respect to the gauge fields of the standard model. $S(A)$ indicates the part of the action for the gauge fields, which is the sum of Yang–Mills-type terms. The function $f(\{n\})$ now depends on all the invariants of the form $n = \sqrt{j^\mu j^\nu \eta_{\mu\nu}}$ made from the currents, including, in principle, terms of the form $\sqrt{j^\mu j^\nu_{(s)} \eta_{\mu\nu}}$, $\sqrt{j^\mu j^\nu_{(a)} \eta_{\mu\nu}}$, etc.

The action (69) is a general action for relativistic (Abelian/non-Abelian) magnetohydrodynamics. The distinction between different types of fluids with the same symmetry is in the choice of the function f (which determines the various partial pressures and hence the equation of state) and "constitutive relations" among the currents. For example, if we have only one species of particles, each carrying an electric charge e, then for the corresponding mass current and electric current we expect the relation $j^\mu_{(e)} = e j^\mu$.

VII. MAGNETOHYDRODYNAMICS WITH SPIN

Most of the terms in the action (69) were already given many years ago in Ref. [5]; see also Ref. [6]. We have also considered a similar action with Wess–Zumino terms added [7,8] to account for anomalies and have shown that the chiral magnetic effect and the chiral vorticity effect are incorporated. The main new point here is the clarification of how the terms associated with the Poincaré symmetry enter. To illustrate how such terms can affect the physics of the fluid, we will consider in some detail a special case of (69) where we take the flow velocity for the mass, spin and electric charge to be the same, i.e., $j^\mu_{(s)} = s j^\mu$, $j^\mu_{(e)} = e j^\mu$. If we have a single species of particles with identical charge we should expect the same velocity for mass and charge. Even so, the spin flow can have a different velocity as spin singlet combinations can form; their transport would affect mass/charge flow but not the spin current. However, for a dilute system where such combinations are unlikely on the scale of coarse graining, having the same flow velocity for spin as well is not unreasonable. This is essentially this special case; we will analyze this in some more detail as it is closely related to the single-particle motion discussed in earlier sections. Thus, we consider the action

$$S = S(A) + \int d^4x [j^\mu (\partial_\mu \theta + \alpha \partial_\mu \beta + e A_\mu) - i \frac{s}{2} j^\mu \mathrm{Tr}(\Sigma_3 \Lambda^{-1} \partial_\mu \Lambda) - f(n, \sigma)]. \tag{70}$$

Anticipating that we will need magnetic moment couplings, we take f to be a function of $n = \sqrt{j^\mu j_\mu}$ and $\sigma = S^{\mu\nu} F_{\mu\nu}$. For obtaining and simplifying the equations of motion, it is convenient to use a slightly different action

$$S = \int d^4x [j^\mu (\partial_\mu \theta + \alpha \partial_\mu \beta + e A_\mu) - \frac{is}{2} j^\mu \mathrm{Tr}(\Sigma_3 \Lambda^{-1} (\partial_\mu + i \gamma_{\mu\nu} \xi^\nu) \Lambda)] - \int d^4x f(n, \sigma) + S(A). \tag{71}$$

In (71), we treat Λ as an arbitrary dynamical variable. The requirement that Λ can be written as BR with the velocity in B being identical to the mass/charge transport velocity $u^\mu = \frac{j^\mu}{\sqrt{j^2}}$ is enforced by the constraint

$$S^{\mu\nu} u_\nu = 0. \tag{72}$$

This is obtained as the equation of motion of the Lagrange multiplier field ξ^ν. First we show that this constraint can indeed give the identity of the flow velocities. Writing out (72), we find

$$\mathrm{Tr}[\Lambda^{-1} \gamma \cdot u \Lambda, \Sigma_3] \Lambda^{-1} \gamma_\nu \Lambda = \Lambda_\nu{}^\alpha \mathrm{Tr}[\Lambda^{-1} \gamma \cdot u \Lambda, \Sigma_3] \gamma_\alpha = 0. \tag{73}$$

