Unitarity of theories containing fractional powers of the d'Alembertian operator

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We examine the unitarity of a class of generalized Maxwell U(1) gauge theories in $(2+1)$ dimensions containing the pseudodifferential operator $\Box^{1-\alpha}$, for $\alpha \in [0,1)$. We show that only QED_3 and its generalization known as pseudo-QED, for which $\alpha = 0$ and $\alpha = 1/2$, respectively, satisfy unitarity. The latter plays an important role in the description of the electromagnetic interactions of charged particles confined to a plane, such as in graphene or in heterojunctions displaying the quantum Hall effect. Possible implications of our results on the role of unitarity in the framework of the AdS/CFT correspondence are briefly pointed out at the end.

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I. INTRODUCTION

Unitarity is an important necessary condition for the consistency of any quantum theory. Consider the time evolution operator $U(t,0)$, defined as

$$|\Psi(t)\rangle = U(t,0)|\Psi(0)\rangle, \tag{1}$$

where $|\Psi(t)\rangle$ is the state vector at instant t. The unitarity of the time-evolution operator, namely the property $U^\dagger U = UU^\dagger = I$, where I is the identity operator, guarantees that the norm of the state vectors, chosen to be equal to one, is preserved in time. Since the state vector can be expanded in the eigenstates of any observable A, it follows that its norm is equal to the sum of the probabilities for the possible outcomes of any measurement of A. Unitarity implies that this sum of probabilities remains equal to one at any time, an essential condition for the probabilistic description of a system. For a time-independent Hamiltonian, we have $U(t,0) = \exp(-iHt)$. Unitarity then implies that the Hamiltonian is a Hermitian operator and therefore the energy eigenvalues are real. This property and the conservation of the sum of probabilities are crucial conditions for the stability of a quantum-mechanical system [1].

Another consequence of the unitarity of the time-evolution operator is that the scattering matrix, which connects the asymptotic states after a scattering event to the ones before it, must also be unitary. Assuming the completeness of the asymptotic states, then it follows that the S-matrix elements form a matrix representation of a unitary scattering operator $S = 1 + iT$. Unitarity of the S operator, namely, $S^\dagger S = 1$, implies

$$i(T^\dagger - T) = T^\dagger T. \tag{2}$$

This relation leads to the optical theorem, which relates the forward scattering amplitude to the total cross section of the scatterer. A very convenient way of testing the consistency of a theory is then provided by the optical theorem, which is satisfied by unitary theories.

In this paper, we examine the unitarity of a class of generalized Maxwell U(1) gauge theories in $(2+1)$ dimensions by using the optical theorem. For an appropriate choice of the gauge, the equations of motion for these theories are $\Box^{1-\alpha}A_\mu = 0$, for any $\alpha \in [0,1)$. We show that only the choices $\alpha = 0$ or $\alpha = 1/2$ corresponding, respectively, to QED_3 and the so-called pseudo-QED (PQED) provide a self-consistent solution to the optical theorem. Particularly, the choice $\alpha = 1/2$ is also consistent with Huygens' principle. The unitarity of PQED is first proven at the tree level, and then for the interacting case.

The outline of this paper is as follows. In Sec. II we revise PQED and propose its generalization to any α. In Sec. III we show that only $\alpha = 0$ or $\alpha = 1/2$ are possible choices in order to obtain a self-consistent solution of the optical theorem. Both cases are considered at the tree level, with no source term in the equation of motion. In Sec. IV we use the random-phase approximation (RPA) approach to show that the version of PQED used to describe the electronic interaction in graphene is also unitary. In Sec. V we adopt perturbation theory up to two loops to show that PQED is a unitary theory.

II. PQED AND ITS GENERALIZATIONS

A. The derivation of PQED

The discovery of condensed matter systems with physical properties that are essentially two-dimensional has fostered the investigation of $(2+1)$-dimensional theories, which could appropriately describe them. Among these we find the GaAs quantum wells exhibiting the quantum Hall effect, the high-T_c cuprates and graphene [2]. In such systems, a crucial issue is the description of the electronic interaction, which naturally is electromagnetic (EM). For this matter, one must consider that the interaction among the electrons is usually mediated by a (spacially) three-dimensional field in spite of the fact that the electron kinematics is confined to a plane. For the sake of convenience, simplicity and elegance, however, it is preferable to provide a completely $(2+1)$-dimensional description of the real electromagnetic interaction among the electrons. This is achieved [3–6] by a theory, called pseudo-QED, which was also used in the bosonization of the massless Dirac field in $(2+1)$ dimensions [7]. Dynamical mass generation for massless electrons also was studied for this model [8].

