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Observables in loop quantum gravity with a cosmological constant

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In many quantum gravity approaches, the cosmological constant is introduced by deforming the gauge group into a quantum group. In three dimensions, the quantization of the Chern-Simons formulation of gravity provided the first example of such a deformation. The Turaev-Viro model, which is an example of a spin-foam model, is also defined in terms of a quantum group. By extension, it is believed that in four dimensions, a quantum group structure could encode the presence of $\Lambda \neq 0$. In this article, we introduce by hand the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$ into the loop quantum gravity (LQG) framework; that is, we deal with $\mathcal{U}_q(\mathfrak{su}(2))$-spin networks. We explore some of the consequences, focusing in particular on the structure of the observables. Our fundamental tools are tensor operators for $\mathcal{U}_q(\mathfrak{su}(2))$. We review their properties and give an explicit realization of the spinorial and vectorial ones. We construct the generalization of the U(N) formalism in this deformed case, which is given by the quantum group $\mathcal{U}_q(\mathfrak{u}(N))$. We are then able to build geometrical observables, such as the length, area or angle operators, etc. We show that these operators characterize a quantum discrete hyperbolic geometry in the three-dimensional LQG case. Our results confirm that a quantum group structure in LQG can be a tool to introduce a nonzero cosmological constant into the theory. Our construction is both relevant for three-dimensional Euclidian quantum gravity with a negative cosmological constant and four-dimensional Lorentzian quantum gravity with a positive cosmological constant.

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I. INTRODUCTION

A. Background

There are different proposals to understand the nature of the cosmological constant Λ. It can be interpreted as encoding some type of vacuum energy (see Refs. [1–3] and references therein) or as a coupling constant just like Newton's constant G. The loop quantum gravity and spin-foam frameworks use the latter interpretation which is motivated by the seminal works of Witten [4], and later those of Fock and Rosly [5], and Alekseev, Grosse, and Schomerus [6,7]. Indeed, in a three-dimensional spacetime, one can rewrite general relativity with a (possibly zero) cosmological constant as a Chern-Simons gauge theory.[1] The general phase-space structure of the theory for any metric signature and sign of Λ can be treated in a nice, unified way [8] using Poisson-Lie groups [9], the classical counterparts of quantum groups. The quantization procedure leads *explicitly* to a quantum group structure. The full construction, from phase space to quantum group, is usually called *combinatorial quantization* [5–7].

We can also quantize three-dimensional gravity using the spin-foam approach. In this approach, three-dimensional (3D) gravity is formulated as a BF theory. When $\Lambda = 0$, this is the well-known Ponzano-Regge model (both Euclidian or Lorentzian), based on the irreducible unitary representations of the relevant gauge group. When $\Lambda \neq 0$, the quantum group structure is introduced by hand. The Ponzano-Regge model is deformed, using irreducible unitary representations of the relevant quantum deformation of the gauge group. This is then called the Turaev-Viro model [10]. The argument consolidating the incorporation of the cosmological constant into a spin-foam model through a quantum group comes from the semiclassical limit. Indeed, the asymptotics of the deformed $\{6j\}_q$ symbol entering into the definition of the Turaev-Viro model goes to the Regge action with a cosmological constant in the regime $\ell_p \ll \ell \ll R = |\Lambda|^{-\frac{1}{2}}$.

The third approach to quantize gravity is the canonical approach, i.e., the loop quantum gravity approach (LQG). In this case, by performing the classical Hamiltonian analysis on general relativity, the cosmological constant only appears in the Hamiltonian constraint. Upon quantization, this means that the kinematical Hilbert space is the same whether or not $\Lambda = 0$. In particular this kinematical Hilbert space (where the Gauss constraint has been solved) is based on the classical relevant gauge group.

Therefore at this stage, quantum groups naturally appear only in the combinatorial quantization of Chern-Simons. Different quantum groups appear according to the metric signature and the sign of the cosmological constant. When $\Lambda \neq 0$, we obtain a q-deformed version of the gauge group[2]

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[1]This is actually an extension of general relativity since degenerate metrics are allowed.

[2]$\mathcal{U}(\mathfrak{g})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ and $\mathcal{U}_q(\mathfrak{g})$ is its q-deformation.

TABLE I. 3D quantum gravity models.

Signature	Λ	Quantum group	QG models
Euclidian	$\Lambda > 0$	$\mathcal{D}(\mathcal{U}_q(\mathfrak{su}(2)))$, $q = e^{i\frac{\ell_p}{R}}$	Chern-Simons $\overset{[11]}{\leftrightarrow}$ Turaev-Viro $\overset{?}{\leftrightarrow}$ LQG
	$\Lambda = 0$	$\mathcal{D}(\mathcal{U}(\mathfrak{su}(2)))$, $\kappa = \ell_p$	Chern-Simons $\overset{[12]}{\leftrightarrow}$ Ponzano-Regge $\overset{[16]}{\leftrightarrow}$ LQG $\overset{[14]}{\leftrightarrow}$ Chern-Simons
	$\Lambda < 0$	$\mathcal{D}(\mathcal{U}_q(\mathfrak{su}(2)))$, $q = e^{\frac{\ell_p}{R}}$	Chern-Simons $\overset{?}{\leftrightarrow}$ Turaev-Viro $\overset{?}{\leftrightarrow}$ LQG
Lorentzian	$\Lambda > 0$	$\mathcal{D}(\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})))$, $q = e^{-\frac{\ell_p}{R}}$	Chern-Simons $\overset{?}{\leftrightarrow}$ Turaev-Viro $\overset{?}{\leftrightarrow}$ LQG
	$\Lambda = 0$	$\mathcal{D}(\mathcal{U}(\mathfrak{sl}(2,\mathbb{R})))$, $\kappa = \ell_p$	Chern-Simons $\overset{[12]}{\leftrightarrow}$ Ponzano-Regge $\overset{[16]}{\leftrightarrow}$ LQG $\overset{[14]}{\leftrightarrow}$ Chern-Simons
	$\Lambda < 0$	$\mathcal{D}(\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R})))$, $q = e^{-i\frac{\ell_p}{R}}$	Chern-Simons $\overset{?}{\leftrightarrow}$ Turaev-Viro $\overset{?}{\leftrightarrow}$ LQG

$\mathcal{U}_q(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the gauge group $G = \mathrm{SL}(2,\mathbb{R})$ in the Lorentzian case [SU(2) in the Euclidean case], with q being a function of the Planck scale and the cosmological radius $R^{-1} = \sqrt{|\Lambda|}$. The deformation parameter q can be real or complex. A nice way to remember what is q in terms of the sign of Λ and the signature is to consider $q = \exp(-\frac{\hbar G\sqrt{\Lambda}}{\sqrt{c^2}})$ and posing $c^2 > 0$ in the Lorentzian case and $c^2 < 0$ in the Euclidian case [11]. Note that this trick gives q or q^{-1}. The full relevant quantum group arising from the combinatorial quantization is $\mathcal{D}(\mathcal{U}_q(\mathfrak{g}))$, the Drinfeld double of $\mathcal{U}_q(\mathfrak{g})$. When $\Lambda = 0$, we get the Drinfeld double $\mathcal{D}(\mathcal{U}(\mathfrak{g}))$ with a noncommutative parameter given by $\kappa = \ell_p$ in units $\hbar = 1 = c$. A list of the different quantum groups relevant for 3D gravity is given in Table I.

Since classically the Chern-Simons formulation and the standard formulation of general relativity are equivalent (modulo the degenerate metrics), we can wonder whether the Chern-Simons combinatorial quantization formalism, LQG, and the spin-foam framework are related in some way. It can be shown explicitly in the Euclidian case, with $\Lambda > 0$, that the Chern-Simons quantum model and the Turaev-Viro model are related; more precisely, the Turaev Viro amplitude is the square of the Chern-Simons amplitude [12]. On the other hand, it seems difficult to relate the LQG formalism, when $\Lambda \neq 0$, to a spin-foam model based on a quantum group if we assume that the LQG kinematical Hilbert space is based on a classical group such as SU(2).

When $\Lambda = 0$, it is also possible to relate the Chern-Simons amplitude and the Ponzano-Regge amplitude [13], which allows one to identify a hidden symmetry given by the Drinfeld double $\mathcal{D}(\mathcal{U}(\mathfrak{g}))$ in the Ponzano-Regge model. Still, when $\Lambda = 0$ explicit links between LQG and the spin-foam framework [14] or between the Chern-Simons combinatorial quantization and LQG [15] have been identified. Note also that we can identify a hidden quantum group structure (the Drinfeld double) in LQG when $\Lambda = 0$ [13,15,16], which is consistent with the other approaches. The different cases for 3D gravity are summarized in Table I. For more details, we refer to the excellent review in Ref. [11].

When dealing with four-dimensional (4D) space-time, there is no Chern-Simons theory to guide us. Hence, it is postulated that the cosmological constant should also be introduced through a quantum group structure. From the spin-foam approach, one then considers the model that one prefers[3] [the Barrett-Crane (BC) or Engle-Pereira-Rovelli-Livine and Freidel-Krasnov (EPRL-FK) model] when $\Lambda = 0$—based on the irreducible unitary representations of the gauge group—and deforms it [19–22]. To argue *a posteriori* that this is the right thing to do, we can look at the asymptotic of the spin-foam amplitude and check that we recover the Regge action with a cosmological constant [23]. It is quite interesting that the current "physical" EPRL spin-foam model defined in the Lorentzian case, with $\Lambda > 0$, leads to a finite amplitude [21,22].

In four dimensions, we are not able to connect the Hamiltonian constraint arising in LQG to a spin-foam model, even when $\Lambda = 0$. Just as in three dimensions, it is not clear at all why a quantum group structure should appear in the LQG framework. Few arguments exist that justify this postulate [24]. We include Table II summarizing the different quantum group models appearing in 4D quantum gravity.

Several remarks can be made at this stage. The partition function of the Plebanski action is invariant under the transformation $\Lambda \to -\Lambda$ [25], which explains why we have the same quantum group for the different signs of the cosmological constant. This change of sign for Λ is equivalent to $q \to q^{-1}$.

In the $\Lambda = 0$ case, the different Lorentzian spin-foam models are based on $\mathrm{SL}(2,\mathbb{C})$, but the spin networks describing the quantum state of space are based on SU(2), which is seen as a subgroup of $\mathrm{SL}(2,\mathbb{C})$. By analogy, the spin-foam models in the "physical" case (Lorentzian, $\Lambda > 0$) are based on $\mathcal{U}_q(\mathfrak{so}(3,1))$ with q *real* [21,22]. We expect then that the spin networks encoding the quantum state of space will be defined in terms of $\mathcal{U}_q(\mathfrak{su}(2))$, which is seen as

[3]We focus here on the EPRL-FK and BC models since their deformation is known. Other spin-foam models exist [17,18], but their quantum group deformation has not been studied (to our knowledge).

TABLE II. 4D quantum gravity models.

Signature	Λ	Quantum group	QG models
Euclidian	$\Lambda > 0$	$\mathcal{U}_q(\mathfrak{so}(4))$, $q = e^{i2\pi\ell_P^2\Lambda}$	BC or EPRL-FK$\overset{?}{\Leftrightarrow}$LQG
	$\Lambda = 0$	?	BC or EPRL-FK$\Leftrightarrow$LQG
	$\Lambda < 0$	$\mathcal{U}_q(\mathfrak{so}(4))$, $q = e^{i2\pi\ell_P^2\Lambda}$	BC or EPRL-FK$\overset{?}{\Leftrightarrow}$LQG
Lorentzian	$\Lambda > 0$	$\mathcal{U}_q(\mathfrak{so}(3,1))$, $q = e^{\ell_P^2\Lambda}$	BC or EPRL-FK$\overset{?}{\Leftrightarrow}$LQG
	$\Lambda = 0$	?	BC or EPRL-FK$\overset{?}{\Leftrightarrow}$LQG
	$\Lambda < 0$	$\mathcal{U}_q(\mathfrak{so}(3,1))$, $q = e^{\ell_P^2\Lambda}$	BC or EPRL-FK$\overset{?}{\Leftrightarrow}$LQG

a sub-Hopf algebra of $\mathcal{U}_q(\mathfrak{so}(3,1))$. By construction, we must then deal with $\mathcal{U}_q(\mathfrak{su}(2))$ with q real. This is the case we shall consider in the following.

We also emphasize in passing that the quantum deformation of the Lorentz group (in three or four dimensions) for complex q is not understood.

B. Motivations

A common feature of 3D and 4D quantum gravity is that it is hard to understand why a q-deformation of the gauge group would appear from the LQG perspective. Since we do not know how to solve the Hamiltonian constraint (for $\Lambda \neq 0$) and since we would like to compare the LQG approach with the well-known models coming from combinatorial quantization formalism and spin foam, we would like to define LQG with a q-deformed group and see what the consequences are. We hope then to identify some hints pointing to the quantum group apparition in this context. In particular, if LQG defined in terms of a quantum group describes quantum curved geometries well, then this is a good sign that this could be a useful theory to consider.

To this aim, we need to understand the structure of the observables associated to spin networks defined using the representations of a quantum group. Not much work has been done in this context: LQG with a quantum group has been explored using the loop variables by Major and Smolin [26–28], and the algebra of cylindrical functions behind the notion of spin networks defined in terms of a quantum group has been studied by Lewandwoski and Okolow [29].

When $\Lambda = 0$, the structure of the observables for a spin network (or an intertwiner) is well understood, thanks to the spinor approach to LQG [30–32]. In particular, it is possible to construct a closed algebra [a $\mathfrak{u}(N)$ Lie algebra, where n is the number of intertwiner legs] that generates all the observables acting on an intertwiner. This approach not only gives some information about the observable structure but it has been applied to different contexts, with many interesting results [30–32]. This formalism has shown that spin networks can be seen as the quantization of classical discrete geometries, the so-called *twisted geometries* [33,34]. It has allowed for the construction of a new Hamiltonian constraint in 3D Euclidian gravity [35], such that the kernel of this constraint is given by the $6j$ symbol, i.e., the Ponzano-Regge amplitude. It has provided the tools needed to implement the simplicity constraints in a rigorous way—using the Gupta-Bleuler method—in order to build a spin-foam model for Euclidian gravity ($\Lambda = 0$) [36].

Generalizing the spinor formalism to the quantum group case will allow us to better understand the quantum gravity regime with a nonzero cosmological constant. Indeed, within this formalism we should be able to construct a Hamiltonian constraint relating Turaev-Viro and LQG [37], and we should be able to determine the relevant phase space for LQG, the space of curved twisted geometries [38].

C. Main results

This generalization of the spinor formalism to the quantum group case is the main result of this paper. We focus on the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$ with q real, which is therefore relevant for both 3D Euclidian gravity with $\Lambda < 0$ and the physical case, i.e., 4D Lorentzian gravity with $\Lambda > 0$.

The key idea for this generalization is the use of *tensor operators*. These are well known in the quantum-mechanical case for SU(2) [39]. Essentially, they are sets of operators that transform well under SU(2), i.e., as a representation. They are known in LQG as *grasping operators*. However, they have not been studied intensively in this context. We show that considering these operators seriously naturally leads to the spinor approach to LQG. These tensor operators can be generalized to the quantum group case (more exactly, they are defined for any quasitriangular Hopf algebra) [40].

Given a $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner with N legs, we identify some sets of operators that transform well under $\mathcal{U}_q(\mathfrak{su}(2))$. Due to the quantum group structure, they are much more complicated than their classical counterparts. In particular, their commutation relations are pretty complicated. We clarify the construction of $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner observables. We show how there exists a fundamental algebra generating all observables, which is a deformation of the $\mathfrak{u}(N)$ algebra. We also discuss the geometric interpretation of some observables for 3D Euclidian LQG with $\Lambda < 0$, pinpointing the fact that the quantum group structure encodes (as expected) the notion of curved discrete geometry. Some of these results were already announced in Ref. [41].

D. Outline of the paper

The paper is organized as follow. In Sec. II, we recall the main features of $\mathcal{U}_q(\mathfrak{su}(2))$, the q-deformed universal enveloping algebra of SU(2), with q real. We recall as well the notion of q-harmonic oscillators which are used to build some tensor operators' explicit realizations.

Section III is a review of tensor operators for $\mathcal{U}_q(\mathfrak{su}(2))$, the essential tools of our construction. Due to the nonlinearity of the quantum group structure, $\mathcal{U}_q(\mathfrak{su}(2))$ tensor operators are more complicated than the standard SU(2) case. In particular, due to the nontrivial nature of the quantum group action, the tensor product of tensor operators is highly nontrivial, which will make the construction of tensor operators acting on different legs of an intertwiner quite cumbersome (but necessary).

Different explicit realizations of tensor operators for $\mathcal{U}_q(\mathfrak{su}(2))$ are given in Sec. IV. We recall the results of Quesne [42] regarding spinor operators, i.e., their definition in terms of q-harmonic oscillators and their commutation relations for spinor operators acting on different legs. We extend this analysis to vector operators, which will be relevant for the construction of the standard geometric operators.

The main results of this paper are presented in Secs. V and VI. We discuss the general construction of observables for a $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner. We construct a new realization of $\mathcal{U}_q(\mathfrak{u}(N))$ in terms of tensor operators, which is also invariant under the action of $\mathcal{U}_q(\mathfrak{su}(2))$. We identify the nonlinear map relating our invariant operators to the standard $\mathcal{U}_q(\mathfrak{u}(N))$ Cartan-Weyl generators. We construct different geometric operators which we interpret in the context of 3D Euclidian LQG with $\Lambda < 0$. We show how we get a quantization of the hyperbolic cosine law, and a quantization of the length and of the area of a triangle. We also pinpoint how the presence of the cosmological constant allows for a notion of a minimum angle.

In the concluding section, we discuss the possible follow-ups of this tensor-operator approach to LQG.

We have also included some appendices to recall the definition of the hyperbolic cosine law, as well as some relevant formulas regarding the $\mathcal{U}_q(\mathfrak{su}(2))$ recoupling coefficients.

II. $\mathcal{U}_q(\mathfrak{su}(2))$ IN A NUTSHELL

A. Definition of $\mathcal{U}_q(\mathfrak{su}(2))$

In this section, we review the salient features of $\mathcal{U}_q(\mathfrak{su}(2))$ (which we shall use extensively) to fix the notations. We consider $\mathcal{U}_q(\mathfrak{su}(2))$, the q-deformation of the universal enveloping algebra of SU(2), with q real, generated by J_z, J_+, J_-. We have the commutation relations

$$[J_z, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = [2J_z], \quad \text{with} \quad [J_z] = \frac{q^{J_z/2} - q^{-J_z/2}}{q^{1/2} - q^{-1/2}}. \tag{1}$$

For $q \to 1$ the right-hand side of the second equation of Eq. (1) approaches $2J_z$ and we thus recover the usual Lie algebra $\mathfrak{su}(2)$. $\mathcal{U}_q(\mathfrak{su}(2))$ is equipped with the structure of a quasitriangular Hopf algebra $(\Delta, \epsilon, S, \mathcal{R})$ [9,43,44]:

(i) The coproduct $\Delta\colon \mathcal{U}_q(\mathfrak{su}(2)) \to \mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2))$ encodes physically the total angular momentum of a two-particle system,

$$\Delta J_z = J_z \otimes \mathbf{1} + \mathbf{1} \otimes J_z, \qquad \Delta J_\pm = J_\pm \otimes q^{J_z/2} + q^{-J_z/2} \otimes J_\pm. \tag{2}$$

Considering the undeformed case, we have

$$(\Delta J_\sigma)|j_1 m_1, j_2 m_2\rangle = (J_\sigma \otimes \mathbf{1} + \mathbf{1} \otimes J_\sigma)|j_1 m_1 j_2 m_2\rangle = (J_\sigma^{(1)} + J_\sigma^{(2)})|j_1 m_1 j_2 m_2\rangle, \quad \text{where } \sigma = +, -, z. \tag{3}$$

In the deformed case, the addition of angular momenta (2) is noncommutative, and hence the addition of q-angular momenta depends on the order in which we set our particles. As we shall see, the braiding constructed using the $\mathcal{R}$-matrix will allow us to relate different orderings.

