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BPS states in supersymmetric chiral models with higher derivative terms

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We study the higher derivative chiral models with four supercharges and Bogomol'nyi–Prasad–Sommerfield (BPS) states in these models. The off-shell Lagrangian generically includes higher powers of the auxiliary fields F, which causes distinct on-shell branches associated with the solutions to the auxiliary fields equation. We point out that the model admits a supersymmetric completion of arbitrary higher derivative bosonic models of a single complex scalar field, and an arbitrary scalar potential can be introduced even without superpotentials. As an example, we present a supersymmetric extension of the Faddeev–Skyrme model without four time derivatives, in contrast to the previously proposed supersymmetric Faddeev–Skyrme-like model containing four time derivatives. In general, higher derivative terms together with a superpotential result in deformed scalar potentials. We find that higher derivative corrections to $1/2$ BPS domain walls and $1/2$ BPS lumps are exactly canceled out, while the $1/4$ BPS lumps (as compact baby Skyrmions) depend on a characteristic feature of the higher derivative models. We also find a new $1/4$ BPS condition for domain wall junctions, which generically receives higher derivative corrections.

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I. INTRODUCTION

Low-energy dynamics of field theories can be described by only light fields such as Nambu–Goldstone modes when one integrates out massive modes. The low-energy effective theories are usually expanded by derivative expansions; thereby, they inevitably contain higher derivative terms of fields. Chiral perturbation theory is such a theory describing low-energy pion dynamics in QCD with a chiral symmetry breaking [1]. The Skyrme model [2], which is a nonlinear sigma model with fourth-order derivative terms, is one of such a class. Supergravity as a low-energy effective theory of string theory should have higher derivative correction terms [3]. Other examples include world-volume effective actions of solitonic objects such as topological solitons in field theories and D-branes in string theories [4]. The effective theory of a D-brane is described by the Dirac–Born–Infeld (DBI) action [5] containing an infinite number of derivatives. Higher derivative field theories are also useful in other areas of physics. In the cosmological context, higher derivative theories are proposed for inflation models such as the K inflation [6] and the Galileon inflation [7]. These higher derivative models are known to admit characteristic soliton solutions such as k defects [8], compactons [9,10], and so on.

On the other hand, supersymmetry is one of the most important tools in modern high-energy physics. It has not only been considered as the most promising candidate to solve the naturalness problem in the Standard Model in the phenomenological side, but also it plays important roles to control quantum corrections in supersymmetric field theories, leading to determining exact low-energy dynamics [11]. When one constructs low-energy effective theories in supersymmetric field theories, one is required to consider higher derivative corrections in a supersymmetric manner. It is, however, not so easy to construct a supersymmetric completion of general higher derivative theories. Off-shell superfield formalisms are useful to write down actions of supersymmetric higher derivative models. In particular, the four-dimensional $\mathcal{N}=1$ superfield formalism that incorporates the chiral superfield Φ is a simple starting point. It is, however, known that not all the off-shell supersymmetric higher derivative models exhibit good physical properties. Off-shell formulations of higher derivative terms often encounter an auxiliary field problem; chiral superfields with space-time derivatives (e.g., $\partial_m\Phi$) sometimes introduce derivative interactions of the auxiliary field F. Consequently, the auxiliary fields become dynamical. It is hard to eliminate them, and the on-shell structure of the action is not obvious. For instance, the chiral Lagrangian of QCD contains the Wess–Zumino–Witten (WZW) term to reproduce the quantum anomaly at low energy. However, a supersymmetric completion of the WZW term proposed in Ref. [12] suffers from this auxiliary field problem [13,14]. It was proposed in Ref. [15] that a supersymmetric WZW term in superspace can be constructed without the auxiliary field problem if the number of chiral superfields is

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doubled.[1] The auxiliary field problem would be more problematic if one were to introduce a superpotential, so one could not introduce a potential.

Nevertheless, supersymmetric higher derivative models of which the building blocks are the chiral superfields are studied in various contexts. Among other things, the chiral models studied in Refs. [18,19] provide a good grounding for studying supersymmetric higher derivative theories. In this model, the auxiliary fields are not accompanied by the space-time derivatives, and therefore they can be eliminated by their equations of motion. In principle, it is possible to write down the explicit on-shell actions of the models. In particular, the scalar potential that shows up after eliminating the auxiliary fields looks more apparent [20]. The coupling of higher derivative chiral models to supergravity was also achieved in this type of model [21,22]. A supersymmetric DBI action was constructed in Ref. [23]. The other examples include a supersymmetric completion of the $P(X,\varphi)$ model [19], the supersymmetric Galileon inflation models [24], and models for the ghost condensation [25]. The same structure appears in quantum effective actions [26,27]. A higher derivative supersymmetric $\mathbb{C}P^1$ model free from the auxiliary field problem was also considered previously as a supersymmetric extension [28,29] of the Faddeev–Skyrme model [30] and a supersymmetric baby Skyrme model [18,31]. The formalism in Refs. [18,19] has been also applied to the construction of manifestly supersymmetric higher derivative corrections to supersymmetric nonlinear realizations [32].

In the former half of this paper, we study higher derivative chiral models developed in Refs. [18,19] in the superfield formalism, where higher derivative terms can be introduced as a tensor with two holomorphic and symmetric indices and two antiholomorphic and symmetric indices. We find a surprising fact that has been overlooked in past studies on the supersymmetric completions of various higher derivative models. The model with a single chiral superfield admits a supersymmetric extension of *arbitrary* bosonic models that consist of a single complex scalar field. As an example, we present a supersymmetric extension of the Faddeev–Skyrme model [30]. The bosonic part of this model does not contain four time derivatives. This is in contrast to the previously proposed supersymmetric extension [28,29] of the Faddeev–Skyrme model that contains an additional four-derivative term that includes four time derivatives. Moreover, we point out that an arbitrary scalar potential can be introduced even without the superpotential. We further work out the higher derivative chiral models with superpotentials. The resulting on-shell Lagrangians are highly nonlinear. We study perturbative analysis revealing the possibility of the ghost kinetic term and deformations of the scalar potential.

Meanwhile, Bogomol'nyi–Prasad–Sommerfield (BPS) topological solitons play important roles in the study of nonperturbative dynamics of supersymmetric field theories since they break and preserve a fraction of supersymmetry, belong to short supermultiplets, and consequently are stable against quantum corrections [33]. When a BPS soliton preserves p/q of supersymmetry, it is called a p/q BPS soliton. For instance, Yang–Mills instantons, BPS monopoles, vortices, lumps, and domain walls [34] are of $1/2$ BPS and composite solitons such as domain wall junctions are of $1/4$ BPS in theories with four supercharges [35–37] and eight supercharges [38] (see Refs. [39–41] as a review for a fraction of supersymmetry for BPS states). BPS solitons remain important in supersymmetric field theories with higher derivative terms. Prime examples of such solitons contain $1/2$ BPS lumps in supersymmetric $\mathbb{C}P^1$ models with a four-derivative term [42]; supersymmetric baby Skyrmions, which are compactons [18,31]; and BPS compactons in K-field theories [43,44]. The higher derivative $\mathbb{C}P^1$ model in Ref. [42] appears as the effective theory of a $1/2$ BPS non-Abelian vortex [45] in supersymmetric theories with eight supercharges. Then, the $1/2$ BPS lumps in the vortex correspond to Yang–Mills instantons in the bulk [46]. While a few examples of BPS solitons in higher derivative supersymmetric theories have been studied thus far, a systematic study of BPS solitons in such theories is needed.