In this section, for the sake of completeness, we review the main steps of the derivation contained in Ref. [3]. We start from standard QED_4, in $(3+1)$ dimensions:

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e j^{\mu}_{3+1}A_{\mu} + \mathcal{L}_m, \tag{3}$$

where j^{μ}_{3+1} and $\mathcal{L}_m$ are, respectively, the electronic current and kinetic Lagrangian. A_μ is the gauge field, and $F^{\mu\nu}$ is the usual field-strength tensor.

The electromagnetic field induces an effective current-current interaction on the electrons, which is captured by the functional (in Euclidean space)

$$Z_{\mathrm{QED}}[j^{\mu}_{3+1}] = Z_0^{-1}\int DA_\mu \exp\left\{-\int d^4\xi \mathcal{L}_{\mathrm{QED}}\right\}, \tag{4}$$

where $\xi = (x, y, z, \tau)$ and Z_0 is a normalization constant which guarantees that $Z[0] = 1$. The functional integration above can be carried out by including a gauge fixing-term, yielding

$$Z_{\mathrm{QED}}[j^{\mu}_{3+1}] = \exp\left\{-\frac{e^2}{2}\int d^4\xi d^4\xi' j^{\mu}_{3+1}(\xi) \times G^{\mu\nu}_{\mathrm{QED}}(\xi-\xi') j^{\nu}_{3+1}(\xi')\right\}, \tag{5}$$

where $G^{\mu\nu}_{\mathrm{QED}}$ is the Euclidean propagator of the electromagnetic field, which is given by

$$G^{\mu\nu}_{\mathrm{QED}}(\xi-\xi') = \delta^{\mu\nu}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot(\xi-\xi')}}{k^2} + \mathrm{gt}, \tag{6}$$

where gt stands for "gauge-dependent terms." These, by the way, do not contribute for Eq. (5).

We now introduce the fact that the electrons are supposed to move on a plane at $z = 0$, thus forming a spacially two-dimensional system. The electronic current, accordingly, is given by

$$j^{\mu}_{3+1}(\xi) = \begin{cases} j^{\mu}(x, y, \tau)\delta(z), & \mu = 0, 1, 2, \\ 0, & \mu = 3. \end{cases} \tag{7}$$

Inserting Eq. (7) into Eq. (5) and integrating over z and z', we get

$$Z_{\mathrm{QED}}[j^{\mu}] = \exp\left\{-\int d^3\eta d^3\eta' j^{\mu}(\eta) \times G^{\mu\nu}_{\mathrm{QED}}(\eta-\eta'; z = z' = 0) j^{\nu}(\eta')\right\}, \tag{8}$$

where $\eta = (x, y, \tau)$ and

$$G^{\mu\nu}_{\mathrm{QED}}(\eta-\eta'; z = z' = 0) = \frac{\delta^{\mu\nu}}{8\pi^2|\eta-\eta'|^2} + \mathrm{gt}. \tag{9}$$

The expression above is the four-dimensional QED Euclidean propagator, calculated at $z = z' = 0$.

Now comes a key step in our derivation. This is the realization that Eq. (9) can be written as a three-dimensional Fourier integral, namely

$$\frac{1}{8\pi^2|\eta-\eta'|^2} = \int \frac{d^3k_{3D}}{(2\pi)^3}\frac{e^{ik_{3D}\cdot(\eta-\eta')}}{4\sqrt{k^2_{3D}}}, \tag{10}$$

and this is the Euclidean propagator of PQED [3], corresponding to the strictly $(2+1)$-dimensional Lagrangian

$$\mathcal{L}_{\mathrm{PQED}} = -\frac{1}{4}F_{\mu\nu}\left[\frac{4}{(-\Box)^{1/2}}\right]F^{\mu\nu} - e j^{\mu}A_\mu + \mathcal{L}_m. \tag{11}$$