(ii) The counit $\epsilon\colon \mathcal{U}_q(\mathfrak{su}(2)) \to \mathcal{U}_q(\mathfrak{su}(2))$ is defined such that $\epsilon(\mathbf{1}) = 1, \epsilon(J_\sigma) = 0$ for $\sigma = +, -, z$.

(iii) The antipode $S\colon \mathcal{U}_q(\mathfrak{su}(2)) \to \mathcal{U}_q(\mathfrak{su}(2))$ encodes in some sense the notion of inverse angular momentum,

$$SJ_z = -J_z, \qquad SJ_\pm = -q^{\pm 1/2} J_\pm. \tag{4}$$

(iv) The $\mathcal{R}$-matrix encodes the "amount" of noncommutativity of the coproduct, i.e., of the addition of angular momenta. Indeed, if we note $\psi\colon \mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2)) \to \mathcal{U}_q(\mathfrak{su}(2)) \otimes \mathcal{U}_q(\mathfrak{su}(2))$ (the permutation), then we have that

$$(\psi \circ \Delta)X = \mathcal{R}(\Delta X)\mathcal{R}^{-1}. \tag{5}$$

In terms of the $\mathcal{U}_q(\mathfrak{su}(2))$ generators, the $\mathcal{R}$-matrix can be written as (using the Sweedler notation [9])

$$\mathcal{R} = \sum \mathcal{R}_{(1)} \otimes \mathcal{R}_{(2)} = q^{J_z \otimes J_z} \sum_{n=0}^{\infty} \frac{(1 - q^{-1})^n}{[n]!} \times q^{n(n-1)/4} (q^{J_z/2} J_+)^n \otimes (q^{-J_z/2} J_-)^n, \tag{6}$$

where $[n]$ denotes the q-number $[n] \equiv \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}$. A cocommutative product would simply mean that

$\mathcal{R} = \mathbf{1} \otimes \mathbf{1}$, which is obtained when $q \to 1$ in Eq. (6). Further properties of the $\mathcal{R}$-matrix are given in Appendix B, in particular its expression in terms of Clebsch-Gordan coefficients.

The noncocommutativity of the coproduct implies that we have a "noncommutative" tensor product. Essentially, we would get a symmetric two-particle system if the permutation of the particle states does not affect the total observable, that is, the permutation leaves the coproduct invariant, $\psi \circ \Delta = \Delta$.

If it is noncocommutative, as in the $\mathcal{U}_q(\mathfrak{su}(2))$ case, we can still define a *deformed* permutation $\psi_{\mathcal{R}}$, thanks to the existence of the $\mathcal{R}$-matrix [40,44],

$$\begin{aligned}\psi_{\mathcal{R}}\colon\ & V \otimes W \to W \otimes V,\\ & v \otimes w \to \psi_{\mathcal{R}}(|v, w\rangle)\\ &\qquad \equiv \psi(\mathcal{R}|v, w\rangle) = \sum \psi(|\mathcal{R}_{(1)} v,\\ & \mathcal{R}_{(2)} w\rangle) = \sum |\mathcal{R}_{(2)} w, \mathcal{R}_{(1)} v\rangle. \end{aligned} \tag{7}$$

Using the key property $(\psi \circ \Delta) X = \mathcal{R}(\Delta X)\mathcal{R}^{-1}$, we have that

$$\begin{aligned}\psi_{\mathcal{R}}(X(|v, w\rangle)) &= \psi(\mathcal{R}X(|v, w\rangle))\\ &= \psi(\mathcal{R}(\Delta X)|v, w\rangle)\\ &= \psi((\psi \circ \Delta X)\mathcal{R}|v, w\rangle)\\ &= (\Delta X)\psi(\mathcal{R}|v, w\rangle)\\ &= X(\psi_{\mathcal{R}}(|v, w\rangle)).\end{aligned}$$

Hence, the tensor product is only symmetric under this deformed notion of permutation. From now on, we shall always consider this deformed permutation $\psi_{\mathcal{R}}$, which is the natural notion of permutation in this quasitriangular context.

The representation theory of $\mathcal{U}_q(\mathfrak{su}(2))$ with q real is very similar to that of $\mathfrak{su}(2)$ [45]. A representation V^j is generated by the vectors $|j, m\rangle$ with $j \in \mathbb{N}/2$ and $m \in \{-j, \ldots, j\}$. The key difference is that the action of the generators on these vectors generates q-numbers,

$$J_z |jm\rangle = m|jm\rangle, \tag{8}$$

$$J_{\pm}|jm\rangle = \sqrt{[j \mp m][j \pm m + 1]}\,|jm \pm 1\rangle. \tag{9}$$

A Casimir operator can be defined as

$$C = J_+ J_- + [J_z][J_z - 1] = J_- J_+ + [J_z][J_z + 1]. \tag{10}$$

The tensor product of vectors $|j_1 m_1, j_2 m_2\rangle$ can be decomposed into a linear combination of vectors using the q-Clebsch-Gordan (CG) coefficients ${}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m}$,

$$\begin{aligned}|j_1 m_1, j_2 m_2\rangle &= \sum_{j,m} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} |jm\rangle,\\ j &= |j_1 - j_2|, \ldots, j_1 + j_2.\end{aligned} \tag{11}$$

Conversely, given a representation V^j of $\mathcal{U}_q(\mathfrak{su}(2))$ we can decompose it along two representations V^{j_1} and V^{j_2} of $\mathcal{U}_q(\mathfrak{su}(2))$ (with $|j_1 - j_2| \le j \le j_1 + j_2$),

$$|jm\rangle = \sum_{m_1, m_2} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} |j_1 m_1, j_2 m_2\rangle. \tag{12}$$

Acting with a generator $J_\sigma (\sigma = +, -, z)$ on the right-hand side of Eq. (11) and with its coproduct on the left-hand side of Eq. (11), we obtain a recursion relation for the CG coefficients [45]. Such recursion relations can be taken as defining the CG coefficients,

$$\begin{aligned}J_z \triangleright |j_1 m_1, j_2 m_2\rangle &= \sum_{j,m} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} J_z \triangleright |jm\rangle \Leftrightarrow \Delta J_z |j_1 m_1, j_2 m_2\rangle\\ &= \sum_{j,m} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} J_z |jm\rangle \Rightarrow m_1 + m_2 = m,\\ J_{\pm} \triangleright |j_1 m_1, j_2 m_2\rangle &= \sum_{j,m} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} J_{\pm} \triangleright |jm\rangle \Leftrightarrow \Delta J_{\pm} |j_1 m_1, j_2 m_2\rangle\\ &= \sum_{j,m} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} J_{\pm} |jm\rangle \Rightarrow q^{-\frac{m_1}{2}} ([j_2 \pm m_2][j_2 \mp m_2 + 1])^{\frac{1}{2}} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 \mp 1 m}\\ &\quad + q^{\frac{m_2}{2}} ([j_1 \pm m_1][j_1 \mp m_1 + 1])^{\frac{1}{2}} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 \mp 1 m_2 m}\\ &= ([j \mp m][j \pm m + 1])^{\frac{1}{2}} {}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m \pm 1}.\end{aligned} \tag{13}$$

We refer to Appendix B for further relevant properties of CG coefficients.

Let us now introduce the notion of an intertwiner for $\mathcal{U}_q(\mathfrak{su}(2))$, which is a fundamental object in LQG. An intertwiner is a vector $|\iota_{j_1 \ldots j_N}\rangle = \sum_{m_i} c_{m_1 \ldots m_N} |j_1 m_1, \ldots, j_N m_N\rangle \in V^{j_1} \otimes \ldots \otimes V^{j_N}$ that is invariant under the action of $\mathcal{U}_q(\mathfrak{su}(2))$,

$$J_\alpha \triangleright |\iota_{j_1\ldots j_N}\rangle = [(\mathbf{1}\otimes\ldots\mathbf{1}\otimes\Delta)\circ\ldots\circ(\mathbf{1}\otimes\Delta)\circ\Delta](J_\alpha)|\iota_{j_1\ldots j_N}\rangle = 0, \quad \alpha=\pm,z. \tag{14}$$

Note that since the coproduct is coassociative, there is no issue of how to compose the coproducts. In the case of $N=3$, Eq. (14) is equivalent to the recursion relations which define the CG coefficients. A normalized 3-valent intertwiner is then uniquely defined by

$$|\iota_{j_1j_2j_3}\rangle = \sum_{m_i}\frac{(-1)^{j_3-m_3}q^{-\frac{m_3}{2}}}{[2j_3+1]^{\frac{1}{2}}}\,{}_q\mathbf{C}^{j_1\ j_2\ j_3}_{m_1m_2-m_3}|j_1m_1,j_2m_2,j_3m_3\rangle.$$

Another ingredient that we shall use extensively in the following sections is the adjoint action of $\mathcal{U}_q(\mathfrak{su}(2))$ on an operator $\mathcal{O}$. It differs from the usual adjoint action of $\mathfrak{su}(2)$ given by a commutator. The $\mathcal{U}_q(\mathfrak{su}(2))$ adjoint action of the generators J_σ is explicitly given by

$$J_z\triangleright\mathcal{O} = [J_z,\mathcal{O}],$$
$$J_\pm\triangleright\mathcal{O} = J_\pm\mathcal{O}q^{-J_z/2} - q^{\pm\frac{1}{2}}q^{-J_z/2}\mathcal{O}J_\pm. \tag{15}$$

The following lemma is useful for relating quantities that are invariant under the adjoint action and the different Casimirs that one can construct. This is especially relevant in our case since the commutator and the adjoint action do not coincide.

Lemma II.1 Let $\mathcal{C}\in\mathcal{U}_q(\mathfrak{su}(2))$ be invariant under the adjoint action; then, $\mathcal{C}$ commutes with the generators J_σ, $\sigma=+,-,z$. Conversely, if $\mathcal{C}\in\mathcal{U}_q(\mathfrak{su}(2))$ commutes with J_σ, then it is invariant under the adjoint action.

B. q-harmonic oscillators and the Schwinger-Jordan trick

To account for the deformation, we consider a pair of q-harmonic oscillators, comprised of annihilation operators $\alpha_i=a,b$, creation operators $\alpha_i^\dagger=a^\dagger,b^\dagger$, and number operators $N_{\alpha_i}=N_a,N_b$, to construct representations of $\mathcal{U}_q(\mathfrak{su}(2))$. They are defined as follows:

$$[\alpha_i,\alpha_j]=[\alpha_i,\alpha_j^\dagger]=0, \quad \text{with} \quad i\neq j,$$
$$[\alpha_i,\alpha_i^\dagger]_{q^{\pm\frac{1}{2}}} = q^{\frac{\mp N_{\alpha_i}}{2}}, \qquad [N_{\alpha_i},\alpha_j^\dagger]=\delta_{ij}\alpha_i^\dagger,$$
$$[N_{\alpha_i},\alpha_j] = -\delta_{ij}\alpha_i, \tag{16}$$

where $[A,B]_{q^n}\equiv AB-q^nBA$. Let us point out that the operator $\alpha_i^\dagger\alpha_i$ is not the number operator N_{α_i}; rather, it is equal to $[N_{\alpha_i}]$. From Eq. (16), we also have that

$$q^{N_{\alpha_i}/2}\alpha_i^\dagger = q^{1/2}\alpha_i^\dagger q^{N_{\alpha_i}/2}, \qquad q^{N_{\alpha_i}/2}\alpha_i = q^{-1/2}\alpha_i q^{N_{\alpha_i}/2},$$
$$\alpha_i^\dagger\alpha_i = [N_{\alpha_i}], \qquad \alpha_i\alpha_i^\dagger = [N_{\alpha_i}+1]. \tag{17}$$

The harmonic oscillator α_i, $\alpha_i^\dagger$, N_{α_i} acts on the Fock space $F_i=\{\sum_{n_i}c_{n_i}|n_i\rangle\}$ with vacuum $|0\rangle$,

$$\alpha_i|0\rangle = 0, \qquad \alpha_i|n_i\rangle = \sqrt{[n_i]}|n_i-1\rangle, \quad \text{with} \quad n_i\geq 1,$$
$$\text{and} \quad \alpha_i^\dagger|n_i\rangle = \sqrt{[n_i+1]}|n_i+1\rangle. \tag{18}$$

The generators of $\mathcal{U}_q(\mathfrak{su}(2))$ can be realized in terms of the pair of q-harmonic oscillators (a,b), their adjoint, and their number operator [46,47],

$$J_z=\frac{1}{2}(N_a-N_b), \qquad J_+=a^\dagger b, \qquad J_-=b^\dagger a,$$
$$C=\left[\frac{1}{2}(N_a+N_b)\right]\left[\frac{1}{2}(N_a+N_b)+1\right]. \tag{19}$$

Using this representation together with Eq. (16), we can recover the commutation relations (1). We can also use the Fock space $F\sim F_a\otimes F_b=\{\sum c_{n_an_b}|n_a,n_b\rangle, c_{n_an_b}\in\mathbb{R}\}$ of this pair of q-harmonic oscillators to generate the representations of $\mathcal{U}_q(\mathfrak{su}(2))$ by setting

$$j=\frac{1}{2}(n_a+n_b), \qquad m=\frac{1}{2}(n_a-n_b). \tag{20}$$

The states $|jm\rangle$ are then homogenous polynomials in the operators $\alpha_i,\alpha_i^\dagger$,

$$|jm\rangle = \frac{(a^\dagger)^{j+m}(b^\dagger)^{j-m}}{\sqrt{[j+m]![j-m]!}}|0,0\rangle. \tag{21}$$

III. TENSOR OPERATORS FOR $\mathcal{U}_q(\mathfrak{su}(2))$

We now introduce the concept of tensor operators. The general definition of tensor operators for a general quasi-triangular Hopf algebra was given in Ref. [40]. We use their formalism in the specific case of $\mathcal{U}_q(\mathfrak{su}(2))$. These objects are the building blocks of our construction of observables for LQG defined with $\mathcal{U}_q(\mathfrak{su}(2))$ as the gauge group. We will show in Sec. V that the use of tensor operators allows us to build any observables associated to an intertwiner (of a quantum or a classical group) in a straightforward manner.

A. Definition and Wigner-Eckart theorem

Definition III.1. Tensor operators [40].

Let V and W be two representations of $\mathcal{U}_q(\mathfrak{su}(2))$ (not necessarily irreducible), and let $L(W)$ be the set of linear maps on W. A tensor operator $\mathbf{t}$ is defined as the intertwinning linear map

$$\mathbf{t}:\ V\to L(W),$$
$$x\to\mathbf{t}(x). \tag{22}$$

If we take $V\equiv V^j$ (the irreducible representation of rank j spanned by vectors $|j,m\rangle$), then we note that $\mathbf{t}(|j,m\rangle)\equiv\mathbf{t}^j_m$. $\mathbf{t}^j=(\mathbf{t}^j_m)_{m=-j\ldots j}$ is called a tensor operator of rank j.

The fact that a tensor operator is an intertwining map for the action of $\mathcal{U}_q(\mathfrak{su}(2))$ means that $\mathbf{t}^j_m$ transforms at the same time as an operator under the adjoint action of $\mathcal{U}_q(\mathfrak{su}(2))$ and as a vector $|jm\rangle$. This is encoded in the *equivariance property*,[4]

$$\begin{aligned} J_z \triangleright \mathbf{t}^j_m &= [J_z, \mathbf{t}^j_m] = m\mathbf{t}^j_m, \\ J_\pm \triangleright \mathbf{t}^j_m &= J_\pm \mathbf{t}^j_m q^{-\frac{J_z}{2}} - q^{\pm\frac{1}{2}} q^{-\frac{J_z}{2}} \mathbf{t}^j_m J_\pm \\ &= \sqrt{[j \mp m][j \pm m + 1]}\mathbf{t}^j_{m\pm1}. \end{aligned} \tag{23}$$

This equivariance property has a very important consequence regarding the matrix elements of $\mathbf{t}^j_m$.

Theorem III.2. Wigner-Eckart theorem [40]:

The matrix elements $\langle j_1, m_1|\mathbf{t}^j_m|j_2, m_2\rangle$ are proportional to the CG coefficients. The constant of proportionality $N^j_{j_1 j_2}$ is a function of j_1, j_2, and j only,

$$\langle j_1, m_1|\mathbf{t}^j_m|j_2, m_2\rangle = N^j_{j_1 j_2\,q}\mathbf{C}^{j\,j_2\,j_1}_{m m_2 m_1}. \tag{24}$$

The proof of the theorem follows from the constraints (23) written for the matrix elements of the tensor operator. These constraints essentially implement the recurrence relations which define the CG coefficients, as given in Eq. (13).

In order to have at least a nonzero matrix element, the j's in the CG coefficients must satisfy the triangular condition. This means in particular that the tensor operator does not have to be realized as a square matrix. Let us consider the cases $j = 0, \frac{1}{2}$, and 1.

(i) The scalar operator $\mathbf{t}^0$ has matrix elements given in terms of ${}_q\mathbf{C}^{0\,j_2\,j_1}_{0 m_2 m_1}$. As a consequence, we must have $j_1 = j_2$ and the scalar operator must be encoded in a square matrix $(2j_1 + 1) \times (2j_1 + 1)$.

(ii) The spinor operator $\mathbf{t}^{\frac{1}{2}}$ matrix elements are given in terms of ${}_q\mathbf{C}^{\frac{1}{2}\,j_2\,j_1}_{m m_2 m_1}$. We must have $j_2 + \frac{1}{2} = j_1$ or $j_2 - \frac{1}{2} = j_1$. The spinor operator cannot be realized by a square matrix. It has to be represented in terms of a rectangular matrix as either $(2j_2 + 2) \times (2j_2 + 1)$ or $(2j_2) \times (2j_2 + 1)$, or a direct sum of the two.

(iii) In a similar way, the vector operator $\mathbf{t}^1$ has matrix elements given by ${}_q\mathbf{C}^{1\,j_2\,j_1}_{m m_2 m_1}$. Hence it must be realized as a matrix as $(2j_2 - 1) \times (2j_2 + 1)$, $(2j_2 + 1) \times (2j_2 + 1)$, or $(2j_2 + 3) \times (2j_2 + 1)$, or a direct sum of some/all of them.

[4]As always we can perform the limit $q \to 1$ to recover the tensor operators for $\mathfrak{su}(2)$. In this case we have

$$[J_z, \mathbf{t}^j_m] = m\mathbf{t}^j_m, \qquad [J_\pm, \mathbf{t}^j_m] = \sqrt{(j \mp m)(j \pm m + 1)}\mathbf{t}^j_{m\pm1}.$$

This transformation is the infinitesimal version of $g\mathbf{t}^j_m g^{-1} = \sum_{m'} \rho^j_{mm'}(g)\mathbf{t}^j_{m'}, g \in \mathrm{SU}(2)$, where ρ is a representation of SU(2).