In the latter half of this paper, we give a general framework to examine BPS states in supersymmetric higher derivative chiral models. Our framework does not only reproduce, in a unified manner, a few remarkable previous studies of the BPS bounds in the supersymmetric higher derivative models admitting BPS baby Skyrmions [18,31], BPS compactons [43,44], and BPS lumps [42] but also includes the more general cases with several new BPS states: $1/2$ BPS domain walls, $1/4$ BPS domain wall junctions, $1/2$ and $1/4$ BPS lumps, and baby Skyrmions. In particular, we find BPS baby Skyrmions found in Ref. [18] to be $1/4$ BPS states. We show that $1/2$ BPS domain walls and $1/2$ BPS lumps do not receive higher derivative corrections while $1/4$ BPS domain wall junctions do.

The organization of this paper is as follows. In Sec. II, we introduce the supersymmetric higher derivative chiral model with four supercharges. We write down the equation of motion for the auxiliary fields and analyze the structure of the on-shell Lagrangians. In particular, we introduce the superpotential, and the deformation of the scalar potential caused by the higher derivative terms is discussed. We then examine BPS states that preserve $1/2$ and $1/4$ of the original supersymmetry in subsequent sections. The $1/2$ BPS domain wall and $1/4$ BPS domain wall junctions are studied in Sec. III, and $1/2$ BPS and $1/4$ BPS lumps are studied in Sec. IV. Section V is devoted to conclusion and discussions. Notations and conventions of superfields are found in the Appendix.

[1] The actual form of the WZW term was derived in Refs. [16,17] and includes a Kähler tensor discussed in the next section.

II. HIGHER DERIVATIVE CHIRAL MODELS

In the first subsection, we present general higher derivative chiral models with multiple chiral superfields. In the second subsection, we further work out the models with a single chiral superfield without and with a superpotential.

A. General chiral models

We consider four-dimensional $\mathcal{N}=1$ supersymmetric higher derivative chiral models that have specific properties. The Lagrangian consists of chiral superfields Φ^i ($i=1,\ldots,N$), for which the component expansion in the chiral base $y^m = x^m + i\theta\sigma^m\bar{\theta}$ is

$$\Phi^i(y,\theta) = \varphi^i(y) + \theta\psi^i(y) + \theta^2 F^i(y), \tag{2.1}$$

where φ^i is the complex scalar field, ψ^i is the Weyl fermion, and F^i is the complex auxiliary field. The notations and conventions of the chiral superfield are found in the Appendix.

The supersymmetric Lagrangian with higher derivative terms is given by

$$\begin{aligned}\mathcal{L} = &\int d^4\theta K(\Phi^i, \Phi^{\dagger\bar{j}}) \\ &+ \frac{1}{16}\int d^4\theta \Lambda_{ik\bar{j}\bar{l}}(\Phi,\Phi^\dagger) D^\alpha\Phi^i D_\alpha\Phi^k \bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} \\ &+ \left(\int d^2\theta W(\Phi^i) + \text{H.c.}\right),\end{aligned} \tag{2.2}$$

where K is the Kähler potential and W is a superpotential as usual. Higher derivative terms are produced by the second term proportional to $\Lambda_{ik\bar{j}\bar{l}}$, which is a (2,2) Kähler tensor symmetric in holomorphic and antiholomorphic indices, of which the components are functions of Φ^i and $\Phi^{\dagger\bar{i}}$ (admitting space-time derivatives acting on them).[2] As we will see, the most important feature of this model is that the auxiliary fields never become dynamical; the equation of motion for the auxiliary fields is an algebraic equation.

Now, we examine the component structure of the model (2.2). The fourth derivative part of the Lagrangian (2.2) has an essential property. This term is evaluated as

$$\begin{aligned}&D^\alpha\Phi^i D_\alpha\Phi^k \bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} \\ &\quad = 16\theta^2\bar{\theta}^2[(\partial_m\varphi^i\partial^m\varphi^k)(\partial_m\bar{\varphi}^{\bar{j}}\partial^m\bar{\varphi}^{\bar{l}}) - 2\partial_m\varphi^i F^k\partial^n\bar{\varphi}^{\bar{j}}\bar{F}^{\bar{l}} \\ &\qquad + F^i\bar{F}^{\bar{j}}F^k\bar{F}^{\bar{l}}] + I_f,\end{aligned} \tag{2.3}$$

where I_f stands for terms that contain fermion fields. Since the bosonic part of the right-hand side of (2.3) saturates the Grassmann coordinate $\theta^2\bar{\theta}^2$, only the lowest component of the tensor $\Lambda_{ik\bar{j}\bar{l}}$ contributes to the bosonic part of the Lagrangian. Therefore, the bosonic part of the Lagrangian (2.2) is

$$\begin{aligned}\mathcal{L}_b = &\frac{\partial^2 K}{\partial\varphi^i\partial\bar{\varphi}^{\bar{j}}}(-\partial_m\varphi^i\partial^m\bar{\varphi}^{\bar{j}} + F^i\bar{F}^{\bar{j}}) + \frac{\partial W}{\partial\varphi^i}F^i + \frac{\partial\bar{W}}{\partial\bar{\varphi}^{\bar{j}}}\bar{F}^{\bar{j}} \\ &+ \Lambda_{ik\bar{j}\bar{l}}(\varphi,\bar{\varphi})[(\partial_m\varphi^i\partial^m\varphi^k)(\partial_n\bar{\varphi}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}) \\ &- \partial_m\varphi^i F^k\partial^m\bar{\varphi}^{\bar{j}}\bar{F}^{\bar{l}} + F^i\bar{F}^{\bar{j}}F^k\bar{F}^{\bar{l}}].\end{aligned} \tag{2.4}$$

This Lagrangian exhibits a higher derivative model that has the following properties: (I) the higher derivative terms are governed by the tensor $\Lambda_{ik\bar{j}\bar{l}}$, and (II) the model is manifestly (off-shell) supersymmetric and Kähler invariant provided that K and W are scalars and $\Lambda_{ik\bar{j}\bar{l}}$ is a tensor. Among other things, the auxiliary fields do not have a space-time derivative,[3] and they are eliminated by the following equation of motion:

$$\frac{\partial^2 K}{\partial\varphi^i\partial\bar{\varphi}^{\bar{j}}}F^i - 2\partial_m\varphi^i F^k\Lambda_{ik\bar{j}\bar{l}}\partial^m\bar{\varphi}^{\bar{l}} + 2\Lambda_{ik\bar{j}\bar{l}}F^iF^k\bar{F}^{\bar{l}} + \frac{\partial\bar{W}}{\partial\bar{\varphi}^{\bar{j}}} = 0. \tag{2.5}$$

This is an algebraic equation and, in principle, solvable. However, Eq. (2.5) is a simultaneous equation of cubic power, and it is hard to find explicit solutions F_i. We comment that when $W=0$ at least $F_i = 0$ is a solution. In this case, the on-shell Lagrangian becomes

$$\mathcal{L}_b = -\frac{\partial^2 K}{\partial\varphi^i\partial\bar{\varphi}^{\bar{j}}}\partial_m\varphi^i\partial^m\bar{\varphi}^{\bar{j}} + \Lambda_{ik\bar{j}\bar{l}}(\partial_m\varphi^i\partial^m\varphi^k)(\partial_n\bar{\varphi}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}). \tag{2.6}$$

In general, there are more solutions other than $F_i = 0$, which we will show explicitly for models with one component field.

B. Chiral models of one component

Now, we consider the single chiral superfield Φ for simplicity. The equation of motion for the auxiliary field becomes

$$K_{\varphi\bar{\varphi}}F - 2F(\partial_m\varphi\partial^m\bar{\varphi} - F\bar{F})\Lambda(\varphi,\bar{\varphi}) + \frac{\partial\bar{W}}{\partial\bar{\varphi}} = 0. \tag{2.7}$$

Here, $K_{\varphi\bar{\varphi}} = \frac{\partial K}{\partial\varphi\partial\bar{\varphi}}$. We solve Eq. (2.7) in the $W=0$ and $W\neq 0$ cases separately.

[2]This tensor term was obtained in Ref. [17] as a part of the supersymmetric Wess–Zumino–Witten term.

