



General coordinate invariance in quantum many-body systems

Tomáš Brauner,^{1,2,*} Solomon Endlich,³ Alexander Monin,³ and Riccardo Penco⁴

¹*Institute for Theoretical Physics, Vienna University of Technology, 1040 Vienna, Austria*

²*Department of Theoretical Physics, Nuclear Physics Institute of the ASCR, 25068 Řež, Czech Republic*

³*Institut de Théorie des Phénomènes Physiques, EPFL, 1015 Lausanne, Switzerland*

⁴*Department of Physics and ISCAP, Columbia University, 10027 New York, USA*

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We extend the notion of general coordinate invariance to many-body, not necessarily relativistic, systems. As an application, we investigate nonrelativistic general covariance in Galilei-invariant systems. The peculiar transformation rules for the background metric and gauge fields, first introduced by Son and Wingate in 2005 and refined in subsequent works, follow naturally from our framework. Our approach makes it clear that Galilei or Poincaré symmetry is by no means a necessary prerequisite for making the theory invariant under coordinate diffeomorphisms. General covariance merely expresses the freedom to choose spacetime coordinates at will, whereas the true, physical symmetries of the system can be separately implemented as “internal” symmetries within the vielbein formalism. A systematic way to implement such symmetries is provided by the coset construction. We illustrate this point by applying our formalism to nonrelativistic s -wave superfluids.

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I. INTRODUCTION

Symmetries play a key role in modern understanding of the fundamental laws of nature. While they paved the way to the discovery of the standard model of elementary particles as well as to a geometric understanding of gravity, they have also proven extremely helpful as a tool for practical computations. In quantum field theory, symmetries of a system can be conveniently encoded in terms of invariance of the generating functional under transformations of a set of background fields, or external perturbations. This approach is particularly fruitful in combination with effective field theory (EFT) techniques, allowing one to connect descriptions of the same physics at vastly different length or momentum scales, possibly based on completely different dynamical degrees of freedom. A prime example of the use of generating functional methods within EFT is the chiral perturbation theory of quantum chromodynamics [1].

Conserved currents associated with physical symmetries can be probed by introducing a set of external gauge fields. Provided gravity is not involved and only internal symmetries are of interest, it is straightforward to make the classical action invariant under simultaneous local transformations of the gauge and matter (that is, all other, nongauge) fields by a suitable choice of transformation rules [2]. This in turn leads to gauge invariance of the generating functional under transformations of the background fields alone [3]. Spacetime symmetries, if desired, can be described by their action on field components in a fixed coordinate grid.

Once gravity enters the game, or when background fields associated with spacetime symmetries are needed, the implementation of symmetries in the generating functional becomes nontrivial, as is evidenced by the recent discussion of general coordinate invariance in nonrelativistic (NR) systems [4–12]. Here, it helps to keep in mind the viewpoint of differential geometry. Namely, treating spacetime as a (differentiable) manifold, physical observables should be viewed as geometric quantities, independent of the choice of local spacetime coordinates. Likewise, the basic building blocks of the action of a given theory are geometric objects such as scalar or vector fields, which can be defined in a coordinate-free manner. From this viewpoint, it is rather clear that general coordinate invariance merely encodes the freedom to choose a coordinate system at will. As physics must be independent of such a choice, true, physical symmetries of the system act directly upon the fields, without a reference to the spacetime coordinates. The vielbein, or frame field, formalism is particularly suited for making this distinction clear: once expressed in terms of the vielbein basis, all observables can be cast in a manifestly coordinate-free fashion.

The objective of this paper is to clarify some of the issues pertinent to general coordinate invariance in NR systems. We do so by insisting on the above differential-geometric picture. A systematic use of the vielbein formalism enables us to keep covariance under coordinate diffeomorphisms manifest at all stages. The noncovariant transformation rules for the background fields [4], including the sources for energy density and current [5], emerge naturally without the need to start from a relativistic theory and perform a NR reduction. We demonstrate in particular that making a given

*brauner@hep.itp.tuwien.ac.at

theory generally coordinate invariant does not require global Poincaré or Galilei symmetry. This is because in our approach, general coordinate invariance is *not* a physical symmetry to start with, and it only determines how the auxiliary background fields enter the action [13].

This basic scheme determines the plan of the paper. In Sec. II, we build a generally covariant framework applicable to a large class of local quantum field theories. As a nontrivial illustration, this is applied in Sec. III to the problem considered in Refs. [4,5], that is, a NR Galilei-invariant field theory with a U(1) internal symmetry. At this stage it could appear that Galilei invariance is imposed by a naive extension of the free Schrödinger field case. The rest of the paper is devoted to its systematic implementation, based on the coset construction [14] and its generalization to spacetime symmetries [15]. First, in Sec. IV we provide a basic overview of the structure of the Galilei group and discuss how it can be implemented within the coset formalism. The next two sections then work out two particular realizations of Galilei symmetry, suitable for the description of a microscopic theory of a charged matter field (Sec. V), and of an EFT for its low-temperature, superfluid phase (Sec. VI). Finally, in Sec. VII we conclude and provide an outlook on possible future application of our formalism.

A. Relation to recent literature

Since several papers have recently appeared which address the same or a similar problem, it is mandatory to clarify the relation of our results to these works. In Ref. [6], a framework based on the Newton-Cartan geometry was put forward as a means to construct actions manifestly invariant under spatial diffeomorphisms. Our approach lends a simple interpretation to the structure introduced therein, both from a mathematical (in terms of the vielbein) and a physical (in terms of sources for momentum density and current) viewpoint.

Formally, the material of Sec. II is related to that of Ref. [7] which, nevertheless, did not interpret the background fields in terms of sources for energy and momentum currents. Our argument in Sec. II is close in spirit to Ref. [9] which, too, showed how to obtain a generally coordinate invariant action from one invariant under global symmetries alone. However, while we systematically use the covariant vielbein resulting in a clear and concise picture, a similar result was obtained in [9] by a combination of an educated guess and “long algebra.”

While this paper was in preparation, Refs. [11,12] appeared which partially overlap with the material presented here, in particular, by illuminating the noncovariant transformation rules for some of the background fields based on the underlying geometry [11], and by systematically using the vielbein formalism [12]. However, we believe that we provide deeper insight in some important points such as the following: (i) the manifestly covariant

definition of the U(1) gauge field; (ii) the importance of spacetime symmetries for the relations between conserved currents; and (iii) the systematic implementation of Galilei symmetry using the coset formalism.

B. Index conventions

Throughout the paper, we use four types of indices: μ, ν, \dots for spacetime coordinates, i, j, \dots for spatial coordinates, A, B, \dots for internal (vielbein) spacetime coordinates, and a, b, \dots for internal spatial coordinates. Where necessary, the temporal coordinate (both internal and spacetime) is denoted by the index 0.

II. GENERAL COORDINATE INVARIANCE

Consider a theory of a set of matter fields, denoted collectively as ψ , and possibly a set of background fields A_μ , coupled to conserved currents of some internal symmetries of the theory. We assume that in flat space, the action of the theory, $S\{\psi, A\}$, is invariant under simultaneous gauge transformations of the matter and gauge fields. Provided there is no anomaly introduced by functional integration, this in turn gives rise to a gauge-invariant generating functional,

$$Z\{A\} \equiv \int \mathcal{D}\psi e^{iS\{\psi, A\}}. \quad (1)$$

We now wish to couple the theory to background spacetime geometry. One possible motivation for doing so may be that the space (time) actually is curved. More often, the physical picture that we will have in mind is rather a system living in flat spacetime. Here then, the spacetime fluctuations serve merely as source fields for conserved currents of spacetime symmetries, analogous to A_μ , which allow for a concise description of correlators of these currents.

A. Vielbein formalism

We are relatively free in the choice of the background fields: different choices may result in generating functionals with different sets of symmetries, which give us access to different observables. It will prove convenient to work with the spacetime vielbein, $e_\mu^A(x)$, representing a fixed local basis in which spacetime tensors can be decomposed. Under an infinitesimal coordinate diffeomorphism, $x'^\mu = x^\mu + \xi^\mu(x)$, the vielbein transforms as a one-form, that is

$$\Delta e_\mu^A = -e_\nu^A \partial_\mu \xi^\nu. \quad (2)$$

For the sake of brevity, we used here a shorthand notation for the form variation of a field, $\Delta\psi(x) \equiv \psi'(x') - \psi(x)$. The total local variation is thus given by $\delta\psi(x) \equiv \psi'(x) - \psi(x) = \Delta\psi(x) - \xi^\mu(x) \partial_\mu \psi(x)$. In flat spacetime, we can always choose the local basis as $e_\mu^A \rightarrow \delta_\mu^A$; hence we

will often be interested in the vielbein fluctuation, $A_\mu^A \equiv \delta_\mu^A - e_\mu^A$. By Eq. (2), this satisfies the transformation rule

$$\Delta A_\mu^A = \partial_\mu \xi^A - A_\nu^A \partial_\mu \xi^\nu. \quad (3)$$

This is exactly what we would expect from a one-form gauge field for local translations induced by coordinate diffeomorphisms. The setup is completed by introducing the dual vielbein, E_A^μ , defined as

$$E_A^\mu e_\mu^B = \delta_A^B. \quad (4)$$

Under diffeomorphisms, this naturally transforms as a vector, $\Delta E_A^\mu = E_A^\nu \partial_\nu \xi^\mu$.