B. Product of tensor operators: scalar product, vector product, and triple product

We now would like to consider the analogue of Eqs. (11) and (12) in terms of tensor operators.

Lemma III.3. Product of tensor operators [40].

Let $\mathbf{t}\colon V \to L(W)$ and $\tilde{\mathbf{t}}\colon V' \to L(W)$ be two tensor operators; then,

$$\mathbf{t}\tilde{\mathbf{t}}\colon V \otimes V' \to L(W), (x, y) \to \mathbf{t}(x)\tilde{\mathbf{t}}(y) \tag{25}$$

is still a tensor operator.

For example, we can decompose a given tensor operator in terms of two other tensor operators, using the CG coefficients,

$$\mathbf{t}^j_m = \sum_{m_1, m_2} {}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1 m_2 m}\mathbf{t}^{j_1}_{m_1}\mathbf{t}^{j_2}_{m_2}. \tag{26}$$

Two specific combinations will be especially relevant for us: the "scalar product" and the "vector product."

1. Scalar product

We define the "scalar product" of two tensor operators as the projection of these operators on the trivial representation. Indeed, considering two tensor operators $\mathbf{t}^{j_1}$ and $\tilde{\mathbf{t}}^{j_2}$, we can combine them using the CG coefficients to build a tensor operator of rank 0, i.e., a scalar operator,

$$\begin{aligned} \mathbf{t}^{j_1} \cdot \tilde{\mathbf{t}}^{j_2} &\equiv \sqrt{[2j_1 + 1]} \sum_{m_1 + m_2 = 0} {}_q\mathbf{C}^{j_1\,j_2\,0}_{m_1 m_2 0}\, \mathbf{t}^{j_1}_{m_1}\tilde{\mathbf{t}}^{j_2}_{m_2} \\ &= \delta_{j_1, j_2}\sum_m (-1)^{j_1 - m} q^{\frac{m}{2}}\mathbf{t}^{j_1}_m \tilde{\mathbf{t}}^{j_1}_{-m}, \end{aligned} \tag{27}$$

In this sense, we can interpret these quantum Clebsch-Gordan coefficients as encoding a (nondegenerate) bilinear form $\mathcal{B}^{(j)}$ defining a scalar product,

$$\begin{aligned} \mathcal{B}^{(j)}(v, w) &= g^{(j)}_{mn} v^m w^n = v \cdot w, \\ g^{(j)}_{mn} &= \sqrt{[2j_1 + 1]}{}_q\mathbf{C}^{j\,j\,0}_{mn0} \\ &= \delta_{m,-n}(-1)^{j-m} q^{\frac{m}{2}} \neq g^{(j)}_{nm}. \end{aligned} \tag{28}$$

To construct a scalar product from the bilinear form $\mathcal{B}$, we usually demand that $\mathcal{B}$ is symmetric, $\mathcal{B}(v, w) = \mathcal{B}(\psi(v, w))$, where ψ is the permutation. However due to the noncocommutativity of the coproduct, we have a nontrivial tensor-product structure. Thus we have to discuss the symmetry with respect to the deformed permutation $\psi_{\mathcal{R}} = \psi \circ \mathcal{R}$. We then have

$$\begin{aligned} v \cdot w &= \mathcal{B}(v, w) = (-1)^{2j} q^{-j(j+1)}\mathcal{B}(\psi_{\mathcal{R}}(v, w)) \\ &= (-1)^{2j} q^{-j(j+1)} w \cdot v. \end{aligned} \tag{29}$$

We notice therefore that—modulo the factor $q^{-j(j+1)}$—if j is an integer we have a (deformed) symmetric bilinear form, whereas in the half-integer case it is (deformed) antisymmetric. This is consistent with the construction when $q \to 1$. Unlike in the classical case, there is an extra factor $q^{-j(j+1)}$ that comes into play. Since we have defined a bilinear form, we can introduce the contravariant and covariant notions. If $|u\rangle = \sum_m u_m |jm\rangle$ is a vector (covariant object), then $\langle u| \equiv \sum_m u_{-m}(-1)^{j-m} q^{\frac{m}{2}} \langle jm|$ will be the covector (contravariant object). This notion can be naturally extended to tensor operators. We have already defined the covariant tensor operators since they transform as vectors. We can introduce the contravariant tensor operators as

$$\mathbf{t}_j^m \equiv (-1)^{j-m} q^{\frac{m}{2}} (\mathbf{t}_{-m}^j)^\dagger, \tag{30}$$

where here $\dagger$ is the standard combination of transpose and complex conjugation. This contravariant notion of tensor operators was actually proposed by Quesne [42].

Finally, given a bilinear form, we can construct the associated notion of an adjoint $\dagger_{\mathcal{B}}$ of an operator A, from $\mathcal{B}(A^{\dagger_{\mathcal{B}}} v, w)) = \mathcal{B}(v, Aw)$. We recall that[5] $g_{mn} = \delta_{m,-n}(-1)^{j-m} q^{\frac{m}{2}}$ is antidiagonal and not symmetric, so we need to be careful. We note that $g^{mn} = (-1)^{-j-m} q^{\frac{m}{2}} \delta_{-m,n}$ is its inverse. Following the adjoint definition, given a bilinear form g_{mn}, we have, for a given operator A,

$$(A^{\dagger_{\mathcal{B}}})^m{}_n = g^{ma} A^d{}_a g_{dn} = ((-1)^{m-n} q^{-\frac{m-n}{2}}) A_{-n}{}^{-m}. \tag{31}$$

2. Vector product

The notion of a "vector product" is defined by associating a vector operator $\hat{\mathbf{t}}^1$ to two vector operators $\mathbf{t}^1, \tilde{\mathbf{t}}^1$ using the CG coefficients,

$$\hat{\mathbf{t}}_m^1 = (\mathbf{t}^1 \wedge \tilde{\mathbf{t}}^1)_m \equiv \sum_{m_1, m_2} {}_q\mathbf{C}^{1\,1\,1}_{m_1 m_2 m} \mathbf{t}^1_{m_1} \tilde{\mathbf{t}}^1_{m_2}. \tag{32}$$

Using their value (recalled in Appendix B), we obtain explicitly

$$\hat{\mathbf{t}}_1 = \sqrt{\frac{[2]}{[4]}} (q^{1/2} \mathbf{t}_1^1 \tilde{\mathbf{t}}_0^1 - q^{-1/2} \mathbf{t}_0^1 \tilde{\mathbf{t}}_1^1),$$

$$\hat{\mathbf{t}}_{-1} = \sqrt{\frac{[2]}{[4]}} (q^{1/2} \mathbf{t}_0^1 \tilde{\mathbf{t}}_{-1}^1 - q^{-1/2} \mathbf{t}_{-1}^1 \tilde{\mathbf{t}}_0^1),$$

$$\hat{\mathbf{t}}_0 = \sqrt{\frac{[2]}{[4]}} (\mathbf{t}_1^1 \tilde{\mathbf{t}}_{-1}^1 - \mathbf{t}_{-1}^1 \tilde{\mathbf{t}}_1^1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \mathbf{t}_0^1 \tilde{\mathbf{t}}_0^1).$$

As we shall see when giving a realization of the vector operators, this vector product is related to the commutation relations of the $\mathfrak{su}(2)$ algebra (when $q = 1$) and to Witten's proposal describing the q-deformation of the $\mathfrak{su}(2)$ algebra [48]. Combining the scalar product with the wedge product, we obtain the generalization of the triple product,

$$(\mathbf{t}^1 \wedge \mathbf{t}^1) \cdot \mathbf{t}^1 \equiv \sum_{m_i} (-1)^{1-m_3} q^{\frac{m_3}{2}} {}_q\mathbf{C}^{1\,1\,1}_{m_1 m_2 m_3} \mathbf{t}^1_{m_1} \mathbf{t}^1_{m_2} \mathbf{t}^1_{-m_3}. \tag{33}$$

This is simply the image of the trivalent intertwiner (15) when restricted to $j_1 = j_2 = j_3 = 1$. The generalization to any j_i is then

$$(\mathbf{t}^{j_1} \wedge \mathbf{t}^{j_2}) \cdot \mathbf{t}^{j_3} \equiv \sum_{m_i} (-1)^{j_3 - m_3} q^{\frac{m_3}{2}} {}_q\mathbf{C}^{j_1\,j_2\,j_3}_{m_1 m_2 m_3} \mathbf{t}^{j_1}_{m_1} \mathbf{t}^{j_2}_{m_2} \mathbf{t}^{j_3}_{-m_3}. \tag{34}$$

In general, given a set of tensor operators, we can use the relevant intertwiner coefficients to construct a scalar operator out of them. Observables for an intertwiner will be the generalization of this construction.

C. Tensor products of tensor operators

The tensor product of tensor operators requires more attention. Indeed, if $\mathbf{t} \in L(W)$ and $\tilde{\mathbf{t}} \in L(W')$ are tensor operators for $\mathcal{U}_q(\mathfrak{su}(2))$, then in general $\mathbf{t} \otimes \tilde{\mathbf{t}}$ will not be a tensor operator for $\mathcal{U}_q(\mathfrak{su}(2))$. To see this, first we recall that we need the coproduct to define the action of the generators J_α on $|j_1 m_1, j_2 m_2\rangle$. For example,

$$\Delta J_+ |j_1 m_1, j_2 m_2\rangle = (J_+ \otimes K + K^{-1} \otimes J_+) |j_1 m_1, j_2 m_2\rangle, \qquad K \equiv q^{\frac{J_z}{2}}. \tag{35}$$

If $\mathbf{t} \otimes \tilde{\mathbf{t}}$ is a (linear) module homomorphism, we then have

$$\begin{aligned}(\mathbf{t} \otimes \tilde{\mathbf{t}})(\Delta J_+ |j_1 m_1, j_2 m_2\rangle) &= (\mathbf{t} \otimes \tilde{\mathbf{t}})(J_+ \otimes K + K^{-1} \otimes J_+ |j_1 m_1, j_2 m_2\rangle) \\ &= (J_+ \triangleright \mathbf{t}^{j_1}_{m_1}) \otimes (K \triangleright \tilde{\mathbf{t}}^{j_2}_{m_2}) + (K^{-1} \triangleright \mathbf{t}^{j_1}_{m_1}) \otimes (J_+ \triangleright \tilde{\mathbf{t}}^{j_2}_{m_2}).\end{aligned} \tag{36}$$

On the other hand, this must be equal to the action of J_+ on $\mathbf{t} \otimes \tilde{\mathbf{t}}$ (which can be seen as a linear map $V \otimes V' \to W \otimes W'$), so that

$$J_+ \triangleright (\mathbf{t} \otimes \tilde{\mathbf{t}}) = (J_+)_{V \otimes V'} (\mathbf{t} \otimes \tilde{\mathbf{t}}) (K^{-1})_{W \otimes W'} - q^{\frac{1}{2}} (K^{-1})_{V \otimes V'} (\mathbf{t} \otimes \tilde{\mathbf{t}}) (J_+)_{W \otimes W'}. \tag{37}$$

We recall that by definition we have

$$(K^{\pm 1})_{W \otimes W'} = \Delta K^{\pm 1} = (K^{\pm 1})_W \otimes (K^{\pm 1})_{W'}, \tag{38}$$

$$(J_+)_{W \otimes W'} = \Delta J_+ = (J_+)_W \otimes (K)_{W'} + (K^{-1})_W \otimes (J_+)_{W'}. \tag{39}$$

[5] We omit the j upper index for simplicity.

If $\mathbf{t}\otimes\tilde{\mathbf{t}}$ is a tensor operator, Eq. (36) must be equal to Eq. (37), which gives (we omit the indices for simplicity)

$$\begin{aligned}&(J_+\mathbf{t}K^{-1}-q^{\frac{1}{2}}K^{-1}\mathbf{t}J_+)\otimes(K\tilde{\mathbf{t}}K^{-1})\\&\quad+(K^{-1}\mathbf{t}K)\otimes(J_+\tilde{\mathbf{t}}K^{-1}-q^{\frac{1}{2}}K^{-1}\tilde{\mathbf{t}}J_+)\\&=J_+\mathbf{t}K^{-1}\otimes K\tilde{\mathbf{t}}K^{-1}+K^{-1}\mathbf{t}K^{-1}\otimes J_+\tilde{\mathbf{t}}K^{-1}\\&\quad-q^{\frac{1}{2}}(K^{-1}\mathbf{t}J_+\otimes K^{-1}\tilde{\mathbf{t}}K+K^{-1}\mathbf{t}K^{-1}\otimes K^{-1}\tilde{\mathbf{t}}J_+).\end{aligned}\tag{40}$$

If $\tilde{\mathbf{t}}=\mathbf{1}$, then Eq. (40) is satisfied for any $\mathbf{t}$, but when $\mathbf{t}=\mathbf{1}$ and $\tilde{\mathbf{t}}\neq\mathbf{1}$, the constraint (40) is not satisfied in general.[6] The problem can be identified with the noncommutativity of the coproduct [40]. Indeed, the operator $\mathbf{1}\otimes\mathbf{t}$ can be seen as being obtained from the permutation of $\mathbf{t}\otimes\mathbf{1}$, but since we are now dealing with a noncommutative tensor product, we need to consider the deformed permutation $\psi_{\mathcal{R}}$ instead of ψ.

Lemma III.4. If $\mathbf{t}$ is a tensor operator of rank j, then ${}^{(1)}\mathbf{t}=\mathbf{t}\otimes\mathbf{1}$ and ${}^{(2)}\mathbf{t}=\psi_{\mathcal{R}}(\mathbf{t}\otimes\mathbf{1})\psi_{\mathcal{R}}^{-1}=\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t})\mathcal{R}_{21}^{-1}$ are tensor operators of rank j.

We extend the construction to an arbitrary number of tensor products,[7]

$$\begin{aligned}{}^{(i)}\mathbf{t}=\;&\mathcal{R}_{ii-1}\mathcal{R}_{ii-2}\ldots\mathcal{R}_{i1}(\mathbf{1}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}\otimes\mathbf{t})\\&\times\mathcal{R}_{i1}^{-1}\ldots\mathcal{R}_{ii-2}^{-1}\mathcal{R}_{ii-1}^{-1}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}.\end{aligned}\tag{41}$$

By abuse of notation, we say that ${}^{(i)}\mathbf{t}$ acts on the ith Hilbert space, even though it is not really the case when $q\neq1$. Note also that if $q=1$, tensor operators that act on different Hilbert spaces will commute, but when $q\neq1$, this will not be the case in general due to the presence of the $\mathcal{R}$-matrices.

When we consider the scalar product of tensor operators living on the same Hilbert space, the $\mathcal{R}$-matrices disappear, which simplifies the calculations.

Lemma III.5. The scalar product of the tensor operators ${}^{(i)}\mathbf{t}^{j_1}$ and ${}^{(i)}\tilde{\mathbf{t}}^{j_2}$ can be reduced to

$${}^{(i)}\mathbf{t}^{j_1}\cdot{}^{(i)}\tilde{\mathbf{t}}^{j_2}=\mathbf{1}\otimes\ldots\mathbf{1}\otimes\mathbf{t}^{j_1}\cdot\tilde{\mathbf{t}}^{j_2}\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}.\tag{42}$$

This lemma follows simply from Eq. (41).

IV. REALIZATION OF TENSOR OPERATORS OF RANK 1/2 AND 1 FOR $\mathcal{U}_q(\mathfrak{su}(2))$

The abstract theory of tensor operators has been summarized above. We want to illustrate the construction by giving some realization of these tensor operators. We know that any representation V^j of $\mathcal{U}_q(\mathfrak{su}(2))$ can be recovered from the fundamental spinor representation $\frac{1}{2}$ and the CG coefficients. In the same way, the most important operators to identify are the spinor operators. If we know them, we can concatenate them using the CG coefficients to obtain any other tensor operators. We first present the realization of the spinor operators using q-harmonic oscillators, followed by the vector operators realized in terms of either the q-harmonic oscillators or the $\mathcal{U}_q(\mathfrak{su}(2))$ generators.

A. Rank-1/2 tensor operators

Rank-$\frac{1}{2}$ tensor operators (i.e., spinor operators) $\mathbf{t}^{\frac{1}{2}}_m$ should be a solution of the following constraints:

$$J_\pm\triangleright\mathbf{t}^{\frac{1}{2}}_\mp=\mathbf{t}^{\frac{1}{2}}_\pm,\qquad J_\pm\triangleright\mathbf{t}^{\frac{1}{2}}_\pm=0,\qquad J_z\triangleright\mathbf{t}^{\frac{1}{2}}_\pm=\pm\frac{1}{2}\mathbf{t}^{\frac{1}{2}}_\pm.\tag{43}$$

Using the Schwinger-Jordan realization of $\mathcal{U}_q(\mathfrak{su}(2))$ generators given in Eq. (19), we can solve these equations and get two solutions $T^{\frac{1}{2}}$ and $\tilde{T}^{\frac{1}{2}}$ satisfying Eq. (43),

$$\begin{aligned}T^{\frac{1}{2}}&=\begin{pmatrix}A^\dagger\\B^\dagger\end{pmatrix}=\begin{pmatrix}a^\dagger q^{N_a/4}\\b^\dagger q^{(2N_a+N_b)/4}\end{pmatrix},\\\tilde{T}^{\frac{1}{2}}&=\begin{pmatrix}\tilde{B}\\\tilde{A}\end{pmatrix}=\begin{pmatrix}q^{(2N_a+N_b+1)/4}b\\-q^{(N_a-1)/4}a\end{pmatrix}.\end{aligned}\tag{44}$$

We recall that a and b are q-harmonic oscillators which satisfy the modified commutation relations (16). We can check that $T^{\frac{1}{2}}$ and $\tilde{T}^{\frac{1}{2}}$ are Hermitian conjugate to each other, according to the modified bilinear form we have defined in Sec. III B 1 [see Eq. (30)]. When looking at the limit $q\to1$, we have

$$T^{\frac{1}{2}}\to\tau^{\frac{1}{2}}=\begin{pmatrix}a^\dagger\\b^\dagger\end{pmatrix},\qquad\tilde{T}^{\frac{1}{2}}\to\tilde{\tau}^{\frac{1}{2}}=\begin{pmatrix}b\\-a\end{pmatrix}.\tag{45}$$

This explicit realization of the tensor operators allows us to explicitly check the Wigner-Eckart theorem, and to identify the normalization of the operators through this realization. In particular, for the q-deformed spinor operator matrix elements, we have

$$\begin{aligned}\langle j_1,m_1|T^{\frac{1}{2}}_m|j_2,m_2\rangle&=\delta_{j_1,j_2+1/2}N^{\frac{1}{2}}_{j_2q}C^{1/2j_2j_1}_{mm_2m_1},\quad\text{with}\\N^{\frac{1}{2}}_{j_2}&=(([d_{j_2}])^{1/2}q^{\frac{j_2}{2}}),\\\langle j_1,m_1|\tilde{T}^{\frac{1}{2}}_m|j_2,m_2\rangle&=\delta_{j_1,j_2-1/2}\tilde{N}^{\frac{1}{2}}_{j_2q}C^{1/2j_2j_1}_{mm_2m_1},\quad\text{with}\\\tilde{N}^{\frac{1}{2}}_{j_2}&=(([d_{j_2}])^{1/2}q^{\frac{1}{4}(2j_2-1)}),\end{aligned}\tag{46}$$

where $m=\pm1/2$ and $d_j=2j+1$. We therefore have two possible realizations of spinor operators in terms of rectangular matrices. Note that the above choice of normalization, $N^{\frac{1}{2}}_{j_2}$ and $\tilde{N}^{\frac{1}{2}}_{j_2}$, can be modified because the spinor operators $T^{\frac{1}{2}}$ and $\tilde{T}^{\frac{1}{2}}$ are defined up to a multiplicative

[6] Note that in the limit $q\to1$, this would be satisfied. Hence $\mathbf{1}\otimes\tilde{\mathbf{t}}$ is a tensor operator for $\mathfrak{su}(2)$.

