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# Topological order and Berry connection for the Maxwell vacuum on a four-torus

Ariel R. Zhitnitsky
*Department of Physics & Astronomy, University of British Columbia, Vancouver, British Columbia V6T 1Z1, Canada*
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We study novel types of contributions to the partition function of the Maxwell system defined on a small compact manifold, such as a torus. These new terms cannot be described in terms of the physical propagating photons with two transverse polarizations. Rather, these novel contributions emerge as a result of tunneling events when transitions occur between topologically different but physically identical vacuum winding states. These new terms give an extra contribution to the Casimir pressure. The infrared physics in the system can be described in terms of the topological auxiliary nonpropagating fields $a_i(\mathbf{k})$ governed by Chern-Simons-like action. The system can be studied in terms of these auxiliary fields precisely in the same way as a topological insulator can be analyzed in terms of Berry's connection $\mathcal{A}_i(\mathbf{k})$. We also argue that the Maxwell vacuum defined on a small four-torus behaves very much in the same way as a topological insulator with $\theta \neq 0$.

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## I. INTRODUCTION: MOTIVATION

The main motivation for present studies is as follows. It has been recently argued [1,2] that if the free Maxwell theory (without any interactions with charged particles) is defined on a small compact manifold, then some novel terms in the partition function will emerge. These terms are not related to the propagating photons with two transverse physical polarizations, which are responsible for the conventional Casimir effect. Rather, these novel terms occur as a result of tunneling events between topologically different but physically identical states. These states play no role when the system is defined in Minkowski space-time $\mathbb{R}_{1,3}$. But these states become important when the system is defined on a finite compact manifold such as torus $\mathbb{T}^4$.

In particular, it has been explicitly shown in [1,2] that these novel terms lead to fundamentally new contributions to the Casimir vacuum pressure, which can not be expressed in terms of conventional propagating physical degrees of freedom. Instead, the new vacuum contributions appear as a result of tunneling events between different topological sectors $|k\rangle$. Mathematically, these sectors emerge as a result of nontriviality of the fundamental group $\pi_1[U(1)] \cong \mathbb{Z}$ when the system is defined on a torus.

A crucial observation for the present studies is as follows. While the Maxwell electrodynamics is the theory of massless particles (photons), the topological portion of the system decouples from dynamics of these massless propagating photons. Indeed, as we discuss below, the total partition function $\mathcal{Z}$ can be represented as a product $\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_{\rm top}$. The conventional partition function $\mathcal{Z}_0$ describing physical photons is not sensitive to the topological sectors $|k\rangle$ of the system which itself is described by $\mathcal{Z}_{\rm top}$. The topological portion of the partition function $\mathcal{Z}_{\rm top}$ behaves very much as topological quantum field theory (TQFT), as argued in [2]. Furthermore, it demonstrates many features of topologically ordered systems, which were initially introduced in the context of condensed matter (CM) systems; see original papers [3–7] and recent reviews [8–12].

In particular, $\mathcal{Z}_{\rm top}$ demonstrates the degeneracy of the system which cannot be described in terms of any local operators. Instead, such a degeneracy can be formulated in terms of some nonlocal operators [2]. Furthermore, our system exhibits some universal subleading corrections to the thermodynamical entropy which cannot be expressed in terms of propagating photons with two physical polarizations. Instead, the corresponding universal contribution to the entropy is expressed in terms of the "instantons" describing the tunneling events between topologically different but physically identical topological sectors $|k\rangle$.

As a result of these similarities, the key question addressed in the present work is as follows. It has been known for some time [4–12] that some key features of topologically ordered systems can be formulated in terms of the so-called Berry's connection in momentum space. Does a similar description exist for the Maxwell vacuum defined on a compact manifold?

To address this question we formulate the topological features of the system in terms of auxiliary fields. Such a formulation exhibits a close mathematical similarity between the auxiliary topological field describing the Maxwell vacuum state and the Berry's connection (which is emergent, not a fundamental field) in topologically ordered CM systems. Such a similarity looks very instructive and suggestive, and further supports our arguments [2] that the ground state of the Maxwell theory defined on a small compact manifold behaves as a TQFT.

 

The structure of our presentation is as follows. In the next section, we review the relevant parts of the two-dimensional Maxwell "empty" theory which does not have any physical propagating degrees of freedom. Still, it demonstrates a number of very nontrivial topological features present in the system. In Sec. III we generalize our description for 4d Maxwell theory defined on a four-torus. In our main section, Sec. IV, we introduce the auxiliary fields which effectively account for the topological sectors of the system. We study the behavior of these auxiliary fields in the far infrared (IR) at small $k \to 0$ in momentum space. We observe a striking similarity of the obtained structure with the analogous formula for the Berry's connection previously derived for many CM topologically ordered systems. This analogy further supports our claim that the ground state of the Maxwell theory belongs to a topologically ordered phase. In our concluding section, Sec. V, we briefly mention possible settings where such unusual topological vacuum features can be experimentally studied. Furthermore, we in fact shall argue that a topological insulator and the topological Maxwell vacuum (studied in this work), while very different in composition, nevertheless behave very much in the same way at large distances.

## II. MAXWELL THEORY IN TWO DIMENSIONS AS TOPOLOGICAL QFT

The 2d Maxwell model has been solved numerous times using very different techniques; see, e.g., [13–15]. It is known that this is an "empty" theory in the sense that it does not support any propagating degrees of freedom in the bulk of space-time. It is also known that this model can be treated as a conventional topological quantum field theory (TQFT). In particular, this model can be formulated in terms of the so-called "BF" action involving no metric. Furthermore, this model exhibits many other features such as fractional edge observables which are typical for TQFT; see, e.g., [14]. We emphasize these properties of the 2d Maxwell theory because the topological portion of the partition function $\mathcal{Z}_{\rm top}$ in our description of the 4d Maxwell system, given in Sec. III, is identically the same as the partition function of the 2d Maxwell system. Such a relation between the two different systems is a result of decoupling of physical propagating photons from the topological sectors in the 4d system.

Our goal here is to review this "empty" 2d Maxwell theory with nontrivial dynamics of the topological sectors when conventional propagating degrees of freedom are not supported by this system.

### A. Partition function and $\theta$ vacua in 2d Maxwell theory

We consider 2d Maxwell theory defined on the Euclidean torus $S^1 \times S^1$ with lengths $L$ and $\beta$ respectively. In the Hamiltonian framework we choose a $A_0 = 0$ gauge along with $\partial_1 A_1 = 0$. This implies that $A_1(t)$ is the only dynamical variable of the system with $E = \dot{A}_1$. The spectrum for $\theta$ vacua is well known [13] and it is given by $E_n(\theta) = \frac{1}{2}(n + \frac{\theta}{2\pi})^2 e^2 L$, such that the corresponding partition function takes the form

$$\mathcal{Z}(V,\theta) = \sum_{n\in\mathbb{Z}} e^{-\frac{e^2 V}{2}(n+\frac{\theta}{2\pi})^2}, \tag{1}$$

where $V = \beta L$ is the two-volume of the system.

We want to reproduce (1) using a different approach based on Euclidean path integral computations because it can be easily generalized to similar computations in 4d Maxwell theory defined on a four-torus. Our goal here is to understand the physical meaning of (1) in terms of the path integral computations.