The covariant vielbein allows us to make any action, defined by a local Lagrangian, invariant under coordinate diffeomorphisms. The only assumption we make is that the Lagrangian is expressed solely in terms of field variables that can be given a geometric meaning without referring to a particular coordinate frame. This means in other words that the Lagrangian is composed out of scalar, vector, and tensor fields and otherwise does not depend on the coordinates explicitly. The construction of the generally covariant Lagrangian out of its counterpart defined in flat spacetime then proceeds in three steps:

- (1) Scalar fields. It is convenient to first turn all field variables into spacetime scalars, which is done by projecting their spacetime tensor components onto the vielbein or its dual. For instance, in the case of a vector field $\tilde{\psi}^\mu$ this amounts to introducing the scalar components $\psi^A \equiv e_\mu^A \tilde{\psi}^\mu$.
- (2) Covariant derivatives. Suppose that for any scalar field ψ (internal indices suppressed), we can construct a covariant derivative $D_\mu \psi$ that transforms under all internal symmetries like ψ itself. Notice that, whenever ψ itself has a vielbein index attached to it (either because it carries spin, or because it is a covariant derivative of another field) this may not be an easy task. Defining the covariant derivative $D_\mu \psi$ in general is done most simply using the coset approach introduced later in this paper. If, however, ψ is a single spacetime scalar, the curved background does not introduce any subtlety. Then, using the dual vielbein, the covariant derivative $D_\mu \psi$ can be projected to a diffeomorphism-scalar derivative,

$$\mathcal{D}_A \psi \equiv E_A^\mu D_\mu \psi. \quad (5)$$

- (3) Volume measure. By applying the above steps iteratively, we can convert any local Lagrangian density into a scalar under diffeomorphisms. Then one can obtain an invariant action by introducing an appropriate volume measure, defined as usual by the determinant of the vielbein: $d\tau d\mathbf{x} \|e_\mu^A\|$.

This procedure should be thought of as the analogy of minimal coupling to gauge fields of internal symmetries. There is no unique way to introduce gauge fields into a given theory with global symmetry. As long as the spacetime geometry is treated merely as a background source, this is not a problem, however: *any* action invariant under diffeomorphisms, which reduces to the same action in a trivial background, or flat spacetime, is equally good.

Hopefully, it is now clear that, contrary to the impression one might get from the literature, making the theory diffeomorphism invariant is the easy part of the problem. The truly nontrivial step in the construction of the action is the implementation of true, physical symmetries, which provides a prescription of how to contract the vielbein indices A . In the following sections of this paper we will elucidate exactly how to systematically encode the physical symmetries inherent in a given system.

B. Nonrelativistic notation

Despite using the Lorentz-like spacetime indices μ and A , we have not made any assumptions about the actual spacetime symmetries of the system, apart from translation invariance in flat spacetime which is implicit in the requirement that the Lagrangian does not depend explicitly on the coordinates. However, since we have in mind primarily applications to NR many-body systems, it may be more transparent to treat spatial and temporal indices separately. Thus, we represent the vielbein as $e_\mu^A = (n_\mu, e_\mu^a)$ and its dual by $E_A^\mu = (V^\mu, E_a^\mu)$. The one-form n_μ can be thought of as defining surfaces of constant time, whereas V^i represents a “velocity” variable [6,16]. The inversion relation (4) is now cast equivalently as

$$\begin{aligned} V^\mu n_\mu &= 1, & V^\mu e_\mu^a &= 0, \\ E_a^\mu n_\mu &= 0, & E_a^\mu e_\mu^b &= \delta_a^b. \end{aligned} \quad (6)$$

Splitting the spatial and temporal indices effectively casts the vielbein in a block-matrix form so that its dual, being a matrix inverse, can be computed using the block-matrix algebra [17]. We thus obtain

$$\begin{aligned} V^\mu &= \frac{1}{n_0} (\delta^{\mu 0} - E_a^\mu e_0^a), & E_a^0 &= -\frac{E_a^i n_i}{n_0}, \\ E_a^i \tilde{e}_i^b &= \delta_a^b, & \text{where } \tilde{e}_i^a &\equiv e_i^a - \frac{e_0^a n_i}{n_0}. \end{aligned} \quad (7)$$

The expressions for V^μ and E_a^μ become explicit once we find the inverse of the reduced spatial vielbein, \tilde{e}_i^a . Using the algebraic identity for the determinant of a block matrix,

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det (D - CA^{-1}B), \quad (8)$$

the volume measure can also be evaluated as

$$dx \|e_\mu^A\| = dx n_0 \|\tilde{e}_i^a\|. \quad (9)$$

C. Background field transformations

Following Eq. (3), we argued that the fluctuations A_μ^A of the vielbein behave as gauge fields for local translations, that is, they act as sources for the energy-momentum density and current. However, since to obtain a vertex function of a conserved current one takes a derivative with respect to the source and subsequently sets it to zero, the precise parametrization of e_μ^A is to a large extent arbitrary. We only require that to linear order in the sources, $n_\mu \simeq \delta_{\mu 0} - \delta_{\mu 0} \Phi - \delta_\mu^i B_i$, denoting the sources for energy density and current as Φ and B_i , respectively. Here we are temporarily adopting the notation of Ref. [5] for easy direct comparison. One practically convenient parametrization is, for instance,

$$n_\mu = e^{-\Phi}(1, -B_i). \quad (10)$$

From the general covariant transformation rule for the vielbein (2), one then obtains by a straightforward manipulation the transformation of the sources,

$$\begin{aligned} \Delta\Phi &= \partial_0 \xi^0 - B_k \partial_0 \xi^k, \\ \Delta B_i &= \partial_i \xi^0 - B_k \partial_i \xi^k + B_i (\partial_0 \xi^0 - B_k \partial_0 \xi^k), \end{aligned} \quad (11)$$

in accord with Eq. (38) of Ref. [5]. While the particular form of these transformation rules is specific to the chosen parametrization (10) and thus inessential, what is important is the way to obtain manifestly covariant expressions by using the vielbein systematically.

D. Comparison with Newton-Cartan formalism

In applications, vielbeins often appear in pairs. Assuming spatial rotational invariance, we expect the covariant derivatives $\mathcal{D}_a \psi$ to enter the action through expressions such as $\delta^{ab} \mathcal{D}_a \psi^\dagger \mathcal{D}_b \psi = \delta^{ab} E_a^\mu E_b^\nu D_\mu \psi^\dagger D_\nu \psi$, so it is natural to introduce a degenerate symmetric tensor

$$g^{\mu\nu} \equiv \delta^{ab} E_a^\mu E_b^\nu. \quad (12)$$

The spatial component of this tensor, $g^{ij} = \delta^{ab} E_a^i E_b^j$, can be inverted, giving rise to a spatial metric,

$$g_{ij} = \delta_{ab} \tilde{e}_i^a \tilde{e}_j^b. \quad (13)$$

Using Eqs. (2) and (11), and noting that in the parametrization (10) we have $\tilde{e}_i^a = e_i^a + e_0^a B_i$, one infers the transformation rule for g_{ij} under diffeomorphisms,

$$\Delta g_{ij} = -g_{kj} \partial_i \xi^k - g_{ik} \partial_j \xi^k - (B_i g_{kj} + B_j g_{ik}) \partial_0 \xi^k, \quad (14)$$

again in accord with Ref. [5].

Three important remarks are in order here. First, in the general case of spacetime diffeomorphisms, using the metric g_{ij} creates more problems than it solves due to its complicated transformation properties. However, the situation dramatically simplifies if one restricts to (possibly time-dependent) spatial diffeomorphisms so that $\xi^0 = 0$, as was originally done in Ref. [4]. One can then set consistently $\Phi = B_i = 0$, upon which the metric g_{ij} transforms properly as a two-form. Spatial indices of other tensors can then be raised and lowered with g^{ij} and g_{ij} as usual.

Second, it is tempting to introduce another degenerate symmetric tensor which, like $g^{\mu\nu}$, does transform covariantly under all spacetime diffeomorphisms,

$$h_{\mu\nu} \equiv \delta_{ab} e_\mu^a e_\nu^b. \quad (15)$$

We use a different symbol to emphasize that this is not an inverse of $g^{\mu\nu}$ (which does not have an inverse to begin with). From their definition, it is easy to see that $h_{\mu\nu}$ and $g^{\mu\nu}$ satisfy a number of relations such as

$$g^{\mu\nu} n_\nu = 0, \quad h_{\mu\nu} V^\nu = 0, \quad g^{\mu\lambda} h_{\lambda\nu} = \delta_\nu^\mu - V^\mu n_\nu, \quad (16)$$

and, together with V^μ and n_μ , reproduce the Newton-Cartan structure of Ref. [6]. Using the relation $V^i = -E_a^i e_0^a e^\Phi$ we see that the field V^μ introduced *ad hoc* in Refs. [6,11] has a simple interpretation: it is associated with the source e_0^a for momentum density.