Since $\Lambda_\nu{}^\alpha$ is invertible, we need $[\Lambda^{-1} \gamma \cdot u \Lambda, \Sigma_3]$ to have zero trace with γ_α, for any α. Further, since $[\Lambda^{-1} \gamma \cdot u \Lambda, \Sigma_3]$ is a linear combination of single powers of γ_μ, we get $[\Lambda^{-1} \gamma \cdot u \Lambda, \Sigma_3] = 0$ or

$$\Lambda^{-1}\gamma\cdot u\Lambda = a\gamma_0 + b\gamma_3. \tag{74}$$

Since $u^\mu u_\mu = 1$, $a = \cosh\omega$, $b = \sinh\omega$, so that we may write $\Lambda^{-1}\gamma\cdot u\Lambda = \Lambda_0^{-1}\gamma_0\Lambda_0$, $\Lambda_0 = \exp(-i\omega\gamma_{03})$. The solution for Λ is thus $\Lambda = BR\Lambda_0$, with B as given in (68). Λ_0 drops out of the action since it commutes with Σ_3 and $\mathrm{Tr}(\Sigma_3\gamma_{03}) = 0$. We may thus drop it from further consideration. Thus, the constraint (72) does enforce the equality of the flow velocities.

The remaining variational equations are

$$\partial_\mu j^\mu = 0, \qquad j^\mu\partial_\mu\alpha = 0, \qquad j^\mu\partial_\mu\beta = 0 \tag{75}$$

$$\begin{aligned} V_\mu - K_\mu + S_{\mu\nu}\xi^\nu &= \frac{\partial f}{\partial n}u_\mu \\ V_\mu &= \partial_\mu\theta + \alpha\partial_\mu\beta + eA_\mu \\ K_\mu &= \frac{is}{2}\mathrm{Tr}(\Sigma_3\Lambda^{-1}\partial_\mu\Lambda) \end{aligned} \tag{76}$$

$$\mathcal{D}S_{\mu\nu} - \frac{2}{n}\frac{\partial f}{\partial\sigma}(S_{\mu\lambda}F^\lambda{}_\nu - S_{\nu\lambda}F^\lambda{}_\mu) + (u_\mu S_{\nu\lambda} - u_\nu S_{\mu\lambda})\xi^\lambda = 0, \tag{77}$$

where $\mathcal{D} = u^\mu\partial_\mu$. The first set (75) arises from varying the action with respect to θ, α and β. The second one, (76), corresponds to the variation with respect to j^μ, and the last one is due to the variation of Λ as in $\delta\Lambda = -i\omega^{\mu\nu}\gamma_{\mu\nu}\Lambda$.

The simplification of these equations will proceed as in the spinless case. It is convenient in what follows to denote $\gamma_{\mu\nu}$ by t_A, taking A, B, C, as composite indices; thus, $t_{12} = \frac{1}{2}\Sigma_3$. The normalization is $\mathrm{Tr}(t_A t_B) = \delta_{AB}$. We can then write

$$\Lambda^{-1}d\Lambda = -it^A\mathcal{E}_A, \qquad \Lambda^{-1}\partial_\mu\Lambda = -it^A\mathcal{E}_{A,M}\partial_\mu\varphi^M, \tag{78}$$

where we denote the parameters of Λ generically by φ^M. This relation gives

$$K_\mu = s\mathcal{E}_{12,M}\partial_\mu\varphi^M. \tag{79}$$

Further, from the definition of $S^A = \frac{s}{2}\mathrm{Tr}(\Sigma_3\Lambda^{-1}t^A\Lambda)$, we find

$$\begin{aligned} f_{ABC}S^A dS^B\wedge dS^C &= s^3 f_{12,MP}f^{12,MN}f^{12,PQ}\mathcal{E}_N\wedge\mathcal{E}_Q \\ &= s^3 f^{12,NQ}\mathcal{E}_N\wedge\mathcal{E}_Q, \end{aligned} \tag{80}$$

where we used the relation $f_{12,MP}f^{12,MN}f^{12,PQ} = f^{12,NQ}$ which may be shown directly from the definitions. Using this equation and taking the curl of K_μ, we find

$$\begin{aligned} \partial_\mu K_\nu - \partial_\nu K_\mu &= -\frac{is}{2}\mathrm{Tr}\Sigma_3[\Lambda^{-1}\partial_\mu\Lambda, \Lambda^{-1}\partial_\nu\Lambda] \\ &= -\frac{1}{s^2}f_{ABC}S^A\partial_\mu S^B\partial_\nu S^C \\ &= \frac{4}{s^2}S_{\alpha\beta}\partial_\mu S^{\lambda\alpha}\partial_\nu S_\lambda{}^\beta. \end{aligned} \tag{81}$$