Inserting Eq. (9) and Eq. (10) into Eq. (8), we can immediately realize that

$$Z_{\mathrm{QED}}[j^{\mu}] = Z_0^{-1}\int DA_\mu \exp\left\{-\int d^3\eta \mathcal{L}_{\mathrm{PQED}}\right\}. \tag{12}$$

The above derivation shows that all the electronic properties determined by QED_4, when projected on a plane are described by a strictly $(2+1)$-dimensional theory, namely PQED. In connection to this point, one could argue whether PQED provides a description of the correlation functions of QED_4. The A_μ correlators are generated by coupling an external source J^{μ}_{3+1} in Eq. (4), namely

$$j^{\mu}_{3+1} \to j^{\mu}_{3+1} + J^{\mu}_{3+1},$$

and subsequently taking functional derivatives of Z_{QED} with respect to this source. Assuming it has the same structure as the electronic current given by Eq. (7), it follows that functional derivatives with respect to the

$(2+1)$-dimensional external source taken in PQED will generate the projected correlators, as it occurred with the two-point function in Eq. (9).

B. Generalized PQED

We will consider here a class of theories in $(2+1)$ dimensions, which contain PQED and QED_3 as particular cases. These are given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\left[\frac{4}{(-\Box)^{\alpha}}\right]F^{\mu\nu} - ej^{\mu}A_{\mu} + \mathcal{L}_m, \tag{13}$$

where $0 \leq \alpha < 1$. For a proper choice of the gauge condition, the U(1) vector field satisfies the equation

$$\Box^{1-\alpha}A^{\mu} = ej^{\mu}, \tag{14}$$

which is pseudodifferential for $\alpha \neq 0$. For $\alpha = 0$, the theory above is just Maxwell QED_3. In the previous section, we have shown that the case $\alpha = 1/2$, namely PQED, is relevant for the description of the electromagnetic interactions of two-dimensional systems. In this case, Eq. (13) provides a full description of the real electromagnetic interaction for electrons confined on a plane [3].

In the above Lagrangian, the first term reads

$$F_{\mu\nu}(\eta)\int d^3\eta' \int \frac{d^3k}{(2\pi)^3}\frac{e^{-ik\cdot(\eta-\eta')}}{(k^2)^{\alpha}}F^{\mu\nu}(\eta'), \tag{15}$$

where $k = (\mathbf{k}, \omega)$ (we excluded the index "3D" for simplicity) and $\eta = (\mathbf{r}, \tau)$. The nonlocality of the propagator is a consequence of the dimensional reduction performed in order to generate the $(3+1)$-dimensional propagator within $(2+1)$-dimensional space. A similar fact occurs when we integrate out parts of the system degrees of freedom as, for instance, in the Caldeira-Leggett model for dissipative quantum mechanics [9].

Nevertheless, in spite of being nonlocal, the theories described by Eq. (13) do respect causality. Indeed, it has been shown that the classic (retarded and advanced) Green functions vanish outside of the light cone for any α, thus preserving causality [5]. For the special case of $\alpha = 1/2$, the classic Green functions reduce to a delta function on the light-cone surface [5]. The interesting consequence of this property is that the theory will obey Huygens' principle in this case [5,10], while QED_3 does not obey it.

We see that the theories described by Eq. (13) satisfy causality despite the apparent nonlocality, but it is not *a priori* obvious whether they respect unitarity. In the present work, we shall test the unitarity of those theories through the application of the optical theorem.

III. UNITARITY AT TREE LEVEL

Let us investigate here the unitarity of the theories given by Eq. (13) by considering the free Feynman propagator (tree level) in connection to the optical theorem. We use the Feynman prescription $k^2 \to k^2 + i\varepsilon$ in order to define the gauge-field propagator corresponding to Eq. (13)

$$G_F^{\mu\nu}(t, \mathbf{r}) = \frac{1}{4}P^{\mu\nu}D_F(t, \mathbf{r}), \tag{16}$$

where

$$P_{\mu\nu} = g_{\mu\nu} - \frac{\partial_{\mu}\partial_{\nu}}{\Box^2} \tag{17}$$

is the transverse projector, $g_{\mu\nu}$ is the Minkowski metric, and $D_F(t, \mathbf{r})$ is the corresponding scalar propagator, in the Minkowski space. Thus, we replace τ by t, and therefore we have

$$D_F(t, \mathbf{r}) = \int \frac{d\omega}{2\pi}\int \frac{d^2k}{(2\pi)^2}\frac{e^{-i\omega t}e^{i\mathbf{k}\cdot\mathbf{r}}}{(\omega^2 - \mathbf{k}^2 + i\varepsilon)^{1-\alpha}}. \tag{18}$$