To proceed with path integral computations one considers the "instanton" configurations on a two-dimensional Euclidean torus with total area $V = L\beta$ described as follows [15]:

$$\int \mathrm{d}^2x Q(x) = k, \qquad eE^{(k)} = \frac{2\pi k}{V}, \tag{2}$$

where $Q = \frac{e}{2\pi} E$ is the topological charge density, $k$ is the integer-valued topological charge in the 2d $U(1)$ gauge theory, and $E(x) = \partial_0 A_1 - \partial_1 A_0$ is the field strength. The action of this classical configuration is

$$\frac{1}{2}\int d^2x E^2 = \frac{2\pi^2 k^2}{e^2 V}. \tag{3}$$

This configuration corresponds to the topological charge $k$ as defined by (2). The next step is to compute the partition function defined as follows:

$$\mathcal{Z}(\theta) = \sum_{k\in\mathbb{Z}} \int \mathcal{D}A^{(k)} e^{-\frac{1}{2}\int d^2x E^2 + \int d^2x L_\theta}, \tag{4}$$

where $\theta$ is the standard theta parameter which defines the $|\theta\rangle$ ground state and which enters the action with topological density operator

$$L_\theta = i\theta \int \mathrm{d}^2x Q(x) = i\theta \frac{e}{2\pi} \int \mathrm{d}^2x E(x). \tag{5}$$

All integrals in this partition function are Gaussian and can be easily evaluated using the technique developed in [15]. The result is

$$\mathcal{Z}(V,\theta) = \sqrt{\frac{2\pi}{e^2 V}} \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2 k^2}{e^2 V} + ik\theta}, \tag{6}$$

where the expression in the exponent represents the classical instanton configurations with action (3) and topological charge (2), while the factor in front is due to the fluctuations; see [1,2] with some technical details and relevant references. While expressions (1) and (6) look different, they are actually identically the same, as the Poisson summation formula states:

$$\mathcal{Z}(\theta) = \sum_{n\in\mathbb{Z}} e^{-\frac{e^2V}{2}(n+\frac{\theta}{2\pi})^2} = \sqrt{\frac{2\pi}{e^2V}} \sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2k^2}{e^2V}+ik\theta}. \quad (7)$$

Therefore, we reproduce the original expression (1) using the path integral approach.

The crucial observation for our present study is that this naively "empty" theory which has no physical propagating degrees of freedom, nevertheless shows some very nontrivial features of the ground state related to the topological properties of the theory. These new properties are formulated in terms of different topological vacuum sectors of the system $|k\rangle$ which have identical physical properties, as they are connected to each other by large gauge transformation operator $\mathcal{T}$ commuting with the Hamiltonian $[\mathcal{T}, H] = 0$. As explained in detail in [1,2] the corresponding dynamics of this "empty" theory represented by partition function (7) should be interpreted as a result of tunneling events between these "degenerate" winding $|k\rangle$ states which correspond to one and the same physical state.

It is known that this model can be treated as TQFT, e.g., it supports edge observables which may assume the fractional values, and shows many other features which are typical for a TQFT; see [14] and references therein. The presence of the topological features of the model can be easily understood from the observation that the entire dynamics of the system is due to the transitions between the topological sectors which themselves are determined by the behavior of surface integrals at infinity $\oint A_\mu dx^\mu$. These sectors are classified by integer numbers and they are not sensitive to specific details of the system, such as the geometrical shape of the system. Therefore, it is not really a surprise that the system is not sensitive to specific geometrical details and can be treated as TQFT. The simplest way to analyze the corresponding topological features of the system is to introduce the topological susceptibility $\chi$ and study its properties; see the next subsection.

### B. Topological susceptibility

The topological susceptibility $\chi$ is defined as follows:

$$\chi \equiv \lim_{k\to 0} \int d^2x\, e^{ikx} \langle TQ(x)Q(0)\rangle, \quad (8)$$

where $Q$ is the topological charge density operator normalized according to Eq. (2). The $\chi$ measures response of the free energy to the introduction of a source term defined by Eq. (5). The computations of $\chi$ in this simple "empty" model can be easily carried out, as the partition function $\mathcal{Z}(\theta)$ defined by (4) is known exactly (7). To compute $\chi$ we should simply differentiate the partition function twice with respect to $\theta$. It leads to the following well-known expression for $\chi$, which is finite in the infinite volume limit [2,15,16]:

$$\chi(V\to\infty) = -\frac{1}{V}\cdot\frac{\partial^2 \ln \mathcal{Z}(\theta)}{\partial\theta^2}\Big|_{\theta=0} = \frac{e^2}{4\pi^2}. \quad (9)$$

A typical value of the topological charge $k$ which saturates the topological susceptibility $\chi$ in the large volume limit is very large, $k \sim \sqrt{e^2V} \gg 1$.

It is important to emphasize that the integrand for the topological susceptibility (8) demonstrates a singular behavior (see [2,15,16] for the details and related references):

$$\langle Q(x)Q(0)\rangle = \frac{e^2}{4\pi^2}\delta^2(x). \quad (10)$$

It represents the nondispersive contact term which cannot be related to any propagating degrees of freedom. In this simplest case of the 2d Maxwell system this comment is quite obvious as 2d Maxwell theory does not support any propagating degrees of freedom. The $\delta^2(x)$ function in (10) should be understood as a total divergence related to the infrared (IR) physics, rather than to ultraviolet (UV) behavior. Indeed,

$$\chi = \frac{e^2}{4\pi^2}\int \delta^2(x) d^2x = \frac{e^2}{4\pi^2}\int d^2x \partial_\mu\left(\frac{x^\mu}{2\pi x^2}\right)$$
$$= \frac{e^2}{4\pi^2}\oint_{S_1\to\infty} dl_\mu \epsilon^{\mu\nu}\left(\frac{x_\nu}{2\pi x^2}\right) = \frac{e^2}{4\pi^2}. \quad (11)$$

In other words, the nondispersive contact term (10) is determined by IR physics at arbitrary large distances rather than UV physics, which can be erroneously assumed to be a source of $\delta^2(x)$ behavior in (10). The computations of this contact term in terms of the delocalized instantons (2) explicitly show that all observables in this system are originated from the IR physics.

One should also remark that the same contact term (9) and its local expression (10) can be also computed using the auxiliary ghost field, the so-called Kogut-Susskind (KS) ghost, as it was originally done in Ref. [17]; see also [2,16] for relevant discussions in the present context. This description in terms of the KS ghost implicitly takes into account the presence of topological sectors in the system. The same property is explicitly reflected by summation over topological sectors $k \in \mathbb{Z}$ in direct computations (4), (6) without introducing any auxiliary fields.

## III. TOPOLOGICAL PARTITION FUNCTION IN 4D

Our goal here is to analyze the Maxwell system on a Euclidean four-torus with sizes $L_1 \times L_2 \times L_3 \times \beta$ in the respective directions. It provides the infrared (IR) regularization of the system. This IR regularization plays a key role in the proper treatment of the topological terms which are related to tunneling events between topologically distinct but physically identical states. First, we want to review the previously known results on the vacuum structure of this system. As the second step, we want to reproduce these known results on the Maxwell vacuum state using a different technique based on the auxiliary fields to be developed in the next section. As we argue in Sec. IV C, precisely these auxiliary topological fields have exactly the same mathematical properties as the emergent Berry's connection in topologically ordered CM systems.

### A. Construction

We follow [1,2] in our construction of the partition function $\mathcal{Z}_{\rm top}$ where it was employed for computation of the corrections to the Casimir effect due to these novel types of topological fluctuations. The crucial point is that we impose the periodic boundary conditions on the gauge $A^\mu$ field up to a large gauge transformation. In what follows we simplify our analysis by considering a clear case with winding topological sectors $|k\rangle$ in the $z$ direction only. The classical configuration in Euclidean space which describes the corresponding tunneling transitions can be represented as follows:

$$\vec{B}_{\rm top} = \vec{\nabla}\times\vec{A}_{\rm top} = \left(0,0,\frac{2\pi k}{eL_1L_2}\right),$$
$$\Phi = e\int dx_1 dx_2 B^z_{\rm top} = 2\pi k \tag{12}$$

in close analogy with the 2d case (2).

The Euclidean action of the system is quadratic and has the following form,

$$\frac{1}{2}\int d^4x\{\vec{E}^2 + (\vec{B}+\vec{B}_{\rm top})^2\}, \tag{13}$$

where $\vec{E}$ and $\vec{B}$ are the dynamical quantum fluctuations of the gauge field. The key point is that the classical topological portion of the action decouples from quantum fluctuations, such that the quantum fluctuations do not depend on topological sector $k$ and can be computed in the topologically trivial sector $k=0$. Indeed, the cross term

$$\int d^4x\vec{B}\cdot\vec{B}_{\rm top} = \frac{2\pi k}{eL_1L_2}\int d^4x B_z = 0 \tag{14}$$

vanishes because the magnetic portion of quantum fluctuations in the $z$ direction, represented by $B_z = \partial_x A_y - \partial_y A_x$, is a periodic function, as $\vec{A}$ is periodic over the domain of integration. This technical remark in fact greatly simplifies our analysis as the contribution of the physical propagating photons is not sensitive to the topological sectors $k$. This is, of course, a specific feature of quadratic action (13), in contrast with non-Abelian and nonlinear gauge field theories where quantum fluctuations of course depend on topological $k$ sectors. The authors of Ref. [18] arrived at the same conclusion (on decoupling of the topological terms from conventional fluctuating photons with nonzero momentum), though in a different context of topological insulators in the presence of the $\theta=\pi$ term.