[3]This is true only for the purely bosonic terms. There are derivative interactions of the auxiliary fields in the fermionic contributions I_f [19]. They are irrelevant when classical configurations of fields are concerned.

1. $W = 0$ case

When there is no superpotential, the equation for the auxiliary field becomes

$$K_{\varphi\bar{\varphi}}F - 2F(\partial_m\varphi\partial^m\bar{\varphi} - F\bar{F})\Lambda = 0. \tag{2.8}$$

Then, the solutions are found to be

$$F = 0, \tag{2.9}$$

$$F\bar{F} = -\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi}. \tag{2.10}$$

There are two different on-shell branches associated with the solutions (2.9) and (2.10).

For the first solution (2.9), the bosonic part of the on-shell Lagrangian is

$$\mathcal{L}_{1b} = -K_{\varphi\bar{\varphi}}\partial_m\varphi\partial^m\bar{\varphi} + (\partial_m\varphi\partial^m\varphi)(\partial_n\bar{\varphi}\partial^n\bar{\varphi})\Lambda. \tag{2.11}$$

The first term is the ordinary kinetic term, and the second term contains higher derivative correction terms. We call this the canonical branch.

An example of the model is the $\mathcal{N} = 1$ supersymmetric DBI action for the world-volume theory of single D3-brane. The corresponding Kähler metric is canonical, $K_{\varphi\bar{\varphi}} = 1$, and the function Λ is given by [23]

$$\Lambda = \frac{1}{1 + A + \sqrt{(1+A)^2 - B}},$$
$$A = \partial_m\Phi\partial^m\Phi^\dagger, \qquad B = \partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger. \tag{2.12}$$

The other examples include a supersymmetric completion of the $P(X, \varphi)$ model [19], the supersymmetric Galileon inflation models [24], and models for the ghost condensation [25].

Another example of Λ that has been overlooked in the literature [18,28,29,31] is

$$\Lambda = \kappa(\partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger)^{-1}\frac{1}{(1+\Phi\Phi^\dagger)^4} \times [(\partial_m\Phi^\dagger\partial^m\Phi)^2 - \partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger], \tag{2.13}$$

where κ is a parameter. Then, with the Fubini–Study metric $K_{\varphi\bar{\varphi}} = \frac{1}{(1+|\varphi|^2)^2}$ for the $\mathbb{C}P^1$ model, the bosonic part of the Lagrangian becomes

$$\mathcal{L}_{1b} = -\frac{\partial_m\varphi\partial^m\bar{\varphi}}{(1+|\varphi|^2)^2} + \kappa\frac{(\partial_m\varphi\partial^m\bar{\varphi})^2 - |\partial_m\varphi\partial^m\varphi|^2}{(1+|\varphi|^2)^4}. \tag{2.14}$$

This is nothing but the Faddeev–Skyrme model [30]. The previous trials to construct an $\mathcal{N} = 1$ supersymmetric extension of the Faddeev–Skyrme model concluded that one needs an extra four-derivative term containing four time derivatives [28,29], while the Lagrangian in Eq. (2.14) does not. It was discussed in Ref. [29] that such a term destabilizes Hopfions (knot solitons). Therefore, the Lagrangian (2.2) provides an $\mathcal{N} = 1$ supersymmetric extension of the Faddeev–Skyrme model without four time derivatives, which is expected to give stable Hopfions.

More generally, since the function Λ is completely arbitrary, one can construct supersymmetric extension of *any* bosonic models that consist of a complex scalar field φ. More surprisingly, we further point out that it is also possible to introduce an arbitrary scalar potential $V(\varphi, \varphi^*)$ even without superpotentials, by choosing Λ as

$$\Lambda = -(\partial_m\Phi\partial^m\Phi\partial_n\Phi^\dagger\partial^n\Phi^\dagger)^{-1}V(\Phi, \Phi^\dagger). \tag{2.15}$$

However, as we will clarify later, superpotentials play an important role when one considers BPS solutions.

On the other hand, for the second solution (2.10), the bosonic part of the on-shell Lagrangian is

$$\mathcal{L}_{2b} = (|\partial_m\varphi\partial^m\varphi|^2 - (\partial_m\varphi\partial^m\bar{\varphi})^2)\Lambda - \frac{(K_{\varphi\bar{\varphi}})^2}{4\Lambda}. \tag{2.16}$$

In this branch, the canonical kinetic term disappears.[4] This model was first studied in Ref. [18] where supersymmetric extensions of the baby Skyrme model are discussed. We note that the second branch (2.16) does not have the smooth limit to the canonical theory ($\Lambda \to 0$). Therefore, we call this the noncanonical branch. Since $F\bar{F}$ should be positive semidefinite, the second solution (2.10) is consistent only in the region

$$-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi} \geq 0. \tag{2.17}$$

We comment on the last term in Eq. (2.16). The term $(K_{\varphi\bar{\varphi}})^2/4\Lambda$ is considered as a scalar potential term since it remains when the function Λ does not depend on fields with space-time derivatives. For a vacuum configuration, the condition (2.17) implies $\Lambda < 0$ for the positive definite Kähler metric $K_{\varphi\bar{\varphi}} > 0$. Then, the scalar potential at a vacuum becomes negative even for the manifestly supersymmetric construction of the model. One resolution of this puzzle is the existence of ghosts, i.e., fields with a kinetic term of the wrong sign. However, it is not obvious whether ghosts exist or not since there is no kinetic term in the Lagrangian (2.16) and no consistent free theory is defined. In that case, K loses its meaning of the Kähler potential, and what determines the sign of the potential energy is the function K. When $K_{\varphi\bar{\varphi}}$ is negative, Λ and the scalar

[4]When Λ is chosen as $\Lambda = -(|\partial_m\varphi\partial^m\varphi|^2 - (\partial_m\varphi\partial^m\bar{\varphi})^2)^{-1}\times \partial_n\varphi\partial^n\bar{\varphi}$, the canonical kinetic term recovers. However quite nonlinear higher derivative terms remain in the Lagrangian due to the factor $1/\Lambda$. This possibility was discussed in the context of higher derivative supergravity models [21].

potential become positive. Actually, choosing the functions of K and Λ appropriately, one can construct scalar potentials that have desired properties [18].

2. $W \neq 0$ case

When $W \neq 0$, one eliminates $\bar{F}$ in (2.7) and obtains the equation for the auxiliary field F:

$$2\Lambda(\varphi,\bar{\varphi})\frac{\partial W}{\partial\varphi}F^3 + \frac{\partial\bar{W}}{\partial\bar{\varphi}}(K_{\varphi\bar{\varphi}} - 2\Lambda(\varphi,\bar{\varphi})\partial_m\varphi\partial^m\bar{\varphi})F + \left(\frac{\partial\bar{W}}{\partial\bar{\varphi}}\right)^2 = 0. \tag{2.18}$$

When there are no higher derivative corrections $\Lambda = 0$, one recovers the ordinary F-term solution $F = -\frac{1}{K_{\varphi\bar{\varphi}}}\frac{\partial\bar{W}}{\partial\bar{\varphi}}$. Since Eq. (2.18) is an algebraic equation of cubic power, the solutions are obtained by the Cardano's method [20],

$$F = \omega^k \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}} + \omega^{3-k}\sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}},$$
$$k = 0, 1, 2, \qquad \omega^3 = 1, \tag{2.19}$$

where ω is a cubic root of unity and p and q are given by

$$p = \frac{1}{2\Lambda(\varphi,\bar{\varphi})}\left(\frac{\partial W}{\partial\varphi}\right)^{-1}\left(\frac{\partial\bar{W}}{\partial\bar{\varphi}}\right)(K_{\varphi\bar{\varphi}} - 2\Lambda(\varphi,\bar{\varphi})\partial_m\varphi\partial^m\bar{\varphi}), \tag{2.20}$$

$$q = \frac{1}{2\Lambda(\varphi,\bar{\varphi})}\left(\frac{\partial W}{\partial\varphi}\right)^{-1}\left(\frac{\partial\bar{W}}{\partial\bar{\varphi}}\right)^2. \tag{2.21}$$