[7] $\mathcal{R}_{ms}=\mathbf{1}^{s-1}\otimes\mathcal{R}_2\otimes\mathbf{1}^{m-s-1}\otimes\mathcal{R}_1$, using the notations of Eq. (6).

function of N_a+N_b. Therefore, $N^{\frac{1}{2}}_{j_2}$ and $\tilde{N}^{\frac{1}{2}}_{j_2}$ can be any function of j_2.

To form observables for a N-valent intertwiner, we need to define spinor operators built from the tensor product of N spinor operators. The explicit realization of the tensor product of spinor operators was discussed in detail by Quesne [42]. The calculation amounts to calculating Eq. (41) for an arbitrary number N of tensor products, in the case of the spinor operators $\mathfrak{t}^{\frac{1}{2}}$.

Now we outline the outcome of this calculation and give the expression of these spinor operators in terms of the q-deformed harmonic oscillators $a_i^\dagger, a_i, N_{a_i}, b_i^\dagger, b_i, N_{b_i}\in F_i\sim F_{a_i}\otimes F_{b_i}$, where the $F_i (i=1,\ldots,N)$ are N independent q-Fock spaces. Let us define the tensor operators ${}^{(i)}T^{\frac{1}{2}}$ and ${}^{(i)}\tilde{T}^{\frac{1}{2}}$ living in $\mathcal{F}\equiv(\otimes_{i=1}^N F_{a_i})(\otimes_{i=1}^N F_{b_i})$ which "act" on the ith Hilbert space,

$$ {}^{(i)}T^{\frac{1}{2}} = \begin{pmatrix} \mathcal{A}_i^\dagger \\ \mathcal{B}_i^\dagger \end{pmatrix}, \qquad {}^{(i)}\tilde{T}^{\frac{1}{2}} = \begin{pmatrix} \tilde{\mathcal{B}}_i \\ \tilde{\mathcal{A}}_i \end{pmatrix}, \quad \text{for } i\in\{1,\ldots,N\}, \tag{47} $$

where

$$ \begin{aligned} {}^{(i)}T^{\frac{1}{2}}_{+} &:= \mathcal{A}_i^\dagger = (\otimes_{k=1}^{i-1}\, q^{\frac{N_{a_k}-N_{b_k}}{4}})a_i^\dagger q^{\frac{N_{a_i}}{4}},\\ {}^{(i)}T^{\frac{1}{2}}_{-} &:= \mathcal{B}_i^\dagger = (\otimes_{k=1}^{i-1}\, q^{\frac{-N_{a_k}+N_{b_k}}{4}})b_i^\dagger q^{\frac{2N_{a_i}+N_{b_i}}{4}} + (q^{\frac{1}{4}}-q^{-\frac{3}{4}})\left[\sum_{l=1}^{i-1}(\otimes_{k=1}^{l-1}\, q^{\frac{-N_{a_k}+N_{b_k}}{4}})a_l b_l^\dagger(\otimes_{k=l+1}^{i-1}\, q^{\frac{N_{a_k}-N_{b_k}}{4}})\right]a_i^\dagger q^{\frac{N_{a_i}}{4}},\\ {}^{(i)}\tilde{T}^{\frac{1}{2}}_{+} &:= \tilde{\mathcal{B}}_i = (\otimes_{k=1}^{i-1}\, q^{\frac{N_{a_k}-N_{b_k}}{4}})q^{\frac{2N_{a_i}+N_{b_i}+1}{4}}b_i,\\ {}^{(i)}\tilde{T}^{\frac{1}{2}}_{-} &:= \tilde{\mathcal{A}}_i = (\otimes_{k=1}^{i-1}\, q^{\frac{-N_{a_k}+N_{b_k}}{4}})(-q^{\frac{N_{a_i}-1}{4}}a_i) + (q^{\frac{1}{4}}-q^{-\frac{3}{4}})\left[\sum_{l=1}^{i-1}(\otimes_{k=1}^{l-1}\, q^{\frac{-N_{a_k}+N_{b_k}}{4}})a_l b_l^\dagger(\otimes_{k=l+1}^{i-1}\, q^{\frac{N_{a_k}-N_{b_k}}{4}})\right]q^{\frac{2N_{a_i}+N_{b_i}+1}{4}}b_i. \end{aligned} $$

These operators will be the building blocks of our construction of $\mathcal{U}_q(\mathfrak{su}(2))$ observables presented in the following section. It will be necessary to have their explicit form in terms of the harmonic oscillators in order to recover the $\mathcal{U}_q(\mathfrak{u}(N))$ structure in Sec. V B.

Note that if $i\neq 1$, the two spinor operators ${}^{(i)}T^{\frac{1}{2}}$ and ${}^{(i)}\tilde{T}^{\frac{1}{2}}$ are *no longer* Hermitian conjugate to each other. Indeed, $(\mathcal{A}_i^\dagger)^\dagger\neq -q^{1/4}\tilde{\mathcal{A}}_i, (\mathcal{B}_i^\dagger)^\dagger\neq q^{-1/4}\tilde{\mathcal{B}}_i, i\in\{2,\ldots,N\}$. To emphasize this lack of Hermiticity, we introduce the notation

$$ \mathcal{C}_i\equiv -q^{1/4}\tilde{\mathcal{A}}_i,\quad \mathcal{D}_i\equiv q^{-1/4}\tilde{\mathcal{B}}_i,\quad \forall\ i\in\{1,\ldots,N\}. \tag{48} $$

That is, we can rewrite the spinor operators ${}^{(i)}\tilde{T}^{\frac{1}{2}}$ as

$$ {}^{(i)}\tilde{T}^{\frac{1}{2}} = \begin{pmatrix} q^{\frac{1}{4}}\mathcal{D}_i \\ -q^{-\frac{1}{4}}\mathcal{C}_i \end{pmatrix}. $$

Quesne has calculated all possible commutation relations between the components of ${}^{(i)}T^{\frac{1}{2}}_{\pm}$, ${}^{(j)}\tilde{T}^{\frac{1}{2}}_{\pm}$ for any $i,j\in\{1,\ldots,N\}$ [42]. First, let us give the commutation relations when $1\leq i=j\leq N$:

$$ \begin{aligned} &\mathcal{B}_i^\dagger\mathcal{A}_i^\dagger = q^{1/2}\mathcal{A}_i^\dagger\mathcal{B}_i^\dagger, \qquad \mathcal{C}_i\mathcal{D}_i = q^{1/2}\mathcal{D}_i\mathcal{C}_i,\\ &\mathcal{D}_i\mathcal{A}_i^\dagger = q^{1/2}\mathcal{A}_i^\dagger\mathcal{D}_i, \qquad \mathcal{C}_i\mathcal{B}_i^\dagger = q^{1/2}\mathcal{B}_i^\dagger\mathcal{C}_i,\\ &\mathcal{C}_i\mathcal{A}_i^\dagger = q\mathcal{A}_i^\dagger\mathcal{C}_i + 1,\\ &\mathcal{D}_i\mathcal{B}_i^\dagger = q\mathcal{B}_i^\dagger\mathcal{D}_i + (q-1)\mathcal{A}_i^\dagger\mathcal{C}_i + 1. \end{aligned} \tag{49} $$

When $1\leq i<j\leq N$, we have

$$ \begin{aligned} &\mathcal{A}_i^\dagger\mathcal{A}_j^\dagger = q^{-1/4}\mathcal{A}_j^\dagger\mathcal{A}_i^\dagger, \quad \mathcal{A}_i^\dagger\mathcal{B}_j^\dagger = q^{1/4}\mathcal{B}_j^\dagger\mathcal{A}_i^\dagger - (q^{3/4}-q^{-1/4})\mathcal{A}_j^\dagger\mathcal{B}_i^\dagger, \quad \mathcal{B}_i^\dagger\mathcal{A}_j^\dagger = q^{1/4}\mathcal{A}_j^\dagger\mathcal{B}_i^\dagger, \quad \mathcal{B}_i^\dagger\mathcal{B}_j^\dagger = q^{-1/4}\mathcal{B}_j^\dagger\mathcal{B}_i^\dagger,\\ &\mathcal{D}_i\mathcal{D}_j = q^{-1/4}\mathcal{D}_j\mathcal{D}_i, \quad \mathcal{D}_i\mathcal{C}_j = q^{1/4}\mathcal{C}_j\mathcal{D}_i - (q^{3/4}-q^{-1/4})\mathcal{D}_j\mathcal{C}_i, \quad \mathcal{C}_i\mathcal{D}_j = q^{1/4}\mathcal{D}_j\mathcal{C}_i, \quad \mathcal{C}_i\mathcal{C}_j = q^{-1/4}\mathcal{C}_j\mathcal{C}_i,\\ &\mathcal{A}_i^\dagger\mathcal{D}_j = q^{-1/4}\mathcal{D}_j\mathcal{A}_j^\dagger, \quad \mathcal{D}_i\mathcal{A}_j^\dagger = q^{-1/4}\mathcal{A}_j^\dagger\mathcal{D}_i, \quad \mathcal{B}_i^\dagger\mathcal{C}_j = q^{-1/4}\mathcal{C}_j\mathcal{B}_i^\dagger, \quad \mathcal{C}_i\mathcal{B}_j^\dagger = q^{-1/4}\mathcal{B}_j^\dagger\mathcal{C}_i, \quad \mathcal{B}_i^\dagger\mathcal{D}_j = q^{1/4}\mathcal{D}_j\mathcal{B}_i^\dagger,\\ &\mathcal{D}_i\mathcal{B}_j^\dagger = q^{1/4}(\mathcal{B}_j^\dagger\mathcal{D}_i + (1-q^{-1})\mathcal{A}_j^\dagger\mathcal{C}_i), \quad \mathcal{A}_i^\dagger\mathcal{C}_j = q^{1/4}(\mathcal{C}_j\mathcal{A}_i^\dagger + (q-1)\mathcal{D}_j\mathcal{B}_i^\dagger), \quad \mathcal{C}_i\mathcal{A}_j^\dagger = q^{1/4}\mathcal{A}_j^\dagger\mathcal{C}_i. \end{aligned} \tag{50} $$

These commutation relations are quite cumbersome and they illustrate that the components of operators acting on different Hilbert spaces do not commute when $q\neq 1$. Obviously, when $q=1$ they simplify a lot.

B. Rank-1 tensor operators

Rank-1 tensor operators (i.e., vector operators) for $\mathcal{U}_q(\mathfrak{su}(2))$ have been identified [40]. These operators are important because in the context of LQG they will encode the notion of a flux operator. We explicitly construct them and provide their commutation relations when they act (or do not act) on different legs.

We can construct them using the spinor operators $T^{\frac{1}{2}}$, $\tilde{T}^{\frac{1}{2}}$ and the CG coefficients,

$$\mathbf{t}^1_m = \sum_{m_1,m_2} {}_q\mathbf{C}^{\frac{1}{2}\frac{1}{2}1}_{m_1m_2m} T^{\frac{1}{2}}_{m_1}\tilde{T}^{\frac{1}{2}}_{m_2}. \quad (51)$$

Using the explicit nonzero CG coefficients given in Appendix B, we have that

$$\mathbf{t}^1_{\pm 1} = T^{\frac{1}{2}}_{\pm}\tilde{T}^{\frac{1}{2}}_{\pm}, \qquad \mathbf{t}^1_0 = \frac{1}{\sqrt{[2]}}(q^{-\frac{1}{4}}T^{\frac{1}{2}}_{+}\tilde{T}^{\frac{1}{2}}_{-} + q^{\frac{1}{4}}T^{\frac{1}{2}}_{-}\tilde{T}^{\frac{1}{2}}_{+}). \quad (52)$$

Explicitly, we obtain that

$$\mathbf{t}^1_1 = q^{-\frac{1}{2}}q^{\frac{(3N_a+N_b)}{4}}a^\dagger b = q^{-1/2}q^{\frac{1}{2}(N_a+N_b)}q^{\frac{J_z}{2}}J_+, \quad (53)$$

$$\mathbf{t}^1_0 = -\frac{1}{[2]^{\frac{1}{2}}}(q^{-1}q^{N_a/2}[N_a] - q^{N_a+N_b/2}[N_b]) = -\frac{q^{-1/2}}{[2]^{\frac{1}{2}}}q^{\frac{1}{2}(N_a+N_b)}(q^{-1/2}J_+J_- - q^{1/2}J_-J_+), \quad (54)$$

$$\mathbf{t}^1_{-1} = -q^{-\frac{1}{2}}q^{(3N_a+N_b)/4}b^\dagger a = -q^{-\frac{1}{2}}q^{\frac{1}{2}(N_a+N_b)}q^{\frac{J_z}{2}}J_-. \quad (55)$$

Once again, we can check that the Wigner-Eckart is satisfied,

$$\langle j_1,m_1|\mathbf{t}^1_l|j_2,m_2\rangle = \delta_{j_1,j_2}N^1_{j_2}\,{}_q\mathbf{C}^{1\,j_2\,j_1}_{lm_2m_1}, \quad \text{with} \quad N^1_{j_2} = -q^{j_2-\frac{1}{2}}\left(\frac{([2j_2][2j_2+2])}{[2]}\right)^{\frac{1}{2}}, \quad (56)$$

and $l \in \{-1,0,1\}$. In this realization, the vector operator is realized as a square matrix. Note that the normalization $N^1_{j_2}$ comes from the chosen spinor normalization (46). For a given vector operator, we can always consider an arbitrary normalization $N^1_{j_2}$.

An important remark is that in the limit $q \to 1$, the components of the vector operator become proportional to the components of the $\mathfrak{su}(2)$ generators,

$$\mathbf{t}^1 \to \tau^1 = \begin{pmatrix} J_+ \\ -\sqrt{2}J_z \\ -J_- \end{pmatrix}. \quad (57)$$

That is, the $\mathfrak{su}(2)$ generators are very simply related to vector operators. Let us now go back to our definition of the generalized scalar product (27) and generalized vector product (32). In the $q=1$ case, the q-deformed CG coefficients of Eqs. (27) and (32) are simply replaced by the standard $\mathfrak{su}(2)$ CG coefficients. In particular, the scalar product is still the projection on the trivial rank and we can define the "norm" of the vector operator τ^1, given by $\tau^1\cdot\tau^1 \equiv \sum_{m_1+m_2=0} C^{1\;1\;0}_{m_1\,m_2\,0}\tau^1_{m_1}\tau^1_{m_2}$. This simplifies to

$$\tau^1\cdot\tau^1 = -2\vec{J}\cdot\vec{J}, \quad (58)$$

where the $\mathfrak{su}(2)$ set of generators $\vec{J}$ is seen as a 3-vector with components $J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = \frac{1}{2i}(J_+ - J_-)$, and $J_z = J_z$, and the "$\cdot$" on the left-hand side of Eq. (58) denotes the standard scalar product of 3-vectors. That is, in the nondeformed case, the norm of the vector operator is proportional to the quadratic Casimir of $\mathfrak{su}(2)$, $\mathcal{C} = \vec{J}\cdot\vec{J}$. The norm of the $\mathcal{U}_q(\mathfrak{su}(2))$ vector operator is by definition a $\mathcal{U}_q(\mathfrak{su}(2))$ invariant, but it is no longer proportional to $|\vec{J}|^2$. Indeed,

$$t^1\cdot t^1 \propto (q^{\frac{1}{2}} - q^{-\frac{1}{2}})^2J_-^2J_+^2 + ([2J_z+4] - [2J_z])J_-J_+ + [2J_z+2][2J_z], \quad (59)$$

where the proportionality coefficient is a function of $q^{\frac{N_a+N_b}{2}}$.

The "vector product" operation in the case $q=1$ can be understood as the commutator of the $\mathfrak{su}(2)$ generators, which is also the natural way to encode the notion of a vector product in LQG. Indeed,

$$(\tau^1\wedge\tau^1)_1 = \frac{1}{\sqrt{2}}(\tau^1_1\tau^1_0 - \tau^1_0\tau^1_1) = [J_z,J_+] = J_+ = \tau^1_1,$$
$$(\tau^1\wedge\tau^1)_{-1} = \frac{1}{\sqrt{2}}(\tau^1_0\tau^1_{-1} - \tau^1_{-1}\tau^1_0) = [J_z,J_-] = -J_- = \tau^1_{-1},$$
$$(\tau^1\wedge\tau^1)_0 = \frac{1}{\sqrt{2}}(\tau^1_1\tau^1_{-1} - \tau^1_{-1}\tau^1_1) = \frac{1}{\sqrt{2}}[J_-,J_+] = -\sqrt{2}J_z = \tau^1_0. \quad (60)$$

Therefore, we see that this vector product can be understood as the commutator of the $\mathfrak{su}(2)$ generators, which is also the natural way to encode the notion of a vector product in LQG.

Using the above realization of the vector operators when $q \neq 1$, one can explicitly check that

$$(\mathbf{t}^1\wedge\mathbf{t}^1)_1 = \sqrt{\frac{[2]}{[4]}}(q^{1/2}\mathbf{t}^1_1\mathbf{t}^1_0 - q^{-1/2}\mathbf{t}^1_0\mathbf{t}^1_1),$$
$$(\mathbf{t}^1\wedge\mathbf{t}^1)_{-1} = \sqrt{\frac{[2]}{[4]}}(q^{1/2}\mathbf{t}^1_0\mathbf{t}^1_{-1} - q^{-1/2}\mathbf{t}^1_{-1}\mathbf{t}^1_0),$$
$$(\mathbf{t}^1\wedge\mathbf{t}^1)_0 = \sqrt{\frac{[2]}{[4]}}(\mathbf{t}^1_1\mathbf{t}^1_{-1} - \mathbf{t}^1_{-1}\mathbf{t}^1_1 + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\mathbf{t}^1_0\mathbf{t}^1_0). \quad (61)$$

Thus, in the quantum group case, the vector product of vector operators is different than the commutation relations defining $\mathcal{U}_q(\mathfrak{su}(2))$. The matrix elements of this new vector operator can be expressed in terms of the matrix elements of $\mathbf{t}^1$,

$$\langle j,m_1|(\mathbf{t}^1\wedge\mathbf{t}^1)_\alpha|j,m_2\rangle=\frac{[2j-1]-[2j+3]}{[2]([4])^{\frac{1}{2}}}q^{j-\frac{1}{2}}\langle j,m_1|\mathbf{t}^1_\alpha|j,m_2\rangle. \tag{62}$$

We see therefore that in the classical case when $q=1$, the generators are related to vector operators, and different structures—such as the adjoint action, the commutator, and the vector product—are encoded in the same way. When $q\neq 1$, all these different degeneracies are actually lifted. We summarize in Table III all the possible relations in the cases of $\mathfrak{su}(2)$ and $\mathcal{U}_q(\mathfrak{su}(2))$.