The classical action for configuration (12) takes the form

$$\frac{1}{2}\int d^4x\vec{B}^2_{\rm top} = \frac{2\pi^2k^2\beta L_3}{e^2L_1L_2}. \tag{15}$$

To simplify our analysis further in computing $\mathcal{Z}_{\rm top}$ we consider a geometry where $L_1, L_2 \gg L_3, \beta$, similar to the construction relevant for the Casimir effect [1,2]. In this case our system is closely related to 2d Maxwell theory by dimensional reduction: taking a slice of the 4d system in the $xy$ plane will yield precisely the topological features of the 2d torus considered in Sec. II. Furthermore, with this geometry our simplification (12) when we consider exclusively the magnetic fluxes in the $z$ direction is justified, as the corresponding classical action (15) assumes minimal possible values. With this assumption we can consider a very small temperature, but still we cannot take a formal limit $\beta\to\infty$ in our final expressions as a result of our technical constraints in the system.

With these additional simplifications the topological partition function becomes [1,2]

$$\mathcal{Z}_{\rm top} = \sqrt{\frac{2\pi\beta L_3}{e^2L_1L_2}}\sum_{k\in\mathbb{Z}} e^{-\frac{2\pi^2k^2\beta L_3}{e^2L_1L_2}} = \sqrt{\pi\tau}\sum_{k\in\mathbb{Z}} e^{-\pi^2\tau k^2}, \tag{16}$$

where we introduced the dimensionless parameter

$$\tau \equiv 2\beta L_3/e^2L_1L_2. \tag{17}$$

Formula (16) is essentially the dimensionally reduced expression for the topological partition function (6) for the 2d Maxwell theory analyzed in Sec. II. One should note that the normalization factor $\sqrt{\pi\tau}$ which appears in Eq. (16) does not depend on topological sector $k$, and essentially it represents our convention of the normalization $\mathcal{Z}_{\rm top}\to 1$ in the limit $L_1L_2\to\infty$ which corresponds to a convenient setup for the Casimir-type experiments, as discussed in [1,2].

### B. External magnetic field

In this section we want to generalize our results for the Euclidean Maxwell system in the presence of the external magnetic field. Normally, in the conventional quantization of electromagnetic fields in infinite Minkowski space, there is no *direct* coupling between fluctuating vacuum photons and an external magnetic field as a consequence of linearity of the Maxwell system. The coupling with fermions generates a negligible effect $\sim\alpha^2 B_{\rm ext}^2/m_e^4$ as the nonlinear Euler-Heisenberg effective Lagrangian suggests; see [1] for the details and numerical estimates. The interaction of the external magnetic field with topological fluctuations (12), in contrast with the coupling with conventional photons, will lead to the effects of order of unity as a result of interference of the external magnetic field with topological fluxes $k$.

The corresponding partition function can be easily constructed for external magnetic field $B_z^{\rm ext}$ pointing along the $z$ direction, as the crucial technical element on decoupling of the background fields from quantum fluctuations assumes the same form (14). In other words, the physical propagating photons with nonvanishing momenta are not sensitive to the topological $k$ sectors, nor to the external uniform magnetic field, similar to our discussions after (14).

The classical action for configuration in the presence of the uniform external magnetic field $B_z^{\rm ext}$ therefore takes the form

$$\frac{1}{2}\int \mathrm{d}^4x(\vec{B}_{\rm ext}+\vec{B}_{\rm top})^2 = \pi^2\tau\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)^2, \qquad (18)$$

where $\tau$ is defined by (17) and the effective theta parameter $\theta_{\rm eff}\equiv eL_1L_2B_{\rm ext}^z$ is expressed in terms of the original external magnetic field $B_{\rm ext}^z$. Therefore, the partition function in the presence of the uniform magnetic field can be easily reconstructed from (16), and it is given by [1,2]

$$\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff}) = \sqrt{\pi\tau}\sum_{k\in\mathbb{Z}}\exp\left[-\pi^2\tau\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)^2\right]. \qquad (19)$$

This system in what follows will be referred to as the topological vacuum ($\mathcal{TV}$) because the propagating degrees of freedom, the photons with two transverse polarizations, completely decouple from $\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})$.

The dual representation for the partition function is obtained by applying the Poisson summation formula (7) such that (19) becomes

$$\mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff}) = \sum_{n\in\mathbb{Z}}\exp\left[-\frac{n^2}{\tau}+in\cdot\theta_{\rm eff}\right]. \qquad (20)$$

Formula (20) justifies our notation for the effective theta parameter $\theta_{\rm eff}$ as it enters the partition function in combination with integer number $n$. One should emphasize that integer number $n$ in the dual representation (20) is not the integer magnetic flux $k$ defined by Eq. (12) which enters the original partition function (16). Furthermore, the $\theta_{\rm eff}$ parameter which enters (19), (20) is not a fundamental $\theta$ parameter which is normally introduced into the Lagrangian in front of the $\vec{E}\cdot\vec{B}$ operator. Rather, this parameter $\theta_{\rm eff}$ should be understood as an effective parameter representing the construction of the $\theta_{\rm eff}$ state for each slice in a four-dimensional system. In fact, there are three such $\theta_{\rm eff}$ parameters representing different slices and corresponding external magnetic fluxes. There are similar three $\theta_i$ parameters representing the external electric fluxes as discussed in [2], such that the total number of $\theta$ parameters classifying the system equals 6, in agreement with the total number of hyperplanes in four dimensions.

## IV. BERRY CONNECTION

The main goal of this section is to argue that our $\mathcal{TV}$ configuration represents the simplest version of a topologically ordered phase very similar to CM systems [4–12]. We want to reformulate the topological features of the system (analyzed in Sec. III) in terms of the Berry's connection and Berry curvature normally computed in momentum space in CM literature. Such a deep relation between the two very different descriptions will demonstrate once again that the ground state for the Maxwell theory defined on a compact manifold exhibits all the features which are normally attributed to a topologically ordered system. We make this relation much more precise by introducing the auxiliary topological fields which can be identified with Berry's connection. With such an interpretation the complex phase in the dual representation (20) can be thought of as the Berry's phase which is known to emerge in many quantum systems.

We start our study in Sec. IV A by reviewing the well-known CM results on the Berry's connection. In Sec. IV B we describe the ground state of the two-dimensional Maxwell theory by using the auxiliary topological fields. We observe a deep mathematical similarity between the Berry's connection computed for CM systems (including the monopole-type behavior in momentum **k** space) and the corresponding formulas computed for the ground state in the Maxwell theory in terms of the auxiliary topological fields. We generalize the corresponding construction to the four-dimensional Maxwell system defined on a four-torus in Sec. IV C.

### A. Berry phase in CM systems

In this subsection we review the computations of the Berry connection in some CM systems. In the context of the topological insulators and quantum Hall systems, the corresponding studies have been carried out in two, three, and four dimensions [4–12]; see also [2,19] with related discussions of the ground state in 2d Maxwell theory.[1]

[1]Not to be confused with conventional CM notations, where it is customary to count the spatial number of dimensions, rather than total number of dimensions. For our 2d system this convention corresponds to (D + 1) Maxwell theory with D = 1. Similar studies have been carried out for topological insulators for D = 1 and D = 3; see, e.g., [18] with many references on the original results. For D = 2 the corresponding computations of the Berry's connection for the quantum Hall systems have been reviewed in [11].