The on-shell Lagrangian is obtained by substituting the solutions of the auxiliary field into the Lagrangian (2.4),

$$\mathcal{L}_b = -\frac{\partial^2 K}{\partial\varphi\partial\bar{\varphi}}\partial_m\varphi\partial^m\bar{\varphi} + (\partial_m\varphi\partial^m\varphi)(\partial_n\bar{\varphi}\partial^n\bar{\varphi})\Lambda + \tilde{F}\bar{\tilde{F}}(-K_{\varphi\bar{\varphi}} + 2\Lambda\partial_m\varphi\partial^m\bar{\varphi}) - 3(\tilde{F}\bar{\tilde{F}})^2\Lambda, \tag{2.22}$$

where $\tilde{F}$ ($\bar{\tilde{F}}$) is one of the solutions in Eq. (2.19). Therefore, there are three different on-shell branches in this model. We note that, although the model is corrected by higher derivative terms, the supersymmetry requires correction terms in the scalar potential that do not contain derivative terms. In particular, the scalar potential of the model is calculated to be

$$V(\varphi) = |\tilde{F}|^2(K_{\varphi\bar{\varphi}} + 3\Lambda^{(0)}|\tilde{F}|^2). \tag{2.23}$$

Here, $\Lambda^{(0)}$ is the function Λ where $\partial_m\varphi = 0$. We note that, even for the manifestly supersymmetric Lagrangian (2.2) with the positive Kähler metric $K_{\varphi\bar{\varphi}}$, a negative scalar potential is possible when $\Lambda^{(0)} < 0$. Again, this fact would be an indication of ghost states in the theory. As we will see below, the on-shell Lagrangian potentially includes ghost states.

Now, we examine the structure of the on-shell Lagrangians in each branch. To see the effects of superpotentials, we write down the explicit on-shell component Lagrangian. In particular, we examine the relation between the positive definiteness of the scalar potential and the ghost states. A similar analysis about the scalar potential was performed in the context of the four-dimensional $\mathcal{N} = 1$ supergravity [21], in which negative potentials are not problematic. On the other hand, negative potentials could be problematic for the rigid supersymmetric case, on which we focus here.

We note that when $W \neq 0$ a solution $F = 0$ is not allowed. We first consider the canonical branch where the solution of the auxiliary field (2.19) has the smooth limit $\Lambda \to 0$ [20]. We look for a perturbative expression of the Lagrangian for small Λ. The solution of the auxiliary field is expanded as

$$F = F_0 + \alpha F_1 + \alpha^2 F_2 + \cdots, \tag{2.24}$$

where α is a parameter associated with the small Λ expansion and F_0 is the solution in $\alpha = 0$ ($\Lambda = 0$). This is given by

$$F_0 = -(K_{\varphi\bar{\varphi}})^{-1}\bar{W}'. \tag{2.25}$$

Here, $W' = \frac{\partial W}{\partial\varphi}$, and $\bar{W}'$ is the complex conjugate of W'. The explicit forms of F_1 and F_2 are obtained iteratively. They are found to be

$$F_1 = \frac{2\Lambda\bar{W}'}{(K_{\varphi\bar{\varphi}})^2}\left[\frac{W'\bar{W}'}{(K_{\varphi\bar{\varphi}})^2} - \partial_m\varphi\partial^m\bar{\varphi}\right], \tag{2.26}$$

$$F_2 = -\frac{4\Lambda^2\bar{W}'}{(K_{\varphi\bar{\varphi}})^7}(\bar{W}'W' - K_{\varphi\bar{\varphi}}\partial_m\varphi\partial^m\bar{\varphi}) \times \{3W'\bar{W}' - (K_{\varphi\bar{\varphi}})^2\partial_m\varphi\partial^m\bar{\varphi}\}. \tag{2.27}$$

Then, substituting these solutions into the auxiliary field F in the Lagrangian (2.22), we obtain the on-shell Lagrangian (we take $\alpha = 1$ for simplicity),

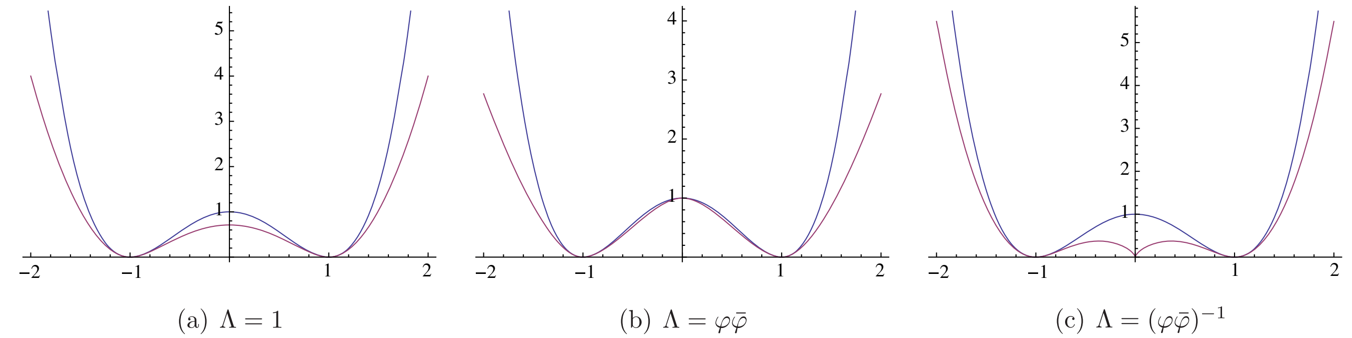

FIG. 1 (color online). Examples of the deformed potentials $V(|\varphi|)$ for $K_{\varphi\bar{\varphi}}, W = \Phi - \frac{1}{3}\Phi^3$. The upper (blue) lines represent the undeformed potentials, while the lower (red) lines are deformed ones. The figures correspond to the $k = 0$ solution.

$$\mathcal{L}_b = -K_{\varphi\bar{\varphi}}\partial_m\varphi\partial^m\bar{\varphi} - \frac{2\Lambda V_0}{K_{\varphi\bar{\varphi}}}\partial_m\varphi\partial^m\bar{\varphi} + \frac{8\Lambda^2 V_0^2}{(K_{\varphi\bar{\varphi}})^3}\partial_m\varphi\partial^m\bar{\varphi} + \Lambda|\partial_m\varphi\partial^m\varphi|^2 - \frac{4V_0\Lambda^2}{(K_{\varphi\bar{\varphi}})^2}(\partial_m\varphi\partial^m\bar{\varphi})^2 - V_0 + \frac{\Lambda V_0^2}{(K_{\varphi\bar{\varphi}})^2} - \frac{4\Lambda^2 V_0^3}{(K_{\varphi\bar{\varphi}})^4} + \mathcal{O}(\alpha^4). \tag{2.28}$$

Here, $V_0 = \frac{1}{K_{\varphi\bar{\varphi}}}|W'|^2$ is the ordinary scalar potential in the supersymmetric chiral models. We note that the scalar potential is deformed by the nonzero Λ and the vacuum structure clearly depends on the structure of the function Λ. The examples of the deformed scalar potentials are found in Fig. 1. The Lagrangian contains an infinite number of the higher derivative terms that are induced by nonzero Λ and W. The structure of the derivative terms is completely determined by supersymmetry. We point out that, even for the canonical kinetic term, it is deformed by Λ. Up to $\mathcal{O}(\Lambda^2)$, it is given by

$$\mathcal{L}_K = -\left[K_{\varphi\bar{\varphi}} + \frac{2\Lambda V_0}{K_{\varphi\bar{\varphi}}} - \frac{8\Lambda^2 V_0^2}{(K_{\varphi\bar{\varphi}})^3}\right]\partial_m\varphi\partial^m\bar{\varphi} + \mathcal{O}(\Lambda^3). \tag{2.29}$$

Since Λ is an arbitrary function, the sign of the kinetic term can be flipped even for the positive-definite Kähler metric $K_{\varphi\bar{\varphi}}$. If the sign of the kinetic term is changed, there appear ghost states in the theory [47]. In that case, the model shows instability caused by the higher derivatives. This fact leads to the nonpositive-semidefinite potential (2.23) even for supersymmetric theories. The sign of the kinetic term depends on the explicit forms of the functions Λ and W. Although it is important and interesting, we do not pursue the (non)existence of the ghost states in this paper. We also note that the metric of the target space of the nonlinear sigma model in the Lagrangian (2.29) does not have to be Kähler anymore even though it is $\mathcal{N} = 1$ supersymmetric.

