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First law of black hole mechanics as a condition for stationarity

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In earlier work, we provided a Hilbert manifold structure for the phase space for the Einstein-Yang-Mills equations, and used this to prove a condition for initial data to be stationary [S. McCormick, Adv. Theor. Math. Phys. 18, 799 (2014)]. Here we use the same phase space to consider the evolution of initial data exterior to some closed 2-surface boundary, and establish a condition for stationarity in this case. It is shown that the differential relationship given in the first law of black hole mechanics is exactly the condition required for the initial data to be stationary; this was first argued nonrigorously by Sudarsky and Wald [Phys. Rev. D 46, 1453 (1992)]. Furthermore, we give evidence to suggest that if this differential relationship holds then the boundary surface is the bifurcation surface of a bifurcate Killing horizon.

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I. INTRODUCTION

In 1992, Sudarsky and Wald SW1 discussed the first law of black hole mechanics in the context of Einstein-Yang-Mills theory. Among other things, they noted that certain surface integrals, associated with the Hamiltonian, were closely related to the first law. From this, it was argued that the differential relationship given by the first law provides a condition for stationarity of the Einstein-Yang-Mills equations. This argument was based on earlier work by Brill, Deser and Fadeev [1], who proposed in the pure Einstein case, that stationary solutions were exactly those solutions that extremize the Arnowitt-Deser-Misner (ADM) mass over the space of solutions. Both arguments were based on Lagrange multipliers; however, neither provided the mathematical machinery required to make such an argument rigorous. The essential missing ingredient, to develop this argument into a mathematical proof, is a manifold structure for the space of solutions.

In 2005, Bartnik [2] provided such a Hilbert manifold structure for the Einstein case, and from this a complete proof of the Brill, Deser and Fadeev argument was given. At first, this may appear to contradict the argument of Sudarsky and Wald, since we have that a solution is stationary if and only if it is a critical point of the mass. However, the case considered by Bartnik has no Maxwell or Yang-Mills fields, and the initial data manifold has a single asymptotic end with no interior boundary; in this case, the first law simply reduces to $dm = 0$. Recently, using similar ideas, the Einstein-Yang-Mills case has been considered by the author [3]; we will refer to this throughout as Paper I, as we make use of several ideas and results from this paper. In Paper I, a suitable phase space for the Einstein-Yang-Mills equations was outlined, and the condition for stationarity in this case becomes

$$dm + V_\infty \cdot dQ_\infty = 0, \tag{1}$$

where V_∞ is the electric potential at infinity and Q_∞ is the total electric charge. Again, Eq. (1) is the appropriate first law in this case.

In this article, we consider evolution exterior to some closed 2-surface boundary, and conclude that the condition for stationarity is again the appropriate version of the first law:

$$dm = \frac{\kappa}{8\pi} dA + \Omega dJ + V \cdot dQ - V_\infty \cdot dQ_\infty, \tag{2}$$

where A is the area, Ω is the angular velocity, J is the angular momentum, V is the electric potential and Q is the electric charge of the boundary surface, respectively. Note that the inclusion of the term $V_\infty \cdot dQ_\infty$, which is not generally included in the first law, permits a nonzero electric potential at infinity. In the Maxwell electrovacuum case, $Q = Q_\infty$, so the expression $(V \cdot dQ - V_\infty \cdot dQ_\infty)$ is equivalent to $\tilde{V} dQ$, where $\tilde{V} = V - V_\infty$ is the potential difference between the boundary surface and infinity. This is then exactly the standard expression for the first law.

An initial data set for the Einstein-Yang-Mills equations is a tuple (g, A, π, ε): a Riemannian metric, a Lie algebra-valued one-form, a symmetric covariant 2-tensor density and a Lie coalgebra-valued vector density on a 3-manifold, $\mathcal{M}$. Here π is the usual momentum conjugate to g, A is the gauge field projected onto $\mathcal{M}$ and ε is its associated momentum, equal to -4 times the Yang-Mills electric field density, E. Throughout we will use both ϵ and E and we also make use of the quantity

$$B^i_a := \frac{1}{2} \epsilon^{ijk} (\nabla_j A_{ak} - \nabla_k A_{aj} + C_{abc} A^b_j A^c_k), \tag{3}$$

the Yang-Mills magnetic field density, where ϵ^{ijk} is the usual antisymmetric tensor density and C_{abc} are the

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structure constants of the Lie algebra, where the indices $a, b, c\ldots$ are Lie algebra indices. Note that the connection in Eq. (3) can be replaced by any torsion-free connection, due to the antisymmetry in ∇A. Throughout, we consider the electric and magnetic fields as viewed by a Gaussian normal set of observers; that is, observers whose worldlines are orthonormal to the Cauchy surface.

The Hilbert manifold structure from Paper I, considered here, consists of initial data sets (g, A, π, ε) with local regularity $H^2 \times H^2 \times H^1 \times H^1$ and an appropriate decay for asymptotic flatness. It is interesting to note that this is exactly the regularity required by the recent work of Klainerman, Rodnianski and Szeftel [4] to ensure that the Cauchy problem for the Einstein equations is well-posed. Furthermore, the regularity assumptions on the Yang-Mills initial data are exactly that required to ensure that the Cauchy for the Yang-Mills equations on a curved background is well-posed, which was recently demonstrated by Ghanem [5]. To the best of the author's knowledge the Cauchy problem for the coupled system has not been considered at this regularity; however, given that each system is well-posed independently, one expects the coupled system to also be well-posed.

The outline of this article is as follows. In Sec. II, we recall the phase space and constraint submanifold from Paper I. Section III introduces the mass, charge and angular momentum definitions, and establishes some properties of these quantities as functions on the phase space. Finally, in Sec. IV, we discuss Hamiltonians and use a Lagrange multiplier argument to establish the condition for stationarity.

II. THE PHASE SPACE

Let $\mathcal{M}$ be a complete, paracompact, connected, oriented 3-manifold that is asymptotically flat in the following sense: there exists a compact set, $K \subset \mathcal{M}$, such that $\mathcal{M}\backslash K = \cup_{i=1}^{N} M_i$, with each M_i diffeomorphic to $\mathbb{R}^3$ minus the closed unit ball; explicitly, there is a collection of diffeomorphisms $\phi_i\colon M_i \to \mathbb{R}^3\backslash\overline{B_1(0)}$. On $\mathcal{M}$, we fix a smooth background metric $\mathring{g}$ such that $\mathring{g} = \phi_{i*}(g_{\mathbb{R}^3})$ on each M_i, the pullback of the Euclidean metric. Further, we define a smooth function $r(x) \geq \frac{1}{2}$ on $\mathcal{M}$, such that $r(x) = |\phi_i(x)|$ on each M_i and $r(x) < 2$ on $\mathcal{M}\backslash K$.