In the simplest D = 1 case the expression for the Berry's phase (which is the accumulated geometric phase of the band electrons under the process when the winding of the gauge field is increased by one unit) can be computed as follows [18]. In the physical $A_0 = 0$ gauge it corresponds to a slow variation of gauge filed $eA_1$ from $\frac{2\pi n}{L}$ to $\frac{2\pi(n+1)}{L}$, where $L$ is the size of a torus along the $x$ direction. The relevant formula is given by [18]

$$\phi_{\rm Berry} = i\int dA_1 \langle\Psi_\theta|\frac{\partial}{\partial A_1}|\Psi_\theta\rangle, \qquad (21)$$

where $|\Psi_\theta\rangle$ is the full wave function of the system which can be expressed in terms of single particle wave functions. One can explicitly demonstrate [18] that $\phi_{\rm Berry} = -2\pi P$ with $P$ being the polarization of the system such that $\theta$ is shifted as follows: $\theta \to (\theta - 2\pi P)$. The key observation in this computation is that the integration over slow-varying gauge fields in Eq. (21) is reduced to integration over allowed momentum $k$ covering the whole Brillouin zone (BZ), i.e.,

$$\phi_{\rm Berry} = i\int_{\rm BZ} dk\langle\Psi_\theta|\frac{\partial}{\partial k}|\Psi_\theta\rangle \equiv \int_{\rm BZ} dk\mathcal{A}(k), \quad (22)$$

where $\mathcal{A}(k)$ is the so-called Berry's connection in the momentum space. A simple technical explanation of this key technical step (related to the change of variables) is that the large gauge transformation formulated in terms of $A_1$ can be expressed in terms of a shift of the momentum $k$ when the system returns to the physically identical (but topologically different) state.

Similar computations can be also carried out for the integer quantum Hall system for D = 2, in which case the corresponding formula for the Berry's connection and Berry's curvature takes the form (see, e.g., [11])

$$\mathcal{A}^j(\mathbf{k}) = \frac{\tau}{2}\frac{\epsilon^{ij}k_i}{\mathbf{k}^2}, \qquad \mathcal{B}(\mathbf{k}) = \frac{\tau}{2}\delta^2(\mathbf{k}), \qquad (23)$$

where $\tau = \pm 1$ describes the degenerate Fermi points with the linear dispersion relation $\epsilon(\mathbf{k}) \sim |\mathbf{k}|$. One can identify the behavior (23) with magnetic monopoles in momentum space with half-integer magnetic charges. As we shall see below in Sec. IV B, a very similar structure also emerges in the description of the ground state of the 2d Maxwell system, when the auxiliary topological fields play the role of the Berry's connection (23).

One should emphasize that in CM literature the corresponding $\mathcal{A}^j(\mathbf{k})$ fields are the emergent gauge fields. The real source for these emergent gauge configurations is the strongly coupled coherent superposition of the physical electrons. In contrast, in our case a formula to be derived below [and which is mathematically identical to Eq. (23)] will arise from the topologically nontrivial gauge configurations of the underlying fundamental gauge theory. In other words, in our case the formula similar to (23) will emerge as a result of the topologically nontrivial vacuum gauge configurations which are present in the system irrespective of the existence of the fermions.

In the following subsection, Sec. IV B, we reformulate the known results about the ground $\theta$ state in the 2d Maxwell system using the topological auxiliary (nonpropagating) fields. The corresponding technique, as we shall see below in Sec. IV C, can be easily generalized to the four-dimensional Maxwell system, which is the main subject of the present work.

### B. Auxiliary topological fields in 2d Maxwell theory

We wish to derive the topological action for the Maxwell system in 2d by using a standard conventional technique exploited, e.g., in [7] for the Higgs model in CM context or in [20] for the so-called weakly coupled "deformed QCD." We shall reproduce below the well-known results for this "empty" 2d system including a nonvanishing expression for the topological susceptibility (9), (10) using the corresponding auxiliary fields in momentum space. It turns out that the corresponding connection and curvature computed using these auxiliary fields play the same role as the Berry's connection and Berry's curvature play in CM systems. To be more precise, the unique topological features of the auxiliary field are precisely the key element which allows us to represent the accumulated geometric phase in terms of the auxiliary field sensitive to the boundary conditions. An explicit demonstration of such a relation between the Berry's phase and auxiliary topological fields is precisely the main subject of this section.

Our starting point is to insert the delta function into the path integral with the field $b(\mathbf{x})$ acting as a Lagrange multiplier,

$$\delta\left[Q(\mathbf{x}) - \frac{e}{2\pi}\epsilon^{jk}\partial_j a_k(\mathbf{x})\right] \sim \int \mathcal{D}[b] e^{i\int d^2\mathbf{x}\, b(\mathbf{x})\cdot[Q(\mathbf{x}) - \frac{e}{2\pi}\epsilon^{jk}\partial_j a_k(\mathbf{x})]}, \qquad (24)$$

where $Q(\mathbf{x}) = \frac{e}{2\pi}E(\mathbf{x})$ in this formula is the topological charge density operator. It will be treated as the original expression for the field operator entering the action (4) with topological term (5). At the same time $a_k(\mathbf{x})$ is treated as a slow-varying external source effectively describing the large distance physics for a given instanton configuration. The insertion (24) of the delta function assumes that the path integral computations must include all the classical $k$-instanton configurations (2), (3), along with quantum fluctuations surrounding them. In other words, we treat $Q(\mathbf{x})$ as a fast degree of freedom, while $a_k(\mathbf{x})$ are considered as slow degrees of freedom representing an external background field.

One should remark here that the corresponding formal manipulation is not a mathematically rigorous procedure, as $a_k(\mathbf{x})$ must be singular somewhere to support nonvanishing topological charges in the system.[2] The presence of such singularity is very similar to emergent singularities in the description of the Berry's connection, Dirac's string, or the Aharonov-Bohm potential. It is not a goal of the present work to search for more rigorous mathematical tools for corresponding problems. The most important argument for us that our procedure represented by Eq. (24) is correct is the fact that the topological susceptibility (29), (30) and the expectation value of the electric field (31), (32) are precisely reproduced when computations are performed with our formal approach utilizing the auxiliary topological fields.

Another point worth mentioning is as follows. As we stated above, the auxiliary field $a_k(\mathbf{x})$ is treated as a slow field, while $Q(\mathbf{x})$ is treated as a fast degree of freedom. At the same time, formally, these fields are proportional to each other, $Q(\mathbf{x}) \sim \epsilon^{jk}\partial_j a_k(\mathbf{x})$, according to (24), and therefore, it is not obvious how these fields could be treated so differently. The answer lies in the observation that our auxiliary fields $a_k(\mathbf{x}), b_z(\mathbf{x})$ are nondynamical fields, have no kinetic terms, and do not propagate, in contrast with conventional gauge fields. Formally, these fields do not have their conjugate momenta, as they are auxiliary nondynamical fields of the system.

The simplest way to understand this construction is through analogy with a well-known and well-understood model in particle physics, the so-called Nambu–Jona-Lasinio model. In this case an auxiliary $\sigma$ field without kinetic term is introduced into the system, analogous to (24). The $\sigma \sim \langle\bar{\psi}\psi\rangle$ field is treated as a slow field and in mean field approximation represents the chiral condensate of the Fermi fields. Our auxiliary fields $a_k(\mathbf{x}), b_z(\mathbf{x})$ should be understood exactly in the same way as the $\sigma$ field is understood in the Nambu–Jona-Lasinio model.

Now we are coming back to our proposed formula (24). Our task now is to integrate out the original fast "instantons" and describe the large distance physics in terms of slow-varying fields $b(\mathbf{x}), a_k(\mathbf{x})$ in the form of the effective action $S_{\rm top}[b, a_k]$ formulated in terms of slow auxiliary fields $b(\mathbf{x}), a_k(\mathbf{x})$. We use the conventional well-established procedure of summation over $k$-instantons reviewed in Sec. II with final result (6). The only new element in comparison with the previous computations is that the fast degrees of freedom must be integrated out in the presence of the new slow-varying background fields $b(\mathbf{x}), a_k(\mathbf{x})$ which appear in Eq. (24). Fortunately, the computations can be easily performed for such external sources. Indeed, one should notice that the background field $b(\mathbf{x})$ enters Eq. (24) exactly in the same manner as external parameter $\theta$ enters (5). Therefore, assuming that $b(\mathbf{x}), a_k(\mathbf{x})$ are slow-varying background fields we arrive at the following expression for the partition function:

$$\mathcal{Z}_{\rm top} = \int \mathcal{D}[b]\mathcal{D}[a] e^{-\frac{e^2}{8\pi^2}\cdot\int d^2\mathbf{x}[\theta + b(\mathbf{x})]^2 - S_{\rm top}}, \qquad (25)$$

where $b(\mathbf{x})$ represents the slow-varying background auxiliary field which is assumed to lie in the lowest $n = 0$ branch, $|b(\mathbf{x})| < \pi$. Correspondingly, in formula (25) we kept only the asymptotically leading term in expansion (1) with $n = 0$ in the large volume limit, $(e^2V) \gg 1$. The topological term $S_{\rm top}[b, a_k]$ in Eq. (25) reads

$$S_{\rm top}[b, a_k] = i\frac{e}{2\pi}\int d^2\mathbf{x}[b(\mathbf{x})\epsilon^{jk}\partial_j a_k(\mathbf{x})]. \qquad (26)$$

Our goal now is to consider the simplest application of the effective low energy topological action (25), (26) we just derived. We want to reproduce the known expression for the topological susceptibility (9), (10) by integrating out the $b$ and $a_k$ fields using low energy effective description (25), (26), rather than an explicit summation over the instantons, which was employed in the original derivation (9), (10). The agreement between these two drastically different approaches will give us confidence that our formal manipulations with the auxiliary fields are a correct and self-consistent procedure. With this confidence, as a next step, we will study the behavior of the auxiliary topological fields in the IR, which corresponds to $k \to 0$ in momentum space. We compare the corresponding formula with the Berry's connection at small $k \to 0$ to observe that both expressions behave in a very similar way at large distances in the IR. Such a similarity allows us to identify the auxiliary field $a_k(\mathbf{x})$ governed by the action (25), (26) with emergent Berry's connection $\mathcal{A}^j(\mathbf{k})$ given by Eq. (23).