Third, we believe that the vielbein formalism offers conceptual as well as practical advantages compared to the Newton-Cartan geometry. The vielbein is obviously a fundamental object, directly related to sources for conserved currents, whereas the tensors $g^{\mu\nu}$ and $h_{\mu\nu}$ are derived. In addition, when correlators of the currents are desired, the generating functional $Z\{A_\mu, n_\mu, V^\mu, g^{\mu\nu}\}$ must be varied subject to constraints of the type (16) [11]. On the contrary, all components of e_μ^A are independent; hence the generating functional expressed as $Z\{A_\mu, e_\mu^A\}$ does not suffer from such problems. Most importantly, however, the Newton-Cartan formalism implicitly assumes spatial rotational symmetry in that it contracts vielbein indices using the Kronecker δ_{ab} . Should we deal with an intrinsically anisotropic system, the use of the vielbein in combination with the treatment of physical symmetries, discussed later in this paper, is mandatory.

III. EXAMPLE: A GALILEI-INVARIANT SYSTEM

In the previous section, we showed how to couple a given theory to fluctuations of spacetime geometry. We now wish to illustrate the general formalism on a concrete example as well as to investigate the constraints that physical symmetries can impose. We therefore consider a class of NR systems, described in flat spacetime by the Lagrangian

$$\mathcal{L} = \frac{i}{2} \psi^\dagger \overleftrightarrow{D}_0 \psi - \frac{\delta^{ij}}{2m} D_i \psi^\dagger D_j \psi - V(\psi^\dagger \psi). \quad (17)$$

Here, ψ is a complex scalar field that interacts with a background U(1) gauge field A_μ via minimal coupling, $D_\mu \psi \equiv (\partial_\mu - iA_\mu) \psi$. We also used the shorthand notation $\psi^\dagger \overleftrightarrow{D}_\mu \psi \equiv \psi^\dagger D_\mu \psi - D_\mu \psi^\dagger \psi$. The model (17) has recently been used to describe the fractional quantum Hall effect [11]. Apart from global spacetime symmetries (spacetime translations, rotations, and Galilei boosts), it is invariant under the local U(1) transformation

$$\Delta \psi = i\theta \psi, \quad \Delta A_\mu = \partial_\mu \theta. \quad (18)$$

In order to cast the Lagrangian in a form independent of the choice of the coordinate frame, we simply replace gauge-covariant derivatives with the diffeomorphism-covariant ones,

$$\mathcal{L} = \frac{i}{2} \psi^\dagger \overleftrightarrow{\mathcal{D}}_0 \psi - \frac{\delta^{ab}}{2m} \mathcal{D}_a \psi^\dagger \mathcal{D}_b \psi - V(\psi^\dagger \psi). \quad (19)$$

We could stop at this point for we have accomplished our goal: to make the theory (17) invariant under coordinate diffeomorphisms. However, it is instructive to make the dependence of the Lagrangian on the individual fields more explicit, as it will help us highlight the difference between diffeomorphisms and actual symmetries.

To this end, we rewrite the Lagrangian (19) in the form put forward in Ref. [5],

$$\begin{aligned} \mathcal{L} = & \frac{i}{2} e^\Phi \psi^\dagger \overleftrightarrow{\tilde{D}}_0 \psi - V(\psi^\dagger \psi) \\ & - \frac{g^{ij}}{2m} (\tilde{D}_i \psi^\dagger + B_i \tilde{D}_0 \psi^\dagger) (\tilde{D}_j \psi + B_j \tilde{D}_0 \psi), \end{aligned} \quad (20)$$

where we used the definition (12). The twisted covariant derivative, $\tilde{D}_\mu \psi \equiv (\partial_\mu - i\tilde{A}_\mu) \psi$, is defined in terms of the appropriately modified electromagnetic field

$$\begin{aligned} \tilde{A}_0 & \equiv A_0 + \frac{1}{2} m V^2 n_0, \\ \tilde{A}_i & \equiv A_i - m V_i + \frac{1}{2} m V^2 n_i, \end{aligned} \quad (21)$$

with the shorthand notation $V_i \equiv g_{ij} V^j$ and $V^2 \equiv V^i V_i$. Notice that the form of the field redefinition (21) is independent of Φ and B_i , and hence of the chosen parametrization of n_μ .

Up to differences in notation, Eq. (21) agrees with similar relations obtained in Refs. [6,11,16]. In contrast to these papers, we *began* from the covariant field A_μ . For the sake of completeness, we write down the transformation rules for \tilde{A}_μ , which are obtained by a straightforward calculation using the properties of its ingredients,

$$\begin{aligned} \Delta \tilde{A}_0 & = \partial_0 \theta - \tilde{A}_\mu \partial_0 \xi^\mu, \\ \Delta \tilde{A}_i & = \partial_i \theta - \tilde{A}_\mu \partial_i \xi^\mu - m e^\Phi g_{ik} \partial_0 \xi^k. \end{aligned} \quad (22)$$

These transformation rules are not a product of intuition or guesswork, but rather descend directly from our fully covariant formalism.

The appearance of the combination \tilde{A}_μ in the action has a very important consequence. In general, taking a functional derivative of the action with respect to a background gauge field gives the corresponding conserved current. Now, A_μ acts as a source for the particle number current j^μ , whereas e_μ^a is a source for the momentum current T_a^μ . Noting that to linear order in the sources, A_i and e_0^a only enter the action (20) through the combination $A_i - m V_i$, immediately leads to the relation

$$T^{0i} = m j^i, \quad (23)$$

valid in the trivial background. In the absence of a source for momentum density, this relation can only be derived by a more-or-less explicit computation [4]. On fairly general grounds, it can be traced to the underlying Galilei symmetry [4,18,19].

The above observation has a very general validity. The appearance of sources only through certain combinations is one of the hallmarks of spacetime symmetries. Whereas here we had to rely on a specific type of Lagrangian, in Sec. V we will see such combinations of sources emerge directly as a consequence of the symmetry. This leads to relations among the corresponding Noether currents, similar to Eq. (23). The generality of such relations was already pointed out in Ref. [19].

As should by now be clear from our construction, each of the terms in Eq. (19) is separately a scalar under coordinate diffeomorphisms. This might seem puzzling since we know that it is Galilei invariance what forces the temporal and spatial derivatives to appear in the particular combination featured in Eq. (19). However, in the vielbein formalism, the physical Galilei symmetry should not be thought of as a finite-dimensional subgroup of the diffeomorphism group. Indeed, a much larger class of local field theories can obviously be made diffeomorphism invariant. Diffeomorphisms merely represent the freedom of a coordinate choice, and any physical symmetry must be imposed on top of general covariance.

In this spirit, Galilei invariance of the theory (19) can be viewed as an internal symmetry acting on the vielbein indices. Under an infinitesimal local Galilei boost with the velocity parameter u^a , the covariant derivatives therein are shifted by

$$\mathcal{D}_0 \psi \rightarrow \mathcal{D}_0 \psi - u^a \mathcal{D}_a \psi, \quad \mathcal{D}_a \psi \rightarrow \mathcal{D}_a \psi + i m u_a \psi. \quad (24)$$

The former rule is identical to a boost transformation of temporal gradients, whereas the latter reflects the fact that

on complex fields, Galilei boosts are realized projectively. In addition, the Lagrangian (19) is obviously preserved by local internal rotations of the vielbein e_μ^a . While here we found the symmetries by a mere educated guess, in Sec. V we will see how they can be imposed systematically using the coset construction.

IV. SPACETIME SYMMETRIES

As repeatedly emphasized above, true physical symmetries can, and should, be treated separately from mere coordinate diffeomorphisms. It is a purpose of this section to set up a systematic procedure that will allow us to do so. We will follow closely the treatment of Ref. [20] and its recent application to the coupling of spontaneously broken symmetries to gravity [21].

The same physics can be equivalently described using the spacetime coordinates x^μ or using coordinates $y^A(x)$ defined in the local frame $e_\mu^A(x)$. In flat spacetime, one can, for instance, choose the vielbein as a global orthonormal basis, and define y^A geometrically as its integral curves, $dy^A = e_\mu^A dx^\mu$. The action of symmetry transformations on the system can be described in such a basis in a way independent of the coordinate chart x^μ . From the point of view of x^μ , even spacetime symmetries therefore appear to be internal. Any given symmetry can now be made local using standard techniques by introducing a set of background gauge fields, one for each symmetry generator. Since spacetime symmetries, in particular translations, act on the coordinates as affine rather than linear transformations though, one has to exercise some care. The implementation of the symmetries can be worked out economically using the method of nonlinear realizations, also known as the coset construction [14,15]. For the reader's convenience, the elements of the coset construction are reviewed in Appendix A.