Next, we recall the weighted Lebesgue and Sobolev spaces, which describe the phase space. The spaces $L^p_\delta(\mathcal{M})$ and $W^{k,p}_\delta(\mathcal{M})$ are defined as the completion of $C^\infty_c(\mathcal{M})$ with respect to the norms

$$\|u\|_{p,\delta} \coloneqq \left(\int_{\mathcal{M}} |u|^p r^{-\delta p-n} \mathring{d\mu}\right)^{1/p} \quad \text{and}$$
$$\|u\|_{k,p,\delta} \coloneqq \sum_{j=0}^{k} \|\mathring{\nabla}^j u\|_{p,\delta-j}, \tag{4}$$

respectively. We use $\circ$ to denote quantities determined by $\mathring{g}$, such as the background Levi-Civita connection, $\mathring{\nabla}$, and the measure, $\mathring{d\mu} = \sqrt{\mathring{g}}dx^3$. Weighted Lebesgue and Sobolev spaces of sections of bundles are defined in the usual way. These weighted spaces have the same local regularity as the usual Lebesgue and Sobolev spaces and behave as $o(r^\delta)$ near infinity on each of the ends, with each successive derivative decaying one power of r faster. We refer to Refs. [6–8] for details on the weighted spaces.

The Yang-Mills gauge group is taken to be a compact Lie group, G, with Lie algebra, $\mathfrak{g}$. We identify $\mathfrak{g}$ with its Lie coalgebra, $\mathfrak{g}^*$, via a positive-definite inner product, γ, which may be taken to be the negative of the Killing form on the semisimple factor and the usual Euclidean inner product on the Abelian factor. The usual decay conditions for asymptotic flatness and the regularity assumptions mentioned above suggest that we impose $(g - \mathring{g}) \in W^{2,2}_{-1/2}$ and $\pi \in W^{1,2}_{-3/2}$, noting that π behaves like a derivative of the metric. Imposing $\varepsilon \in W^{1,2}_{-3/2}$ enforces the usual $\frac{1}{r^2}$ fall off of the electric field in electromagnetism; however, the appropriate domain for A is less obvious. The Lie algebra, $\mathfrak{g}$, is split into its center, $\mathfrak{z}$, and a γ-orthogonal subspace, $\mathfrak{k}$. Then A is decomposed into $A = A_{\mathfrak{z}} + A_{\mathfrak{k}}$, with $A_{\mathfrak{z}}$ valued in $\mathfrak{z}$ and $A_{\mathfrak{k}}$ valued in $\mathfrak{k}$. The domain for A is taken to be such that $A_{\mathfrak{z}} \in W^{2,2}_{-1/2}$ and $A_{\mathfrak{k}} \in W^{2,2}_{-3/2}$.

The decay conditions on A are chosen such that the gauge-covariant derivative, $\hat{D} \coloneqq \partial + [A, \cdot] \sim \partial + A_{\mathfrak{k}}$, behaves analogously to the usual covariant derivative at infinity; that is, $\hat{D}\theta = \partial\theta + o(r^{-3/2})\theta$. Although it may appear somewhat unnatural to require this condition for the analysis, such a condition is in fact required to ensure that the total charge is well-defined [9]. It should be noted that this condition also puts the electric and magnetic fields on equal footing. In the language of physics, this condition is that the Yang-Mills fields are asymptotic to photon fields before vanishing.

Formally, the phase space from Paper I is given by

$$\mathcal{F} \coloneqq \mathcal{G}^+ \times \mathcal{K} \times \mathcal{A} \times \mathcal{E},$$

where

$$\mathcal{G}^+ \coloneqq \{g | (g - \mathring{g}) \in W^{2,2}_{-1/2}(S_2), g > 0\}, \qquad \mathcal{K} \coloneqq W^{1,2}_{-3/2}(S^2 \otimes \Lambda^3),$$
$$\mathcal{A} \coloneqq W^{2,2}_{-1/2}(T^*\mathcal{M} \otimes \mathfrak{z}) \oplus W^{2,2}_{-3/2}(T^*\mathcal{M} \otimes \mathfrak{k}), \qquad \mathcal{E} \coloneqq W^{1,2}_{-3/2}(T\mathcal{M} \otimes \mathfrak{g}^* \otimes \Lambda^3).$$

In the above, S_2 and S^2 are the spaces of symmetric covariant and contravariant tensors on $\mathcal{M}$, respectively, and we denote by Λ^k, the bundle of k-forms on $\mathcal{M}$.

We also define the spaces

$$\mathcal{N} := L^2_{-1/2}(\Lambda^0 \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0),$$
$$\mathcal{N}^* := L^2_{-5/2}(\Lambda^3 \times T^*\mathcal{M} \otimes \Lambda^3 \times \mathfrak{g}^* \otimes \Lambda^3).$$

Throughout this article, we use the following conventions for indices on different spaces:

$\mathcal{M}, \mathbb{R}^3$	Latin lower case, mid-alphabet	$i, j, \ldots$
${}^4\mathcal{M}, \mathbb{R}^{3,1}$	Greek lower case, mid-alphabet	$\mu, \nu \ldots$
$\mathfrak{g}$	Latin lower case, early alphabet	$a, b \ldots,$
${}^4P, (\mathbb{R}^{3,1} \oplus \mathfrak{g})$	Greek lower case, early alphabet	$\alpha, \beta \ldots$

where ${}^4\mathcal{M}$ is the spacetime in which $\mathcal{M}$ sits, and ${}^4P$ is a G-bundle over ${}^4\mathcal{M}$, which is associated with the Yang-Mills fields. By a slight abuse of notation, we will write $\xi^\alpha = (\xi^0, \xi^i, \xi^a) = (\xi^\mu, \xi^a)$ to indicate a $(4+n)$-dimensional object, and identify the components with appropriate projections. For example, if ξ^α is a section of T^4P, we consider ξ^0 to be a scalar function, ξ^i to be a vector field over $\mathcal{M}$, and $\xi^a \in \mathfrak{g}$.