The extension of $\mathbf{t}^1$ to ${}^{(i)}\mathbf{t}^1$, for $i\in\{1,\cdots,N\}$, can be done either through Eq. (41) or by using the spinor operators ${}^{(i)}T^{\frac{1}{2}}_{m_1}$ and ${}^{(i)}\tilde{T}^{\frac{1}{2}}_{m_2}$ given in Eq. (47),

$$ {}^{(i)}\mathbf{t}^1_m=\sum_{m_1,m_2}{}_q\mathbf{C}^{\frac{1}{2}\frac{1}{2}1}_{m_1m_2m}\,{}^{(i)}T^{\frac{1}{2}}_{m_1}\,{}^{(i)}\tilde{T}^{\frac{1}{2}}_{m_2}\to\begin{cases}{}^{(i)}\mathbf{t}^1_1=q^{\frac{1}{4}}\mathcal{A}_i^\dagger\mathcal{D}_i=\mathcal{A}_i^\dagger\tilde{\mathcal{B}}_i,\\ {}^{(i)}\mathbf{t}^1_0=\frac{1}{\sqrt{[2]}}(q^{\frac{1}{2}}\mathcal{B}_i^\dagger\mathcal{D}_i-q^{-\frac{1}{2}}\mathcal{A}_i^\dagger\mathcal{D}_i)=\frac{1}{\sqrt{[2]}}(q^{\frac{1}{4}}\mathcal{B}_i^\dagger\tilde{\mathcal{B}}_i+q^{-\frac{1}{4}}\mathcal{A}_i^\dagger\tilde{\mathcal{A}}_i),\\ {}^{(i)}\mathbf{t}^1_{-1}=-q^{-\frac{1}{4}}\mathcal{B}_i^\dagger\mathcal{C}_i=\mathcal{B}_i^\dagger\tilde{\mathcal{A}}_i.\end{cases} \tag{63}$$

Explicitly, in terms of the $\mathcal{U}_q(\mathfrak{su}(2))$ generators we have

$$\begin{aligned}
{}^{(i)}\mathbf{t}^1_1&=q^{\sum_{k=1}^{i-1}{}^{(k)}J_z}\,{}^{(i)}J_+q^{\frac{{}^{(i)}J_z}{2}}q^{\frac{N_{a_i}+N_{b_i}}{2}},\\
{}^{(i)}\mathbf{t}^1_0&=\frac{1}{\sqrt{[2]}}\left[-q^{-\frac{1}{2}}(q^{-\frac{1}{2}}{}^{(i)}J_+{}^{(i)}J_--q^{\frac{1}{2}}{}^{(i)}J_-{}^{(i)}J_+)q^{\frac{N_{a_i}+N_{b_i}}{2}}+(q^{\frac{1}{2}}-q^{-\frac{1}{2}})(1+q^{-\frac{1}{2}})\sum_{l=1}^{i-1}\left[q^{(\frac{{}^{(l)}J_z}{2}+\sum_{k=l+1}^{i-1}{}^{(k)}J_z)}{}^{(l)}J_-\right]{}^{(i)}J_+q^{\frac{{}^{(i)}J_z}{2}}q^{\frac{N_{a_i}+N_{b_i}}{2}}\right],\\
{}^{(i)}\mathbf{t}^1_{-1}&=-q^{-1}q^{-\sum_{k=1}^{i-1}({}^{(k)}J_z)}\,{}^{(i)}J_-q^{\frac{{}^{(i)}J_z}{2}}q^{\frac{N_{a_i}+N_{b_i}}{2}}-q^{-1}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})\sum_{l=1}^{i-1}\left(q^{-\sum_{k=1}^{l-1}({}^{(k)}J_z+\frac{{}^{(l)}J_z}{2})}{}^{(l)}J_-\right)(q^{-\frac{1}{2}}{}^{(i)}J_+{}^{(i)}J_--q^{\frac{1}{2}}{}^{(i)}J_-{}^{(i)}J_+)q^{\frac{N_{a_i}+N_{b_i}}{2}}\\
&\quad+q^{-1}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^2\left(\sum_{l=1}^{i-1}q^{-\sum_{k=1}^{l-1}\frac{{}^{(k)}J_z}{2}}{}^{(l)}J_-q^{\sum_{k=l+1}^{i-1}\frac{{}^{(k)}J_z}{2}}\right)^2{}^{(i)}J_+q^{\frac{{}^{(i)}J_z}{2}}q^{\frac{N_{a_i}+N_{b_i}}{2}}.
\end{aligned}$$

With this choice of normalization inherited from Eq. (56), the commutation relations between the ${}^{(i)}\mathbf{t}^1_m$ are quite complicated. For $1\leq i<j\leq N$,

TABLE III. Relations between vector operators and generators.

	$\mathfrak{su}(2)$	$\mathcal{U}_q(\mathfrak{su}(2))$
Generators	J_σ, $\sigma=\pm,z$ with commutation relations $[J_+,J_-]=2J_z$ $[J_\pm,J_z]=\mp J_\pm$	J_σ, $\sigma=\pm,z$ with commutation relations $[J_+,J_-]=\frac{q^{J_z}-q^{-J_z}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}$ $[J_\pm,J_z]=\mp J_\pm$
Vector operators	$\tau^1=\begin{pmatrix}J_+\\ -\sqrt{2}J_z\\ -J_-\end{pmatrix}$	$\mathbf{t}^1\propto\begin{pmatrix}q^{\frac{J_z}{2}}J_+\\ -\frac{1}{\sqrt{[2]}}(q^{-1/2}J_+J_--q^{1/2}J_-J_+)\\ -q^{\frac{J_z}{2}}J_-\end{pmatrix}$
Adjoint action	$J_\sigma\triangleright\mathcal{O}=[J_\sigma,\mathcal{O}]$ for $\sigma=\pm,z$	$J_\pm\triangleright\mathcal{O}=J_\pm\mathcal{O}q^{-\frac{J_z}{2}}-q^{\pm\frac{1}{2}}q^{-\frac{J_z}{2}}\mathcal{O}J_\pm$, $J_z\triangleright\mathcal{O}=[J_z,\mathcal{O}]$
"Scalar product" (· defined by Eq. (27)	$\tau^1\cdot\tau^1=-2\mathcal{C}=-2\lvert\vec{J}\rvert^2$ where $\mathcal{C}$ is the quadratic Casimir of $\mathfrak{su}(2)$.	$\mathbf{t}^1\cdot\mathbf{t}^1=I$ where I is a $\mathcal{U}_q(\mathfrak{su}(2))$ invariant; $\lvert\vec{J}\rvert^2$ is not a Casimir for $\mathcal{U}_q(\mathfrak{su}(2))$.
"Vector product" (∧ defined by Eq. (32)	$(\tau^1\wedge\tau^1)_{\pm1}=[J_z,J_\pm]$, $(\tau^1\wedge\tau^1)_z=\frac{1}{\sqrt{2}}[J_-,J_+]$.	$(\mathbf{t}^1\wedge\mathbf{t}^1)_\alpha=\hat{\mathbf{t}}^1_\alpha=$ vector operator; not simply related to the commutators between generators of $\mathcal{U}_q(\mathfrak{su}(2))$.

$$\begin{aligned}
&{}^{(i)}\mathbf{t}^1_1{}^{(j)}\mathbf{t}^1_1 = q^{-1}{}^{(j)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_1, \qquad {}^{(i)}\mathbf{t}^1_1{}^{(j)}\mathbf{t}^1_0 = {}^{(j)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_1 + (q^{-1}-q){}^{(j)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_0,\\
&{}^{(i)}\mathbf{t}^1_1{}^{(j)}\mathbf{t}^1_{-1} = q{}^{(j)}\mathbf{t}^1_{-1}{}^{(i)}\mathbf{t}^1_1 - (q-1)[2]{}^{(j)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_0 + (q^2-1)(1-q^{-1}){}^{(j)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_{-1},\\
&{}^{(i)}\mathbf{t}^1_0{}^{(j)}\mathbf{t}^1_1 = {}^{(j)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_0, \qquad {}^{(i)}\mathbf{t}^1_0{}^{(j)}\mathbf{t}^1_0 = {}^{(j)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_0 - (q^{\frac{1}{2}}-q^{-\frac{1}{2}})(q+1)(1+q^{-1}){}^{(j)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_{-1},\\
&{}^{(i)}\mathbf{t}^1_0{}^{(j)}\mathbf{t}^1_{-1} = {}^{(j)}\mathbf{t}^1_{-1}{}^{(i)}\mathbf{t}^1_0 + (q^{-1}-q){}^{(j)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_{-1},\\
&{}^{(i)}\mathbf{t}^1_{-1}{}^{(j)}\mathbf{t}^1_1 = q{}^{(j)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_{-1}, \qquad {}^{(i)}\mathbf{t}^1_{-1}{}^{(j)}\mathbf{t}^1_0 = {}^{(j)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_{-1}, \qquad {}^{(i)}\mathbf{t}^1_{-1}{}^{(j)}\mathbf{t}^1_{-1} = q^{-1}{}^{(j)}\mathbf{t}^1_{-1}{}^{(i)}\mathbf{t}^1_{-1}.
\end{aligned} \tag{64}$$

For $i = j \in \{1, \ldots, N\}$,

$$\begin{aligned}
{}^{(i)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_0 &= q^{-1}{}^{(i)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_1 + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{\sqrt{[2]}}\mathcal{E}_{ii}{}^{(i)}\mathbf{t}^1_1 + q^{-1}\sqrt{[2]}{}^{(i)}\mathbf{t}^1_1,\\
{}^{(i)}\mathbf{t}^1_1{}^{(i)}\mathbf{t}^1_{-1} &= q^{-1}{}^{(i)}\mathbf{t}^1_{-1}{}^{(i)}\mathbf{t}^1_1 + \frac{(q^{-1}-1)}{[2]}({}^{(i)}\mathbf{t}^1_0)^2 + \frac{2(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}{[2]^{\frac{3}{2}}}{}^{(i)}\mathbf{t}^1_0\mathcal{E}_{ii} + \frac{2q^{-1}}{\sqrt{[2]}}{}^{(i)}\mathbf{t}^1_0 - \frac{q^{-1}(q^{\frac{1}{2}}-q^{-\frac{1}{2}})}{[2]}\mathcal{E}_{ii},
\end{aligned} \tag{65}$$

$$ {}^{(i)}\mathbf{t}^1_0{}^{(i)}\mathbf{t}^1_{-1} = q^{-1}{}^{(i)}\mathbf{t}^1_{-1}{}^{(i)}\mathbf{t}^1_0 + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{[2]}\mathcal{E}_{ii} + q^{-1}\sqrt{[2]}{}^{(i)}\mathbf{t}^1_{-1}, \tag{66}$$

where $\mathcal{E}_{ii} \coloneqq -q^{\frac{1}{4}}\mathcal{A}^\dagger_i\tilde{\mathcal{A}}_i + q^{-\frac{1}{4}}\mathcal{B}^\dagger_i\tilde{\mathcal{B}}_i$ is a $\mathcal{U}_q(\mathfrak{su}(2))$ invariant (see Sec. V B) and it commutes with any ${}^{(i)}\mathbf{t}^1_\alpha, (\alpha = \pm, z)$.

V. OBSERVABLES FOR THE INTERTWINER SPACE

As emphasized in the Introduction, we are focusing on the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$ with q real, which is relevant for both 3D Euclidian gravity with $\Lambda < 0$ and the physical case, i.e., 4D Lorentzian gravity with $\Lambda > 0$.

A. General construction and properties of intertwiner observables

From now on, we consider the space of N-valent intertwiners with N legs ordered from 1 to N. Let us consider n tensor operators ${}^{(\alpha)}\mathbf{t}^{J_\alpha}$ of respective rank J_α, associated with the αth leg of the vertex, built from Eq. (41). To construct an observable, i.e., a scalar operator, we can use the same combination that would appear in the definition of an intertwiner built out from the vectors $|J_\alpha, m_\alpha\rangle$. Indeed, if $|\iota_{J_1\ldots J_n}\rangle = \sum_m c^{J_1\ldots J_n}_{m_1\ldots m_n}|J_1 m_1, \ldots J_n m_n\rangle$, then

$$ I^{J_1\ldots J_n} = \sum_{m_i} c^{J_1\ldots J_n}_{m_1\ldots m_n}{}^{(1)}\mathbf{t}^{J_1}_{m_1}\ldots{}^{(n)}\mathbf{t}^{J_n}_{m_n} \tag{67}$$

will be a scalar operator. Similarly as for intertwiners, the bivalent and trivalent ones are the simplest and we can write them explicitly,

$$\begin{aligned}
I^{J_\alpha J_\beta} &\equiv {}^{(\alpha)}\mathbf{t}^{J_\alpha}\cdot{}^{(\beta)}\mathbf{t}^{J_\beta} = \delta_{J_\alpha,J_\beta}\sum_m(-1)^{J_\alpha-m}q^{\frac{m}{2}}{}^{(\alpha)}\mathbf{t}^{J_\alpha}_m{}^{(\beta)}\mathbf{t}^{J_\beta}_{-m}\\
&\equiv I^{J_\alpha}_{\alpha\beta},
\end{aligned} \tag{68}$$

$$\begin{aligned}
I^{J_\alpha J_\beta J_\gamma} &\equiv ({}^{(\alpha)}\mathbf{t}^{J_\alpha}\wedge{}^{(\beta)}\mathbf{t}^{J_\beta})\cdot{}^{(\gamma)}\mathbf{t}^{J_\gamma}\\
&= \sum_{m_i}(-1)^{J_\gamma-m_3}q^{\frac{m_3}{2}}{}_q\mathbf{C}^{J_\alpha J_\beta J_\gamma}_{m_1 m_2 m_3}{}^{(\alpha)}\mathbf{t}^{J_\alpha}_{m_1}{}^{(\beta)}\mathbf{t}^{J_\beta}_{m_2}{}^{(\gamma)}\mathbf{t}^{J_\gamma}_{-m_3}.
\end{aligned} \tag{69}$$

We recognize the generalized notions of the scalar product and the triple product, respectively.

This construction works well for operators acting on an intertwiner; however, in the general LQG context, we need to deal with spin networks, so we need to consider the tensor product of such intertwiners $|\iota_{j_1\ldots j_N}\rangle \otimes |\iota'_{j_{1'}\ldots j_{N'}}\rangle \otimes \ldots$. Although the tensor product is not commutative, we do not need to use the deformed permutation to define an operator acting on any intertwiner of the tensor product. Indeed, since an intertwiner is a $\mathcal{U}_q(\mathfrak{su}(2))$-invariant vector, the tensor product involving such invariant vectors is commutative.

More explicitly, we have seen earlier that if $\mathbf{t}$ is a tensor operator, then $\mathbf{1}\otimes\mathbf{t}$ will not in general be a tensor operator. However, if $\mathbf{1}\otimes\mathbf{t}$ is restricted to act on some invariant vectors $|\iota\rangle\otimes|\iota'\rangle$, then $\mathbf{1}\otimes\mathbf{t}$ will still be a tensor operator.

To see this, let us consider the invariant vectors $|\iota\rangle\otimes|\iota'\rangle \in W\otimes W'$. We recall that an invariant vector means that

$$ J_\pm|\iota\rangle = 0, \qquad K^{\pm1}|\iota\rangle = |\iota\rangle. \tag{70}$$

Let us first determine the transformation of $\mathbf{1}\otimes\mathbf{t}$ as a representation of $\mathcal{U}_q(\mathfrak{su}(2))$ [that is, Eq. (36)] when acting on the vectors $|\iota\rangle\otimes|\iota'\rangle$,

$$\begin{aligned}
&((J_+K^{-1} - q^{\frac{1}{2}}K^{-1}J_+)\otimes K\tilde{\mathbf{t}}K^{-1}\\
&\quad + \mathbf{1}\otimes(J_+\tilde{\mathbf{t}}K^{-1} - q^{\frac{1}{2}}K^{-1}\tilde{\mathbf{t}}J_+))|\iota\rangle\otimes|\iota'\rangle\\
&= \mathbf{1}\otimes(J_+\triangleright\tilde{\mathbf{t}})|\iota\rangle\otimes|\iota'\rangle.
\end{aligned} \tag{71}$$

If $\mathbf{1}\otimes\mathbf{t}$ transforms well when restricted to the invariant vectors $|\iota\rangle\otimes|\iota'\rangle$, we must recover the same outcome as

Eq. (71) when considering $\mathbf{1}\otimes\mathbf{t}$ transforming as an operator [that is, Eq. (37)],

$$(J_+K^{-1}\otimes K\tilde{\mathbf{t}}K^{-1}+\mathbf{1}\otimes J_+\tilde{\mathbf{t}}K^{-1} - q^{\frac{1}{2}}(K^{-1}J_+\otimes K^{-1}\tilde{\mathbf{t}}K+K^{-2}\otimes K^{-1}\tilde{\mathbf{t}}J_+))|\iota\rangle\otimes|\iota'\rangle = \mathbf{1}\otimes(J_+\triangleright\tilde{\mathbf{t}})|\iota\rangle\otimes|\iota'\rangle. \tag{72}$$

On the right-hand side of the two above equations, we have used Eq. (70) to recover $\mathbf{1}\otimes(J_+\triangleright\tilde{\mathbf{t}})|\iota\rangle\otimes|\iota'\rangle$. A similar calculation can be made for the action of J_- and K. Hence, the operator $\mathbf{1}\otimes\tilde{\mathbf{t}}$ transforms as a tensor operator of the same rank as $\tilde{\mathbf{t}}$ when restricted to act on an invariant state $|\iota\rangle\otimes|\iota'\rangle$. This means that we can just focus on the observables associated to one intertwiner, and if we look at another intertwiner *a priori* we do not need to order the vertices, unless we look at observables that live on both intertwiners at the same time.