This integral has been calculated in Ref. [5] (see Appendix A therein), yielding

$$D_F(t, \mathbf{r}) = -\frac{C_{\alpha}}{(t^2 - \mathbf{r}^2 - i\epsilon)^{1/2+\alpha}}, \tag{19}$$

where

$$C_{\alpha} = \frac{2^{2\alpha-1/2}}{(2\pi)^{3/2}}\frac{\Gamma(\alpha+1/2)}{\Gamma(1-\alpha)}.$$

In order to probe the unitarity of the theories described by Eq. (13), let us first consider the scalar field. Later on we shall return to the vector-field case.

Taking the amplitude corresponding to the operator (2) evaluated between states $|i\rangle$ and $|f\rangle$, which is written as $\langle i|T|f\rangle = (2\pi)^3\delta^3(k_i - k_f)D_{if}$ and introducing a complete set of intermediate states $|x\rangle$ on the rhs, the above unitarity condition becomes

$$D_{if}^{*} - D_{if} = -i\sum_x \int d\Phi(2\pi)^3\delta^3(k_i - k_f)(D_{ix}^{*}D_{xf}), \tag{20}$$

where $d\Phi$ is the phase-space factor, which is needed for dimensional reasons and also to ensure that the sum over the intermediate states corresponds to the identity. The equation above is known as the generalized optical theorem.

Now, for $i \to f$, the amplitude D_{ii} becomes the Feyman propagator,

$$D_{ii} = D_F(t-t', \mathbf{r}-\mathbf{r}')$$

which is given by Eq. (19). Notice that, in the Heisenberg picture $D_F(t-t', \mathbf{r}-\mathbf{r}') = \langle \mathbf{r}, t|\mathbf{r}', t'\rangle$.

The unitarity condition, therefore, would lead to the equation

$$\begin{aligned} D_F^*(t,\mathbf{r}) - D_F(t,\mathbf{r}) &= -i\int d\Phi(2\pi)^3\delta^3(0)\int\frac{dt_x}{2\pi}\int\frac{d^2r_x}{(2\pi)^2} \\ &\quad\times D_F^*(t_x,\mathbf{r}_x)D_F(t-t_x,\mathbf{r}-\mathbf{r}_x). \end{aligned} \tag{21}$$

Our strategy to test unitarity of a given theory will be to check whether the corresponding propagator satisfies the optical theorem. For this purpose, we Fourier transform the above equation to energy-momentum space,

$$D_F^*(\omega,\mathbf{k}) - D_F(\omega,\mathbf{k}) = -i\mathcal{T}^\gamma D_F^*(\omega,\mathbf{k})D_F(\omega,\mathbf{k}), \tag{22}$$

where $D_F(\omega,\mathbf{k})$ is promptly obtained from Eq. (18). In the above expression, we used the fact that the phase-space integral combined with $\delta^3(0)$ yields $\mathcal{T}^\gamma$, where $\mathcal{T}$ is the characteristic time of the system and $\gamma = -2(1-\alpha)$ (see Appendix A).

Defining $\chi_\alpha = (\omega^2 - \mathbf{k}^2 + i\varepsilon)^{1-\alpha}$, we can write the equation above as

$$\frac{2\mathrm{Im}(\chi_\alpha)}{\chi_\alpha^*\chi_\alpha} = \frac{\mathcal{T}^{-2(1-\alpha)}}{\chi_\alpha^*\chi_\alpha}. \tag{23}$$

For unitarity to be respected, we must have

$$2\mathrm{Im}(\chi_\alpha) = \mathcal{T}^{-2(1-\alpha)}. \tag{24}$$

However, since the rhs is a constant, for the above condition to be consistent, $\mathrm{Im}(\chi_\alpha)$ must also be a constant, in the limit $\varepsilon \to 0$. In other words, in that limit the lhs cannot be a function of $\lambda = \omega^2 - \mathbf{k}^2$ for Eq. (24) to be consistent.