A. Poincaré invariance

In order to set the stage, we review here briefly the case of relativistic systems, invariant under the Poincaré group [20]. We wish to describe its action in the space of local coordinates y^A , scalar under diffeomorphisms. While Lorentz transformations, generated by the operator of angular momentum J_{AB} , act on them linearly, spacetime translations, generated by the momentum operator P_A , act on y^A by a shift. Geometrically, one can view the Minkowski space spanned by y^A as the coset space \mathcal{G}/\mathcal{H} , where \mathcal{G} is the Poincaré group and \mathcal{H} its Lorentz subgroup. From the general expression (A3), the coset space can be represented by the matrix field

$$U(x) \equiv e^{iy^A(x)P_A}. \quad (25)$$

The action of a group element $\mathfrak{g} \in \mathcal{G}$ is then defined as in Eq. (A4). This immediately reproduces the expected

behavior under Lorentz transformations, $y^A \rightarrow \Lambda^A_B y^B$ [in which case $\mathfrak{h}(y, \mathfrak{g}) = \mathfrak{g}$], as well as under spacetime translations, $y^A \rightarrow y^A + a^A$ [in which case $\mathfrak{h}(y, \mathfrak{g}) = \mathbb{1}$].

The virtue of the coset construction is that it admits a straightforward gauging of the symmetry. To this end, one introduces a set of one-form gauge fields R_μ^{AB} for the generators J_{AB} , and S_μ^A for P_A . Put together in the matrix field, $\mathcal{A}_\mu \equiv \frac{1}{2} R_\mu^{AB} J_{AB} + S_\mu^A P_A$, they transform according to Eq. (A6). The last element of the construction is the Maurer-Cartan (MC) form (A7). It can be evaluated explicitly using the commutation relations of the Poincaré algebra. Projecting it back to the two subspaces of generators, one finds that $\omega_\mu^{J,AB} = -R_\mu^{AB}$, whereas [20]

$$\omega_\mu^{P,A} = \partial_\mu y^A - S_\mu^A - R_\mu^{AB} y_B. \quad (26)$$

The latter quantity is invariant under translations generated by P_A . It transforms as a vector under the Lorentz group \mathcal{H} and can be interpreted as the covariant vielbein, that is, $\omega_\mu^{P,A} \equiv e_\mu^A$. The connection $\omega_\mu^{J,AB}$, on the other hand, allows one to define a covariant derivative of matter fields ψ that transforms nontrivially under the Lorentz group \mathcal{H} via Eq. (A10), i.e.

$$D_\mu \psi \equiv \left[\partial_\mu - \frac{i}{2} R_\mu^{AB} \mathcal{R}(J_{AB}) \right] \psi. \quad (27)$$

The above construction leads to a generating functional $Z\{e_\mu^A, R_\mu^{AB}\}$, invariant separately under internal gauge spacetime translations and Lorentz transformations, as well as under coordinate diffeomorphisms. The coordinates y^A can be fixed arbitrarily and do not represent dynamical degrees of freedom; hence they also appear in the generating functional. Nevertheless, they only enter together with S_μ^A and R_μ^{AB} through the combination defined by Eq. (26). At this stage, R_μ^{AB} is still an independent background field. If desired, it can be eliminated in a way that respects all the symmetries, for instance, by setting to zero the field-strength tensor associated with ω_μ^P , that is, the torsion tensor,

$$\omega_{\mu\nu}^{P,A} = D_\mu \omega_\nu^{P,A} - D_\nu \omega_\mu^{P,A}. \quad (28)$$

This determines R_μ^{AB} in terms of derivatives of e_μ^A , giving it a value typical for a spin connection [20],

$$R_\mu^{AB} = -\frac{1}{2} [e^{\nu A} (\partial_\mu e_\nu^B - \partial_\nu e_\mu^B) + e_{\mu C} e^{\nu A} e^{\lambda B} \partial_\lambda e_\nu^C - (A \leftrightarrow B)]. \quad (29)$$

After this reduction, the generating functional depends solely on e_μ^A and is still constrained separately by the internal translations and spacetime diffeomorphisms. This reproduces the algorithm for construction of invariant

actions outlined in Sec. II. Additionally, however, the coset construction tells us that covariant derivatives for fields with spin must be defined according to Eq. (27).

Notice that the internal coordinates y^A were introduced merely as a device that allows us to treat spacetime symmetries separately from coordinate diffeomorphisms. At the end of the day, they can be fixed at will; a natural choice in flat spacetime is $y^A = \delta_\mu^A x^\mu$. Upon fixing the internal coordinates, geometric objects such as vectors or tensors become ‘‘pinned’’ to the coordinate grid; spacetime symmetries and coordinate diffeomorphisms then become linked to each other. However, all general arguments can be carried out without such gauge fixing.

B. Galilei invariance

We now proceed to our main object of interest in this section: the Galilei symmetry. The generators of the Galilei algebra will be denoted as J_{ab} (rotations), N_a (boosts), P_a (space translations), and H (time translations). In the case of a single quantum-mechanical particle, the commutator $[P_a, N_b]$ has a central charge, causing the Galilei group to be realized projectively [22]. In a many-body system composed of particles of the same mass m , the Lie algebra can be closed by introducing the operator of particle number Q . All nontrivial commutators of the Galilei algebra then read

$$\begin{aligned} [J_{ab}, J_{cd}] &= i(\delta_{ac}J_{bd} + \delta_{bd}J_{ac} - \delta_{ad}J_{bc} - \delta_{bc}J_{ad}), \\ [J_{ab}, P_c] &= i(\delta_{ac}P_b - \delta_{bc}P_a), \\ [J_{ab}, N_c] &= i(\delta_{ac}N_b - \delta_{bc}N_a), \\ [P_a, N_b] &= imQ\delta_{ab}, \\ [H, N_a] &= iP_a. \end{aligned} \quad (30)$$

One can associate with the above generators the gauge fields R_μ^{ab} (rotations), B_μ^a (boosts), S_μ^a (spatial translations), T_μ (time translations), and A_μ (particle number). They can all be collected in the matrix gauge field

$$\mathcal{A}_\mu \equiv \frac{1}{2}R_\mu^{ab}J_{ab} + B_\mu^aN_a + S_\mu^aP_a + T_\mu H + A_\mu Q. \quad (31)$$

The transformation of the gauge fields under local symmetry transformations is again given by Eq. (A6). Choosing a parametrization of the Galilei group in terms of the transformation parameters α^{ab} , u^a , a^a , b , and θ ,

$$\mathfrak{g} = \exp\left(\frac{i}{2}\alpha^{ab}J_{ab} + iu^aN_a + ia^aP_a + ibH + i\theta Q\right), \quad (32)$$

we obtain explicit expressions for infinitesimal field transformations [23],

$$\begin{aligned} \Delta R_\mu^{ab} &= \partial_\mu\alpha^{ab} - R_\nu^{ab}\partial_\mu\xi^\nu + \alpha^a{}_cR_\mu^{cb} + \alpha^b{}_cR_\mu^{ac}, \\ \Delta B_\mu^a &= \partial_\mu u^a - B_\nu^a\partial_\mu\xi^\nu + \alpha^a{}_bB_\mu^b - u_bR_\mu^{ab}, \\ \Delta S_\mu^a &= \partial_\mu a^a - S_\nu^a\partial_\mu\xi^\nu + \alpha^a{}_bS_\mu^b - a_bR_\mu^{ab} + u^aT_\mu - bB_\mu^a, \\ \Delta T_\mu &= \partial_\mu b - T_\nu\partial_\mu\xi^\nu, \\ \Delta A_\mu &= \partial_\mu\theta - A_\nu\partial_\mu\xi^\nu + mu_aS_\mu^a - ma_aB_\mu^a, \end{aligned} \quad (33)$$

where the internal vector indices are lowered and raised by the Kronecker metrics δ_{ab} and δ^{ab} . By construction, all gauge fields transform as one-forms under coordinate diffeomorphisms. For the sake of future reference, we add explicit expressions for the field-strength tensor, defined in the matrix form as $\mathcal{A}_{\mu\nu} \equiv \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]$. By projecting it to the respective generators, one obtains the field strengths of the individual gauge fields,

$$\begin{aligned} R_{\mu\nu}^{ab} &= \partial_\mu R_\nu^{ab} - \partial_\nu R_\mu^{ab} + R_\mu^a{}_cR_\nu^{cb} - R_\nu^a{}_cR_\mu^{cb}, \\ B_{\mu\nu}^a &= \partial_\mu B_\nu^a - \partial_\nu B_\mu^a - R_{\mu b}^aB_\nu^b + R_{\nu b}^aB_\mu^b, \\ S_{\mu\nu}^a &= \partial_\mu S_\nu^a - \partial_\nu S_\mu^a - R_{\mu b}^aS_\nu^b + R_{\nu b}^aS_\mu^b - B_\mu^aT_\nu + B_\nu^aT_\mu, \\ T_{\mu\nu} &= \partial_\mu T_\nu - \partial_\nu T_\mu, \\ A_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu - mB_\mu^aS_{\nu a} + mB_\nu^aS_{\mu a}. \end{aligned} \quad (34)$$

The transformation rules for the components of the field-strength tensor are obtained from Eq. (33) by dropping all terms containing a derivative of the parameters α^{ab} , u^a , a^a , b , θ .