If we have many legs in our intertwiner, it might be cumbersome to calculate the terms ${}^{(i)}\mathbf{t}^J$ and ${}^{(j)}\mathbf{t}^J$ and then calculate the observable I^J_{ij}, since we have to make extensive use of the deformed permutations, and a lot of CG coefficients (or $\mathcal{R}$-matrices) appear. If we know the matrix elements of I^J_{12} and I^J_{21}, we can construct all the other terms by induction. We know that by definition

$$I^J_{12}=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}(\mathbf{t}^J_{m_1}\otimes\mathbf{1})\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{21}, \tag{73}$$

$$I^J_{13}=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}(\mathbf{t}^J_{m_1}\otimes\mathbf{1}\otimes\mathbf{1}) \times\mathcal{R}_{32}\mathcal{R}_{31}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}, \tag{74}$$

$$I^J_{23}=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_1}\otimes\mathbf{1}) \times\mathcal{R}^{-1}_{21}\mathcal{R}_{32}\mathcal{R}_{31}(\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}. \tag{75}$$

We can construct the observable I^J_{13} from I^J_{12} by permuting 2 with 3, using the braided permutation ψ_{23} defined in Eq. (7). Upon this permutation, we have in particular that $\mathcal{R}_{21}$ becomes $\mathcal{R}_{31}$,

$$\begin{aligned}\psi_{23}I^J_{12}\psi^{-1}_{23}&=\sqrt{[2J+1]}\sum_{m_i}\mathcal{R}_{32}\,{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}(\mathbf{t}^J_{m_1}\otimes\mathbf{1}\otimes\mathbf{1})\\&\quad\times\mathcal{R}_{31}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}\\&=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}(\mathbf{t}^J_{m_1}\otimes\mathbf{1}\otimes\mathbf{1})\\&\quad\times\mathcal{R}_{32}\mathcal{R}_{31}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}\\&=I^J_{13}.\end{aligned} \tag{76}$$

We have used the fact that $\mathcal{R}_{23}$ and $\mathbf{t}^J_{m_1}\otimes\mathbf{1}\otimes\mathbf{1}$ commute. This can be extended to arbitrary I^J_{1j}. Now we would like to consider the construction of I^J_{23} from I^J_{12}. As a matter of fact, we can start from I^J_{13} and permute 1 and 2 using the deformed permutation ψ_{12},

$$\psi_{12}I^J_{13}\psi^{-1}_{12}=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_1}\otimes\mathbf{1}) \times\mathcal{R}_{31}\mathcal{R}_{32}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{32}\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{21}. \tag{77}$$

To simplify this expression, we use the Yang-Baxter equation,

$$\mathcal{R}_{dc}\mathcal{R}_{db}\mathcal{R}_{cb}=\mathcal{R}_{cb}\mathcal{R}_{db}\mathcal{R}_{dc}, \tag{78}$$

with $c=2, b=1, d=3$. We then have

$$\begin{aligned}\psi_{12}I^J_{13}\psi^{-1}_{12}&=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_1}\otimes\mathbf{1})\\&\quad\times\mathcal{R}_{31}\mathcal{R}_{32}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{21}\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}\\&=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_1}\otimes\mathbf{1})\\&\quad\times\mathcal{R}_{31}\mathcal{R}_{32}\mathcal{R}^{-1}_{21}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32},\end{aligned} \tag{79}$$

where we used the fact that $\mathcal{R}^{-1}_{21}$ commutes with $\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2}$. We use again the Yang-Baxter equation (80) for the product of $\mathcal{R}$-matrices in the middle of the above expression,

$$\mathcal{R}^{-1}_{21}\mathcal{R}_{32}\mathcal{R}_{31}\mathcal{R}_{21}=\mathcal{R}_{31}\mathcal{R}_{32}, \tag{80}$$

$$\begin{aligned}\psi_{12}I^J_{13}\psi^{-1}_{12}&=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_1}\otimes\mathbf{1})\\&\quad\times\mathcal{R}^{-1}_{21}\mathcal{R}_{32}\mathcal{R}_{31}\mathcal{R}_{21}v^{-1}_{21}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}\\&=\sqrt{[2J+1]}\sum_{m_i}{}_q\mathbf{C}^{J\,J\,0}_{m_1m_2 0}\mathcal{R}_{21}(\mathbf{1}\otimes\mathbf{t}^J_{m_1}\otimes\mathbf{1})\\&\quad\times\mathcal{R}^{-1}_{21}\mathcal{R}_{32}\mathcal{R}_{31}(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{t}^J_{m_2})\mathcal{R}^{-1}_{31}\mathcal{R}^{-1}_{32}\\&=I^J_{23}.\end{aligned} \tag{81}$$

This is the relevant expression for I^J_{23}. Hence, we can obtain any I^J_{ij} with $i<j$ using the braided permutation, starting from I^J_{12}. A similar argument applies to constructing the terms I^J_{ji} with $i<j$. We can obtain them by induction using the braided permutation, starting from the first term I^J_{21}.

Now that we have provided a general rule and some tricks to construct observables, it is natural to answer the following questions:

(i) Can we generate any observables from a fundamental algebra of observables?

(ii) What is the physical meaning and the implications of some of the key observables defined in the $\mathcal{U}_q(\mathfrak{su}(2))$ context?

We explore these questions now.

B. $\mathcal{U}_q(\mathfrak{u}(N))$ formalism for LQG defined over $\mathcal{U}_q(\mathfrak{su}(2))$

We want to construct the "smallest" observables. It is therefore natural to consider the observables built from the scalar product of spinor operators (47). Since we have two types of spinor operators, we have different possible combinations,

$$\mathcal{E}_{\alpha\beta} \equiv -{}^{(\alpha)}T^{\frac{1}{2}} \cdot {}^{(\beta)}\tilde{T}^{\frac{1}{2}}, \qquad \mathcal{G}^\dagger_{\alpha\beta} \equiv -{}^{(\alpha)}T^{\frac{1}{2}} \cdot {}^{(\beta)}T^{\frac{1}{2}}$$
$$\mathcal{F}_{\alpha\beta} \equiv -{}^{(\alpha)}\tilde{T}^{\frac{1}{2}} \cdot {}^{(\beta)}\tilde{T}^{\frac{1}{2}}. \tag{82}$$

Note that since the operators on different legs do not commute, we could *a priori* choose a different order of T and $\tilde{T}$ in the definition of $\mathcal{E}_{\alpha\beta}$. However, one can show that choosing the order leads to the same operator modulo a constant factor. This factor comes from the (deformed) symmetry of the scalar product as well as the commutation relations between the spinor operators acting on different legs.

Let us focus on the operators $\mathcal{E}_{\alpha\beta}$. Consider first the spinor operators that act on the same leg $\alpha = \beta = i$,

$$\mathcal{E}_{ii} \coloneqq -{}^{(i)}T^{\frac{1}{2}} \cdot {}^{(i)}\tilde{T}^{\frac{1}{2}} = -q^{\frac{1}{4}}\mathcal{A}_i^\dagger \tilde{\mathcal{A}}_i + q^{-\frac{1}{4}}\mathcal{B}_i^\dagger \tilde{\mathcal{B}}_i. \tag{83}$$

Having in mind Lemma III.5, we can forget about the tensor product, and the only relevant action is on the leg i; therefore,

$$\mathcal{E}_{ii}|\iota_{j_1\ldots j_N}\rangle = [2j_i]|\iota_{j_1\ldots j_N}\rangle. \tag{84}$$

Consider now the spinor operators that act on different legs i and j,

$$\mathcal{E}_{ij} = \mathcal{A}_i^\dagger \mathcal{C}_j + \mathcal{B}_i^\dagger \mathcal{D}_j. \tag{85}$$

The action of $\mathcal{E}_{12} = \mathcal{A}_1^\dagger \mathcal{C}_2 + \mathcal{B}_1^\dagger \mathcal{D}_2$ on a trivalent intertwiner is given by

$$\begin{aligned}\mathcal{E}_{12}|\iota_{j_1j_2j_3}\rangle = &-\tilde{N}^{\frac{1}{2}}_{j_2} N^{\frac{1}{2}}_{j_1}(-1)^{j_1+j_2+j_3-1} q^{-\frac{3}{4}j_1}\\ &\times\sqrt{[2j_1+2][2j_2]}\begin{Bmatrix} j_2-\frac{1}{2} & \frac{1}{2} & j_2\\ j_1 & j_3 & j_1+\frac{1}{2}\end{Bmatrix}\\ &\times|i_{j_1+\frac{1}{2}j_2-\frac{1}{2}j_3}\rangle,\end{aligned} \tag{86}$$

with the normalization choice

$$N^{\frac{1}{2}}_j = [d_j]^{\frac{1}{2}} q^{\frac{j}{2}}, \qquad \tilde{N}^{\frac{1}{2}}_j = [d_j]^{\frac{1}{2}} q^{\frac{j}{2}-\frac{1}{4}}.$$

The other operators $\mathcal{E}_{ij}$ $(i, j \in \{1, 2, 3\})$ can be constructed using the tricks described in the previous section. In a similar way, we get

$$\begin{aligned}q^{\frac{3}{4}}\mathcal{F}_{12}|i_{j_1j_2j_3}\rangle = &-\tilde{N}^{\frac{1}{2}}_{j_2}\tilde{N}^{\frac{1}{2}}_{j_1}(-1)^{j_1+j_2+j_3} q^{\frac{1}{2}(j_2+1)}\\ &\times\sqrt{[2j_1][2j_2]}\begin{Bmatrix} j_2-\frac{1}{2} & \frac{1}{2} & j_2\\ j_1 & j_3 & j_1-\frac{1}{2}\end{Bmatrix}\\ &\times|i_{j_1-\frac{1}{2}j_2-\frac{1}{2}j_3}\rangle,\\ q^{\frac{1}{4}}\mathcal{G}^\dagger_{12}|i_{j_1j_2j_3}\rangle = &-N^{\frac{1}{2}}_{j_2}N^{\frac{1}{2}}_{j_1}(-1)^{j_1+j_2+j_3} q^{-\frac{1}{2}(j_1+\frac{3}{4})}\\ &\times\sqrt{[2j_1+2][2j_2+2]}\\ &\times\begin{Bmatrix} j_2+\frac{1}{2} & \frac{1}{2} & j_2\\ j_1 & j_3 & j_1+\frac{1}{2}\end{Bmatrix}|i_{j_1+\frac{1}{2}j_2+\frac{1}{2}j_3}\rangle.\end{aligned} \tag{87}$$

When we perform the limit $q \to 1$, the operators $\mathcal{A}_i^\dagger$, $\mathcal{B}_i^\dagger$, $\mathcal{C}_i$, and $\mathcal{D}_i$ become, respectively, $a_i^\dagger$, $b_i^\dagger$, a_i, and b_i, that is, the standard harmonic oscillators' operators. Hence in this limit, the operators $\mathcal{E}_{ij}$ become $a_i^\dagger a_j + b_i^\dagger b_j$, which are the generators E_{ij} of a $\mathfrak{u}(n)$ Lie algebra, written using the Schwinger-Jordan representation. In a similar way, the operators $\mathcal{F}_{ij}, \mathcal{G}_{ij}$ become, respectively, F_{ij} and $F^\dagger_{ij}$, defined as follows:

$$\mathcal{G}^\dagger_{ij} \overset{q\to 1}{\to} a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger = F^\dagger_{ij}, \qquad \mathcal{F}_{ij} \overset{q\to 1}{\to} a_i b_j - b_i a_j = F_{ij}. \tag{88}$$

We recognize the operators E, F, and $F^\dagger$, which are the basis of the U(N) formalism [30–32]. They appear very naturally in our framework.

It is then natural to demand if the operators $\mathcal{E}_{ij}$ are the generators $\mathcal{U}_q(\mathfrak{u}(N))$. First, let us recall the definition of $\mathcal{U}_q(\mathfrak{u}(N))$ Cartan-Weyl generators [45]. We have, respectively, the raising, diagonal, and lowering operators $\mathfrak{E}_{ii+1}$, $\mathfrak{E}_i$, and $\mathfrak{E}_{i-1i}$, with the following commutation relations:

$$\begin{aligned}[][\mathfrak{E}_{ii}, \mathfrak{E}_{jj}] &= 0,\\ [\mathfrak{E}_{ii}, \mathfrak{E}_{jj+1}] &= (\delta_{ij} - \delta_{ij+1})\mathfrak{E}_{jj+1},\\ [\mathfrak{E}_{ii}, \mathfrak{E}_{j-1j}] &= (\delta_{ij+1} - \delta_{ij})\mathfrak{E}_{j-1j},\\ [\mathfrak{E}_{ii+1}, \mathfrak{E}_{j-1j}] &= \delta_{ij}(\mathfrak{E}_i - \mathfrak{E}_{i+1}).\end{aligned}$$

The other generators are constructed by induction,

$$\mathfrak{E}_{ij} = q^{\frac{1}{2}\mathfrak{E}_{j-1}}(\mathfrak{E}_{ij-1}\mathfrak{E}_{j-1j} - q^{\frac{1}{2}}\mathfrak{E}_{j-1,j}\mathfrak{E}_{ij-1}), \qquad j > i+1, \tag{89}$$

$$\mathfrak{E}_{ji} = q^{-\frac{1}{2}\mathfrak{E}_{j-1}}(\mathfrak{E}_{jj-1}\mathfrak{E}_{j-1i} - q^{-\frac{1}{2}}\mathfrak{E}_{j-1,i}\mathfrak{E}_{jj-1}), \qquad j > i+1. \tag{90}$$

Note that $\mathfrak{E}_{ij}$ is not necessarily the adjoint of $\mathfrak{E}_{ji}$ due to the presence of q. The coproduct is defined as follows:

$$\begin{aligned}\Delta\mathfrak{E}_i &= \mathfrak{E}_i \otimes \mathbf{1} + \mathbf{1} \otimes \mathfrak{E}_i,\\ \Delta\mathfrak{E}_{ii+1} &= \mathfrak{E}_{ii+1} \otimes q^{\mathfrak{E}_i+\mathfrak{E}_{i+1}} + q^{\mathfrak{E}_i+\mathfrak{E}_{i+1}} \otimes \mathfrak{E}_{ii+1}.\end{aligned} \tag{91}$$

The coproduct for the other generators is obtained by induction.

The Schwinger-Jordan map allows us to express these generators in terms of N q-harmonic oscillators a_i,

$$\mathfrak{E}_{ij} = a_i a_j^\dagger, \qquad \mathfrak{E}_i = \frac{1}{2}(N_i - N_{i+1}). \tag{92}$$

To have the representation of these generators in terms of N pairs of q-harmonic oscillators (a_i, b_i), we use the coproduct:

$$\mathfrak{E}_{ii} \coloneqq N_{a_i} + N_{b_i}, \tag{93}$$

$$\begin{aligned}\mathfrak{E}_{i,i+p} \coloneqq{}& a_i^\dagger a_{i+p} q^{\frac{N_{b_i}+2\left(\sum_{l=1}^{p-1} N_{b_{i+l}}\right)-N_{b_{i+p}}}{4}} \\ &+ q^{\frac{-N_{a_i}+2\left(\sum_{l=1}^{p-1} N_{a_{i+l}}\right)+N_{a_{i+p}}}{4}} b_i^\dagger b_{i+p} \\ &+ (q^{-\frac{1}{4}} - q^{\frac{3}{4}}) \sum_{k=1}^{p-1} \left(q^{\frac{N_{a_{i+k}}+2\left(\sum_{l=k+1}^{p-1} N_{a_{i+l}}\right)+N_{a_{i+p}}}{4}} \right. \\ &\left. \times a_i^\dagger a_{i+k} q^{\frac{N_{b_i}+2\left(\sum_{l=1}^{k-1} N_{b_{i+l}}\right)+N_{b_{i+k}}}{4}} b_{i+k}^\dagger b_{i+p} \right),\end{aligned} \tag{94}$$

$$\begin{aligned}\mathfrak{E}_{i+p,i} \coloneqq{}& a_i a_{i+p}^\dagger q^{\frac{N_{b_i}-2\left(\sum_{l=1}^{p-1} N_{b_{i+l}}\right)-N_{b_{i+p}}}{4}} \\ &+ q^{\frac{-N_{a_i}-2\left(\sum_{l=1}^{p-1} N_{a_{i+l}}\right)+N_{a_{i+p}}}{4}} b_i b_{i+p}^\dagger + (q^{\frac{1}{4}} - q^{-\frac{3}{4}}) \\ &\times \sum_{k=1}^{p-1} \left(q^{\frac{-N_{a_i}-2\left(\sum_{l=1}^{k-1} N_{a_{i+l}}\right)-N_{a_{i+k}}}{4}} \right. \\ &\left. \times a_{i+p}^\dagger a_{i+k} q^{\frac{-N_{b_{i+k}}-2\left(\sum_{l=k+1}^{p-1} N_{b_{i+l}}\right)-N_{b_{i+p}}}{4}} b_{i+k}^\dagger b_i \right).\end{aligned} \tag{95}$$

Using the definition of the $\mathfrak{E}_{ij}$ in terms of the q-harmonic oscillators (a_i, b_i) deduced from the expression of the spinor operators, we can identify a nonlinear relationship between these $\mathfrak{E}_{ij}$ and the $\mathcal{E}_{ij}$,

$$\begin{aligned}\mathcal{E}_{ii} &= q^{-\frac{1}{2}} q^{\frac{\mathfrak{E}_{ii}}{2}} [\mathfrak{E}_{ii}], \\ \mathcal{E}_{i,i+1} &= q^{\frac{\mathfrak{E}_{i+1,i+1}}{4}} \mathfrak{E}_{i,i+1}, \\ \mathcal{E}_{i+1,i} &= q^{\frac{\mathfrak{E}_{ii}}{2}} \mathfrak{E}_{i+1,i} q^{\frac{\mathfrak{E}_{i+1,i+1}}{4}}, \\ \mathcal{E}_{i,i+p} &= q^{\frac{-\sum_{l=1}^{p-1} \mathfrak{E}_{i+l,i+l} + \mathfrak{E}_{i+p,i+p}}{4}} \mathfrak{E}_{i,i+p}, \\ \mathcal{E}_{i+p,i} &= q^{\frac{2\mathfrak{E}_{ii} + \sum_{l=1}^{p-1} \mathfrak{E}_{i+l,i+l}}{4}} \mathfrak{E}_{i+p,i} q^{\frac{\mathfrak{E}_{i+p,i+p}}{4}}.\end{aligned} \tag{96}$$

To have a nonlinear redefinition of the generators is something common when dealing with quantum groups. For example, there exist different realizations of $\mathcal{U}_q(\mathfrak{su}(2))$, all related by a nonlinear redefinition of the generators [49]. Biedenharn also recalled different definitions of the generators of $\mathcal{U}_q(\mathfrak{u}(N))$ related by nonlinear transformations in Ref. [45]. For some choice of generators, the commutation relation might take a simpler shape but the coproduct would be more complicated, and vice versa. The key point here is that we have found that the intertwiner carries a representation of $\mathcal{U}_q(\mathfrak{u}(N))$, and this generalizes the results of Refs. [30,32].

In the classical case, when $q = 1$, it was shown that the intertwiner carries an *irreducible* $\mathfrak{u}(n)$ representation [31]. A similar result also holds here. A cumbersome proof can probably be obtained by looking at the Casimirs of $\mathcal{U}_q(\mathfrak{u}(N))$. We do not want to follow this route. Instead, we would like to recall the seminal results by Jimbo, Rosso, and Lustzig [50–52] which essentially stated that all the finite-dimensional representations of the deformation $\mathcal{U}_q(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ (where $\mathfrak{g}$ is any complex simple Lie algebra) are completely reducible. The irreducible representations can be classified in terms of highest weights, and in particular they are *deformations* of the irreducible representations of $\mathcal{U}(\mathfrak{g})$ when q is *not* a root of unity. We can extend this result to the semi-simple case and to $\mathcal{U}_q(\mathfrak{u}(N))$ in particular (see Sec. 2.5 of Ref. [45], for example). Now we know that when $q = 1$ the intertwiner is an irreducible representation of $\mathfrak{u}(n)$; hence, by deforming the enveloping algebra, the representation of $\mathcal{U}_q(\mathfrak{u}(N))$ carried by the $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner must remain an irreducible representation. As a consequence, the $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner must carry an irreducible representation of $\mathcal{U}_q(\mathfrak{u}(N))$, just as in the classical case.