In order to verify this condition, we introduce a polar representation for χ_α, namely, $\chi_\alpha = (\rho e^{i\theta})^{1-\alpha}$, with $\rho^2(\lambda) = \lambda^2 + \varepsilon^2$ and $\theta(\lambda) = \sin^{-1}(\varepsilon/\rho)$. Then, we require that

$$\frac{d}{d\lambda}\mathrm{Im}(\chi_\alpha) = \frac{d}{d\lambda}\rho\sin[(1-\alpha)\theta] = 0. \tag{25}$$

Calculating the derivative, we obtain

$$\tan[\theta(\lambda)(1-\alpha)] = \tan[\theta(\lambda)], \tag{26}$$

which has an obvious solution $\alpha = 0$. Indeed, it is clear that for this value of α, $\mathrm{Im}(\chi_0) = \varepsilon$ and therefore it is independent of λ.

A less obvious solution is $\alpha = 1/2$, which is valid because in this case Eq. (26) admits a solution $\theta(\lambda) = 2\pi - \varepsilon$, which is compatible with the definition of $\theta(\lambda)$. In this case, we also find $\mathrm{Im}(\chi_{1/2}) = \varepsilon$ (see Appendix B).

We conclude that, for the theories with $\alpha = 0$ and $\alpha = 1/2$, the two sides of Eq. (23) would coincide consistently by identifying 2ε with $\mathcal{T}^{-2}$. For other values of α, $\mathrm{Im}(\chi_\alpha)$ would depend on λ and, therefore, we would not be able to find a consistent solution of Eq. (24) satisfying the generalized optical theorem.

The demonstrations provided above were meant for the scalar theories associated with Eq. (13). The corresponding results for the vector propagator (16) then, follow straightforwardly by making $\mathcal{T}^{-2(1-\alpha)}/4 \to \mathcal{T}'^{-2(1-\alpha)}$ and from the fact that the transverse projector has the property: $P^2 = P$.

We conclude that out of the class of theories described by Eq. (13), only the ones with $\alpha = 0$ and $\alpha = 1/2$, namely QED_3 and PQED are unitary.

IV. UNITARITY OF PQED IN THE RPA

Next, we consider PQED, the case for which $\alpha = 1/2$. As we have seen, it describes the EM interaction of the particles coupled to it. Having graphene in mind we describe the electrons as massless Dirac fermions undergoing the EM interaction mediated by the gauge field A_μ. The Lagrangian in this case reads [11]

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}\left[\frac{4}{\sqrt{-\Box}}\right]F^{\mu\nu} + \bar\psi(i\partial + e\gamma^\mu A_\mu)\psi, \tag{27}$$

where e is the dimensionless coupling constant, ψ is the Dirac field, and γ^μ are Dirac matrices which can be either two- or four-dimensional, since we are in $(2+1)$ dimensions.

The corrections to the gauge-field propagator are expressed in terms of the the vacuum polarization $\Pi_{\mu\nu}(p)$. The one-loop expression for this has been calculated in Ref. [12] and is given by

$$\Pi_{\mu\nu}(k) = \sqrt{k^2}\left[-\frac{e^2}{16}P_{\mu\nu} + \frac{e^2}{2\pi}\left(n + \frac{1}{2}\right)\epsilon_{\mu\nu\alpha}\frac{k^\alpha}{\sqrt{k^2}}\right], \tag{28}$$

where n is an integer. The result above is for two-dimensional Dirac matrices.

According to Eq. (18), the free gauge-field propagator in momentum space reads

$$G_{0,\mu\nu}(k) = \frac{1}{4}\frac{P_{\mu\nu}(k)}{\sqrt{k^2}}. \tag{29}$$

We include the vacuum-polarization corrections by using the RPA, where the corrected propagator is given by the geometrical series

$$G_{\mu\nu} = G_{0,\mu\alpha}[\delta_{\alpha,\nu} + \Pi^{\alpha\beta}G_{0,\beta\nu} + \Pi^{\alpha\beta}G_{0,\beta\sigma}\Pi^{\sigma\gamma}G_{0,\gamma\nu} + \cdots]. \tag{30}$$

Because of the peculiar momentum dependence of the vacuum-polarization tensor, the corrected propagator has basically the same momentum dependence as the free one

$$G_{\mu\nu}(k) = \frac{1}{\sqrt{k^2 + i\epsilon}}\left(A_1 P_{\mu\nu}(k) + A_2 \frac{\epsilon_{\mu\nu\alpha}k^\alpha}{\sqrt{k^2}}\right), \tag{31}$$

where A_1 and A_2 are constants depending on the coefficients of the vacuum-polarization tensor. Note that we use the Feynman prescription as we did before. Unitarity of the theory is guaranteed provided the optical theorem (20) is still respected.