Recall the constraint map, $\Phi\colon \mathcal{F} \to \mathcal{N}^*$, given by

$$\Phi_0(g, A, \pi, \varepsilon) = \left(\frac{1}{2}(\pi^k_k)^2 - \pi^{ij}\pi_{ij} - \left(\frac{1}{8}\varepsilon^k_a\varepsilon^a_k + 2B^k_aB^a_k\right)\right)g^{-1/2} + R\sqrt{g}, \quad (5)$$

$$\Phi_i(g, A, \pi, \varepsilon) = 2\nabla^j\pi_{ij} - \varepsilon^j_a(\mathring{\nabla}_i A^a_j - \mathring{\nabla}_j A^a_i) + \mathring{\nabla}_j(\varepsilon^j_a)A^a_i, \quad (6)$$

$$\Phi_a(g, A, \pi, \varepsilon) = -\mathring{\nabla}_j\varepsilon^j_a - C^c_{ab}A^b_j\varepsilon^j_c. \quad (7)$$

The momentum constraint (6) differs from that considered in Paper I by the term $\Phi_a A^a_i$. This difference amounts to a difference in interpretation of the nondynamical degree of freedom associated with Φ_a. As this is simply the addition of another constraint, the results of Paper I clearly remain valid. Also note that in Paper I, $\mathcal{M}$ was considered to have only a single asymptotic end; however, this was for simplicity of presentation rather than technical reasons. It is clear that the entire phase space analysis is valid for multiple asymptotic ends; the full analysis may be found in Chaper 4 of the author's doctoral thesis [10]. In particular, for a given source, $s \in \mathcal{N}^*$, the level set

$$\mathcal{C}(s) := \{(g, A, \pi, \varepsilon) \in \mathcal{F} | \Phi(g, A, \pi, \varepsilon) = s\},$$

has a Hilbert manifold structure; we call this the constraint submanifold. We demonstrate that the energy-momentum and other quantities are not defined on all of $\mathcal{F}$. We therefore consider the energy, momentum, angular momentum and charge as functions on constraint submanifolds with $s \in L^1$.

III. MASS, CHARGE AND ANGULAR MOMENTUM

In this section, we discuss the quantities relevant to the first law. Some of these quantities are defined at a particular end while others are defined on some surface corresponding to a horizon. In order to do this, an artificial boundary to one of the ends is introduced as follows. Let Σ be a closed 2-surface such that $\mathcal{M}\backslash\Sigma$ consists of two connected components, one of which contains only a single end, M_0. We denote by $\mathcal{M}_0$, the connected component of $\mathcal{M}\backslash\Sigma$ containing M_0.

The ADM energy-momentum covector, $\mathbb{P}_\mu(g, \pi) = (\mathbb{P}_0, \mathbb{P}_i) = (m_0, p_i)$, is given by

$$16\pi m_0 := \oint_{S_\infty} \mathring{g}^{jk}(\mathring{\nabla}_k g_{ij} - \mathring{\nabla}_i g_{jk})dS^i, \quad (8)$$

$$16\pi p_i := 2\oint_{S_\infty} \pi_{ij}dS^j, \quad (9)$$

where S_∞ is understood as the limit of increasingly large spheres. Throughout, the unit normal vector associated with the surface element dS is to be understood as pointing in the direction of infinity in M_0. The $\mathfrak{g}$-valued total Yang-Mills electric charge is given by

$$16\pi Q_{\infty a} := 4\oint_{S_\infty} E^i_a dS_i = -\oint_{S_\infty} \varepsilon^i_a dS_i, \quad (10)$$

and we write $\mathbb{P}_a = Q_{\infty a}$, so that the tuple $\mathbb{P}_\alpha := (\mathbb{P}_0, \mathbb{P}_i, \mathbb{P}_a) \in \mathbb{R}^{3,1} \oplus \mathfrak{g}^*$ can be identified with the asymptotic value of a section of ${}^4P$. The charge Q_Σ associated with Σ is defined analogously:

$$16\pi Q_{\Sigma a} := 4\oint_{\Sigma} E^i_a dS_i = -\oint_{\Sigma} \varepsilon^i_a dS_i. \quad (11)$$

Let $\xi^\mu_\infty \in \mathbb{R}^{3+1}$ be identified with some timelike vector, corresponding to the tangent to the worldline of an observer at spatial infinity. We also let $\xi^a_\infty \in \mathfrak{g}$ correspond to the asymptotic value of the electric potential, which we will assume to be constant. A total measure of the energy, viewed by this observer at spatial infinity, is then given by $\xi_\infty \cdot (E, p_i, Q_a)$, which will be more convenient to work with than the tuple (E, p_i, Q_a) itself. In order to write this as the integral of a divergence, we need to extend ξ_∞ to a section of $T^4P \cong \Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M})$.

Near infinity, $\xi_\infty \in \mathbb{R}^{3,1} \oplus \mathfrak{g}$ may be identified with some smooth section,

$$\tilde{\xi}_\infty \in C^\infty(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M})),$$

such that $\overset{\circ}{\nabla}\tilde{\xi}_\infty = 0$. We then say a smooth section, $\hat{\xi}_\infty \in C^\infty(\Lambda^0(\mathcal{M}) \times T\mathcal{M} \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}))$, is a constant translation near infinity representing ξ_∞, if $\hat{\xi}_\infty = \tilde{\xi}_\infty$ on $E_{2\hat{R}}$ and vanishes on $B_{\hat{R}}$, for some $\hat{R} > 2$, where $B_R \coloneqq \{x \in \mathcal{M} | r(x) < R\}$ and $E_R \coloneqq \mathcal{M}_0 \backslash \overline{B_R}$. While a representation of ξ_∞ is not unique, the difference between two distinct representations is smooth and compactly supported. This lets us prescribe asymptotics for ξ, but we would also like to prescribe some boundary values on Σ; for this, we fix a smooth section, $\hat{\xi}_\Sigma$, with support near Σ. We then define $\xi_{\rm ref} \coloneqq \hat{\xi}_\infty + \hat{\xi}_\Sigma$ to encapsulate both boundary conditions.

We define the spaces

$$W^{2,2}_{\xi_{\rm ref}} \coloneqq \{\xi | (\xi - \xi_{ref}) \in W^{2,2}_{-1/2c}(\Lambda^0(\mathcal{M}_0) \times T\mathcal{M}_0 \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}_0))\}, \tag{12}$$

$$L^2_{\xi_\infty} \coloneqq \{\xi | (\xi - \hat{\xi}_\infty) \in L^2_{-1/2}(\Lambda^0(\mathcal{M}_0) \times T\mathcal{M}_0 \times \mathfrak{g} \otimes \Lambda^0(\mathcal{M}_0))\}, \tag{13}$$

where $W^{2,2}_{-1/2c}$ is the completion of C^∞_c with respect to the $W^{2,2}_{-1/2}$ norm. Elements of these spaces may be interpreted as sections of ${}^4P$, restricted to $\mathcal{M}_0$, with prescribed asymptotics and boundary values on Σ.