Finally, we can discuss the Hermiticity property of the scalar operators we have constructed. Indeed, we expect an observable to be self-adjoint. The operators $\mathcal{E}_{ij}$ are not self-adjoint, but this should not come as a surprise. Indeed, the classical operators E_{ij} are not Hermitian either. However, the adjoint $(E_{ij})^\dagger = E_{ji}$ is still a generator. This means that we can do a linear change of basis $E_{ij} \to E_{ij} + (E_{ij})^\dagger$ in the $\mathfrak{u}(n)$ basis to construct self-adjoint generators. This is actually how the formalism was initially introduced in Ref. [30]. The Cartan-Weyl generators $\mathfrak{E}_{ij}$, when expressed in terms of the harmonic oscillators, satisfy a similar property, namely $\mathfrak{E}_{ij}^\dagger = \mathfrak{E}_{ji}$ [45]. As a consequence, from the $\mathcal{E}_{ij}$, we can do a (nonlinear) change of basis and construct the relevant Hermitian $\mathcal{U}_q(\mathfrak{u}(N))$ generators [which will be $\mathcal{U}_q(\mathfrak{su}(2))$ invariant] using the maps (96).

VI. GEOMETRIC INTERPRETATION OF SOME OBSERVABLES IN THE LQG CONTEXT

In LQG with $\Lambda = 0$, the intertwiner is understood as the fundamental chunk of quantum space. For a two-dimensional (2D) space, it is dual to a face, whereas in three dimensions it is dual to a polyhedron. The intertwiner is invariant under the action of $\mathfrak{su}(2)$, and hence the observables should be invariant under the adjoint action of $\mathfrak{su}(2)$. We see that the use of tensor operators allows us to construct such observables in a direct manner: we need to construct

TABLE IV. Geometry of the flat and hyperbolic triangles.

	Flat case, $\Lambda = 0$	Hyperbolic case, $\Lambda < 0$, $R = \lvert\Lambda\rvert^{-\frac{1}{2}}$
Closure constraint	$\sum_i \vec{n}_i = 0$	To be determined [38]
Edge length	$\lvert\vec{n}_i\rvert = \ell_i$	$\lvert\vec{n}_i\rvert = \sinh\frac{\ell_i}{R}$
Cosine law	$\cos\theta_a = -\hat{n}_b\cdot\hat{n}_c = -\frac{\ell_a^2-\ell_b^2-\ell_c^2}{2\ell_b\ell_c}$	$\cos\theta_a = -\hat{n}_b\cdot\hat{n}_c = \frac{-\cosh\frac{\ell_a}{R}+\cosh\frac{\ell_b}{R}\cosh\frac{\ell_c}{R}}{\sinh\frac{\ell_b}{R}\sinh\frac{\ell_c}{R}}$
Area	$\mathcal{A}^2 = \frac{1}{4}(s(s-\ell_a)(s-\ell_b)(s-\ell_c))$	$\sin^2\frac{\mathcal{A}}{2R^2} = \frac{\sinh(\frac{s}{2R})\sinh(\frac{s-\ell_a}{2R})\sinh(\frac{s-\ell_b}{2R})\sinh(\frac{s-\ell_c}{2R})}{\cosh^2\frac{\ell_a}{2R}\cosh^2\frac{\ell_b}{2R}\cosh^2\frac{\ell_c}{2R}}$

operators that transform as a scalar under the adjoint action of $\mathfrak{su}(2)$. We have seen in the previous section how this formalism can be extended to the quantum group case $\mathcal{U}_q(\mathfrak{su}(2))$ in a direct manner. When $\Lambda = 0$, some observables have a clear geometrical meaning. We have for example the quantum version of the angle, the length, etc. We now explore the generalization of these geometric operators in three dimensions, namely, in the Euclidian case[8] with $\Lambda < 0$.

For simplicity we are going to focus on the three-leg intertwiner. When $\Lambda = 0$, we know that it encodes the quantum state of a triangle. Let us quickly recall the main geometric features of a triangle, either flat or hyperbolic.

Classically, a *flat* triangle can be described by the normals $\vec{n}_i$, $i = a, b, c$ to its edges, such that $|\vec{n}_i| = \ell_i$ is the edge length. To have a triangle the normals need to sum up to zero; this is the closure constraint. All the geometric information of the triangle can then be expressed in terms of these normals, as recalled in Table IV.

Now, let us consider a hyperbolic triangle Fig. 1. Its edges are geodesics in the 2D hyperboloid of radius R. Unlike the flat triangle, a hyperbolic triangle can be characterized by its three angles θ_i or the three lengths ℓ_i of its edges. The hyperbolic cosine laws relate the edge lengths and the angles (see Table IV). The area $\mathcal{A}$ of the triangle is given in terms of the angles,

$$\mathcal{A} = (\pi - (\theta_a + \theta_b + \theta_c))R^2. \tag{97}$$

In order to make the limit to the flat case easier, we can encode all this information in terms of the normals. Note however that due to the curvature, we have a different tangent space at each point of the edge. The tangent vectors and their normal are therefore not living in the same vector space for different points. In the curved case, we shall consider the normals $\vec{n}_i$ at each vertex of the triangle. As a direct consequence, the closure constraint in the curved case is subtler than in the flat case. We postpone the study of this constraint to a detailed analysis of the relevant phase space in Ref. [38]. We recall in Table IV the main geometric features of the flat and hyperbolic triangles, in terms of the normals. We use the notation $s = \frac{1}{2}(\ell_a + \ell_b + \ell_c)$.

[8]Note that Barrett also explored some aspects of quantum curved geometries in the Euclidian case with $\Lambda > 0$ using a different approach [53].

The quantization of the flat triangle can be done very naturally. The quantum state is given by the three-leg SU(2) intertwiner. We associate the normalized normals $\vec{n}_i$ to the flux operators ${}^{(i)}\vec{J}$, which we now know are related to the SU(2) vector operators ${}^{(i)}\tau^1$ (cf. Sec. IV B). This provides a direct quantization of all the geometric data: the closure constraint, length, angles, and area (see Ref. [54] for a recent review of these results).

We now consider a $\mathcal{U}_q(\mathfrak{su}(2))$ three-leg intertwiner $|\iota_{j_b j_c j_a}\rangle$. The ordering we choose for the legs is fixed, as we have already emphasized. We would like to check whether it encodes the quantum state of a hyperbolic triangle. We use the $\mathcal{U}_q(\mathfrak{su}(2))$ tensor operators to probe the geometry of this state of geometry. Since we are in the 3D framework with a negative cosmological constant, we take $q = e^{\lambda}$, with $\lambda = \frac{\ell_p}{R}$, and $\Lambda^{-1} = -R^2$.

A. Angle operator

Since we know that the angles completely specify the hyperbolic triangle, we can focus first on operators characterizing angles. In analogy with the nondeformed case, we define the scalar product of the vector operators ${}^{(i)}\hat{\mathbf{t}}^1$ and ${}^{(j)}\hat{\mathbf{t}}^1$, with the chosen normalization $\hat{N}^1_{j_i} = 1$ and $i \neq j$. We look at the action of this operator on the three-leg intertwiner $|\iota_{j_b j_c j_a}\rangle$. For simplicity we focus on ${}^{(b)}\hat{\mathbf{t}}^1 \cdot {}^{(c)}\hat{\mathbf{t}}^1$, since we know how to recover the other types of operators from this one using tricks developed in Sec. V A:

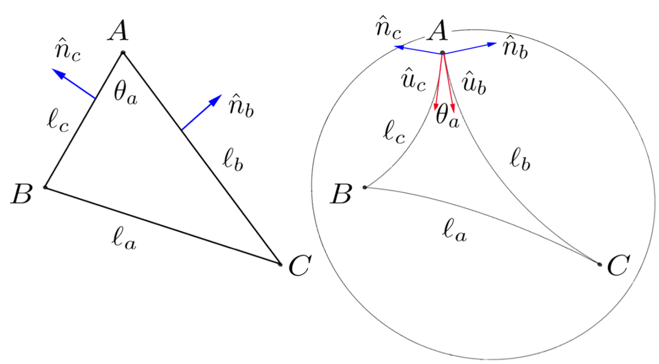

FIG. 1 (color online). The hyperbolic triangle is represented in the Poincaré disc. The (outgoing) normals $\hat{n}_i$ are defined in the tangent plane at the vertex of the triangle as the orthogonal vectors to the tangent vectors $\hat{u}_i$.

$$
\begin{aligned}
{}^{(b)}\hat{\mathbf{t}}^1 \cdot {}^{(c)}\hat{\mathbf{t}}^1 |\iota_{j_b j_c j_a}\rangle &= -q \frac{\cosh\frac{\lambda}{2}\cosh((j_a+\frac{1}{2})\lambda) - \cosh((j_b+\frac{1}{2})\lambda)\cosh((j_c+\frac{1}{2})\lambda))}{\sqrt{(\sinh(j_b\lambda))(\sinh((j_b+1)\lambda))(\sinh(j_c\lambda))(\sinh((j_c+1)\lambda))}} |\iota_{j_b j_c j_a}\rangle \\
&= -q \frac{\cosh\frac{\lambda}{2}\cosh((j_a+\frac{1}{2})\lambda) - \cosh((j_b+\frac{1}{2})\lambda)\cosh((j_c+\frac{1}{2})\lambda))}{\sqrt{(\sinh^2((j_b+\frac{1}{2})\lambda) - \sinh^2\frac{\lambda}{2})(\sinh^2((j_c+\frac{1}{2})\lambda) - \sinh^2\frac{\lambda}{2})}} |\iota_{j_b j_c j_a}\rangle, \qquad (98)
\end{aligned}
$$

where we have used $q = e^\lambda$ and $\sinh(j\lambda))(\sinh((j+1)\lambda)) = \sinh^2((j+\frac{1}{2})\lambda) - \sinh^2\frac{\lambda}{2}$. We recognize in Eq. (98) a quantization of the hyperbolic cosine law, provided we consider the quantization of the length edge given by $\ell \to (j+\frac{1}{2})\ell_p$. Note that the factors $\sinh^2\frac{\lambda}{2}$ in the denominator and $\cosh\frac{\lambda}{2}$ in the numerator can be interpreted as ordering ambiguity factors, arising from the respective quantization of $\sinh\frac{\ell_i}{R}$ and $\cosh\frac{\ell_i}{R}$.

In the limit $q \to 1$, we recover the quantized cosine law for a flat triangle [55] expressed in terms of the quantized normals, modulo an overall sign and a factor of $\frac{1}{2}$,

$$
\begin{aligned}
&{}^{(b)}\hat{\tau}^1 \cdot {}^{(c)}\hat{\tau}^1 |\iota_{j_a j_b j_c}\rangle \\
&= -\left(\frac{j_a(j_a+1) - j_b(j_b+1) - j_c(j_c+1)}{\sqrt{j_b(j_b+1)j_c(j_c+1)}} + \mathcal{O}(\lambda^2)\right)|\iota_{j_b j_c j_a}\rangle. \qquad (99)
\end{aligned}
$$

From the construction of the vector operators in Sec. IV B, we know that

$$
\begin{aligned}
&{}^{(b)}\hat{\tau}^1 \cdot {}^{(c)}\hat{\tau}^1 |\iota_{j_a j_b j_c}\rangle \\
&= -\frac{2}{\sqrt{j_b(j_b+1)j_c(j_c+1)}} {}^{(b)}\vec{J} \cdot {}^{(c)}\vec{J} |\iota_{j_a j_b j_c}\rangle.
\end{aligned}
$$

This allows us to identify the source of the discrepancy for the factor of $\frac{1}{2}$ and the overall sign. In particular, the global minus sign in Eqs. (98) and (99) with respect to the flat/hyperbolic cosine law simply comes from the definition of the scalar product we have used.

Since ${}^{(i)}\vec{J}$ is interpreted in the LQG formalism as the quantized normal to the edge of the triangle, in the deformed case we interpret ${}^{(b)}\hat{\mathbf{t}}^1$ and ${}^{(c)}\hat{\mathbf{t}}^1$ as the quantized normals of the edges AC and AB, respectively, at the vertex A of the hyperbolic triangle.

We can play with the normalization of the vector operators to have a better defined hyperbolic law. Indeed, we notice that both Eq. (98) and Eq. (99) diverge when $j = 0$. Instead of taking the vector operator ${}^{(i)}\hat{\mathbf{t}}^1$ with the normalization $N^1_{j_i} = 1$, we can consider ${}^{(i)}\tilde{\mathbf{t}}^1$ with the normalization

$$
\begin{aligned}
\tilde{N}^1_j &\equiv \frac{\sqrt{\sinh(j\lambda)\sinh((j+1)\lambda)}}{\sinh((j+\frac{1}{2})\lambda)} \\
&\overset{q\to 1}{\to} \frac{\sqrt{j(j+1)}}{j+\frac{1}{2}}. \qquad (100)
\end{aligned}
$$

In this case the cosine laws become well behaved for small j,

$$
{}^{(b)}\tilde{\mathbf{t}}^1 \cdot {}^{(c)}\tilde{\mathbf{t}}^1 |\iota_{j_b j_c j_a}\rangle = -q \frac{\cosh\frac{\ell_p}{2\ell_c}\cosh((j_a+\frac{1}{2})\lambda) - \cosh((j_b+\frac{1}{2})\lambda)\cosh((j_c+\frac{1}{2})\lambda))}{\sinh((j_b+\frac{1}{2})\lambda)\sinh((j_c+\frac{1}{2})\lambda)} |\iota_{j_b j_c j_a}\rangle. \qquad (101)
$$

When dealing with a nonzero cosmological constant and the Planck length, one can expect (using dimensional analysis) to have a minimum angle [56]. This can now be explicitly checked. Setting $j_a = 0$, we must have $j_b = j_c = j$ since we are dealing with an intertwiner, and the quantum cosine law (98) gives

$$
\theta^{\min}_a(j) = \arccos\left(-q\frac{\cosh^2\frac{\lambda}{2} - \cosh^2((j+\frac{1}{2})\lambda)}{\sinh^2((j+\frac{1}{2})\lambda) - \sinh^2\frac{\lambda}{2}}\right), \qquad (102)
$$

which means that there is a nonzero minimum angle. When $\ell_p \to 0$ (classical limit) or $R \to \infty$ (flat quantum limit), Eq. (102) tends to 0, so we recover that the triangle is degenerate.

As expected, the angle observables can be expressed in terms of the $\mathcal{U}_q(\mathfrak{su}(2))$ generators,

$$
\begin{aligned}
i > j, \qquad {}^{(i)}\hat{\mathbf{t}}^1 \cdot {}^{(j)}\hat{\mathbf{t}}^1 &= \left(q^{-\frac{3}{2}}(-\mathcal{E}_{ij}\mathcal{E}_{ji} + \mathcal{E}_{ii}) + \frac{1}{[2]}\mathcal{E}_{ii}\mathcal{E}_{jj}\right), \\
i < j, \qquad {}^{(i)}\hat{\mathbf{t}}^1 \cdot {}^{(j)}\hat{\mathbf{t}}^1 &= \left(q^{\frac{1}{2}}(-\mathcal{E}_{ij}\mathcal{E}_{ji} + \mathcal{E}_{ii}) + \frac{q^2}{[2]}\mathcal{E}_{ii}\mathcal{E}_{jj}\right).
\end{aligned}
$$

B. Length operator

The length operator is obtained by looking at the norm of the *unnormalized* vector operator ${}^{(i)}\mathbf{t}^1$ with the normalization N^i_j,

$$ {}^{(i)}\mathbf{t}^1 \cdot {}^{(i)}\mathbf{t}^1 |\iota_{j_b j_c j_a}\rangle = (N^1_{j_i})^2 |\iota_{j_b j_c j_a}\rangle, \quad i = a, b, c. \tag{103} $$

By inspecting the classical and quantum hyperbolic cosine law (and keeping in mind that ${}^{(i)}\mathbf{t}^1$ encodes the quantization of the normal), it is natural to take

$$ N^1_j = \sqrt{\sinh^2\left(\left(j+\frac{1}{2}\right)\lambda\right) - \sinh^2\left(\frac{\lambda}{2}\right)} \quad \text{or} \quad \tilde{\mathbf{N}}^1_j = \sinh\left(\left(j+\frac{1}{2}\right)\lambda\right). \tag{104} $$

The normalization $\tilde{N}^1_j$ leads to the regularized hyperbolic cosine law (101). We note therefore that the norm of the vector operator corresponds to a function of the length operator. The length is quantized, with the eigenvalue $(j+\frac{1}{2})\ell_p$, as we have argued previously. The norm of the vector operator can be expressed in terms of the $\mathcal{E}$ operators,

$$ {}^{(i)}\mathbf{t}^1 \cdot {}^{(i)}\mathbf{t}^1 = \frac{1}{[2]}(-q\mathcal{E}_i^2 - (1+q^{-1})\mathcal{E}_i). \tag{105} $$

C. "Area" operator

In the flat case, one expresses the square of the area of the triangle in terms of a cosine and the norm of the normals; in this way, the operator is easy to quantize using vector operators [57],

$$ \mathcal{A}^2 = \frac{1}{4}(|\vec{n}_b|^2|\vec{n}_c|^2 - (\vec{n}_c\cdot\vec{n}_c)^2). \tag{106} $$

We proceed in the same manner as in the hyperbolic case. We do not consider the square of the area, but rather the square of the sine of the area. Indeed, the area of a hyperbolic triangle is given in terms of the triangle angles [Eq. (97)]. There are various ways to express functions of the area in terms of the edge lengths [58]. A convenient one will be

$$ \sin^2\frac{\mathcal{A}}{2R^2} = \frac{\sinh(\frac{s}{2R})\sinh(\frac{s-\ell_a}{2R})\sinh(\frac{s-\ell_b}{2R})\sinh(\frac{s-\ell_c}{2R})}{\cosh^2\frac{\ell_a}{2R}\cosh^2\frac{\ell_b}{2R}\cosh^2\frac{\ell_c}{2R}}, \tag{107} $$

where $s = \frac{1}{2}(\ell_a+\ell_b+\ell_c)$. Of course, in the flat limit ($R\to\infty$) we recover Heron's formula (see Table IV).

Playing with the cosine laws, we can express $\sin^2\frac{\mathcal{A}}{2R^2}$ only in terms of the normals,

$$ \begin{aligned} \sin^2\frac{\mathcal{A}}{2R^2} &= \frac{1}{4}\frac{\sinh^2\frac{\ell_b}{R}\sinh^2\frac{\ell_c}{R}(1-\cos^2\theta_a)}{(\cosh^2\frac{\ell_a}{2R}\cosh^2\frac{\ell_b}{2R}\cosh^2\frac{\ell_c}{2R})} \\ &= 2\frac{|\vec{n}_b|^2|\vec{n}_c|^2 - (\vec{n}_b\cdot\vec{n}_c)^2}{(1+\sqrt{1+|\vec{n}_a|^2})(1+\sqrt{1+|\vec{n}_b|^2})(1+\sqrt{1+|\vec{n}_c|^2})}. \end{aligned} \tag{108} $$

There is no difficulty in quantizing this expression since it only involves scalar products and norms of normals, which upon quantization become operators that are diagonal and functions of the Casimir operator. There is therefore no ordering issue anywhere. The area also has a discrete spectrum.

VII. OUTLOOK

A. Summary

Let us summarize the main results of our paper. We have recalled the definition of tensor operators for $\mathcal{U}_q(\mathfrak{su}(2))$, with q real, which is the relevant case to study Euclidian 3D LQG with $\Lambda < 0$ and Lorentzian $3+1$-dimensional LQG with $\Lambda > 0$.