The physical meaning of some of the gauge fields introduced above is immediately clear, as they represent sources for the respective conserved currents: S_μ^a for momentum, T_μ for energy, and A_μ for the particle number current. In Sec. II, we already interpreted fluctuations of the spacetime vielbein as sources for the momentum and energy currents. One of our first tasks therefore will be to compute the MC form to check whether the covariant vielbein obtained from it is compatible with our earlier definition. Furthermore, as we will see, the gauge field R_μ^{ab} will give us access to spin degrees of freedom analogously to the relativistic case. Finally, the gauge field B_μ^a in principle couples to the conserved currents for Galilei boosts, although it does not seem to admit a simple physical interpretation.

C. Choice of the coset space

Before applying the strategy of Sec. IVA to describe Galilei-invariant systems, it is worth pausing and commenting on the various ways in which a coset construction for the Galilei group can be implemented consistently with unbroken rotational symmetry. While for a relativistic system in vacuum the choice of the coset space \mathcal{G}/\mathcal{H} is essentially unique, the commutation relations of the Galilei algebra (30) admit several consistent options. Given that

spacetime translations are always nonlinearly realized, there are altogether four different options for the choice of generators of \mathcal{H} :

- (1) $\{J_{ab}, N_a, Q\}$. This option corresponds to using a coset parametrization identical to the relativistic one given in Eq. (25). While mathematically consistent, it turns out to be impractical: due to the commutation relation $[P_a, N_b] = imQ\delta_{ab}$, the coset generators $\{P_a, H\}$ do not span a representation of the subgroup \mathcal{H} . As a consequence, the coset construction cannot be applied directly as the $\omega_{\perp\mu}$ part of the MC form does not transform covariantly like in Eq. (A9).
- (2) $\{J_{ab}, N_a\}$. Here one chooses the generators of \mathcal{H} as in the Poincaré case. The remaining generators $\{P_a, H, Q\}$ form a representation of \mathcal{H} which is reducible but indecomposable [24]. It is still possible to build manifestly Galilei-covariant expressions using such a representation provided one embeds the Galilei group in the Lorentz group of a spacetime with one extra dimension (see Ref. [25] and references therein). Within this formalism, the coset generators $\{P_a, H, Q\}$ define a vector of \mathcal{H} with an extra component due to Q . As a result, tensor fields (such as gauge fields) now have more components, some of which are unphysical and have to be eliminated in a covariant manner.
- (3) $\{J_{ab}, Q\}$. Thanks to Q 's inclusion in the subgroup \mathcal{H} , this option allows the straightforward inclusion of charged matter fields. Additionally, it automatically leads to the correct Galilei transformation for such fields, whereby a time-dependent coordinate shift is accompanied by a change of phase of the field. Since Galilei boosts N_a belong to the coset space \mathcal{G}/\mathcal{H} , we will have to introduce a set of fields $v^a(x)$ that account for their nonlinear realization. The resulting EFT framework naturally reproduces a version of Schrödinger field theory with an auxiliary velocity field which makes the equations of motion of first order in derivatives. This choice will be discussed in detail in Sec. V.
- (4) $\{J_{ab}\}$. In this scheme, internal U(1) transformations generated by Q are realized nonlinearly and consequently one needs to introduce another field, $\pi(x)$. The boost fields v^a are still present, but in this case can be eliminated algebraically by a set of covariant conditions, known as the inverse Higgs constraints [19,26,27]. Since the coset fields can always be interpreted as Nambu-Goldstone (NG) fields of spontaneously broken symmetry, this setup is ready-made for an EFT description of superfluid phases of matter, in which particle number is spontaneously broken. This implementation of the coset construction will be described in Sec. VI.

V. MICROSCOPIC THEORY

Here we develop the framework corresponding to a subgroup \mathcal{H} that is generated by J_{ab} and Q . After making this choice for the coset space, the mathematical structure of the setup is completely dictated by symmetry. We therefore first work out the details and only then interpret what we found.

Following closely the general discussion in Appendix A, we parametrize the coset space \mathcal{G}/\mathcal{H} as

$$U(x) \equiv e^{iy^a(x)P_a} e^{iz(x)H} e^{iv^a(x)N_a}. \quad (35)$$

The gauged MC form is defined as usual by Eq. (A7). In order to evaluate it explicitly, we make use of the conjugation relations that follow from the commutators (30),

$$\begin{aligned} e^{-i\alpha\cdot P} J_{ab} e^{i\alpha\cdot P} &= J_{ab} - \alpha_a P_b + \alpha_b P_a, \\ e^{-i\alpha\cdot P} N_a e^{i\alpha\cdot P} &= N_a + m\alpha_a Q, \\ e^{-i\alpha\cdot N} P_a e^{i\alpha\cdot N} &= P_a - m\alpha_a Q, \\ e^{-i\alpha\cdot N} H e^{i\alpha\cdot N} &= H - \alpha\cdot P + \frac{1}{2}m\alpha^2 Q, \end{aligned} \quad (36)$$

where we used the shorthand notation $\mathbf{u}\cdot\mathbf{v} \equiv \delta_{ab}u^a v^b$. The individual components of the MC form, projected onto the respective generators, read

$$\begin{aligned} \omega_\mu^{J_{ab}} &= -R_\mu^{ab}, \\ \omega_\mu^Q &= -A_\mu - m\mathbf{y}\cdot\mathbf{B}_\mu - m\mathbf{v}\cdot\partial_\mu\mathbf{y} \\ &\quad + m v_a (S_\mu^a + R_\mu^{ab} y_b + z B_\mu^a) + \frac{1}{2}m\mathbf{v}^2(\partial_\mu z - T_\mu), \end{aligned} \quad (37)$$

for the generators of \mathcal{H} , and

$$\begin{aligned} \omega_\mu^{N_a} &= \partial_\mu v^a - B_\mu^a - R_\mu^{ab} v_b, \\ \omega_\mu^{P_a} &= \partial_\mu y^a - (S_\mu^a + R_\mu^{ab} y_b + z B_\mu^a) - v^a(\partial_\mu z - T_\mu), \\ \omega_\mu^H &= \partial_\mu z - T_\mu, \end{aligned} \quad (38)$$

for the generators of \mathcal{G}/\mathcal{H} .

A. Fields and symmetry transformations

In order to understand the structure just introduced, it is important to work out the transformation properties of all the ingredients. The canonical transformation of coset fields is given by Eq. (A4), whereas that of the MC form by Eq. (A9). From these general results, it is easy to deduce the following transformation properties.

Spatial rotations. These are by construction realized linearly on the coset fields: z is a scalar while y^a and v^a are vectors. In this case, $\mathfrak{h} = \mathfrak{g} = e^{\frac{i}{2}\alpha^{ab}J_{ab}}$. The components

of the MC form ω_μ^Q and ω_μ^H are scalars while ω_μ^{Na} and ω_μ^{Pa} are vectors under rotations. Finally, ω_μ^{Jab} transforms as a tensor gauge field, just like $-R_\mu^{ab}$.

U(1) charge transformations. Here we have again $\mathfrak{h} = \mathfrak{g} = e^{i\theta Q}$; all coset fields are trivially invariant. Likewise, all components of the MC form remain U(1) invariant except for ω_μ^Q which behaves as a gauge field,

$$\delta_Q \omega_\mu^Q = -\partial_\mu \theta. \quad (39)$$

Space and time translations. In the parametrization given by Eq. (35), the coset fields transform simply,

$$\delta_{P,H} y^a = a^a, \quad \delta_{P,H} z = b, \quad \delta_{P,H} v^a = 0. \quad (40)$$

Consequently, $\mathfrak{h} = \mathbb{1}$ and all components of the MC form are trivially invariant under translations.