Next, we study the effect of the superpotential in the noncanonical branch associated with the solution (2.10). Since we cannot take the $\Lambda \to 0$ limit, we consider the small W perturbation around $W = 0$. The solution of the auxiliary field is expanded as

$$F = F_0' + \beta F_1' + \beta^2 F_2' + \cdots, \tag{2.30}$$

where β is a parameter associated with the small W expansion and

$$F_0' = \sqrt{-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi}}. \tag{2.31}$$

Here, we choose a real solution of F_0. Using the $U(1)_R$ symmetry, we make the superpotential be real and positive. Then, the solutions F_1' and F_2' are found to be

$$F_1' = -\frac{W'}{4\Lambda}\left(-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial\bar{\varphi}\right)^{-1}, \tag{2.32}$$

$$F_2' = -\frac{3(W')^2}{32\Lambda^2}\left(-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial\bar{\varphi}\right)^{-\frac{5}{2}}. \tag{2.33}$$

The on-shell Lagrangian is

$$\mathcal{L}_b = (|\partial_m\varphi\partial^m\bar{\varphi}|^2 - (\partial_m\varphi\partial^m\bar{\varphi})^2)\Lambda - \frac{(K_{\varphi\bar{\varphi}})^2}{4\Lambda} - 2(K_{\varphi\bar{\varphi}}V_0)^{\frac{1}{2}}\left(-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi}\right)^{\frac{1}{2}} - \frac{K_{\varphi\bar{\varphi}}V_0}{16\Lambda}\left(-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi}\right)^{-1} + \mathcal{O}(\beta^3). \tag{2.34}$$

We can observe that the scalar potential $(K_{\varphi\bar{\varphi}})^2/4\Lambda$ is deformed by the superpotential W.

Finally, a comment is in order. We started from the four-dimensional theory. However, the lower-dimensional models, such as three-dimensional $\mathcal{N} = 2$ and two-dimensional $\mathcal{N} = (2, 2)$ theories can be easily obtained by the dimensional reduction. Actually, the $W = 0$ case corresponds to the three-dimensional $\mathcal{N} = 2$ models discussed in Ref. [18].

III. BPS DOMAIN WALLS AND THEIR JUNCTION

In this and the next sections, we study BPS configurations in the supersymmetric higher derivative chiral models discussed in the previous section. Since we consider models with scalar fields, we focus on the BPS domain walls and lumps in the following.

BPS equations that preserve a fraction of supersymmetry are obtained from the condition that the on-shell supersymmetry transformation of the fermion vanishes $\delta_\xi^{\text{on}}\psi_\alpha = 0$. Here, δ_ξ^{on} represents the on-shell supersymmetry transformation by parameters ξ_α, $\bar{\xi}^{\dot{\alpha}}$. The off-shell supersymmetry transformation δ_ξ^{off} of the fermion is given by

$$\delta_\xi^{\text{off}}\psi_\alpha = i\sqrt{2}(\sigma^m)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}\partial_m\varphi + \sqrt{2}\xi_\alpha F. \qquad (3.1)$$

By substituting a solution of the auxiliary field F into $\delta_\xi^{\text{off}}\psi_\alpha = 0$ and assuming a specific field configuration together with appropriate Killing spinor conditions on ξ_α, $\bar{\xi}^{\dot{\alpha}}$, we find corresponding on-shell BPS equations. Since there is the variety of branches associated with the solutions F in our model, we study each branch separately.

A. 1/2 BPS domain walls

When a scalar field model with an ordinary canonical kinetic term has a potential with several vacua, there is a domain wall solution that interpolates between these vacua. We look for 1/2 BPS domain wall solutions in the higher derivative model (2.2). We consider domain wall configurations of the complex scalar field φ. Namely, the field depends on the one direction $\varphi = \varphi(x^1)$. We first consider the case in which the superpotential exists. In this case, the solution $F = 0$ is not allowed. Therefore, we generically consider the $F \neq 0$ branch. The Killing spinor condition for the 1/2 BPS domain wall configuration is [34]

$$\xi_\alpha = -ie^{i\eta}(\sigma^1)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}}. \qquad (3.2)$$

Here, η is a phase factor. Then, the off-shell BPS equation is

$$\partial_1\varphi = e^{i\eta}F. \qquad (3.3)$$

By plugging a solution in (2.19) into the right-hand side of Eq. (3.3) and arranging the resulting condition by $\partial_1\varphi$, we obtain the on-shell BPS condition. Here, instead of that, we use the equation of motion for the auxiliary field F in order to observe the universal property of the three solutions (2.19). Substituting the BPS condition (3.3) into the equation of motion for F, we obtain

$$K_{\varphi\bar{\varphi}}e^{-i\eta}\partial_1\varphi + \{-2e^{-i\eta}\partial_1\varphi\cdot\partial_1\varphi\partial_1\bar{\varphi} + 2e^{-2i\eta}(\partial_1\varphi)^2e^{i\eta}\partial_1\bar{\varphi}\}\Lambda + \frac{\partial\bar{W}}{\partial\bar{\varphi}} = 0. \qquad (3.4)$$

The higher derivative terms including Λ cancel out, and we obtain the on-shell BPS equation

$$K_{\varphi\bar{\varphi}}\partial_1\varphi + e^{i\eta}\frac{\partial\bar{W}}{\partial\bar{\varphi}} = 0. \qquad (3.5)$$

Equation (3.5) is nothing but the ordinary (without higher derivative terms) BPS condition for the domain wall. This result suggests that, even for the existence of the three different on-shell branches in the model, the BPS domain wall cannot distinguish them. Furthermore, the on-shell energy density of the domain wall is evaluated as

$$\begin{aligned}\mathcal{E} &= K_{\varphi\bar{\varphi}}|\partial_1\varphi|^2 - |\partial_1\varphi|^4\Lambda - |\partial_1\varphi|^2(-K_{\varphi\bar{\varphi}} + 2\Lambda|\partial_1\varphi|^2) \\ &\quad + 3|\partial_1\varphi|^4\Lambda \\ &= -e^{-i\eta}\partial_1 W + \text{H.c.}\end{aligned} \qquad (3.6)$$

The last expression gives the tension of the ordinary BPS domain wall. Therefore, we conclude that all the higher derivative corrections to the solutions and energy are canceled out in the BPS domain walls. This is a consequence of the fact that the configuration depends on the one direction. It is easy to confirm that the solutions to the BPS condition (3.3) together with the equation of motion for the auxiliary field (3.5) satisfy the full equation of motion for the scalar field[5]:

$$\begin{aligned}&-\frac{\partial^3 K}{\partial\varphi\partial^2\bar{\varphi}}(|\partial_m\varphi|^2 - |F|^2) + \frac{\partial^2\bar{W}}{\partial\bar{\varphi}^2}\bar{F} + [|\partial_m\varphi\partial^m\varphi|^2 - 2|F|^2|\partial_m\varphi|^2 + |F|^4]\frac{\partial\Lambda}{\partial\bar{\varphi}} \\ &- \partial_m\left[-K_{\varphi\bar{\varphi}}\partial^m\varphi + 2\Lambda((\partial_n\varphi)^2\partial^m\bar{\varphi} - |F|^2\partial^m\varphi) + \{|\partial_n\varphi\partial^n\varphi|^2 - 2|F|^2|\partial_n\varphi|^2 + |F|^4\}\frac{\partial\Lambda}{\partial(\partial_m\bar{\varphi})}\right] = 0.\end{aligned} \qquad (3.7)$$