We have shown how tensor operators are the natural choice for constructing observables for a $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner. These operators are the key to studying LQG defined in terms of a quantum group, as they provide sets of operators that transform well under the quantum group. We have generalized the $\mathrm{U}(N)$ formalism to the quantum group $\mathcal{U}_q(\mathfrak{su}(2))$. That is, we have shown how we can construct a closed algebra of observables [i.e., invariant under $\mathcal{U}_q(\mathfrak{su}(2))$] which can be related to the quantum group $\mathcal{U}_q(\mathfrak{u}(N))$. This means that the $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner carries a $\mathcal{U}_q(\mathfrak{u}(N))$ representation, which we argued must be irreducible. We have constructed the natural generalization of the LQG geometric operators and interpreted them in the 3D Euclidian setting. We have shown that a three-leg $\mathcal{U}_q(\mathfrak{su}(2))$ intertwiner encodes the quantum state of a hyperbolic triangle. We have also shown how the presence of a cosmological constant leads to a notion of the minimum angle, as expected [56]. These results provide new evidence for the use of the quantum group as a tool to encode the cosmological constant in the LQG formalism.

We note that the use of tensor operators can also be useful for dealing with lattice Yang-Mills theories built with $\mathcal{U}_q(\mathfrak{su}(2))$ as the gauge group. In particular, it would be interesting to see how tensor operators can be useful for implementing the observables found in Ref. [59]. In fact, there are a number of interesting routes open for exploration.

B. Hyperbolic polyhedra

We have studied the geometric operators in the context of 3D LQG. We have shown that they induce a quantum hyperbolic geometry. These operators should also be interpreted in the $3+1$-dimensional LQG case. The vector operator acting on a leg i would be interpreted as the quantization of the normal of the ith face of the polyhedron. The squared norm of the vector operator acting on each leg would now be interpreted as a *function* of the squared area operator. This implies that in this case we still expect to have a discrete spectrum for the (squared) area. The angle operator would now encode the quantization of the dihedral angle, i.e., the angle between normals. One could then construct the analogue of the squared volume operator, using the triple product between vector operators. Following the intuition gained from looking at the area operator for the triangle, we would then expect to get a *function* of the volume of the hyperbolic polyhedron. We leave the properties of such an operator to further investigations, as well as other interesting geometric operators we could construct to probe the quantum geometry of hyperbolic polyhedra.

C. Other signatures and other signs for Λ

When defining tensor operators, we have focused on $\mathcal{U}_q(\mathfrak{su}(2))$ with q real. This choice provided the relevant structure to study the physical case, $3+1$-dimensional LQG with $\Lambda > 0$. However, there are a number of other cases to study. At the classical level, with $q = 1$, we could explore the construction of tensor operators for $\mathrm{SL}(2,\mathbb{R})$, which would be relevant for Lorentzian $2+1$-dimensional LQG with $\Lambda = 0$. Interestingly, the Wigner-Eckart theorem has not been defined for $\mathrm{SL}(2,\mathbb{R})$; that is, there is no general formula for tensor operators transforming as $\mathrm{SL}(2,\mathbb{R})$ (nonunitary) finite-dimensional and discrete representations.[9] This work is in progress [60]. It would then be relevant to discuss the quantum group version of this structure, which would be relevant for $2+1$-dimensional Lorentzian gravity with $\Lambda \neq 0$.

Another interesting case to explore would be $\mathcal{U}_q(\mathfrak{su}(2))$ when q is a root of unity, which would be relevant for 3D Euclidan LQG with $\Lambda > 0$. We have not considered this case here, as $\mathcal{U}_q(\mathfrak{su}(2))$ when q is a root of unity is not a quasitriangular Hopf algebra, but rather a quasi-Hopf algebra. This means that the construction in Ref. [40] does not apply directly. On the other hand, the representation theory of $\mathcal{U}_q(\mathfrak{su}(2))$ when q is a root of unity can be trimmed of the unwanted features so that its recoupling theory can be well under control [9]. This is why the Turaev Viro model can still be defined as it is. It is then quite likely that we can define the tensor operators in this case, in terms of their matrix elements, which would be proportional to the Clebsch-Gordan coefficients. We leave this for future investigations.

[9] More precisely, there exists a definition of such tensor operators acting on the unitary (infinite-dimensional) discrete representation, provided by harmonic oscillators (the Shwinger-Jordan trick). There is no such definition for operators acting on unitary (infinite-dimensional) continuous representations.

D. Phase-space structure

One of our key results is that the quantum group spin networks can be used in the LQG context to introduce the cosmological constant. Recent developments have shown that spin networks can be seen as quantum states of flat discrete geometries, when $\Lambda = 0$. The classical phase-space structure behind spin networks is nicely described by the "twisted geometries" framework. Since we have identified the meaning of the quantum geometric operators in the quantum group case, we can provide some guiding lines for identifying the relevant phase-space structure, i.e., the notion of *curved twisted geometries*. In particular, one knows that the classical analogue of a quantum group is a Poisson-Lie group, so we can expect to use this structure to define the curved twisted geometries. This work is in progress [38]. Understanding how curved twisted geometries appear is interesting, but it is not enough. The important question to solve is how one can connect the canonical Hamiltonian analysis of general relativity and these curved twisted geometries. Understanding this will provide an explanation of why the cosmological constant appears already at the kinematical level, and not only in the Hamiltonian constraint as the canonical Hamiltonian analysis would indicate. We leave this important question for future investigations.

E. Hamiltonian constraint

LQG and spin foams are supposed to be two facets of the same theory. This can only be shown explicitly in the $\Lambda = 0$ case in three dimensions [14]. Recently, a Hamiltonian constraint was constructed using the spinor formalism [35]. It has been designed to encode a recursion relation on the $6j$ symbol, and hence by construction it relates the Ponzano-Regge model to the LQG approach. Now that we have generalized the spinor approach to the quantum group case, we can construct a q-deformed version of this Hamiltonian constraint. It would essentially encode the recursion relation of the q-deformed $6j$ symbol. Hence this new q-deformed Hamiltonian constraint would relate the Turaev-Viro model and LQG with a cosmological constant. This work is in progress [37].

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APPENDIX A: HYPERBOLIC COSINE LAW

Consider the upper 2D hyperboloid H^2_+, embedded in $\mathbb{R}^3$, with curvature $-R^{-2} = \Lambda$, where R is the radius of curvature,

$$H^2_+ = \{\vec{x} \in \mathbb{R}^3, x_1 > 0, x_i \eta^{ij} x_j = x_1^2 - x_2^2 - x_3^2 = |\vec{x}|^2 = R^2\}. \tag{A1}$$

On H^2_+, we consider three points A, B, C and the geodesics joining them: we obtain a hyperbolic triangle. Without loss of generality, we can always assume that A sits at the origin of H^2_+, that is, as a point of $\mathbb{R}^3$ it is given by the vector $\vec{A} = (R, 0, 0)$. The points B and C are then obtained from $\vec{A}$ by performing a boost L_c, L_b with rapidities c and b, respectively. Explicitly,

$$\vec{B} = L_c \vec{A}, \qquad \vec{C} = L_b \vec{A}. \tag{A2}$$

As a consequence, we have $\langle \vec{A}, \vec{A} \rangle = |\vec{A}|^2 = |\vec{B}|^2 = |\vec{C}|^2 = R^2$.

Consider the normalized space-like vectors $\hat{u}_{AB}, \hat{u}_{AC} \in T_A H^2_+$, which is the tangent plane of H^2_+ at the point A. They are the tangent vectors to the geodesics joining A to B and A to C, respectively. By construction, these vectors are orthogonal to $\vec{A}$,

$$\hat{u}_{AB} = \frac{\vec{B} - \frac{1}{R^2}\langle \vec{A}, \vec{B}\rangle \vec{A}}{|\vec{B} - \frac{1}{R^2}\langle \vec{A}, \vec{B}\rangle \vec{A}|}, \qquad \hat{u}_{AC} = \frac{\vec{C} - \frac{1}{R^2}\langle \vec{A}, \vec{C}\rangle \vec{A}}{|\vec{C} - \frac{1}{R^2}\langle \vec{A}, \vec{C}\rangle \vec{A}|}. \tag{A3}$$

Since we are dealing with a homogeneous space, we express the lengths ℓ_i of the geodesic arcs using the dimensionful parameter R, such that $\ell_c = Rc$, $\ell_b = Rb$, and $\ell_a = Ra$.

By definition, we know that the angle between two geodesics that intersect is defined in terms of the angle between the tangent vectors. If we focus in particular on the angle α between the arcs AB and AC, we have

$$\cos\alpha = \langle \hat{u}_{AB}, \hat{u}_{AC} \rangle. \tag{A4}$$

Using the expression for the tangent vectors, we obtain the hyperbolic cosine law,

$$\cos\alpha = \frac{-\cosh\frac{\ell_a}{R} + \cosh\frac{\ell_b}{R}\cosh\frac{\ell_c}{R}}{\sinh\frac{\ell_b}{R}\sinh\frac{\ell_c}{R}}. \tag{A5}$$

In the flat case, by performing the limit $R \to \infty$ in Eq. (A5) we recover the Al-Kashi rule,

$$\cos\alpha = \frac{-\ell_a^2 + (\ell_b^2 + \ell_c^2)}{2\ell_b \ell_c}. \tag{A6}$$

APPENDIX B: USEFUL FORMULAS

These formulas are taken from Ref. [45].

1. q-Clebsch-Gordan

An explicit expression of the q-Clebsch-Gordan coefficients in the van der Waerden form is given as

$${}_q\mathbf{C}^{j_1\, j_2\, j}_{m_1 m_2 m} := \delta_{m, m_1+m_2} q^{\frac{1}{4}(j_1+j_2-j)(j_1+j_2+j+1)+\frac{1}{2}(j_1 m_2 - j_2 m_1)} \Delta(j_1 j_2 j) \tag{B1}$$

$$\times ([j_1+m_1]![j_1-m_1]![j_2+m_2]![j_2-m_2]![j+m]![j-m]![2j+1])^{\frac{1}{2}} \tag{B2}$$

$$\times \sum_n \frac{(-1)^n q^{-\frac{n}{2}(j_1+j_2+j+1)}}{[n]![j_1+j_2-j-n]![j_1-m_1-n]![j_2+m_2-n]![j-j_2+m_1+n]![j-j_1-m_2+n]!}, \tag{B3}$$

where the triangle function Δ is given by

$$\Delta(abc) := \left(\frac{[a+b-c]![a-b+c]![-a+b+c]!}{[a+b+c+1]!}\right)^{\frac{1}{2}}. \tag{B4}$$

For $q \to 1$ the q-Clebsch-Gordan coefficients reduce to the usual CG coefficients in the van der Waerden form.

The q-Clebsch-Gordan coefficients have two orthogonality relations,

$$\sum_{m_1,m_2} {}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1 m_2 m}\, {}_q\mathbf{C}^{j_1\,j_2\,j'}_{m_1 m_2 m'} = \delta_{jj'}\delta_{mm'}, \tag{B5}$$

$$\sum_{j,m} {}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1 m_2 m}\, {}_q\mathbf{C}^{j_1\,j_2\,j}_{m'_1 m'_2} = \delta_{m_1 m'_1}\delta_{m_2 m'_2}. \tag{B6}$$

Note that in the first equation we have assumed that j_1, j_2, and j satisfy the triangle conditions.

The q-Clebsch-Gordan coefficients have some symmetries; we list here those that are most relevant for the current work:

$${}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1 m_2 m} = (-1)^{j_1+j_2-j}\, {}_{q^{-1}}\mathbf{C}^{j_1\,j_2\,j}_{-m_1 -m_2 -m}, \tag{B7}$$

$${}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1 m_2 m} = (-1)^{j_1+j_2-j}\, {}_{q^{-1}}\mathbf{C}^{j_2\,j_1\,j}_{m_2 m_1 m}, \tag{B8}$$

$${}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1 m_2 m} = (-1)^{j-j_2-m_1} q^{\frac{m_1}{2}} \sqrt{\frac{[2j+1]}{[2j_2+1]}}\, {}_q\mathbf{C}^{j_1\,j\,j_2}_{-m_1 m m_2}. \tag{B9}$$

We list below the values of some specific CG coefficients:

$${}_q\mathbf{C}^{j_1 j_2\,0}_{m_1 m_2 0} = \delta_{j_1,j_2}\delta_{m_1,-m_2} \frac{(-1)^{j_1-m_1} q^{\frac{m_1}{2}}}{\sqrt{[2j_1+1]}}, \tag{B10}$$

$${}_q\mathbf{C}^{111}_{101} = q^{1/2}\sqrt{\frac{[2]}{[4]}}, \qquad {}_q\mathbf{C}^{111}_{011} = -q^{-1/2}\sqrt{\frac{[2]}{[4]}},$$

$${}_q\mathbf{C}^{1\,1\,1}_{-10-1} = -q^{-1/2}\sqrt{\frac{[2]}{[4]}}, \qquad {}_q\mathbf{C}^{1\,\,1\,\,1}_{0-1-1} = q^{1/2}\sqrt{\frac{[2]}{[4]}},$$

$${}_q\mathbf{C}^{1\,\,11}_{1-10} = \sqrt{\frac{[2]}{[4]}}, \qquad {}_q\mathbf{C}^{1\,11}_{-110} = -\sqrt{\frac{[2]}{[4]}},$$

$${}_q\mathbf{C}^{111}_{000} = \sqrt{\frac{[2]}{[4]}}\left(q^{\frac{1}{2}} - q^{-\frac{1}{2}}\right), \tag{B11}$$

$${}_q\mathbf{C}^{\frac{1}{2}\frac{1}{2}1}_{\frac{1}{2}\frac{1}{2}1} = 1 = {}_q\mathbf{C}^{\frac{1}{2}\,\frac{1}{2}\,\,1}_{-\frac{1}{2}-\frac{1}{2}-1}, \qquad {}_q\mathbf{C}^{\frac{1}{2}\,\frac{1}{2}1}_{\frac{1}{2}-\frac{1}{2}0} = \frac{q^{-\frac{1}{4}}}{\sqrt{[2]}},$$

$${}_q\mathbf{C}^{\frac{1}{2}\,\frac{1}{2}1}_{-\frac{1}{2}\frac{1}{2}0} = \frac{q^{\frac{1}{4}}}{\sqrt{[2]}}. \tag{B12}$$

2. $q-6j$ symbol

The $q-6j$ symbol is invariant under the rescaling $q\to q^{-1}$. It satisfies the following orthogonality relation:

$$\sum_j \begin{Bmatrix} b & c & j \\ k & a & n \end{Bmatrix}\begin{Bmatrix} a & b & m \\ c & k & j \end{Bmatrix} = \delta_{mn}. \tag{B13}$$

The contraction of two $q-6j$ symbols can give another one, which is a useful property for us:

$$\sum_m (-1)^{a+b+c+k-j-m-n} q^{\frac{1}{2}(a(a+1)+b(b+1)+c(c+1)+k(k+1)-j(j+1)-m(m+1)-n(n+1))} \begin{Bmatrix} a & b & m \\ c & k & j \end{Bmatrix}\begin{Bmatrix} a & c & n \\ k & b & m \end{Bmatrix} = \begin{Bmatrix} a & c & n \\ b & k & j \end{Bmatrix}.$$

It has some symmetries when permuting some of its elements,

$$\begin{Bmatrix} a & b & m \\ c & k & j \end{Bmatrix} = \begin{Bmatrix} c & k & m \\ a & b & j \end{Bmatrix}. \tag{B14}$$

A specific value of the $q-6j$ symbol that is relevant to us is

$$\begin{Bmatrix} j_1 & j_1 & 1 \\ j_2 & j_2 & j_3 \end{Bmatrix} = (-1)^{j_1+j_2+j_3} \frac{[j_2+j_3-j_1][j_1+j_3-j_2] - [j_1+j_2-j_3][j_1+j_2+j_3+2]}{([2j_1][2j_1+1][2j_1+2][2j_2][2j_2+1][2j_2+2])^{\frac{1}{2}}}. \tag{B15}$$

3. $\mathcal{R}$-matrix and deformed permutation

The $\mathcal{R}$-matrix for $\mathcal{U}_q(\mathfrak{su}(2))$ can be expressed in terms of the q-Clebsch-Gordan coefficients,

$$(\mathcal{R}^{j_1 j_2})^{m_1 m_2}_{m'_1 m'_2} = \sum_{j,m} q^{-\frac{1}{2}(j_1(j_1+1)+j_2(j_2+1)-j(j+1))}\, {}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1\,m_2\,m}\, {}_{q^{-1}}\mathbf{C}^{j_1\,j_2\,j}_{m'_1\,m'_2\,m} \tag{B16}$$

$$= \sum_{j,m} (-1)^{j_1+j_2-j} q^{-\frac{1}{2}(j_1(j_1+1)+j_2(j_2+1)-j(j+1))}\, {}_q\mathbf{C}^{j_1\,j_2\,j}_{m_1\,m_2\,m}\, {}_q\mathbf{C}^{j_2\,j_1\,j}_{m'_2\,m'_1\,m}, \tag{B17}$$

with $m_1 + m_2 = m'_1 + m'_2$ and $m'_1 - m_1 \geq 0$ (this is zero otherwise). The second equation has been obtained using the symmetries of the q-Clebsch-Gordan coefficients.

The inverse of the $\mathcal{R}$-matrix is obtained from the above formulas by setting $q \to q^{-1}$,

$$(\mathcal{R}^{-1\,j_1 j_2})^{m_1 m_2}_{m'_1 m'_2} = \sum_{j,m} (-1)^{j_1+j_2-j} q^{\frac{1}{2}(j_1(j_1+1)+j_2(j_2+1)-j(j+1))} {}_{q^{-1}}\mathbf{C}^{j_1\, j_2\, j}_{m_1\, m_2\, m}\, {}_{q^{-1}}\mathbf{C}^{j_2\, j_1\, j}_{m'_2\, m'_1\, m} \quad \text{(B18)}$$

$$= \sum_{j,m} (-1)^{j_1+j_2-j} q^{\frac{1}{2}(j_1(j_1+1)+j_2(j_2+1)-j(j+1))} {}_{q}\mathbf{C}^{j_2\, j_1\, j}_{m_2\, m_1\, m}\, {}_{q}\mathbf{C}^{j_1\, j_2\, j}_{m'_1\, m'_2\, m}. \quad \text{(B19)}$$

One can check that this is true by evaluating $\mathcal{R}^{-1}\mathcal{R}$ and using the orthogonality properties of the q-Clebsch-Gordan coefficients. Furthermore, we can check that when $q \to 1$, we recover that the $\mathcal{R}$-matrix is simply the identity map [for this one uses the classical version of Eq. (B8) and the orthogonality relation (B6)].

We are interested in the deformed permutation $\psi_{\mathcal{R}} = \psi\mathcal{R}$ ($\psi_{\mathcal{R}}^{-1} = \mathcal{R}^{-1}\psi$), which means that instead of considering $\mathcal{R}^{j_1 j_2}$ ($\mathcal{R}^{-1\,j_1 j_2}$), we consider $\mathcal{R}^{j_2 j_1}$ ($\mathcal{R}^{-1\,j_2 j_1}$). The relevant formula for $\mathcal{R}^{j_2 j_1}$ is obtained from Eq. (B17) by exchanging j_1 and j_2.

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