Galilei boosts. This is the most interesting transformation. By a direct computation using Eq. (36), we find that z is boost invariant whereas

$$\delta_N y^a = z u^a, \quad \delta_N v^a = u^a. \quad (41)$$

As expected, z and y^a behave exactly as time and a spatial coordinate, while v^a behaves as a velocity. Furthermore, we find that $\mathfrak{h} = e^{im(u \cdot y + \frac{1}{2} z u^2) Q}$. Since this matrix defines the transformation rule for matter fields, we recover the well-known fact that for any charged field, a Galilei boost is accompanied by a change of its phase. Due to the form of \mathfrak{h} , we finally observe that all components of the MC form are boost invariant except for ω_μ^Q which changes under an infinitesimal boost as

$$\delta_N \omega_\mu^Q = -m \partial_\mu (\mathbf{u} \cdot \mathbf{y}). \quad (42)$$

B. Invariant actions

The coset construction provides us with all the ingredients we need to build manifestly invariant actions. As to the covariant vielbein e_μ^A , we naturally identify

$$n_\mu = \omega_\mu^H, \quad e_\mu^a = \omega_\mu^{Pa}. \quad (43)$$

Given a matter field ψ , the \mathcal{H} -components of the MC form allow us to define its covariant derivative,

$$D_\mu \psi = \left[\partial_\mu + iq \omega_\mu^Q - \frac{i}{2} R_\mu^{ab} \mathcal{R}(J_{ab}) \right] \psi, \quad (44)$$

where q is the electric charge of ψ , and \mathcal{R} the (spin) representation of the rotation group in which it transforms. The dual vielbein E_A^μ can in turn convert $D_\mu \psi$ into a spacetime scalar; see Eq. (5).

We have managed to identify the building blocks that were used in Secs. II and III to construct the action. However, the coset construction tells us much more.

Namely, we now know how to contract the vielbein indices: they have to be summed over in a way that preserves the linearly realized subgroup \mathcal{H} , that is, rotations and U(1) phase redefinitions. Notice that Galilei invariance is already built in automatically, thanks to the presence of the coset field v^a .

For a complex scalar field ψ , there is a unique Lagrangian that respects all the symmetries and contains just one derivative,

$$\mathcal{L} = \frac{i}{2} \psi^\dagger \overleftrightarrow{\mathcal{D}}_0 \psi - V(\psi^\dagger \psi). \quad (45)$$

This should be contrasted with Eq. (19). In order to see that the two Lagrangians describe equivalent theories, we have to deal with the boost NG field v^a hidden in the definition of E_A^μ and ω_μ^Q . To highlight the precise way this NG field enters the Lagrangian (45), it is convenient to introduce the ‘‘bare’’ vielbein with v^a removed,

$$\bar{e}_\mu^a \equiv \partial_\mu y^a - (S_\mu^a + R_\mu^{ab} y_b + z B_\mu^a), \quad (46)$$

so that $e_\mu^a = \omega_\mu^{Pa} = \bar{e}_\mu^a - v^a n_\mu$. Denoting the ‘‘bare’’ dual vielbein, inverse to \bar{e}_μ^a and n_μ , analogously as \bar{E}_a^μ , \bar{V}^μ , it is easy to verify the relations

$$E_a^\mu = \bar{E}_a^\mu, \quad V^\mu = \bar{V}^\mu + v^a \bar{E}_a^\mu. \quad (47)$$

Finally, define the modified gauge field $\bar{A}_\mu \equiv A_\mu + m \mathbf{y} \cdot \mathbf{B}_\mu$ so that $-V^\mu \omega_\mu^Q = V^\mu \bar{A}_\mu + \frac{1}{2} m v^2$. The Lagrangian (45) thus acquires the explicit form

$$\mathcal{L} = \frac{i}{2} V^\mu \psi^\dagger \overleftrightarrow{\partial}_\mu \psi + V^\mu \psi^\dagger \bar{A}_\mu \psi + \frac{1}{2} m v^2 \psi^\dagger \psi - V(\psi^\dagger \psi). \quad (48)$$

The field v^a is obviously not dynamical because it enters Eq. (48) without any derivatives. Upon integrating v^a out using its equation of motion [16], the Lagrangian (19) is recovered, provided \bar{e}_μ^a and \bar{A}_μ are identified with the vielbein and the U(1) gauge field introduced therein [28].

It is integrating out the boost NG field v^a that eventually ensures that all the ingredients introduced in Sec. III have appropriate transformation rules under Galilei boosts and appear in the Lagrangian in the right combinations. While e_μ^A is boost invariant, it follows from Eq. (47) that $\delta_N \bar{V}^\mu = -u^a \bar{E}_a^\mu$. Likewise, it is easy to check the transformation rule for \bar{A}_μ ,

$$\Delta \bar{A}_\mu = \partial_\mu (\theta + m \mathbf{u} \cdot \mathbf{y}) - \bar{A}_\nu \partial_\mu \xi^\nu - m u_a \bar{e}_\mu^a. \quad (49)$$

From here, one immediately recovers the previously guessed rule (24) for covariant derivatives together with the phase factor induced by Galilei boosts as appropriate for the complex field ψ .

C. Relations among conserved currents

As we have just shown, the coset formalism exactly reproduces the results of Sec. III. Moreover, we now have access to a wealth of information due to the presence of independent sources for translation, rotation, boost, and electromagnetic currents. Equations (37) and (38) ensure that the sources appear only through certain specific combinations. For instance, A_μ always appears together with the boost source in the linear combination \bar{A}_μ , defined above, while S_μ^a always enters through the combination

$$\bar{S}_\mu^a \equiv S_\mu^a + R_\mu^{ab} y_b + z B_\mu^a, \quad (50)$$

so that $\bar{e}_\mu^a = \partial_\mu y^a - \bar{S}_\mu^a$. The combined diffeomorphism gauge transformation of this field reads

$$\Delta \bar{S}_\mu^a = \partial_\mu (a^a + \alpha^{ab} y_b + z u^a) - \bar{S}_\nu^a \partial_\mu \xi^\nu - \alpha_b^a \bar{e}_\mu^b - u^a n_\mu, \quad (51)$$

which indicates that \bar{S}_μ^a transforms as a gauge field with respect to the total coordinate shift, including the contributions from rotations and boosts.

Let us initially assume that the theory does not contain any fields with spin so that ω_μ^J does not contribute, and that ω_μ^N can be disregarded (see Sec. VI on this point). After integrating out the auxiliary field v^a , the action then quite generally depends on the sources only through n_μ , \bar{e}_μ^a , and \bar{A}_μ . This eventually leads to the relations

$$M^{\mu ij} = x^i T^{\mu j} - x^j T^{\mu i}, \quad B^{\mu i} = t T^{\mu i} - m x^i j^\mu \quad (52)$$

among the particle number current j^μ , momentum current $T^{\mu i}$, boost current $B^{\mu i}$, and angular momentum current $M^{\mu ij}$, valid in Cartesian coordinates in flat spacetime [19]. These identities directly reflect the fact that a local rotation can be compensated by a local translation, and a local Galilei boost can be compensated by a combination of a local translation and a local U(1) transformation [29]. Note that the identity (23) is not a relation between currents, equating momentum density to the U(1) current. It is also a consequence of Galilei symmetry though, and can be understood as a consistency condition associated with the above identity for $B^{\mu i}$ [19].

D. Spin connection and torsion

So far, we have simplified the discussion by considering solely fields without spin. However, the coset construction provides a concise description of the general case. To that end, it is suitable to trade the components of the MC form for their gauge-covariant combinations, given by the field-strength tensor, $\omega_{\mu\nu} \equiv \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + i[\omega_\mu, \omega_\nu]$. After some manipulations using Eqs. (34), (37), and (38), its components can be given the explicit form,

$$\begin{aligned} \omega_{\mu\nu}^{Jab} &= -R_{\mu\nu}^{ab}, \\ \omega_{\mu\nu}^O &= -A_{\mu\nu} - m y_a B_{\mu\nu}^a + m v_a (S_{\mu\nu}^a + R_{\mu\nu}^{ab} y_b + z B_{\mu\nu}^a) \\ &\quad - \frac{1}{2} m v^2 T_{\mu\nu}, \\ \omega_{\mu\nu}^{Na} &= -B_{\mu\nu}^a - R_{\mu\nu}^{ab} v_b, \\ \omega_{\mu\nu}^{Pa} &= -S_{\mu\nu}^a - R_{\mu\nu}^{ab} y_b - z B_{\mu\nu}^a + v^a T_{\mu\nu}, \\ \omega_{\mu\nu}^H &= -T_{\mu\nu}. \end{aligned} \quad (53)$$

In analogy with the relativistic case, we can therefore interpret $\omega_{\mu\nu}^{Jab}$ as the spatial curvature tensor and $\omega_{\mu\nu}^H$ as the temporal torsion tensor (both up to a sign).

The other components of the MC form depend on the dynamical field v^a so we have to be more careful. We define the spatial torsion tensor by stripping off the v^a -term from $\omega_{\mu\nu}^{Pa}$,

$$\mathcal{S}_{\mu\nu}^a \equiv S_{\mu\nu}^a + R_{\mu\nu}^{ab} y_b + z B_{\mu\nu}^a. \quad (54)$$

It is easy to check using Eq. (33) that $\mathcal{S}_{\mu\nu}^a$ is covariant under internal rotations and invariant under internal spacetime translations and U(1) transformations. The only transformation that affects it is the internal Galilei boost, under which $\delta_N \mathcal{S}_{\mu\nu}^a = u^a T_{\mu\nu}$. As could have been expected, $\mathcal{S}_{\mu\nu}^a$ together with $T_{\mu\nu}$ transform as a vector under Galilei boosts.