Throughout, we will chose $\hat{\xi}^0_\Sigma \equiv 0$, and we may then define the energy-momentum covector by its pairing with with a vector at infinity, as follows:

$$16\pi\xi^0_\infty \mathbb{P}_0(g) = \int_{\mathcal{M}_0} (\hat{\xi}^0_\infty \overset{\circ}{g}{}^{ik} \overset{\circ}{g}{}^{jl} (\overset{\circ}{\nabla}_k \overset{\circ}{\nabla}_l g_{ij} - \overset{\circ}{\nabla}_i \overset{\circ}{\nabla}_k g_{jl}) \tag{14}$$

$$+ \overset{\circ}{g}{}^{ik} \overset{\circ}{g}{}^{jl} \overset{\circ}{\nabla}_k \hat{\xi}^0_\infty (\overset{\circ}{\nabla}_l g_{ij} - \overset{\circ}{\nabla}_i g_{jl})) \sqrt{\overset{\circ}{g}}, \tag{15}$$

$$16\pi\xi^i_\infty \mathbb{P}_i(\pi) = \int_{\mathcal{M}_0} (2\xi^i_{\rm ref} \overset{\circ}{\nabla}_j \pi^j_i + 2\pi^{ij} \overset{\circ}{\nabla}_i \xi_{{\rm ref}\, j} + \overset{\circ}{\nabla}_i (\varepsilon^i_a A^a_j) \xi^j_{\rm ref} + \varepsilon^i_a A^a_j \overset{\circ}{\nabla}_i \xi^j_{\rm ref}) + \oint_\Sigma (2\xi^i_\Sigma \pi^j_i - \varepsilon^j_a A^a_i \xi^i_\Sigma) dS_j. \tag{16}$$

Note that while Eq. (16) contains the terms (g, A, ε), the quantity $\mathbb{P}_i$ only depends on π; the boundary terms on Σ combine with the bulk integral to give a boundary integral at infinity, which removes the dependence on g as $g = \overset{\circ}{g} + o(r^{-1/2})$, and the Yang-Mills terms at infinity vanish [see Eq. (34)], leaving only a π dependence. When $\hat{\xi}^i_\Sigma$ agrees with a rotational Killing field, the integral over Σ in Eq. (16) is proportional to the angular momentum. This leads us to define a generalized notion of angular momentum,

$$16\pi\tilde{J}_{\xi_{\rm ref}}(g, A, \pi, \varepsilon) \coloneqq -\oint_\Sigma (2\hat{\xi}^i_\Sigma \pi^j_i - \varepsilon^j_a A^a_i \hat{\xi}^i_\Sigma) dS_j. \tag{17}$$

Note that we follow the sign convention of Wald [11]. The second term in Eq. (17), corresponding to the angular momentum of the Yang-Mills fields, is nonstandard and appears to have been first considered by Sudarsky and Wald [12]; however, they considered the integration to be performed at infinity. It will be important for us to use a quasilocal definition of angular momentum instead. While this is useful for our purposes, we do not argue here that this gives a suitable quasilocal definition of angular momentum in general. There is a great deal of literature on the problem of quasilocal mass and angular momentum (see Ref. [13] and references therein).

To write the electric charge as a bulk integral, we will fix a choice of the Lagrange multiplier, $\xi^a_{\rm ref}$, with $\hat{\xi}_\Sigma = \xi_\Sigma \in \mathfrak{g}$, constant. Similar to the above, we have

$$16\pi(\xi^a_\infty \mathbb{P}_a - \xi^a_\Sigma Q_{\Sigma a}) = 4\int_{\mathcal{M}_0} (\xi^a_{\rm ref} \overset{\circ}{\nabla}_i E^i_a + E^i_a \overset{\circ}{\nabla}_i \xi^a_{\rm ref}). \tag{18}$$

Lemma III.1. Let χ be a vector field on $\mathcal{M}$ with $\|\chi\|_{L^\infty(\Sigma)} < \infty$. The maps $Q_\Sigma\colon \mathcal{F} \to \mathfrak{g}^*$ and $\tilde{J}_\chi\colon \mathcal{F} \to \mathbb{R}$ are smooth.

Proof.—By considering any function $\varphi \in C^\infty_c(\mathcal{M})$ with $\varphi \equiv 1$ on Σ, the Sobolev trace theorem gives

$$|Q_\Sigma| \le c\|E\|_{L^1(\Sigma)} = \|\varphi E\|_{L^1(\Sigma)} \le c\|\varphi E\|_{L^2(\Sigma)} \le c\|E\|_{1,2,-3/2}. \tag{19}$$

We estimate $\tilde{J}_\chi$ similarly:

$$\tilde{J}_\chi \le c(\|\chi\|_{L^2(\Sigma)} \|\pi\|_{1,2,-3/2} + \|\chi\|_{L^\infty(\Sigma)} \|\varphi A\|_{L^2(\Sigma)} \|\varphi\varepsilon\|_{L^2(\Sigma)}) \le c\|\chi\|_{L^\infty(\Sigma)} (\|\pi\|_{1,2,-3/2} + \|A\|_{1,2,-1/2} \|\varepsilon\|_{1,2,-3/2}).$$

Since Q_Σ and $\tilde{J}_\chi$ are bounded and linear, smoothness follows. □

Theorem III.2. For an integrable source, $s \in L^1$, the map $\mathbb{P}\colon \mathcal{C}(s) \to \mathbb{R}^{3,1} \oplus \mathfrak{g}^*$ is smooth.