Next, we consider the case in which $W = 0$. Even for this case, there is the scalar potential $(K_{\varphi\bar{\varphi}})^2/4\Lambda$ in the Lagrangian (2.16). This branch corresponds to the $F \neq 0$ solution (2.10). Substituting the off-shell BPS

[5] We have assumed that Λ does not depends on the second space-time derivatives or higher of φ.

condition (3.3) into the equation of motion for the auxiliary field (2.8) and assuming $F \neq 0$, the on-shell BPS condition becomes

$$K_{\varphi\bar{\varphi}} = 0. \tag{3.8}$$

This condition never provides the domain wall equation. When there is no ghost, Eq. (3.8) is just a vacuum condition of the scalar potential $(K_{\varphi\bar{\varphi}})^2/4\Lambda$. Therefore, although there is a scalar potential in the noncanonical branch, superpotentials are necessary for 1/2 BPS domain wall solutions. We also note that the 1/2 BPS domain wall solution to Eq. (3.5) interpolates between "vacua" specified by the superpotential $W' = 0$ as its tension stands for. We stress that the condition $W' = 0$ does not always imply vacua of the scalar potential, especially in the noncanonical branch. The BPS domain walls remain intact even for the deformation of the scalar potential. Although there are other vacua that originate from the singularity of the function Λ [see Fig. 1(c)], domain walls that interpolate these vacua are not BPS and break all the supersymmetry.

B. 1/4 BPS domain wall junctions

We next consider 1/4 BPS domain wall junctions [35]. The scalar field depends on the two spacial directions x^1 and x^2. First, we consider the $W \neq 0$ case. We impose the Killing spinor conditions on the supersymmetry parameters,

$$\frac{1}{2}(\sigma^1 + i\sigma^2)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} = 0, \quad \frac{1}{2}(\sigma^1 - i\sigma^2)_{\alpha\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} = ie^{-i\eta}\xi_\alpha, \tag{3.9}$$

where η is a phase factor. Then, we obtain the BPS condition from the supersymmetry transformation (3.1),[6]

$$\bar{\partial}\varphi = e^{i\eta}F. \tag{3.10}$$

Here, F in the right-hand side is one of the solutions in (2.19). This is the 1/4 BPS condition. Substituting the condition (3.10) into the equation of motion (2.7) for the auxiliary field, we obtain the on-shell BPS equation on the scalar field:

$$K_{\varphi\bar{\varphi}}\bar{\partial}\varphi - \bar{\partial}\varphi(|\partial\varphi|^2 - |\bar{\partial}\varphi|^2)\Lambda + e^{i\eta}\frac{\partial\bar{W}}{\partial\bar{\varphi}} = 0. \tag{3.11}$$

When $\Lambda = 0$, the on-shell BPS equation (3.11) becomes that of the ordinary BPS domain wall junctions [35] of which the analytic solutions are studied in Ref. [37]. Different from the 1/2 BPS domain wall case, the higher derivative corrections do not cancel in Eq. (3.11). The solutions are deformed from the ones in Ref. [37] in general and depend on the explicit form of the function Λ.

Now, we confirm that the BPS solutions to (3.10) satisfy the full equation of motion for the scalar field (3.7). Using the BPS condition (3.10), we find the following terms in (3.7) vanish:

$$|\partial_m\varphi\partial^m\varphi|^2 - 2|F|^2|\partial_m\varphi|^2 + |F|^4 = 0. \tag{3.12}$$

By using the BPS equation and the equation of motion for the auxiliary field, we find that the other terms in (3.7) also vanish:

$$\begin{aligned} &-\frac{\partial^3 K}{\partial\varphi\partial^2\bar{\varphi}}(|\partial_m\varphi|^2 - |F|^2) + \frac{\partial^2\bar{W}}{\partial\bar{\varphi}^2}\bar{F} \\ &-\partial_m\left[-\frac{\partial^2 K}{\partial\varphi\partial\bar{\varphi}}\partial^m\varphi + 2\Lambda((\partial_n\varphi)^2\partial^m\bar{\varphi} - |F|^2\partial^m\varphi)\right] = 0. \end{aligned} \tag{3.13}$$

Therefore, we conclude that the solutions to the deformed BPS equation (3.11) actually satisfy the full equation of motion in Eq. (3.7).

The energy density of the domain wall junction is evaluated as

$$\begin{aligned} \mathcal{E} &= K_{\varphi\bar{\varphi}}\partial_i\varphi\partial_i\bar{\varphi} - (\partial_i\varphi\partial_i\varphi)(\partial_j\bar{\varphi}\partial_j\bar{\varphi})\Lambda \\ &\quad - |F|^2(-K_{\varphi\bar{\varphi}} + 2\Lambda\partial_i\varphi\partial_i\bar{\varphi}) + 3|F|^4\Lambda \\ &= \frac{1}{2}K_{\varphi\bar{\varphi}}(|\partial\varphi|^2 - |\bar{\partial}\varphi|^2) - 2\,\mathrm{Re}\left[e^{-i\eta}\frac{\partial W}{\partial\bar{z}}\right]. \end{aligned} \tag{3.14}$$

This is nothing but the expression of the ordinary (without higher derivative terms) domain wall junctions. After integration over the (x^1, x^2) plane, the first term gives the junction charge, and the second term gives the tension of the domain walls. They are evaluated on the boundary at the infinity of the (x^1, x^2) plane. Again, the junction charge and the domain wall tension are solely determined by the asymptotic boundary conditions of the scalar field and the superpotential and do not depend on the function Λ. Although the expression of the Bogomol'nyi bound of the energy is not deformed by the higher derivative terms, we stress that the solutions of the 1/4 BPS domain wall junction are potentially deformed in general.

Finally, we examine 1/4 BPS domain wall junctions in the $W = 0$ noncanonical branch. The Killing spinor and the off-shell BPS conditions are given by Eqs. (3.9) and (3.10). The solution of the auxiliary field is given in Eq. (2.10). Then, the on-shell BPS equation is found to be

$$\frac{1}{2}(|\partial\varphi|^2 - |\bar{\partial}\varphi|^2) = \frac{K_{\varphi\bar{\varphi}}}{2\Lambda}. \tag{3.15}$$

Equation (3.15) is supplemented by the consistency condition in Eq. (2.17). Again, the higher derivative corrections

[6] We define the complex coordinate $z = \frac{1}{2}(x^1 + ix^2)$ and derivatives $\partial = \frac{\partial}{\partial z} = \partial_1 - i\partial_2$. $\bar{\partial}$ is the complex conjugate of ∂.

to the on-shell $1/4$ BPS condition are not canceled. We will comment on this equation in the next section.

IV. BPS LUMPS AND BABY SKYRMIONS

Next, we consider lumps in $W = 0$ higher derivative models. We look for the BPS equation for lumps that depend on x^1 and x^2. Recall that for the $W = 0$ case the solutions of the auxiliary field are given by

$$F = 0, \tag{4.1}$$

$$F = e^{i\alpha}\sqrt{-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \partial_m\varphi\partial^m\bar{\varphi}}, \tag{4.2}$$

where α is a phase factor. There are the canonical and noncanonical branches associated with the solutions in Eqs. (4.1) and (4.2), respectively. In the following subsections, we examine BPS lump equations in each branch.