To proceed with this task, we compute the topological susceptibility at $\theta = 0$ as follows:

$$\langle Q(\mathbf{x})Q(\mathbf{0})\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[b]\mathcal{D}[a]e^{-S_{\rm tot}[b,a_k]} \frac{e^2}{4\pi^2}\cdot[\epsilon^{jk}\partial_j a_k(\mathbf{x}), \epsilon^{j'k'}\partial_{j'}a_{k'}(\mathbf{0})], \qquad (27)$$

where $S_{\rm tot}[b, a_k]$ determines the dynamics of auxiliary $b$ and $a_k$ fields, and it is given by

$$S_{\rm tot}[b, a_k] = \int d^2\mathbf{x}\left[\frac{e^2}{8\pi^2}b^2(\mathbf{x}) + i\frac{e}{2\pi}b(\mathbf{x})\epsilon^{jk}\partial_j a_k(\mathbf{x})\right]. \qquad (28)$$

The obtained Gaussian integral (27) over $\int\mathcal{D}[b]$ can be explicitly executed, and we are left with the following integral over $\int\mathcal{D}[a]$:

[2]I am thankful to the anonymous referee for pointing this out.

$$\langle Q(\mathbf{x})Q(\mathbf{0})\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}[a]e^{-\frac{1}{2}\int d^2\mathbf{x}[\epsilon^{jk}\partial_j a_k(\mathbf{x})]^2} \cdot \frac{e^2}{4\pi^2}[\epsilon^{jk}\partial_j a_k(\mathbf{x}), \epsilon^{j'k'}\partial_{j'} a_{k'}(\mathbf{0})]. \quad (29)$$

The integral (29) is also Gaussian and can be explicitly evaluated with the following final result:

$$\langle Q(\mathbf{x})Q(\mathbf{0})\rangle = \frac{e^2}{4\pi^2}\delta^2(\mathbf{x}),$$
$$\int d^2\mathbf{x}\langle Q(\mathbf{x})Q(\mathbf{0})\rangle = \frac{e^2}{4\pi^2}. \quad (30)$$

A few comments are in order. First, formula (30) precisely reproduces our previous expression (9), (10) derived by explicit summation over fluxes-instantons, and without even mentioning any auxiliary topological fields $b(\mathbf{x}), a_k(\mathbf{x})$. It obviously demonstrates a self-consistency of our formal manipulations with auxiliary topological fields. As we shall see below, the reformulation of the system in terms of the auxiliary topological fields is extremely useful for studying some other (very nontrivial) topological features of the gauge system.

Secondly, the expression (30) for the topological susceptibility represents the contact nondispersive term which cannot be associated with any physical propagating degrees of freedom as we discussed in Sec. II B. The nature of this contact term can be understood in terms of the tunneling transitions between topologically different but physically identical $|k\rangle$ states. As we already mentioned in Sec. II B the same contact term can be also understood in terms of the propagating Kogut-Susskind ghost [17], which effectively describes the tunneling transitions in terms of an auxiliary Kogut-Susskind ghost which, however, does not belong to the physical Hilbert space; see [16] for the details in the given context.

To proceed with our task of establishing the relation between the topological auxiliary fields and the Berry's connection, we want to compute the expectation value for the topological charge density operator $\langle Q\rangle \equiv \langle \frac{e}{2\pi}E\rangle$ at nonvanishing $\theta \neq 0$. The corresponding computations can be easily performed using the same technique described above. The only new element which occurs is the necessity to compute the path integral at nonvanishing $\theta$, as the entire final result will be proportional to $\theta$; see Eq. (31) below. However, the presence of $\theta$ in the effective action does not produce any technical difficulties as the emergent path integral remains to be the Gaussian integral determined by the quadratic action (25) even for nonvanishing $\theta$. The corresponding computation at $\theta \neq 0$ can be easily executed by a conventional shift of variables $b(\mathbf{x}), a_k(\mathbf{x})$. The result is

$$\lim_{\mathbf{k}\to 0}\int d^2\mathbf{x}e^{i\mathbf{k}\mathbf{x}}\langle Q(\mathbf{x})\rangle = \lim_{\mathbf{k}\to 0}\left(\frac{e}{2\pi}\right)\int d^2\mathbf{x}e^{i\mathbf{k}\mathbf{x}}\langle \epsilon^{ij}\partial_i a_j(\mathbf{x})\rangle = \frac{ie^2\theta}{4\pi^2}V, \quad (31)$$

where $V$ is the total volume of the system playing the role of the IR regulator in all computations in the 2d Maxwell system, as reviewed in Sec. II. The obtained formula (31) reproduces the well-known result that a nonvanishing $\theta$ corresponds to nonvanishing background electric field $E \equiv \epsilon^{ij}\partial_i a_j$ in the system [21],

$$\langle E\rangle_{\mathrm{Eucl}} = \frac{ie\theta}{2\pi}, \qquad \langle E\rangle_{\mathrm{Mink}} = \frac{e\theta}{2\pi}; \quad (32)$$

see also [2] with some comments in the given context.

The nonvanishing expectation value of the gauge invariant operator (31) is a highly nontrivial phenomenon, as the operator $Q(\mathbf{x})$ itself is a total divergence. Naively, all correlation functions with operator $Q(\mathbf{x})$, including the expectation value of $\langle Q(\mathbf{x})\rangle$ itself, must vanish in the $k \to 0$ limit, as there are no physical massless degrees of freedom in the system. We know that this naive conclusion is incorrect as well-established results (8), (9), (31), (32) explicitly show. The loophole in the aforementioned naive conclusion is related to the generating of the nondispersive (contact) contributions which cannot be formulated in terms of any physical propagating degrees of freedom. The same IR physics, as we already mentioned, can be also described in terms of the massless Kogut-Susskind ghost [17] which effectively (implicitly) describes the tunneling transitions between the topological $|k\rangle$ sectors.[3]

Such a strong IR sensitivity implies that the Fourier transform of the auxiliary topological field $a_j$ saturating the expectation value (31) has the singular behavior at small momentum $k \to 0$:

$$a^j(\mathbf{k}\to 0) \equiv \frac{1}{V}\int d^2\mathbf{x}e^{i\mathbf{k}\mathbf{x}}a^j(\mathbf{x}) \to \left(\frac{e\theta}{2\pi}\right)\frac{\epsilon^{ij}k_i}{\mathbf{k}^2}, \quad (33)$$

in spite of the fact that the system does not support any physical massless propagating degrees of freedom, which erroneously can be associated with the pole (33).

[3]Furthermore, one can argue that the topological auxiliary field $a_i(\mathbf{k})$ introduced above can be expressed in terms of the Kogut-Susskind ghost. Apparently, such a relation is a very generic feature of many gauge theories. In fact, an analogous relation can be explicitly worked out in four-dimensional gauge theory, in the so-called weakly coupled "deformed QCD" where the auxiliary topological fields, similar to $a_i(\mathbf{x}), b(\mathbf{x})$ fields from (28), are related to the Veneziano ghost [20]. The Veneziano ghost was postulated in QCD long ago [22] with the sole purpose of saturating the nondispersive (contact) term in topological susceptibility, similar in structure to Eq. (30). As it is known, this contact term plays the key role in the resolution of the so-called $U(1)_A$ problem in QCD [22,23].