Proof.—$\mathbb{P}_0$ is exactly of the form considered by Bartnik [2], except that the integrals are over a manifold with boundary in our case. However, this difference does not affect Bartnik's proof that $\mathbb{P}_0$ is smooth so the result applies here also. As $\mathbb{P}_i$ differs from Bartnik's definition by Yang-Mills terms and $16\pi\tilde{J}_{\xi_{\rm ref}}$, we cannot directly apply his results; we instead show smoothness as follows. Lemma III.1 shows that $\tilde{J}_{\xi_{\rm ref}}$ is smooth, so we must only consider the bulk (volume) integral in Eq. (16). Note that the second and fourth terms in the bulk integral defining $\mathbb{P}_i$ [Eq. (16)] are clearly bounded, as $\nabla\xi$ has bounded support. We also have

$$\int_{\mathcal{M}_0} 2\xi^i_{\mathrm{ref}}\mathring{\nabla}_j\pi^j_i \leq c\|\xi^i_{\mathrm{ref}}\|_{\infty,0}\|\mathring{\nabla}\cdot\pi\|_{1,-3},$$

which is then controlled, as follows, by the fact that we have an integrable source. Recalling the difference of connections tensor,

$$\tilde{\Gamma}^i_{jk} \coloneqq \Gamma^i_{jk} - \mathring{\Gamma}^i_{jk} = \frac{1}{2}g^{il}(\mathring{\nabla}_j g_{lk} + \mathring{\nabla}_k g_{jl} - \mathring{\nabla}_l g_{jk}),$$

and making use of the momentum constraint (6), we have

$$\begin{aligned}\|\mathring{\nabla}\cdot\pi\|_{1,-3} &\leq c(\|\nabla\cdot\pi\|_{1,-3} + \|\tilde{\Gamma}\pi\|_{1,-3})\\ &\leq c(\|s\|_{1,-3} + \|\varepsilon\mathring{\nabla}A\|_{1,-3} + \|A\mathring{\nabla}\varepsilon\|_{1,-3} + \|\tilde{\Gamma}\|_{2,-3/2}\|\pi\|_{2,-3/2})\\ &\leq c(\|s\|_{1,-3} + \|\varepsilon\|_{2,-3/2}\|\mathring{\nabla}A\|_{2,-3/2}\\ &\quad + \|A\|_{2,-1/2}\|\mathring{\nabla}\varepsilon\|_{2,-5/2} + \|\mathring{\nabla}g\|_{2,-3/2}\|\pi\|_{2,-3/2})\\ &\leq c(\|s\|_{1,-3} + \|\varepsilon\|_{1,2,-3/2}\|A\|_{1,2,-1/2} + \|\mathring{\nabla}g\|_{2,-3/2}\|\pi\|_{2,-3/2}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\int_{\mathcal{M}_0}\xi^j_{\mathrm{ref}}\mathring{\nabla}_i(\varepsilon^i_a A^a_j) &\leq c\|\xi_{\mathrm{ref}}\|_{\infty,0}(\|A\mathring{\nabla}\varepsilon\|_{1,-3} + \|\varepsilon\mathring{\nabla}A\|_{1,-3})\\ &\leq c\|\xi_{\mathrm{ref}}\|_{\infty,0}(\|A\|_{2,-1/2}\|\mathring{\nabla}\varepsilon\|_{2,-5/2} + \|\varepsilon\|_{2,-3/2}\|\mathring{\nabla}A\|_{2,-3/2}).\end{aligned} \tag{20}$$

Since the bulk integral is linear in each of the variables and bounded, smoothness follows; that is, $\mathbb{P}_i$ is smooth. □

The remaining component, $\xi^a_\infty\mathbb{P}_a$, consists of a bulk integral plus the term $\xi^a_\Sigma Q_{\Sigma a}$ [Eq. (18)]; the latter is again smooth by Lemma III.1 and the bulk integral is estimated similarly to the above. The second term in the bulk integral is clearly bound again as $\mathring{\nabla}\xi_{\mathrm{ref}}$ has bounded support, while the first term makes use of the Gauss constraint (7) and the fact that the source is integrable:

$$\begin{aligned}\int_{\mathcal{M}_0}\xi^a_{\mathrm{ref}}\mathring{\nabla}_i E^i_a &\leq c(\|\xi_{\mathrm{ref}}\|_{\infty,0}\|\mathring{\nabla}\cdot E\|_{1,-3})\\ &\leq c\|\xi_{\mathrm{ref}}\|_{\infty,0}(\|s\|_{1,-3} + \|A_{\mathfrak{k}}\varepsilon\|_{1,-3})\\ &\leq c\|\xi_{\mathrm{ref}}\|_{\infty,0}(\|s\|_{1,-3} + \|A_{\mathfrak{k}}\|_{2,-3/2}\|\varepsilon\|_{2,-3/2}).\end{aligned}$$

It follows that $\mathbb{P}$ is smooth.

IV. HAMILTONIANS AND THE FIRST LAW

It is well known that the source-free evolution equations can be succinctly written as

$$\frac{d}{dt}\begin{bmatrix} g\\ A\\ \pi\\ \varepsilon\end{bmatrix} = -\begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{bmatrix}\circ D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi), \tag{21}$$

where $D\Phi^*_{(g,A,\pi,\varepsilon)}$ is the formal adjoint of the linearization of Φ, and t is interpreted as the flow parameter of a vector field on ${}^4P$, identified with ξ (see, for example, Refs. [14,15]). The flow of ξ is interpreted as a simultaneous time evolution and continuous change of gauge. Equation (21) motivates Moncrief's result, equating stationary solutions with initial data satisfying $D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) = 0$, where ξ^μ corresponds to a time translation at infinity [16,17] (see also the subsequent work by Arms, Marsden and Moncrief in the Einstein-Yang-Mills case [15]). We call such an initial data set a *generalized stationary initial data set*.

If the formal adjoint agrees with the true adjoint, then these evolution equations correspond exactly to Hamilton's equations for the usual ADM Hamiltonian,

$$\mathcal{H}^{\mathrm{ADM}(\xi)}(g, A, \pi, \varepsilon) \coloneqq -\int_{\mathcal{M}} \xi \cdot \Phi(g, A, \pi, \varepsilon). \tag{22}$$

Unfortunately, this is not the case when $\mathcal{M}$ is an asymptotically flat manifold as the formal adjoint differs from the true adjoint by a collection of boundary terms unless ξ vanishes sufficiently fast at infinity. In order to generate the correct equations of motion, the first variation of the Hamiltonian density must be of the form

$$DH_{(g,A,\pi,\varepsilon)}(h, b, p, f) = \Xi \cdot (h, b, p, f), \tag{23}$$

for some $\Xi \in T^*_{(g,A,\pi,\varepsilon)}\mathcal{F}$.