The propagator above can be conveniently written as

$$G_{\mu\nu}(k) = C_{\mu\nu}(k)D_F(k), \tag{32}$$

where

$$C_{\mu\nu}(k) = A_1 P_{\mu\nu}(k) + A_2 \frac{\epsilon_{\mu\nu\alpha}k^\alpha}{\sqrt{k^2}}, \tag{33}$$

with $D_F(k)$ given by Eq. (18) for $\alpha = 1/2$.

The optical theorem now reads

$$\begin{aligned} &G^*_{\mu\nu}(t,\mathbf{r}) - G_{\mu\nu}(t,\mathbf{r}) \\ &\quad = -i\int d\Phi(2\pi)^3\delta^3(0)\int\frac{dt_x}{2\pi}\int\frac{d^2r_x}{(2\pi)^2} \\ &\quad\times G^*_{\mu\alpha}(t_x,\mathbf{r}_x)G_{\alpha\nu}(t-t_x,\mathbf{r}-\mathbf{r}_x). \end{aligned} \tag{34}$$

Next, we adopt the same strategy as for the noninteracting case and perform a Fourier transform on both sides of the above equation, again, considering that the Fourier transform of a convolution is a product. We obtain

$$G^*_{\mu\nu}(\omega,\mathbf{k}) - G_{\mu\nu}(\omega,\mathbf{k}) = -i\mathcal{T}^{-1}G^*_{\mu\alpha}(\omega,\mathbf{k})G_{\alpha\nu}(\omega,\mathbf{k}). \tag{35}$$

The lhs of Eq. (35) is given by

$$\frac{C_{\mu\nu}(k)2i\mathrm{Im}(\chi_{1/2})}{[(\omega^2 - \mathbf{k}^2)^2 + \epsilon^2]^{1/2}}, \tag{36}$$

whereas the rhs of Eq. (35) reads

$$\frac{-i\mathcal{T}^{-1}C_{\mu\alpha}(k)C_{\alpha\nu}(k)}{[(\omega^2 - \mathbf{k}^2)^2 + \epsilon^2]^{1/2}}, \tag{37}$$

where

$$C^2_{\mu\nu}(k) = (A_1^2 - A_2^2)P_{\mu\nu}(k) - 2A_1A_2\frac{\epsilon_{\mu\nu\alpha}k^\alpha}{\sqrt{k^2}}. \tag{38}$$

We now consider Eq. (36) and Eq. (37). Since both are proportional to the operators $P_{\mu\nu}(k)$ and $\epsilon_{\mu\nu\alpha}k^\alpha/\sqrt{k^2}$, therefore, we have to compare the corresponding coefficients of both terms. Using the result of Appendix B, we conclude that the optical theorem will be obeyed and consequently, unitarity will be preserved, provided we make the choices

$$(2\varepsilon)^{1/2} = \frac{A_1^2 - A_2^2}{2A_1}\mathcal{T}^{-1}, \tag{39}$$

in the $P_{\mu\nu}(k)$ term and

$$(2\varepsilon')^{1/2} = A_1\mathcal{T}^{-1}, \tag{40}$$

in the $\epsilon_{\mu\nu\alpha}k^\alpha/\sqrt{k^2}$ term.

This concludes our proof of the unitarity of PQED of massless electrons in the RPA.

V. BEYOND THE RPA

Within the RPA, the one-loop expression for the vacuum-polarization tensor, Eq. (28) is used in the geometrical series that corrects the free propagator of the gauge field. This approach can be improved by adding the two-loop correction for the vacuum-polarization tensor, as calculated by Teber [4],

$$\Pi^{(2)}_{\mu\nu}(k) = -\frac{\sqrt{k^2}}{16}\left(\frac{92 - 9\pi^2}{18\pi}\right)\alpha_g P_{\mu\nu}, \tag{41}$$

where $\alpha_g \approx 300/137 = 2.189$ is the fine-structure constant of graphene. Considering that $(92 - 9\pi^2)/18\pi \approx 0.056$, we see that the two-loop correction is sensible. There is no correction to the Chern-Simon term due the Coleman-Hill theorem [13].