The source of this pole is obviously related to the same topological instantonlike long ranged configurations (2) saturating the contact term in the topological susceptibility (11). The singular behavior (33), which simply represents a nonvanishing expectation value (31), (32), obviously implies that the integral in momentum space around $\mathbf{k} \sim 0$ does not vanish. Indeed, using $k \to 0$ behavior (33), one arrives at the following relation,

$$\frac{1}{e}\oint_{|\mathbf{k}|\to 0} a^j(\mathbf{k}) dk_j = \frac{1}{e}\int d^2\mathbf{k}[\epsilon^{ij}\partial_{k_i} a_j(\mathbf{k})] = \theta \int d^2\mathbf{k}\partial_{k_i}\left(\frac{k_i}{2\pi\mathbf{k}^2}\right) = \theta\int d^2\mathbf{k}\delta^2(\mathbf{k}) = \theta, \qquad (34)$$

which essentially represents the same well-known statement about the nonvanishing gauge invariant expectation value (31), (32), but written in different terms involving the auxiliary topological fields in momentum space.

From (34) one can easily recognize that the auxiliary field $\frac{1}{e}a^j(\mathbf{k})$ in momentum space strongly resembles the Berry's connection (23), while $\frac{1}{e}\epsilon^{ij}\partial_{k_i} a_j(\mathbf{k})$ can be thought as the Berry's curvature discussed previously in CM physics; see, e.g., [11] for review. The fundamental difference between the analysis of our system and the computations of the Berry phase in CM literature is that the Berry connection (23) in CM systems is a collective phenomenon with accumulation of the geometric phase of the band electrons. It is represented, as a matter of convenience rather than necessity, in terms of the emergent gauge field $\mathcal{A}_i(\mathbf{k})$. In contrast, in our case, the topological fields $a_i(\mathbf{k})$ represent some fundamental (though auxiliary, nonpropagating) fields describing the ground state of the underlying gauge theory. These fields are present in the system even without any matter fields. The topological features of the auxiliary fields in our case emerge as a result of the summation of the topological sectors in path integral formulation rather than a result of a complex interaction of the band electrons in CM systems.

Nevertheless, as we observed above, there is very strong mathematical similarity between these two, physically very different, entities. These similarities, in particular, include the following features: while $a_i(\mathbf{k})$ and $\mathcal{A}_i(\mathbf{k})$ are gauge-dependent objects, the corresponding integrals (22) and (34) are gauge invariant (modulo $2\pi$) observables describing the same property related to the polarization. The $2\pi$ periodicity for all observables in both systems also has a very simple physical explanation. For our system the $2\pi$ periodicity follows from the partition function (1), (7), while in CM context [11,18] the $2\pi$ periodicity corresponds to the adiabatic process when the many body wave function returns to its physically identical (but topologically different) state. Furthermore, the main features of the systems are formulated in terms of global rather than local behavior, as formulas (22) and (34) suggest. One should comment here that explicit computations of the Berry's connection for a specific CM system very often require some tedious microscopic local computations, though the final result in fact describes the global behavior of the system, not sensitive to any local characteristics.

We conclude this section with the following general comment. We have not produced any new physical results in this section, as the relevant questions in 2d QED, such as the expectation value of the electric field at nonzero $\theta$ [represented by Eqs. (31), (32)] or a nondispersive (contact) contribution to the topological susceptibility (30), were computed long time ago.[4] Our contribution in this section is much more modest. We reproduced these known results by using a different technique: we expressed the relevant correlation functions in terms of the auxiliary topological fields $a_i(\mathbf{k})$. We established the physical meaning of these fields and argued that these auxiliary objects play the same role as Berry's connection $\mathcal{A}_i(\mathbf{k})$ in CM systems.

As we shall discuss below, the technical tools developed and tested in this subsection (by reproducing the known results) will be very useful in our study of a similar phenomena in physically relevant four-dimensional Maxwell theory formulated on the torus. This mathematical similarity occurs as a result of dimensional reduction (to be used below) which essentially translates the corresponding 4d problems into the 2d analysis developed in the present section.

### C. Auxiliary topological fields in 4d Maxwell system

We wish to derive the topological action for the 4d Maxwell system by using the same technique exploited in the previous subsection, Sec. IV B. Our starting point is to insert the delta function, similar to Eq. (24), into the path integral with the field $b^z(\mathbf{x})$ acting as a Lagrange multiplier

$$\delta[B^z(\mathbf{x}) - \epsilon^{zjk}\partial_j a_k(\mathbf{x})] \sim \int \mathcal{D}[b_z] e^{iL_3\beta\int d^2\mathbf{x} b_z(\mathbf{x})\cdot[B^z(\mathbf{x})-\epsilon^{zjk}\partial_j a_k(\mathbf{x})]}, \qquad (35)$$

where $B^z(\mathbf{x})$ in this formula is treated as the original expression for the field operator entering the action (13), including all classical $k$-instanton configurations (12), (15) and quantum fluctuations surrounding these classical configurations. In other words, we treat $B^z(\mathbf{x})$ as fast degrees of freedom. At the same time, $a_k(\mathbf{x})$ is treated as a slow-varying external source effectively describing the large distance physics for a given instanton configuration. Our task now is to integrate out the original fast "fluxes" (12), (15) and describe the large distance physics in terms of

[4] In particular, formula (30) can be derived using the Kogut-Susskind ghost formalism [17].

slow-varying fields $b_z(\mathbf{x}), a_k(\mathbf{x})$ in the form of the effective action similar to (28) derived for the 2d system. The physical meaning of these formal manipulations is explained in Sec. IV B after Eq. (24), and we shall not repeat it here.

To proceed with computations, we use the same procedure by summation over $k$-instantons as described in Sec. III. The only new element in comparison with the previous computations is that the fast degrees of freedom must be integrated out in the presence of the new slow-varying background fields $b_z(\mathbf{x}), a_k(\mathbf{x})$ which appear in Eq. (35). Fortunately, the computations can be easily performed if one notices that the background field $b_z(\mathbf{x})$ enters Eq. (35) exactly in the same manner as the external magnetic field enters (19). Therefore, assuming that $b_z(\mathbf{x}), a_k(\mathbf{x})$ are slow-varying background fields we arrive at the following expression for the partition function for our $\mathcal{TV}$ system:

$$\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff}) = \sqrt{\pi\tau}\sum_{k\in\mathbb{Z}}\int \mathcal{D}[b_z]\mathcal{D}[a]e^{-S-S_{\rm top}}, \quad (36)$$

where quadratic action $S[b_z, a_k]$ is defined as

$$S[b_z, a_k] = \pi^2\tau\int_{\mathbb{T}_2}\frac{d^2\mathbf{x}}{L_1L_2}\left(k+\frac{\phi(\mathbf{x})+\theta_{\rm eff}}{2\pi}\right)^2, \quad (37)$$

while the topological term $S_{\rm top}[b_z, a_k]$ in Eq. (36) reads

$$S_{\rm top}[b_z, a_k] = iL_3\beta\int_{\mathbb{T}_2} d^2\mathbf{x}[b_z(\mathbf{x})\epsilon^{zjk}\partial_j a_k(\mathbf{x})]. \quad (38)$$

In formula (37) we rescale the slow-varying background auxiliary $b_z$ field such that $\phi(\mathbf{x}) \equiv eL_1L_2b_z(\mathbf{x})$. Parameter $\theta_{\rm eff} \equiv eL_1L_2B_z^{\rm ext}$ represents the external magnetic field while $\mathbb{T}_2$ represents the two tori defined on the (1,2) plane.

The topological action (38) in all respects is very similar to the topological action derived for the 2d system (26). Therefore, we anticipate that all consequences discussed in Sec. IV B for the 2d system will have their analogues in the 4d system, including the relation between the Berry's connection and auxiliary fields in momentum space in the IR at $\mathbf{k}\to 0$.