In the pure gravity case with no interior boundary, Regge and Teitelboim [18] demonstrated that the ADM mass must be added to the ADM Hamiltonian in order to obtain a Hamiltonian density satisfying (23). In Paper I, where we considered the Einstein-Yang-Mills case with no interior boundary, we also added a charge term, corresponding to the additional Yang-Mills energy. However, problems arise when one looks at the evolution exterior to some boundary and the addition of these extra terms does not suffice.

Computing the variation of the ADM Hamiltonian density (22) yields

$$\begin{aligned} DH^{\mathrm{ADM}(\xi)}_{(g,A,\pi,\varepsilon)}(h, b, p, f) = &- D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) \cdot (h, b, p, f) + \nabla^i((\xi^0(\mathring{\nabla}_i \mathrm{tr}_g h - \nabla^j h_{ij}) \\ &+ \mathring{\nabla}^j(\xi^0)h_{ij} - \mathrm{tr}_g h \mathring{\nabla}_i(\xi^0))\sqrt{g} - 2\xi^j p_{ij} + \xi^a f_{ai} - 2\pi^k_i h_{jk}\xi^j \\ &+ \pi^{jk}h_{jk}\xi_i - \epsilon_{ijk}b^{ak}B^j_a\xi^0\sqrt{g} - \varepsilon_{ia}b^a_j\xi^j + \xi_i\varepsilon^j_a b^a_j - f_{ia}A^a_j\xi^j). \end{aligned} \tag{24}$$

The first term is exactly of the form we require [Eq. (23)]; however, the cumbersome divergence term does not vanish in general and therefore the evolution is not Hamiltonian. Fortunately, it does have the following geometric interpretation to be exploited. Assume Σ is the bifurcation surface of a bifurcate Killing horizon, ξ^μ is the stationary Killing field and ϕ^μ is the rotational Killing field tangent to $\mathcal{M}$ with 2π-periodic orbits; we then have $\xi^\mu + \Omega\phi^\mu \equiv 0$ on Σ for some constant Ω, which is to be interpreted as the angular velocity of the horizon. The zeroth law of black hole mechanics states that the surface gravity $\kappa = \frac{1}{2} n^i \nabla_i \xi^0$ is constant on Σ, where n^i is the unit normal to Σ pointing towards infinity in M_0. We also ask that the electric potential, $V^a = \xi^a$ be constant at infinity and on Σ. In this case, the expression (24) becomes

$$\begin{aligned} \int_{\mathcal{M}_0} DH^{\mathrm{ADM}(\xi)}_{(g,A,\pi,\varepsilon)}(h, b, p, f) = &- \int_{\mathcal{M}_0} D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) \cdot (h, b, p, f) - 16\pi Dm_{(g,\pi)}(h, p) \\ &+ 2\kappa DAr_{\Sigma g}(h) + 16\pi\Omega DJ_{\Sigma(g,A,\pi,\varepsilon)}(h, b, p, f) + 16\pi(V_\Sigma \cdot DQ_{\Sigma(\varepsilon)}(f) - V_\infty \cdot DQ_{\infty(\varepsilon)}(f)), \end{aligned} \tag{25}$$

where $\mathrm{Ar}_\Sigma(g)$ is the surface area Σ and $m(g, \pi) = \sqrt{-\mathbb{P}^\mu \mathbb{P}_\mu}$ is the total mass. Note that the fact that $\mathbb{P}^\mu$ is timelike follows from the positive mass theorem, assuming the dominant energy condition (see Theorem 11.2 of Ref. [19]). Compare this to the first law of black hole mechanics, which states that for perturbations to a stationary solution the following variational formula holds:

$$\delta m = \frac{\kappa}{8\pi}\delta \mathrm{Ar}_\Sigma + \Omega \delta J + V \cdot \delta Q. \tag{26}$$

This motivates an interesting result of Ashtekar, Fairhurst and Krishnan [20] in the framework of isolated horizons. They considered the ADM Hamiltonian on a manifold with an interior boundary representing an isolated horizon, and demonstrated that the validity of the first law is a necessary and sufficient condition for the evolution to be Hamiltonian. However, we take a different approach regarding these additional terms corresponding to the first law. A new Hamiltonian is introduced, *à la* Regge and Teitelboim, that gives the correct equations of motion somewhat more generally, and the first law plays quite a different role. We define the modified Hamiltonian,

$$\mathcal{H}^{\mathrm{RT}(\xi)}(g, A, \pi, \varepsilon) \coloneqq 16\pi(\xi_\infty \cdot \mathbb{P} + \tilde{J}_\xi - \xi^a_\Sigma Q_{\Sigma a}) - \int_{\mathcal{M}_0} \xi \cdot \Phi, \tag{27}$$

for some $\xi \in W^{2,2}_{\xi_{\mathrm{ref}}}$. As before, we fix ξ_{ref} on Σ such that $\xi^0_{\mathrm{ref}} = 0$, ξ^a_{ref} is constant and ξ^i_{ref} is tangent to Σ. Note that

$(\xi_\infty \cdot \mathbb{P} + \tilde{J}_\xi - \xi^a_\Sigma Q_{\Sigma a})$ only depends on the boundary and asymptotic values of ξ, so the Hamiltonian essentially acts as a Lagrange function; extremizing the Hamiltonian is equivalent to extremizing $(\xi_\infty \cdot \mathbb{P} + \tilde{J}_\xi - \xi^a_\Sigma Q_{\Sigma a})$ subject to the constraints being satisfied, where ξ with the prescribed boundary and asymptotic conditions acts as the Lagrange multiplier. This is the fundamental idea behind Theorem IV.8, below.

Note that the first and last terms in Eq. (27) are divergent in general; however, following Bartnik [2] (see also, Paper I), we combine the integrals and the dominant terms of each cancel out. This leads us to the regularized Hamiltonian,

$$\begin{aligned}\mathcal{H}^\xi(g,A,\pi,\varepsilon) := &\int_{\mathcal{M}_0}(\xi^\alpha_{\rm ref}-\xi^\alpha)\Phi_\alpha + \int_{\mathcal{M}_0}\xi^0_{\rm ref}(\mathring{g}^{ki}\mathring{g}^{lj}\mathring{\nabla}_k\mathring{\nabla}_l g_{ij} - \mathring{\Delta}(\mathrm{tr}_{\mathring{g}}g)\sqrt{\mathring{g}} - \Phi_0)\\ &+\int_{\mathcal{M}_0}\mathring{g}^{ik}\mathring{g}^{lj}\mathring{\nabla}_k(\xi^0_{\rm ref})(\mathring{\nabla}_j g_{ij} - \mathring{\nabla}_i \mathrm{tr}_{\mathring{g}}g)\sqrt{\mathring{g}}\\ &+\int_{\mathcal{M}_0}\xi^i_{\rm ref}(\mathring{\nabla}_j(2\pi^j_i + \varepsilon^j_a A^a_i) - \Phi_i) + \int_{\mathcal{M}_0}(2\pi^i_j + \varepsilon^i_a A^a_j)\mathring{\nabla}_i\xi^j_{\rm ref}\\ &-\int_{\mathcal{M}_0}\xi^a_{\rm ref}(\mathring{\nabla}_i\varepsilon^i_a - \Phi_a) - \int_{\mathcal{M}_0}\varepsilon^i_a\mathring{\nabla}_i\xi^a_{\rm ref},\end{aligned} \tag{28}$$

which is defined on all of $\mathcal{F}$.