Taking the curl of (76) and contracting with u^μ and using (75) and (81), we get

$$\begin{aligned} \mathcal{D}(f'u_\nu) - \partial_\nu f' &= eu^\lambda F_{\lambda\nu} - \frac{4}{s^2}S_{\alpha\beta}\mathcal{D}S^{\lambda\alpha}\partial_\nu S_\lambda{}^\beta + \mathcal{D}(S_{\nu\lambda}\xi^\lambda) \\ &\quad - u^\mu\partial_\nu(S_{\mu\lambda}\xi^\lambda), \end{aligned} \tag{82}$$

where $f' = (\partial f/\partial n)$. The equation for the spin density, namely, (77), remains as it is for now. The key issue, however, is that the constraint (72) must be preserved by the evolution equations. The requirement is that

$$\mathcal{D}(S_{\mu\nu}u^\nu) = 0 \tag{83}$$

on the constrained subspace with $S_{\mu\nu}u^\nu = 0$. Using (77) and (82), this can be written as

$$\begin{aligned} S_{\mu\nu}\Big[& u_\lambda F^{\lambda\nu}X(n,\sigma) + \mathcal{D}(S^{\nu\lambda}\xi_\lambda) - u^\rho\partial^\nu(S_{\rho\lambda}\xi^\lambda) \\ & - \frac{4}{s^2}S_{\alpha\beta}(\mathcal{D}S^{\lambda\alpha})\partial^\nu S_\lambda{}^\beta + \partial^\nu f' + f'\xi^\nu\Big] = 0, \end{aligned} \tag{84}$$

where

$$X(n,\sigma) = e - \frac{2}{n}\frac{\partial f}{\partial n}\frac{\partial f}{\partial\sigma}. \tag{85}$$

The vector inside the bracket in (84) is orthogonal to $S_{\mu\nu}$, so it can be written as a linear combination of u^ν and W^ν, where W^ν is the normalized Pauli–Lubanski vector defined as

$$W^\mu = -\frac{1}{2s}\epsilon^{\mu\nu\alpha\beta}u_\nu S_{\alpha\beta}, \qquad W\cdot W = -1. \tag{86}$$

The following relations, which are easily verified, are useful for further simplification,

$$S_{\mu\nu} = s\epsilon_{\mu\nu\rho\sigma}W^\rho u^\sigma \Rightarrow S_{\mu\nu}u^\nu = S_{\mu\nu}W^\nu = W\cdot u = 0 \tag{87}$$

$$S_{\mu\nu}S^{\nu\rho} = -s^2(\delta_\mu^\rho + W_\mu W^\rho - u_\mu u^\rho). \tag{88}$$

By virtue of these relations, a vector ω^ν, satisfying $S_{\mu\nu}\omega^\nu = 0$, is of the form $\omega^\mu = u^\mu(u\cdot\omega) - W^\mu(W\cdot\omega)$. Applying this to (84) we get

$$X(n,\sigma)[u_\lambda F^{\lambda\nu} + W^\nu W_\rho u_\lambda F^{\lambda\rho}] + \mathcal{D}(S_{\nu\lambda}\xi^\lambda) - u_\rho \partial^\nu (S^{\rho\lambda}\xi_\lambda) + W^\nu[W_\rho \mathcal{D}(S^{\rho\lambda}\xi_\lambda) - u_\rho W_\sigma \partial^\sigma (S^{\nu\lambda}\xi_\lambda)] - \frac{4}{s^2} S_{\alpha\beta}(\mathcal{D}S^{\lambda\alpha})(\partial^\nu S_\lambda{}^\beta + W^\nu W_\sigma \partial^\sigma S^\beta_\lambda) + \partial^\nu f' + W^\nu W_\rho \partial^\rho f' - u^\nu \mathcal{D} f' + f'\xi^\nu = 0. \quad (89)$$