Observe that, remarkably, the two-loop correction has precisely the same functional dependence as the one-loop one. As a consequence, the only effect of the two-loop correction to the vacuum polarization is to redefine the constant A_1 in Eq. (31). Therefore, it immediately follows that the optical theorem, and consequently, unitarity are respected in the two-loop extension of the RPA.

VI. CONCLUSIONS

We have tested the unitarity of a class of field theories in $2+1$ dimensions containing fractional powers $(1-\alpha)$ of the d'Alembertian operator, which despite being nonlocal, respect causality. QED_3 and PQED are particular cases, respectively, with $\alpha = 0$ and $\alpha = 1/2$.

Our strategy is to verify whether the propagator satisfies the optical theorem. We first considered the free propagator for generic α and showed that only when $\alpha = 0$ and $\alpha = 1/2$, namely, for QED_3 and PQED, unitarity is respected.

We then considered the case of PQED coupled to massless Dirac fermions, which is the model for graphene. We have shown that the propagator corrected both within the RPA and in its two-loop extension satisfies the optical theorem, and hence unitarity is preserved in both cases.

The fact that unitarity holds only for two theories among the ones studied here, namely, PQED and QED_3 may have far-reaching consequences. Since the former is a conformal invariant gauge theory in $2+1$ dimensions, it is conceivably related to a gravity theory with an anti–de Sitter solution in $3+1$ dimensions, in the framework of the AdS/CFT correspondence. QED_3, conversely, is not conformal invariant. Within this approach, therefore, it could be somehow related to the conformal anomaly [14]. We shall explore these ideas elsewhere.

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APPENDIX A: THE PHASE-SPACE FACTOR

Here we are going to determine the phase-space factor [1]. Let us consider Eq. (21) and write

$$\int d\Phi (2\pi)^3 \delta^3(0) \equiv \mathcal{T}^\gamma, \tag{A1}$$

where $\mathcal{T}$ is the characteristic time scale of the system. For dimensional reasons, we have $\gamma + 3 = 2(\alpha + 1/2)$ and consequently $\gamma = -2(1-\alpha)$. This justifies the γ dependence in Eq. (22).

APPENDIX B: THE $\mathrm{Im}(\chi_\alpha)$

Let us show here that, for $\alpha = 0, 1/2$, indeed, the expression of $\mathrm{Im}(\chi_\alpha)$ relevant for the optical theorem, is given by $\varepsilon, \varepsilon^{1/2}$, respectively, and therefore just depends on ε.

Using $\chi_\alpha \equiv (\omega^2 - \mathbf{k}^2 + i\varepsilon)^{1-\alpha}$, we have, for $\alpha = 0$, $\chi_0 = (\omega^2 - \mathbf{k}^2 + i\varepsilon)$ and evidently $\mathrm{Im}(\chi_0) = \varepsilon \propto \mathcal{T}^{-2}$.

For the case $\alpha = 1/2$, notice that the condition for the optical theorem to be satisfied is

$$\frac{2\mathrm{Im}(\chi_{1/2})}{[(\omega^2 - \mathbf{k}^2)^2 + \varepsilon^2]^{1/2}} = \frac{K\mathcal{T}^{-1}}{[(\omega^2 - \mathbf{k}^2)^2 + \varepsilon^2]^{1/2}}, \tag{B1}$$

for some dimensionless constant K. Squaring this equation and multiplying both the numerators by ε, we obtain that both sides are proportional to $\delta(\omega^2 - \mathbf{k}^2)$. As a consequence, we must equate the numerators at $\omega^2 - \mathbf{k}^2 = 0$, namely,

$$2\mathrm{Im}(\chi_{1/2})|_{\omega^2 = \mathbf{k}^2} = (2\varepsilon)^{1/2} = K\mathcal{T}^{-1}, \tag{B2}$$

which completes the proof for $\alpha = 1/2$.

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