A. 1/2 BPS lumps

We first focus on the canonical branch associated with the solution (4.1). Hopfions in the supersymmetric higher derivative $\mathbb{C}P^1$ model of this type were discussed before [28,29]. BPS lumps in the supersymmetric higher derivative $\mathbb{C}P^1$ model were discussed in Ref. [42]. BPS lumps in the higher derivative $\mathbb{C}P^n$ nonlinear sigma models were discussed in a different context without supersymmetry [48].

In this branch, the BPS lump equation is obtained by imposing the first condition in (3.9) on the spinor $\bar{\xi}^{\dot{\alpha}}$, as can be seen in, e.g., Refs. [39–41]. Then, the BPS equation for lumps is given by [49]

$$\bar{\partial}\varphi = 0. \tag{4.3}$$

This is nothing but the ordinary $1/2$ BPS lump condition. This is confirmed by the Bogomol'nyi bound of the energy density. For the canonical branch, we have the energy density

$$\begin{aligned}\mathcal{E} &= K_{\varphi\bar{\varphi}}|\partial_i\varphi|^2 - |\partial_i\varphi\partial_i\varphi|^2\Lambda \\ &= |\bar{\partial}\varphi|^2(K_{\varphi\bar{\varphi}} - |\partial\varphi|^2\Lambda) - iK_{\varphi\bar{\varphi}}\varepsilon_{ij}\partial_i\varphi\partial_j\bar{\varphi} \\ &\geq -iK_{\varphi\bar{\varphi}}\varepsilon_{ij}\partial_i\varphi\partial_j\bar{\varphi},\end{aligned} \tag{4.4}$$

where we have assumed the condition $\Lambda \leq K_{\varphi\bar{\varphi}}/|\partial\varphi|^2$ for the positive-semidefiniteness of the energy $\mathcal{E}$. The right-hand side is nothing but the topological charge density for the $1/2$ BPS lump. The energy bound is saturated provided the condition (4.3) is satisfied. Then, we find that the higher derivative corrections to the solutions and the energy bound are canceled out in this branch. It is confirmed that solutions to Eq. (4.3) satisfy the full equation of motion for the scalar field (3.7). When we consider the Fubini–Study metric for the $\mathbb{C}P^1$ model and take the function Λ as

$$K_{\varphi\bar{\varphi}} = \frac{1}{(1+|\varphi|^2)^2}, \qquad \Lambda = \frac{1}{(1+|\varphi|^2)^4}, \tag{4.5}$$

the bound (4.4) becomes just the BPS bound obtained in the context of the effective theory on a non-Abelian vortex [42].

In summary, although the Lagrangian contains higher derivative corrections, the $1/2$ BPS lump solution to Eq. (4.3) (which is a holomorphic function with appropriate boundary conditions) does not receive any corrections in the canonical branch (2.11).

B. 1/4 BPS lumps as compact baby Skyrmions

We next consider the noncanonical branch. Since this is associated with the $F \neq 0$ solution (4.2) even for $W = 0$, we need to impose both of the two conditions in (3.9) in order to obtain the BPS equation from the variation of the fermion. Then, the BPS equation is

$$\bar{\partial}\varphi = e^{i\eta'}\sqrt{-\frac{K_{\varphi\bar{\varphi}}}{2\Lambda} + \frac{1}{2}(|\partial\varphi|^2 + |\bar{\partial}\varphi|^2)}, \tag{4.6}$$

where $\eta' = \eta + \alpha \in \mathbb{R}$ is a phase factor. This is the $1/4$ BPS equation. Again, the BPS lump does not cancel the higher derivative corrections generally. We can make the deformed BPS equation (4.6) into the following form:

$$\frac{1}{2}(|\partial\varphi|^2 - |\bar{\partial}\varphi|^2) = \frac{K_{\varphi\bar{\varphi}}}{2\Lambda}. \tag{4.7}$$

We confirm that the solutions to the BPS equation (4.6) satisfy the full on-shell equation of motion for the scalar field (3.7). In the noncanonical branch, we have the Bogomol'nyi completion of the energy:

$$\begin{aligned}\mathcal{E} &= -(|\partial_i\varphi\partial_i\varphi|^2 - (\partial_i\varphi\partial_i\bar{\varphi})^2)\Lambda + \frac{(K_{\varphi\bar{\varphi}})^2}{4\Lambda} \\ &= \Lambda\left[\frac{1}{2}(|\partial\varphi|^2 - |\bar{\partial}\varphi|^2) - \frac{K_{\varphi\bar{\varphi}}}{2\Lambda}\right]^2 + \frac{K_{\varphi\bar{\varphi}}}{2}(|\partial\varphi|^2 - |\bar{\partial}\varphi|^2) \\ &\geq -iK_{\varphi\bar{\varphi}}\varepsilon_{ij}\partial_i\varphi\partial_j\bar{\varphi}.\end{aligned} \tag{4.8}$$

Since we have $\Lambda > 0$ for static configurations from the consistency condition (2.17) of the solution, the energy bound is saturated by the topological charge density of lumps provided that the BPS condition (4.7) is satisfied. It is obvious that the expression of the topological charge is not corrected by the higher derivative terms.

In the noncanonical branch, the Lagrangian does not contain ordinary canonical kinetic term. An example of such a kind of noncanonical model is the extremal (BPS) baby Skyrme model. The model consists of the fourth

TABLE I. BPS states in the $W = 0$ and $W \neq 0$ higher derivative chiral models. Corresponding solutions of the auxiliary field F (whether they vanish or not) are also presented.

	1/2 BPS	1/4 BPS
$W = 0$	Lumps ($F = 0$)	Compact lumps ($F \neq 0$)
$W \neq 0$	Domain walls ($F \neq 0$)	Deformed domain wall junctions ($F \neq 0$)

derivative term and potential terms in $(2 + 1)$ dimensions. More concretely, if we take the Kähler potential and Λ as in (4.5), then the Lagrangian (2.16) is nothing but the fourth derivative part of the baby Skyrme model with an irrelevant constant term. In Ref. [18], the authors found specific Kähler potentials and constructed the potentials of the baby Skyrme model. Actually, the condition (4.7) was first found in the supersymmetric baby Skyrme model [18]. Equation (4.7) is the same as the one found in the previous section, Eq. (3.15). The difference of solutions is specified by boundary conditions. However, the energy bound in (4.8) suggests that there are no BPS domain walls junctions in the noncanonical branch. The example of solutions to Eq. (4.7) are the compact baby Skyrmions [9,10] that are solitons with compact support. The BPS states in the higher derivative chiral models are summarized in Table I.

V. CONCLUSION AND DISCUSSIONS

In this paper, we have studied BPS states in the four-dimensional $\mathcal{N} = 1$ supersymmetric higher derivative chiral model of which the Lagrangian is given in Eq. (2.2). The model is governed by a (2,2) Kähler tensor $\Lambda_{ij\bar{k}\bar{l}}$ symmetric in holomorphic and antiholomorphic indices, in addition to the Kähler potential K and the superpotential W. They are functions of the chiral superfields Φ^i. In particular, the tensor $\Lambda_{ij\bar{k}\bar{l}}$ determines the higher derivative interactions of the models. A specific feature of the model is that the auxiliary fields F^i do not have space-time derivatives on them and can be eliminated by their equation of motion algebraically. One can explicitly write down the on-shell Lagrangian of the model at least for a single chiral superfield. Since the equation of motion for the auxiliary fields is no longer a linear equation, there are several on-shell branches in this model. This fact deserves new nontrivial BPS equations that include higher derivative corrections.