Equation (89) is to be understood as the equation determining ξ_ν. Further, from the way ξ_ν it enters the action (71), it is clear that, without loss of generality, we can take ξ^ν to be orthogonal to u_ν and W_ν as well. We can solve (89) as a series expansion in gradient terms and powers of the external field F, namely,

$$\xi^\nu = \xi^\nu_{(1)} + \xi^\nu_{(2)} + \cdots, \quad (90)$$

where $\xi^\nu_{(1)}$ contains terms linear in gradients or in F; $\xi^\nu_{(2)}$ contains terms quadratic in gradients or quadratic in F or linear in gradients and F and so on. In this way, we find

$$\xi^\nu_{(1)} = \frac{1}{f'}[u^\nu \mathcal{D} f' - \partial^\nu f' - W^\nu W_\rho \partial^\rho f' - X(n,\sigma)[u_\lambda F^{\lambda\nu} + W^\nu W_\rho u_\lambda F^{\lambda\rho}]] \quad (91)$$

$$\xi^\nu_{(2)} = -\frac{1}{f'}\Big[\mathcal{D}(S^{\nu\lambda}\xi_{\lambda(1)}) - u_\rho \partial^\nu (S^{\rho\lambda}\xi_{\lambda(1)}) + W^\nu[W_\rho \mathcal{D}(S^{\rho\lambda}\xi_{\lambda(1)}) - u_\rho W_\sigma \partial^\sigma (S^{\rho\lambda}\xi_{\lambda(1)})] - \frac{4}{s^2} S_{\alpha\beta}(\mathcal{D}S^{\lambda\alpha}_{(1)})(\partial^\nu S_\lambda{}^\beta + W^\nu W_\sigma \partial^\sigma S^\beta_\lambda)\Big]. \quad (92)$$

We can now substitute these expressions into (77) and (82) to obtain the equations of motion for the spin density $S_{\mu\nu}$ and the fluid velocity u_ν. The expressions are quite involved, so at this point we will simplify the equations by imposing the condition

$$X(n,\sigma) = 0 \Rightarrow e = \frac{2}{n}\frac{\partial f}{\partial n}\frac{\partial f}{\partial \sigma}. \quad (93)$$

This determines the dependence of f on σ in terms of its dependence on n. As discussed earlier, in the absence of the external field F, we take $f(n) = n + V(n)$. [$V(n) = 0$ corresponds to the pressureless case.] One can find a power series solution for $f(n,\sigma)$, satisfying (93), of the form

$$f(n,\sigma) = n + V(n) + \frac{en\sigma}{2[1+V'(n)]} + \mathcal{O}(\sigma^2). \quad (94)$$

This can be thought of as the analog of the requirement of $g = 2$ in the single-particle case. In the nonrelativistic limit, $V'(n) \ll 1$, and we see from the term linear in σ that the magnetic moment is proportional to the charge density en and corresponds to $g = 2$. The condition (93) characterizes a special kind of fluid, perhaps the most relevant case, since $g - 2$ is usually very small. Nevertheless, we emphasize that this is a specialization; one can always use the general solution (91), (92) for more general types of fluids.

Using (91) and (92) in (77) and (82) and keeping only terms linear in F and gradients we find the following expressions for the equations of motion:

$$\mathcal{D}(f' u_\nu) - \partial_\nu f' = e\Big[u^\lambda F_{\lambda\nu} - \frac{4}{s^2 f'}\partial_\nu S^{\lambda\beta}(SFS - FSS)_{\lambda\beta} - \frac{1}{f'^2}(S_{\nu\alpha}F^\alpha_\lambda - S_{\lambda\alpha}F^\alpha_\nu)\partial^\lambda f'\Big] + \cdots \quad (95)$$

$$\mathcal{D}S_{\mu\nu} = \frac{1}{f'}[S_\mu{}^\lambda(eF_{\lambda\nu} + G_{\lambda\nu}) - S_\nu{}^\lambda(eF_{\lambda\mu} + G_{\lambda\mu})] - \frac{4e}{s^2 f'^2}(u_\mu S_\nu{}^\lambda - u_\nu S_\mu{}^\lambda)\partial_\lambda S^{\rho\beta}(SFS - FSS)_{\rho\beta} + \frac{e}{f'^3}[s^2[(u_\mu F^\lambda_\nu - u_\nu F^\lambda_\mu) + (u_\mu W_\nu - u_\mu W_\nu)W^\rho F^\lambda_\rho] + u_\mu (SFS)_{\nu\lambda} - u_\nu (SFS)_{\mu\lambda}]\partial_\lambda f' + \cdots \quad (96)$$

$$G_{\lambda\nu} = u_\lambda \partial_\nu f' - u_\nu \partial_\lambda f'$$
$$(SFS - FSS)_{\lambda\beta} = S_\lambda{}^\rho F_{\rho\tau} S^\tau{}_\beta - F_\lambda{}^\rho S_{\rho\tau} S^\tau{}_\beta. \quad (97)$$

We have ignored the gradients of the external field, so that this is valid for almost uniform fields, or as the first set of terms in an expansion in terms of gradients of the fields.