Before we proceed with computations to establish such a connection, we want to make the following comment. The topological term (38) which emerges as an effective description of our system is in fact a Chern-Simons-like topological action. In our simplified setting we limited ourselves by considering the fluxes along the $z$ direction only. It is natural to assume that a more general construction would include fluxes in all three directions, which would lead to a generalization of action (38). Therefore, it is quite natural to expect that the action in this case would assume a Chern-Simons-like form $i\beta\int_{\mathbb{T}_3} d^3\mathbf{x}[\epsilon^{ijk}b_i(\mathbf{x})\partial_j a_k(\mathbf{x})]$ which replaces (38). A similar structure in CM systems is known to describe a topologically ordered phase. Therefore, it is not really a surprise that we observed in [2] some signatures of the topological order in the Maxwell system defined on a compact manifold. The emergence of the topological Chern-Simons action (38) further supports this basic claim that the Maxwell system on a compact manifold belongs to a topologically ordered phase, as the auxiliary topological fields entering (38) play the same role as the Berry's connection in topologically ordered CM systems.

Now we can follow the same procedure which we tested for the 2d system in Sec. IV B to compute the expectation value of the magnetic field at nonvanishing $\theta_{\rm eff}$. The corresponding result is known [1]: it has been derived by using conventional computation of the path integral by summation over all "fluxes-instantons." Our goal now is to reproduce this result by using the auxiliary topological fields governed by the action (38). We follow the same procedure as before and define the induced magnetic field in the system in the conventional way:

$$\begin{aligned}\langle B^z_{\rm ind}(\tau,\theta_{\rm eff})\rangle &= -\frac{1}{\beta V}\frac{\partial \ln \mathcal{Z}_{\rm top}(\tau,\theta_{\rm eff})}{\partial B_{\rm ext}}\\ &= \frac{2\pi}{eL_1L_2}\left\langle k+\frac{\theta_{\rm eff}+\phi(\mathbf{x})}{2\pi}\right\rangle, \end{aligned} \quad (39)$$

where the last expectation value must be evaluated using the partition function (36). The corresponding Gaussian integral over auxiliary $\phi(\mathbf{x})$ field can be easily executed with the result

$$\langle B^z_{\rm ind}(\tau,\theta_{\rm eff})\rangle = \lim_{\mathbf{k}\to 0}\int\frac{d^2\mathbf{x}}{L_1L_2}e^{i\mathbf{kx}}\langle -i\epsilon^{zjk}\partial_j a_k(\mathbf{x})\rangle, \quad (40)$$

where the corresponding expectation value $\langle\ldots\rangle$ should be computed using the following partition function determined by the action $S_{\rm tot}[a_k]$ (which includes both the quadratic and topological terms):

$$\begin{aligned}S_{\rm tot}[a_k] &= \frac{L_3\beta}{2}\int_{\mathbb{T}_2} d^2\mathbf{x}(\epsilon^{zjk}\partial_j a_k(\mathbf{x}))^2\\ &\quad - i(2\pi k+\theta_{\rm eff})\frac{L_3\beta}{eL_1L_2}\int_{\mathbb{T}_2} d^2\mathbf{x}(\epsilon^{zjk}\partial_j a_k(\mathbf{x})).\end{aligned} \quad (41)$$

The path integral (40) is Gaussian and can be executed by a conventional shift of variables in the action $S_{\rm tot}[a_k]$ defined by (41):

$$(\epsilon^{zjk}\partial_j a'_k(\mathbf{x})) = (\epsilon^{zjk}\partial_j a_k(\mathbf{x})) - i\frac{(2\pi k+\theta_{\rm eff})}{eL_1L_2}. \quad (42)$$

Exact evaluation of the Gaussian path integral (40) with action (41) leads to the following final result for $\langle B^z_{\rm ind}\rangle$:

$$\langle B^z_{\rm ind}\rangle = \frac{2\pi}{eL_1L_2}\frac{\sqrt{\pi\tau}}{\mathcal{Z}_{\rm top}}\sum_{k\in\mathbb{Z}}\left(k+\frac{\theta_{\rm eff}}{2\pi}\right)e^{-\pi^2\tau(k+\frac{\theta_{\rm eff}}{2\pi})^2}, \quad (43)$$

where the partition function $\mathcal{Z}_{\rm top}$ for our $\mathcal{TV}$ system in this formula[5] is determined by Eq. (19). As expected, the expression (43) exactly reproduces the corresponding formula derived in Ref. [1] by explicit summation over fluxes-instantons. We reproduced the results of Ref. [1] using a drastically different technique, as our computations (43) in this section are based on calculation of the path integral defined by the partition function (36) formulated in terms of the auxiliary topological fields $b_z(\mathbf{x}), a_i(\mathbf{x})$. Agreement between the two approaches obviously supports the consistency of our formal manipulations with the path integral and auxiliary fields.

An important new point (which could not be seen within the computational technique of Ref. [1]) is the expression (40) for the induced field $\langle B^z_{\rm ind}\rangle$ in terms of the auxiliary object $a_i(\mathbf{x})$. As we shall see in a moment, precisely this connection allows us to identify the auxiliary topological nonpropagating field $a_i(\mathbf{k})$ in momentum space with the Berry's connection $\mathcal{A}_i(\mathbf{k})$ from Sec. IV A, as both entities have very similar properties.

Before we proceed to establish such a connection, we would like to make few comments. First, as one can see from (43) the expression for $\langle B^z_{\rm ind}\rangle$ accounts for the total field in the system, including the external field as well as the induced field due to the interference of the external field with the topological fluxes (12). However, in the absence of the external field $\theta_{\rm eff}\equiv eL_1L_2B^z_{\rm ext}=0$ the contributions to the expectation value (43) from the fluxes with positive and negative signs cancel each other, and $\langle B^z_{\rm ind}\rangle$ vanishes. For $\theta_{\rm eff}\neq 0$ the cancellation does not hold, and the field $\langle B^z_{\rm ind}\rangle \neq 0$ will be obviously induced.

The effect must vanish when the tunneling transitions due to the fluxes are suppressed at $e\to 0$ which corresponds to large $\tau\gg 1$. It is very instructive to see how it happens. The corresponding expression which is valid for $\tau\gg 1$ and small but finite $\theta_{\rm eff}\ll 1$ reads

$$\langle B^z_{\rm ind}\rangle = \frac{\theta_{\rm eff}}{eL_1L_2}\left[1-\frac{4\pi e^{-\pi^2\tau}\sinh(\pi\tau\theta_{\rm eff})}{\theta_{\rm eff}}\right]. \quad (44)$$

One can explicitly see from Eq. (44) that the tunneling effects are suppressed in the large $\tau$ limit, and the magnetic field in this case in the system is entirely determined by an external source, $\langle B^z_{\rm ind}\rangle \to B^z_{\rm ext}$ at $\tau\gg 1$, as expected.

The key point for the present analysis is the expression (40) for $\langle B^z_{\rm ind}\rangle$ in terms of the auxiliary fields $a_k(\mathbf{x})$. This formula in all respects is very similar to expression (31) previously analyzed in the 2d system. One can follow the same logic of that analysis to arrive at the conclusion that the auxiliary field $a_k(\mathbf{x})$ can be thought as the Berry's connection [similar to (33) from the 2d analysis] with the following singular behavior at small $\mathbf{k}\to 0$ in momentum space,

$$a^i(\mathbf{k}\to 0)\equiv \frac{1}{L_1L_2}\int\frac{d^2\mathbf{x}}{2\pi}e^{i\mathbf{k}\mathbf{x}}a^i(\mathbf{x}) = \langle B^z_{\rm ind}\rangle\frac{\epsilon^{zij}k_j}{2\pi\mathbf{k}^2}\Rightarrow\left(\frac{\theta_{\rm eff}}{eL_1L_2}\right)\frac{\epsilon^{zij}k_j}{2\pi\mathbf{k}^2}, \quad (45)$$

where in the last line we use the asymptotical behavior (44) which is valid for large $\tau\gg 1$.

The behavior (45) also suggests that $\epsilon^{zij}\partial_{k_i}a_j(\mathbf{k})\sim\delta^2(\mathbf{k})$ plays the same role as the Berry's curvature in CM physics; see (23) and [11] for review. One should emphasize that these similarities in the IR behavior in two very different systems should not be considered as a pure mathematical curiosity. In fact, there is a very deep physical reason why these two, naively unrelated entities must behave very similarly in the IR. Indeed, as it is known the Berry's phase in CM systems effectively describes the variation of the $\theta$ parameter $\theta\to\theta-2\pi P$ as a result of the coherent influence of strongly interacting fermions which polarize the system, i.e., $P=\pm 1/2$; see, e.g., [18]. The auxiliary topological field $a^i(\mathbf{x})$ in our $\mathcal{TV}$ system with similar IR behavior essentially describes the same physics. To be more precise, the interference between the external magnetic field and fluxes leads to the magnetic polarization formulated in terms of $a^i(\mathbf{x})$ fields, similar to the generation of polarization $P=\pm 1/2$ in CM systems expressed in terms of the Berry's connection $\mathcal{A}^i(\mathbf{x})$. Our key observation is that the polarization features of the $\mathcal{TV}$ system in our case are represented by Eqs. (40), (45). These equations play the same role as Eqs. (22), (23) in CM systems.