To further check the consistency of the definition (54), we note that $\mathcal{S}_{\mu\nu}^a$ can be expressed in terms of the MC form rather than in terms of the field-strength tensor $\mathcal{A}_{\mu\nu}$ in a way independent of v^a ,

$$\mathcal{S}_{\mu\nu}^a = -\partial_\mu \bar{e}_\nu^a + \partial_\nu \bar{e}_\mu^a + R_{\mu b}^a \bar{e}_\nu^b - R_{\nu b}^a \bar{e}_\mu^b + n_\mu B_\nu^a - n_\nu B_\mu^a. \quad (55)$$

Barring the appearance of the boost source B_μ^a , the generating functional of the EFT depends on the vielbein n_μ , \bar{e}_μ^a , electromagnetic source \bar{A}_μ , and the spin connection R_μ^{ab} . Owing to the fact that Eq. (55) is algebraic in R_μ^{ab} , the latter can be traded for the spatial torsion $\mathcal{S}_{\mu\nu}^a$ [12]. Since the spatial torsion depends on the choice of reference frame, covariant constraints on the background can be obtained by setting either the temporal torsion, or both the temporal and the spatial torsion to zero. In the latter case, the spin connection can be expressed in terms of derivatives of the vielbein as in Eq. (29).

E. Rotationally invariant systems

The formalism developed above allows a streamlined construction of effective actions invariant under internal U(1) symmetry, spacetime translations, spatial rotations, and Galilei boosts. In real condensed matter systems, boost invariance is often broken at a much higher energy scale than other symmetries. It is therefore illustrative to inspect

how our results modify in this case, where the low-energy EFT is invariant under spatial rotations but not under the boosts.

Relaxing the constraints imposed by boost invariance is, in fact, extremely simple. In all the above formulas we have to discard the external source B_μ^a as well as the coset field v^a . The MC form thereby reduces to

$$\begin{aligned}\omega_\mu^{Jab} &= -R_\mu^{ab}, & \omega_\mu^Q &= -A_\mu, \\ \omega_\mu^{Pa} &= \partial_\mu y^a - \bar{S}_\mu^a, & \omega_\mu^H &= \partial_\mu z - T_\mu,\end{aligned}\quad (56)$$

where $\bar{S}_\mu^a = S_\mu^a + R_\mu^{ab} y_b$. The temporal torsion $T_{\mu\nu}$ and spatial torsion $S_{\mu\nu}^a$ become completely decoupled, depending just on n_μ and on \bar{e}_μ^a and R_μ^{ab} , respectively. Equation (55) becomes identical to a Euclidean version of its relativistic counterpart. If desired, it can be used to eliminate the spin connection in favor of the vielbein as usual. Local invariant Lagrangians are constructed as before by imposing the linearly realized symmetry \mathcal{H} , consisting of spatial rotations and internal U(1) transformations.

VI. SUPERFLUID EFFECTIVE THEORY

In the previous section, we showed how to construct invariant actions for charged matter fields. The setup therefore provides a suitable tool for the discussion of symmetries of microscopic theories of electrons and nuclei, atoms or molecules. Consider now a system which becomes superfluid at sufficiently low temperatures. Provided there are no other gapless modes in the spectrum, we then expect the low-energy physics to be dominated by the ensuing NG boson. In this case, both Galilei boosts and the particle number U(1) are spontaneously broken by the physical ground state, and the framework based on the isotropy subgroup \mathcal{H} generated by J_{ab} becomes appropriate. In this section, we describe the main differences compared to the coset construction carried out in the previous section. The coset construction for a nonrelativistic superfluid in flat spacetime has also been discussed recently in Ref. [30].

On account of having an extra broken generator, the associated NG field, $\pi(x)$, must be added to the coset element (35),

$$U(x) \equiv e^{iy^a(x)P_a} e^{iz(x)H} e^{iv^a(x)N_a} e^{i\pi(x)Q}. \quad (57)$$

Since Q commutes with all other generators, this modification has a limited impact on the results derived in Sec. V. The only change to the MC form is an extra term $\partial_\mu \pi$ in ω_μ^Q , so that now [cf. Eq. (37)]

$$\tilde{\omega}_\mu^Q = \partial_\mu \pi - \bar{A}_\mu - m\mathbf{v} \cdot \mathbf{e}_\mu - \frac{1}{2} m\mathbf{v}^2 n_\mu. \quad (58)$$

The transformation rules of the fields change accordingly. First, under U(1) charge transformations, $\delta_Q \pi = \theta$ so that

$\mathfrak{h} = 1$ and consequently all components of the MC form are trivially invariant under U(1). Second, under small Galilei boosts, one now has $\delta_N \pi = m\mathbf{u} \cdot \mathbf{y}$. Again, $\mathfrak{h} = 1$ and the MC form is completely boost invariant.

The transformation rules have thus become extremely simple. Spacetime translations, U(1) transformations, and Galilei boosts are all realized by shifts of their respective coset fields (plus the corrections $\delta_N \pi$ and $\delta_N y^a$, induced by the boosts), leaving the MC form invariant. Spatial rotations, on the other hand, act linearly on all the fields except for the gauge field R_μ^{ab} , as expected.

The EFT now contains two dynamical NG fields: π and v^a . However, it is well known that there are no physical gapless states in superfluids that could be associated with the spontaneously broken Galilei boosts. The low-energy spectrum only contains one gapless state, to which both the U(1) current and the boost currents couple [31]. In the EFT, this is reflected by the fact that the v^a field is not protected by symmetry from acquiring a mass term [19,27]. It therefore does not contribute to low-energy physics even if it was initially introduced as an independent degree of freedom.

To obtain an EFT for the physical mode π (in the presence of background fields) alone, one eliminates v^a by integrating it out, or by imposing an algebraic inverse Higgs constraint [26]. The latter is more convenient as the elimination is performed on the level of the covariant constituents of the theory, without the need to know the details of its action. While the precise form of the inverse Higgs constraint may sometimes be ambiguous, in this case a convenient choice is $E_a^\mu \tilde{\omega}_\mu^Q = 0$, leading to

$$v_a = \frac{1}{m} E_a^\mu D_\mu \pi, \quad (59)$$

where we set $D_\mu \pi \equiv \partial_\mu \pi - \bar{A}_\mu$. Plugging this back into the expression (47) for the dual vielbein, it becomes

$$E_a^\mu = \bar{E}_a^\mu, \quad V^\mu = \bar{V}^\mu + \frac{1}{m} g^{\mu\nu} D_\nu \pi, \quad (60)$$

where we used the degenerate metric $g^{\mu\nu}$, introduced in Eq. (12). Since the low-energy spectrum contains no gapped matter states on which ω_μ^J could act, the only building blocks at our disposal are

$$\begin{aligned}\omega_\mu^{Na} &= -B_\mu^a + \frac{1}{m} D_\mu (E^{a\nu} D_\nu \pi), \\ V^\mu \tilde{\omega}_\mu^Q &= \bar{V}^\mu D_\mu \pi + \frac{g^{\mu\nu}}{2m} D_\mu \pi D_\nu \pi,\end{aligned}\quad (61)$$

where, in the last term on the right-hand side of the first line, $D_\mu \psi^a \equiv \partial_\mu \psi^a - R_{\mu b}^a \psi^b$.

In a power-counting scheme where $\partial_\mu \pi$ counts as order zero and any additional derivative acting on the fields increases the order [32], ω_μ^N will not contribute at the lowest

order in derivatives. The leading-order action will then be given by [33]

$$S = \int dt dx n_0 \sqrt{\|g\|} P(v^\mu \tilde{\omega}_\mu^Q), \quad (62)$$

where P defines the thermodynamic pressure of the system at zero temperature as a function of the chemical potential for the $U(1)$ symmetry. This generalizes the action found in Ref. [4] to an arbitrary spacetime background, including sources for momentum density and energy density and current.

VII. CONCLUSIONS

In this paper, we have developed a framework that casts a given Lagrangian field theory in a form manifestly invariant under arbitrary coordinate reparametrizations, and at the same time provides a transparent treatment of symmetries of the system. By making systematic use of the vielbein formalism, we naturally reproduced in Secs. II and III the Newton-Cartan structure, recently introduced in the studies of the fractional quantum Hall effect. (See also Ref. [34] for a study of Newton-Cartan geometry with torsion in a different context.) Utilizing the coset construction, we then showed in Secs. IV, V, and VI how the physical symmetries can be implemented separately from coordinate diffeomorphisms. The discussion in Sec. VB suggests that it is more natural to construct Galilei-invariant actions with the additional auxiliary velocity field v^a . Upon integrating this out, the action takes a form that seems difficult to obtain directly in a systematic manner.

Although our approach might seem intimidating due to the amount of notation and algebra involved, it is in fact extremely simple conceptually: once the basic scheme is set up, all the algebraic manipulations are enforced by symmetry, and do not require any further insight or guesswork. To conclude, we make several comments on the potential extensions and applications of our approach in the form of final remarks.