Proposition IV.1. The regularized Hamiltonian, $\mathcal{H}_\xi\colon\ \mathcal{F}\to\mathbb{R}$, is well-defined and smooth.

Proof.—This Hamiltonian is exactly of the form considered in Paper I, except that the integrals are performed over a manifold with boundary here, and we have the additional momentum terms, $\int_{\mathcal{M}_0}\xi^i_{\rm ref}\mathring{\nabla}_j(\varepsilon^j_a A^a_i)$ and $\int_{\mathcal{M}_0}\varepsilon^i_a A^a_j\mathring{\nabla}_i\xi^j_{\rm ref}$. As above, the fact that the manifold has a boundary does not affect the proof at all. Up to the addition of some additional Yang-Mills momentum terms, we conclude $\mathcal{H}^\xi$ is smooth from Theorem 4.4 of Paper I. The additional momentum terms,

$$\int_{\mathcal{M}_0}\xi^i_{\rm ref}\varepsilon^j_a A^a_i + \varepsilon^i_a A^a_j\mathring{\nabla}_i\xi^j_{\rm ref},$$

are linear in their arguments so they simply must be shown to be bounded to demonstrate that they too are smooth. The latter momentum term is clearly bound since $\mathring{\nabla}\xi_{\rm ref}$ has bounded support and the former is estimated by Eq. (20). □

An immediate corollary of Theorem 4.2 from Paper I is the following.

Proposition IV.2. For $\xi\in W^{2,2}_{-1/2c}$, we have

$$\int_{\mathcal{M}_0}\xi\cdot D\Phi_{(g,A,\pi,\varepsilon)}(h,b,p,f) = \int_{\mathcal{M}_0}(h,b,p,f)\cdot D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi), \tag{29}$$

for all $(h,b,p,f)\in T_{(g,A,\pi,\varepsilon)}\mathcal{F}$.

Proof.—The difference, $(h,b,p,f)\cdot D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) - \xi\cdot D\Phi_{(g,A,\pi,\varepsilon)}(h,b,p,f)$, is easily computed to give

$$\begin{aligned}&\nabla^i((\xi^0(\mathring{\nabla}_i\mathrm{tr}_g h - \nabla^j h_{ij}) + \mathring{\nabla}^j(\xi^0)h_{ij} - \mathrm{tr}_g h\mathring{\nabla}_i(\xi^0))\sqrt{g} - 2\xi^j p_{ij} + \xi^a f_{ai})\\ &\quad - \nabla^i(2\pi^k_i h_{jk}\xi^j - \pi^{jk}h_{jk}\xi_i + \epsilon_{ijk}b^{ak}B^j_a\xi^0\sqrt{g} + \varepsilon_{ia}b^a_j\xi^j - \xi_i\varepsilon^j_a b^a_j + f_{ai}\xi^j A^a_j).\end{aligned}$$

The integral of this divergence is then expressed as surface integrals at infinity and on Σ. The terms at infinity vanish by Theorem 4.2 of Paper I and the terms on Σ vanish by the hypothesis $\xi\in W^{2,2}_{-1/2c}$. We do have the extra term, $f_{ai}\xi^j A^a_j$, not considered in Paper I; however, this clearly vanishes by the same argument. □

Proposition IV.3. For $\xi\in W^{2,2}_{\xi_{\rm ref}}$, the variation of the regularized Hamiltonian is given by

$$D\hat{\mathcal{H}}^\xi[h,b,p,f] = -\oint_\Sigma(\mathring{\nabla}^j(\xi^0)h_{ij} - \mathrm{tr}_g h\mathring{\nabla}_i(\xi^0))\sqrt{g}dS^i - \int_{\mathcal{M}_0}D\Phi^*(\xi)\cdot(h,b,p,f). \tag{30}$$

Proof.—We consider the terms in Eq. (28) separately. By Proposition IV.2, the variation of the first integral in Eq. (28) becomes

$$\int_{\mathcal{M}_0}(h,b,p,f)\cdot D\Phi^*(\xi_{\mathrm{ref}}-\xi).$$

The variation of the second and third terms combine to give

$$\int_{\mathcal{M}_0}\{\mathring{g}^{ik}\mathring{\nabla}_k(\xi^0_{\mathrm{ref}}\mathring{g}^{jl}(\mathring{\nabla}_l h_{ij}-\mathring{\nabla}_i h_{jl}))\sqrt{\mathring{g}}-\nabla^i(\xi^0_{\mathrm{ref}}(\nabla^j h_{ij}-\nabla_i \mathrm{tr}_g h))\sqrt{g} \\ +\nabla^i(h_{ij}\nabla^j\xi^0_{\mathrm{ref}}-\mathrm{tr}_g h\nabla_i\xi^0_{\mathrm{ref}})\sqrt{g}-(h,b,p,f)\cdot D\Phi^*_0(\xi^0_{\mathrm{ref}})\}. \tag{31}$$

Then the first two terms in the above combine to give a total divergence,

$$-\oint_{\mathcal{M}_0}\mathring{\nabla}_k(g^{ik}\xi^0_{\mathrm{ref}}g^{jl}(\nabla_l h_{ij}-\nabla_i h_{jl})\left(\sqrt{g}-\sqrt{\mathring{g}}\right)+(g^{ik}-\mathring{g}^{ik})\xi^0_{\mathrm{ref}}g^{jl}(\nabla_l h_{ij}-\nabla_i h_{jl})\sqrt{\mathring{g}} \\ +\mathring{g}^{ik}\mathring{g}^{jl}\xi^0_{\mathrm{ref}}((\nabla_l-\mathring{\nabla}_l)h_{ij}-(\nabla_i-\mathring{\nabla}_i)h_{jl})\sqrt{\mathring{g}}), \tag{32}$$

which is rewritten as surface integrals, both at infinity and on Σ. The integral at infinity is identical to that considered by Bartnik [2] and therefore vanishes by the same argument, while the surface integral on Σ vanishes since $\xi^0_\Sigma=0$. The third term in Eq. (31) is again a divergence, but only gives a boundary term on Σ since $\mathring{\nabla}\xi_{\mathrm{ref}}$ has bounded support. This boundary term on Σ is then exactly the surface integral in Eq. (30).