The first equation in this set, (95), is the analog of the Euler equation with the Lorentz force on the right-hand side, as expected for magnetohydrodynamics. (The term magnetohydrodynamics is often used for the more restricted case where the electric field $\vec{E}$ is related to the magnetic field $\vec{B}$ via $\vec{E} + \vec{v} \times \vec{B} = 0$, which plays the role of Ohm's law for a plasma. We are using the term in a more general sense. The specialization to $\vec{B}$ via $\vec{E} = \vec{v} \times \vec{B}$ can be easily made at any stage.) The appearance of a term involving the gradient of the spin density on the right-hand side of this equation is not surprising since a term like $S^{\mu\nu}F_{\mu\nu}$ in the Hamiltonian would be like a contribution to the potential energy and we should expect its gradient in the equation of motion.

The second equation (96) describes the flow (or precession) of the spin density. What is novel and interesting is that this equation shows a precession term $S_\mu{}^\lambda G_{\lambda\nu} - S_\nu{}^\lambda G_{\lambda\mu}$ in addition to the usual precession effect due to $e(S_\mu{}^\lambda F_{\lambda\nu} - S_\nu{}^\lambda F_{\lambda\mu})$, even in the absence of gradients for $S^{\alpha\beta}$. Since $G_{\lambda\nu}$ involves gradients of the pressure and energy density, we see that nonuniform pressure and energy density in magnetohydrodynamics can generate precessional motion for spin density. Notice that we may rewrite the Euler equation (95) also as

$$f'\mathcal{D}u_\nu = \left[u^\lambda(eF_{\lambda\nu} + G_{\lambda\nu}) - \frac{4}{s^2 f'}\partial_\nu S^{\lambda\beta}(SFS - FSS)_{\lambda\beta}\right] + \cdots \quad (98)$$

Thus, $G_{\lambda\nu}$ plays a role similar to that of $F_{\lambda\nu}$ in this equation, so its presence in the spin precession equation (96) is not entirely surprising.

VIII. DISCUSSION

The group-theoretic formulation of fluid dynamics has been used to describe fluids with non-Abelian charges and to include anomaly effects. We explore this formulation further in this paper. The focus here is to clarify the role of Poincaré group, rather than internal symmetries. For this, we started by considering relativistic charged-particle dynamics in some detail. The minimal symplectic form, as given by the Poincaré group, along with the Lorentz force is shown to imply a gyromagnetic ratio of 2. (We give a clearer definition of what is meant by minimal in the text.) A similar result was found some time ago in $2+1$ dimensions. In that case, it is known that variants of the symplectic form can accommodate the anomalous magnetic moment. We show that a similar result holds in $3+1$ dimensions as well. We analyze the canonical structure and also show how the one-particle wave equation with the correct magnetic moment and spin-orbit interactions can be obtained upon quantization.

The extension to fluids is then considered, and the general group-theoretic framework is clarified further. The main result may be summarized as Eq. (69) which gives the general form of the action for a fluid with Poincaré symmetry and an internal symmetry corresponding to group G. This action with the addition of a suitable Wess–Zumino term for anomalies should describe general fluid dynamics with anomalous symmetries as well. The derivatives can also be made Levi–Civitá covariant to accommodate gravitational effects. Variants of (69) have been used previously, with and without anomalies, to describe a number of phenomena [5,7,8,13,18].

We also considered another special case, namely, the extension of standard magnetohydrodynamics (Maxwell field coupled to charged fluids) to include spin effects. The nature of this theory is dictated purely on symmetry grounds by the Poincaré or Lorentz group. The equation for the fluid shows new spin precession effects due to the gradients of pressure and energy density. There are also corrections to the Euler equation depending on the gradients of the spin density in the presence of electric and magnetic fields.

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