First, we used for illustrative purposes only the simplest type of model: a NR Galilei-invariant system with a $U(1)$ internal symmetry. This $U(1)$ symmetry plays a special role in that it enters the commutation relations of the Galilei algebra (30). As a consequence, the associated gauge field A_μ enters the action in a nontrivial manner. We anticipate that other, possibly non-Abelian, internal symmetries can be added straightforwardly, as suggested in Ref. [7]. This is obvious in the coset formalism, worked out in Sec. V: adding an extra set of generators that commute with the Galilean algebra results in a separate contribution to the MC form which does not interfere with already existing structure.

Second, the formalism can straightforwardly be applied to any local field theory regardless of its symmetries and particle composition. This underlines the fact that there is very little physical content in general coordinate invariance alone, expressing merely the freedom to choose an arbitrary

coordinate system for the description of physical observables. An interesting direction of future work would be to apply the formalism developed here to systems where spacetime symmetries such as rotations or translations actually are spontaneously broken. This would represent a synthesis of the approach of Refs. [35,36], where EFTs for NG modes of spacetime symmetries were studied, with that of Refs. [37,38] which applied the generating functional technique to spontaneously broken internal symmetries.

Third, note that in two spatial dimensions, the Galilei algebra possesses another, exotic central charge [39], possibly related to the two-dimensional spin. While the inclusion of such a central charge by means of an additional $U(1)$ generator in the coset construction seems straightforward, it would be interesting to investigate its physical consequences for, say, two-dimensional superfluids.

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APPENDIX: OVERVIEW OF THE COSET CONSTRUCTION

Suppose that the action of a given theory is invariant under the Lie group \mathcal{G} . It is customary to assume that elementary fields that enter the action span multiplets of this group, that is, the symmetry transformations from \mathcal{G} act on them linearly. However, this is not always the case. For instance, spacetime translations act on the coordinates x^μ by a shift, $x^\mu = x^\mu + \xi^\mu$. Likewise, NG fields of a spontaneously broken symmetry transform by a similar shift under the broken transformations, which in turn guarantees the low-energy theorems for the associated NG bosons [40]. As soon as a symmetry is realized nonlinearly, invariant actions cannot be obtained by usual tensor methods, that is, by taking a product of fields and then

contracting their indices with invariant tensors of the symmetry group. The coset construction [14] solves the problem, and we outline here its essentials needed in the body of the paper.

We assume that the group \mathcal{G} corresponds to an internal symmetry in the sense that it does not affect the spacetime coordinates x^μ , and its action on the fields does not depend on x^μ explicitly. This is reasonable for, as stressed repeatedly throughout the paper, true symmetries can be implemented in a way that does not refer to coordinate reparametrizations, the latter merely representing the freedom to choose a coordinate system. We further need to know the subgroup $\mathcal{H} \subset \mathcal{G}$ which is realized linearly on all the fields.

To find a nonlinear realization of the full group \mathcal{G} , one first defines its action on the coset space \mathcal{G}/\mathcal{H} . This space consists of all (mutually disjoint) sets of the form $\chi_{\mathfrak{g}} \equiv \{\mathfrak{g}\mathfrak{h} | \mathfrak{h} \in \mathcal{H}\}$ with fixed $\mathfrak{g} \in \mathcal{G}$ [41]. For each such coset χ , one can pick a fixed representative, U_χ . In other words, every element $\mathfrak{g} \in \mathcal{G}$ can be uniquely decomposed as

$$\mathfrak{g} = U_{\chi_{\mathfrak{g}}} \mathfrak{h}_{\mathfrak{g}}, \quad (\text{A1})$$

where $\mathfrak{h}_{\mathfrak{g}} \in \mathcal{H}$. The action of the group on the coset space is then defined by left multiplication

$$U_{\chi_{\mathfrak{g}'}} \xrightarrow{\mathfrak{g}} U_{\chi_{\mathfrak{g}\mathfrak{g}'}} = \mathfrak{g} U_{\chi_{\mathfrak{g}}} \mathfrak{h}_{\mathfrak{g}'} \mathfrak{h}_{\mathfrak{g}\mathfrak{g}'}^{-1}. \quad (\text{A2})$$

Denote now the generators of \mathcal{H} temporarily as T_α and the generators of \mathcal{G}/\mathcal{H} as T_a . At least in some neighborhood of unity, one can represent a given coset as

$$U(\pi) \equiv e^{i\pi^a T_a}, \quad (\text{A3})$$

where the objects π^a serve as local coordinates on the coset space \mathcal{G}/\mathcal{H} . The transformation rule (A2) can then be given the more familiar form,

$$U(\pi) \xrightarrow{\mathfrak{g}} U(\pi') = \mathfrak{g} U(\pi) \mathfrak{h}(\pi, \mathfrak{g})^{-1}. \quad (\text{A4})$$

To implement this construction in a quantum field theory, the coset coordinates are interpreted as fields, $\pi^a(x)$. In case of spontaneous symmetry breaking, the subgroup \mathcal{H} corresponds to the symmetry of the physical vacuum and π^a represent the ensuing NG bosons. However, the construction is more general and also applies to nonlinear realizations of symmetries such as translations which are not necessarily spontaneously broken. The geometric fields π^a then rather play the role of arbitrary but fixed functions that specify the local coordinate frame.

Once the coset structure is made local, it is natural to promote the action of the symmetry group \mathcal{G} to a local one as well, that is, to make \mathfrak{g} coordinate dependent. To achieve manifest covariance under such gauge symmetry,

a set of background gauge fields is needed, which can be put together in the one-form matrix-valued variable

$$\mathcal{A}_\mu = \mathcal{A}_\mu^\alpha T_\alpha + \mathcal{A}_\mu^a T_a. \quad (\text{A5})$$

The group \mathcal{G} is assumed to act upon it as usual in non-Abelian gauge theory,

$$\mathcal{A}_\mu \xrightarrow{\mathfrak{g}} \mathfrak{g} \mathcal{A}_\mu \mathfrak{g}^{-1} + i\mathfrak{g} \partial_\mu \mathfrak{g}^{-1}. \quad (\text{A6})$$

The basic building block of the coset construction is the MC form, whose gauge-covariant version reads

$$\omega_\mu \equiv -iU^{-1}(\partial_\mu - i\mathcal{A}_\mu)U. \quad (\text{A7})$$

It decomposes as

$$\omega_\mu = \omega_{\parallel\mu} + \omega_{\perp\mu} \equiv \omega_{\parallel\mu}^\alpha T_\alpha + \omega_{\perp\mu}^a T_a. \quad (\text{A8})$$

Using Eqs. (A4) and (A6), and assuming that the generators T_a span a representation of \mathcal{H} [42], it is easy to verify the transformation rules of the MC form,

$$\omega_{\parallel\mu} \xrightarrow{\mathfrak{g}} \mathfrak{h} \omega_{\parallel\mu} \mathfrak{h}^{-1} - i\mathfrak{h} \partial_\mu \mathfrak{h}^{-1}, \quad \omega_{\perp\mu} \xrightarrow{\mathfrak{g}} \mathfrak{h} \omega_{\perp\mu} \mathfrak{h}^{-1}, \quad (\text{A9})$$

where \mathfrak{h} is defined by Eq. (A4).

So far we have only discussed the fields π^a whose presence is enforced by the geometry of the coset space. Nongeometric, or matter, fields can be added to the construction at will though. We will denote such fields collectively as $\psi(x)$. By assumption, they transform under \mathcal{H} in some linear (not necessarily irreducible) representation \mathcal{R} : $\psi \xrightarrow{\mathfrak{h}} \mathcal{R}(\mathfrak{h})\psi$. This prescription can be promoted to a nonlinear realization of the whole group \mathcal{G} by using the matrix \mathfrak{h} from Eq. (A4): $\psi \xrightarrow{\mathfrak{g}} \mathcal{R}(\mathfrak{h}(\pi, \mathfrak{g}))\psi$. Upon such a symmetry transformation, ψ acquires additional coordinate dependence through the π -dependence of \mathfrak{h} even if \mathfrak{g} itself is global. A covariant derivative of ψ is constructed with the help of $\omega_{\parallel\mu}$, which transforms as a gauge connection of \mathcal{H} ,

$$D_\mu \psi \equiv [\partial_\mu + i\mathcal{R}(\omega_{\parallel\mu})]\psi. \quad (\text{A10})$$

Invariant Lagrangians can be assembled using standard tensor methods out of ψ , $\omega_{\perp\mu}$ and their covariant derivatives. In fact, even Lagrangians invariant only up to a surface term, leading to topological actions of the Wess-Zumino type [36,37,43,44], can be obtained from the building blocks, provided by the MC form. Note that $\omega_{\perp\mu}$ can play a dual role: (i) as a covariant derivative of π^a in case these represent dynamical NG fields of a spontaneously broken symmetry; and (ii) as a covariant vielbein in case the functions $\pi^a(x)$ represent local coordinates that are not dynamical and can be chosen at will.

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