This close analogy (mathematical and physical), in fact, may have some profound observational and experimental consequences as an electrically charged probe inserted into our system characterized by $\theta_{\rm eff}$ would behave very much in the same way as a probe inserted into a CM system characterized by a nonvanishing Berry's phase. In other words, our $\mathcal{TV}$ system must demonstrate a number of unusual features which are typical for topologically ordered phases in CM systems. One of these properties, the degeneracy of the system, which cannot be described in terms of any local operator (but rather is characterized by a nonlocal operator) has been already established [2]. There

[5]We note that $k$-independent numerical factor $\sqrt{\pi\tau}$ enters both Eqs. (19) and (43). This numerical factor simply represents our normalization's convention and does not affect the computations of any expectation values, such as (43). Our normalization corresponds to the following behavior of the topological partition function: $\mathcal{Z}_{\rm top}\to 1$ in the limit $L_1L_2\to\infty$. Such a convention corresponds to the geometry of the original Casimir setup experiment; see [1] for the details.

must be other interesting experimentally observable effects in 4d Maxwell theory, similar to a number of profound effects which are known to occur in topologically ordered phases in CM systems [4–12]. We leave this subject for future studies.

## V. CONCLUSION AND FUTURE DIRECTIONS

In this work we discussed a number of very unusual features exhibited by the Maxwell theory formulated on a four-torus, which was coined the topological vacuum ($\mathcal{TV}$). All these features are originated from the topological portion of the partition function $\mathcal{Z}_{\rm top}(\tau, \theta_{\rm eff})$ and cannot be formulated in terms of conventional $E\&M$ propagating photons with two physical transverse polarizations. In other words, all effects discussed in this paper have a nondispersive nature.

The computations of the present work along with previous calculations of Refs. [1,2] imply that the extra energy (and entropy), not associated with any physical propagating degrees of freedom, may emerge in the gauge systems if some conditions are met. This fundamentally new type of energy emerges as a result of dynamics of pure gauge configurations at arbitrary large distances. This unique feature of the system when an extra energy is not related to any physical propagating degrees of freedom was the main motivation for a proposal [16,24,25] that the vacuum energy of the Universe may have, in fact, precisely such nondispersive nature.[6] This proposal when an extra energy cannot be associated with any propagating particles should be contrasted with a conventional description when an extra vacuum energy is always associated with some new physical propagating degree of freedom, such as the inflaton.

The main motivation for the present studies is to test these ideas (about a fundamentally new type of vacuum energy) using a simple quantum field theory setting which nevertheless preserves the crucial element, the degeneracy of the topological sectors, responsible for this novel type of energy. This simplest possible setting can be realized in the Maxwell theory formulated on a four-torus. Most importantly, the effect with this simplest setting can be, in principle, tested in a tabletop experiment if the corresponding boundary conditions can be somehow imposed in real physical systems. Otherwise, our construction should be considered as the simplest possible 4d QFT model when extra vacuum energy is generated. The crucial point is that this extra vacuum energy cannot be associated with any physical propagating degrees of freedom, as argued in the present work.

Essentially, the proposal [16,24,25] identifies the observed vacuum energy with the Casimir type energy, which however is originated not from dynamics of the physical massless propagating degrees of freedom, but rather, from the dynamics of the topological sectors which are always present in gauge systems, and which are highly sensitive to arbitrary large distances. The vacuum energy in this case can be formulated in terms of the auxiliary topological fields which are similar in spirit to $b_z(\mathbf{x}), a_k(\mathbf{x})$ fields from (28), (38) and which effectively describe the dynamics of the topological sectors in the expanding background [25]. As we discussed at length in Sec. IV C these auxiliary topological fields play the same role as the Berry's connection in CM systems. The $b_z(\mathbf{x}), a_k(\mathbf{x})$ fields do not propagate, but they do contribute to the vacuum energy. It would be very exciting if this new type of the vacuum energy not associated with propagating particles could be experimentally measured in a laboratory, as we advocate in this work.

Aside from testing the ideas on vacuum energy of the Universe, the Maxwell system studied in the present work is an interesting system on its own. Indeed, being a "free" Maxwell theory, it nevertheless shows a number of very unusual features which are normally attributed to a CM system in a topologically ordered phase. In particular, it shows the degeneracy of the system which cannot be detected by any local operators, but is characterized by a nonlocal operator [2]. Furthermore, in the present work we argued that the auxiliary topological fields $b_z(\mathbf{x}), a_k(\mathbf{x})$ fields in 4d Maxwell system behave very much in the same way as the Berry's connection in CM systems. More than that, a charged probe particle inserted into our system would feel the topological features of the $b_z(\mathbf{x}), a_k(\mathbf{x})$ fields in the same way as a probe inserted into a CM system characterized by a nontrivial Berry's connection $\mathcal{A}^i(\mathbf{x})$.

Therefore, it would be very exciting if one could find a system where a charged probe inserted into our four-torus (filled by vacuum) would behave similarly to a probe inserted into a much more complicated CM system, where the corresponding nontrivial Berry's connection is emergent as a result of a coherent many body physics.

The simplest possible setup we can imagine is as follows. Normally, in condensed matter literature, one considers a junction between a conventional insulator ($\mathcal{I}$) and topological insulator ($\mathcal{TI}$). One can also consider the $\mathcal{TI}$ which is sandwiched between two conventional insulators; i.e., one can consider a system like $\mathcal{I}-\mathcal{TI}-\mathcal{I}$. Our claim essentially is that the $\mathcal{TI}$ in this system can be replaced by the $\mathcal{TV}$ configuration considered in this work. In other

[6]There are two instances in the evolution of the Universe when the vacuum energy plays a crucial role. The first instance is identified with the inflationary epoch when the Hubble constant $H$ was almost constant, which corresponds to the de Sitter type behavior $a(t) \sim \exp(Ht)$ with exponential growth of the size $a(t)$ of the Universe. The second instance, when the vacuum energy plays a dominating role, corresponds to the present epoch when the vacuum energy is identified with the so-called dark energy $\rho_{\rm DE}$, which constitutes almost 70% of the critical density. In the proposal [16,24,25], the vacuum energy density can be estimated as $\rho_{\rm DE} \sim H\Lambda_{\rm QCD}^3 \sim (10^{-4}~{\rm eV})^4$, which is amazingly close to the observed value.

words, one considers a system like $\mathcal{I}-\mathcal{TV}-\mathcal{I}$. Our claim is that this system would behave very much as the $\mathcal{I}-\mathcal{TI}-\mathcal{I}$, because $\mathcal{TV}$ behaves very much in the same way as a $\mathcal{TI}$, as advocated in this work.[7] These similarities include such nontrivial features as the degeneracy (characterized by a nonlocal operator), the Berry's connection, and the presence of the effective $\theta_{\mathrm{eff}}$ state, among many others things. Therefore, it is natural to expect that while $I-\mathcal{TI}-\mathcal{I}$ and $\mathcal{I}-\mathcal{TV}-\mathcal{I}$ systems are very different in composition, the behavior of these systems will be very much the same in the IR. We leave this exciting subject on possible applications of our $\mathcal{TV}$ system, which we believe belongs to a topologically ordered phase, for future investigations.

[7]The conducting (or superconducting) edges of the $\mathcal{TV}$ portion in $\mathcal{I}-\mathcal{TV}-\mathcal{I}$ sandwiched in the form of a closed circle support the periodic boundary conditions up to the large gauge transformations. This freedom in terms of large gauge transformation, or what is the same, a finite probability to form the "fluxes-instantons" in the bulk of the system, eventually leads to the emergence of the partition function (16) with all its consequences discussed in the present work. fluxes-instantons (12) also satisfy the boundary condition $B_{\perp}=0$ on the superconducting edges.

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