When there is no superpotential, there are two distinct on-shell branches. One is the canonical branch associated with the solution $F = 0$. An example of this model is the supersymmetric DBI model [23]. We have shown that this branch, in fact, allows a supersymmetric extension of any bosonic models of complex scalar fields. We have exhibited the explicit function Λ, which corresponds to the supersymmetric extension of the Faddeev–Skyrme model without four time derivatives, which is in contrast to the previous studies [28,29] concluding that such a term is necessary for supersymmetry. The other branch is the noncanonical one corresponding to the solution $F \neq 0$. In this branch, the ordinary canonical kinetic term disappears, and the Lagrangian starts from the forth-order derivative terms. An example of this model is the extremal (BPS) baby Skyrme model. This branch was discussed in Refs. [18,19]. Although the $W = 0$ case has been essentially discussed in the literature [18,19], things get more involved when one introduces a superpotential W. In this case, a solution $F = 0$ is not allowed. There are three on-shell branches associated with the three different solutions of the auxiliary field equation [20–22]. The resulting on-shell Lagrangians have highly nonlinear expressions. Perturbative analysis reveals the possibility of ghost kinetic terms and deformations of the scalar potential.

Even though the on-shell Lagrangian is complicated and becomes highly nonlinear in the $W \neq 0$ case, one can derive the off-shell BPS conditions from the supersymmetry transformation of fermions. These conditions are supplemented by the equation of motion for the auxiliary field giving rise to the on-shell conditions. We have analyzed the properties of BPS states. For the $1/2$ BPS domain wall case, the higher derivative corrections are exactly canceled out in the $W \neq 0$ case. The solution to the BPS equation satisfies the full equation of motion for the scalar field. We have shown that the tension of the domain wall does not receive any higher derivative corrections. In the $W = 0$ noncanonical branch, the $1/2$ BPS condition does not provide the domain wall equation. For the $1/4$ BPS domain wall junction in the $W \neq 0$ case, the on-shell BPS equation receives higher derivative corrections. This is a new $1/4$ BPS equation for domain wall junctions. The solution is deformed by the higher derivative effects, and it is confirmed that the solution satisfies the full equation of motion. The expression of the energy bound is shown to be the same as the ordinary (without higher derivative terms) theory, namely, the sum of the junction charge and the tension. For lump configurations in the $W = 0$ case, there are two on-shell BPS equations. One is the $1/2$ BPS lumps associated with the $F = 0$ solution, where all the derivative corrections are canceled out. The other is the $1/4$ BPS lumps associated with the $F \neq 0$ solution. The on-shell BPS equation is deformed by the higher derivative corrections. This is nothing but the equation studied in Ref. [18]. An example of solutions to this equation is compactons in the extremal (BPS) baby Skyrme model.

While we were able to solve explicitly auxiliary field equations (2.5) only for one chiral superfield, reducing the third-order algebraic equation (2.7), the multicomponent equation (2.5) has yet to be solved. When the target space has a large isometry, it should be possible to solve it. Construction of more general target spaces, for instance, a higher derivative $\mathbb{C}P^n$ model and its BPS solitons, remains as a future problem.

While we have exhausted all BPS states that are already known in conventional $\mathcal{N}=1$ supersymmetric theories without higher derivatives, there may still remain unknown BPS states particular for higher derivative theories. In fact, 1/4 baby BPS Skyrmions do not exist in conventional theories. A sine-Gordon kink inside a domain wall (corresponding to a baby Skyrmion in the bulk) [50], a baby Skyrmion inside a domain wall (corresponding to a three-dimensional Skyrmion in the bulk) [51], or a baby Skyrmion string ending on a domain wall [52] or stretched between domain walls [40,53] is one of possibilities of composite BPS states.

It should be important to generalize our formalism to theories with extended supersymmetries such as eight supercharges. Although only four out of eight supercharges are manifestly realized in the $\mathcal{N}=1$ superfield formalism, this is still useful to study the off-shell effective theory of BPS solitons in models with eight supercharges [54]. Supersymmetric theories with eight supercharges are known to admit plenty of composite BPS states [39,40]. In particular, a classification of all possible BPS states in supersymmetric theories with eight supercharges was given in Ref. [41]. It is an interesting future problem to explore which BPS states (do not) receive higher derivative corrections.

As this problem concerns, 1/2 BPS topological solitons in theories with eight supercharges preserve four supercharges on their world volume. Off-shell effective actions of the 1/2 BPS domain wall and vortex were obtained in $d=3+1$, $\mathcal{N}=1$ superfield formalism at the leading order [54]. The formulation presented in this paper should be useful to obtain the off-shell action of higher derivative corrections to these effective actions. For instance, as mentioned in Eqs. (4.3) and (4.4), the $\mathbb{C}P^1$ model with a four-derivative term appearing as the effective theory of a non-Abelian vortex admits 1/2 BPS lumps [42], corresponding to Yang–Mills instantons in the bulk [46]. In the same way, an $SU(N)$ principal chiral model with the Skyrme term appears [55] on a non-Abelian domain wall [56]. The off-shell higher derivative corrections to the effective theories on these solitons are some of future directions.

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APPENDIX: NOTATION AND CONVENTIONS

We use the notation of the textbook of Wess and Bagger [57]. The component expansion of the $\mathcal{N}=1$ chiral superfield in the x basis is

$$\Phi(x,\theta,\bar{\theta}) = \varphi + i\theta\sigma^m\bar{\theta}\partial_m\varphi + \frac{1}{4}\theta^2\bar{\theta}^2\Box\varphi + \theta^2 F, \quad \text{(A1)}$$

where only the bosonic components are presented. The supercovariant derivatives are defined as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m,$$
$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m. \quad \text{(A2)}$$

The sigma matrices are $\sigma^m = (\mathbf{1},\vec{\tau})$. Here, $\vec{\tau} = (\tau^1, \tau^-2, \tau^3)$ are Pauli matrices.

The bosonic component of the supercovariant derivatives of Φ^i are

$$D^\alpha\Phi^i D_\alpha\Phi^j = -4\bar{\theta}^2\partial_m\varphi^i\partial^m\varphi^j + 4i(\theta\sigma^m\bar{\theta})(\partial_m\varphi^i F^j + F^i\partial_m\varphi^j) - 4\theta^2 F^i F^j$$
$$+ 2\theta^2\bar{\theta}^2(\Box\varphi^i F^j + F^i\Box\varphi^j - \partial_m\varphi^i\partial^m F^j - \partial_m F^i\partial^m\varphi^j), \quad \text{(A3)}$$

$$\bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{i}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{j}} = -4\theta^2\partial_m\bar{\varphi}^{\bar{i}}\partial^m\bar{\varphi}^{\bar{j}} - 4i(\theta\sigma^m\bar{\theta})(\partial_m\bar{\varphi}^{\bar{i}}\bar{F}^{\bar{j}} + \bar{F}^{\bar{i}}\partial_m\bar{\varphi}^{\bar{j}}) + 4\bar{\theta}^2\bar{F}^{\bar{i}}\bar{F}^{\bar{j}}$$
$$+ 2\theta^2\bar{\theta}^2(\bar{F}^{\bar{i}}\Box\bar{\varphi}^{\bar{j}} + \Box\bar{\varphi}^{\bar{i}}\bar{F}^{\bar{j}} - \partial_m\bar{\varphi}^{\bar{i}}\partial^m\bar{F}^{\bar{j}} - \partial_m\bar{F}^{\bar{i}}\partial^m\bar{\varphi}^{\bar{j}}), \quad \text{(A4)}$$

$$D^\alpha\Phi^i D_\alpha\Phi^k\bar{D}_{\dot{\alpha}}\Phi^{\dagger\bar{j}}\bar{D}^{\dot{\alpha}}\Phi^{\dagger\bar{l}} = 16\theta^2\bar{\theta}^2\left[(\partial_m\varphi^i\partial^m\varphi^k)(\partial_m\bar{\varphi}^{\bar{j}}\partial^m\bar{\varphi}^{\bar{l}}) - \frac{1}{2}(\partial_m\varphi^i F^k + F^i\partial_m\varphi^k)(\partial^n\bar{\varphi}^{\bar{j}}\bar{F}^{\bar{l}} + \bar{F}^{\bar{j}}\partial^n\bar{\varphi}^{\bar{l}}) + F^i\bar{F}^{\bar{j}}F^k\bar{F}^{\bar{l}}\right]. \quad \text{(A5)}$$

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