The variation of the fourth and fifth terms in Eq. (28) gives

$$\int_{\mathcal{M}_0}\{2\mathring{\nabla}_i(\xi^j_{\mathrm{ref}}p^i_j)+2\mathring{\nabla}_j(\xi^i_{\mathrm{ref}}\pi^{jk}h_{ki})+\nabla_i(\varepsilon^i_a b^a_j\xi^j_{\mathrm{ref}})+\nabla_i(f^i_a\xi^j_{\mathrm{ref}}A^a_j)-2\nabla_i(\xi^j_{\mathrm{ref}}p^i_j)-2\nabla_i(\pi^{ki}h_{jk}\xi^j_{\mathrm{ref}})-\nabla_i(\varepsilon^i_a b^a_j\xi^j_{\mathrm{ref}}) \\ -\nabla_i(f^i_a\xi^j_{\mathrm{ref}}A^a_j)+\nabla_i(\xi^i_{\mathrm{ref}}\varepsilon^j_a b^a_j)-(h,b,p,f)\cdot D\Phi^*_i(\xi^i_{\mathrm{ref}})\}. \tag{33}$$

Since p, π, f and ε are densities, the divergences above do not depend on the connection used and thus the first two lines in Eq. (33) cancel exactly. The surface integral on Σ arising from the remaining divergence in Eq. (33) vanishes, since ξ^i_{ref} is tangent to Σ and the surface integral at infinity is shown to vanish as follows. Let $S_R=\{x\in M_0|r(x)=R\}$ and–noting that b and ξ_{ref} are continuous by the Sobolev-Morrey embedding—we have

$$\begin{aligned}\left|\oint_{S_R}\xi^i_{\mathrm{ref}}\varepsilon^j_a b^a_j dS_i\right| &\lesssim \|b\|_{\infty(S_R)}\|\xi_{\mathrm{ref}}\|_{\infty(S_R)}\oint_{S_R}|\varepsilon|dS \\ &\lesssim o(r^{-1/2})O(1)R^{1/2}\|\varepsilon\|_{1,2,-3/2} \\ &= o(1),\end{aligned} \tag{34}$$

where we have made use of the estimate,

$$\oint_{S_R}|u|dS\le cR^{1/2}\|u\|_{1,2,-3/2},$$

from Ref. [2] (Theorem 4.4). It follows that $\oint_\infty \xi^i_{\mathrm{ref}}\varepsilon^j_a b^a_j dS_i=0$ and therefore the variation of the fourth and fifth terms in Eq. (28) reduces to

$$-\int_{\mathcal{M}_0}(h,b,p,f)\cdot D\Phi^*_i(\xi^i_{\mathrm{ref}}). \tag{35}$$

Finally, the variation of the sixth and seventh terms in Eq. (28) is given by

$$\int_{\mathcal{M}_0}-\mathring{\nabla}_i(\hat{\xi}^a_\infty f^i_a)+\nabla_i(\hat{\xi}^a_\infty f^i_a)-(h,b,p,f)\cdot D\Phi^*_a(\hat{\xi}^a_\infty). \tag{36}$$

Since f is a density, the divergences again do not depend on the connection and therefore the first two terms in Eq. (36) cancel exactly, leaving

$$-\int_{\mathcal{M}}(h,b,p,f)\cdot D\Phi^*_a(\hat{\xi}^a_\infty). \tag{37}$$

Assembling all of the pieces completes the proof. □

If Σ is indeed the bifurcation surface of a bifurcate Killing horizon, corresponding to the Killing vector $\xi+\xi_{\mathrm{ref}}$, then $\xi^0=0$ on Σ and the surface gravity, $\kappa=\frac{1}{2}n^i\mathring{\nabla}_i(\xi^0)$, is constant; it follows that $\mathring{\nabla}(\xi^0)$ is normal to Σ. Making use of coordinates adapted to Σ, the surface integral in Eq. (30) becomes

$$\begin{aligned} -\oint_\Sigma (\mathring{\nabla}^j(\xi^0)h_{ij} - \mathrm{tr}_g h\mathring{\nabla}_i(\xi^0))\sqrt{g}dS^i &= -\oint_\Sigma (g^{j3}\mathring{\nabla}_3(\xi^0)h_{ij}n^i - h^k_k\mathring{\nabla}_3(\xi^0))\sqrt{g}dS \\ &= \oint_\Sigma \mathring{\nabla}_3(\xi^0)h^A_A\sqrt{g}dS \\ &= 2\kappa dAr_\Sigma, \end{aligned} \tag{38}$$

where the index "3" refers to the direction normal to Σ, while $A = 1, 2$ are tangential.

It can be seen from Proposition IV.3 that this new Hamiltonian gives the correct equations of motion when $\mathring{\nabla}\xi^0 \equiv 0$ on Σ, or when Σ is the bifurcation surface of a bifurcate Killing Horizon and g is a critical point of the area functional of Σ. There does not appear to be an obvious way to further modify the Hamiltonian such that the correct equations of motion are generated in general.

To prove the main theorem, we will need to make use of the following generalization of the method of Lagrange multipliers to Banach manifolds (see Theorem 6.3 of Ref. [2]).

Theorem IV.4. Suppose $K: B_1 \to B_2$ is a C^1 map between Banach manifolds, such that $DK_u: T_uB_1 \to T_{K(u)}B_2$ is surjective, with closed kernel and closed complementary subspace for all $u \in K^{-1}(0)$. Let $f \in C^1(B_1)$ and fix $u \in K^{-1}(0)$; then, the following statements are equivalent:

(i) For all $v \in \ker DK_u$, we have

$$Df_u(v) = 0. \tag{39}$$

(ii) There is $\lambda \in B_2^*$ such that for all $v \in B_1$,

$$Df_u(v) = \langle \lambda, DK_u(v)\rangle, \tag{40}$$

where $\langle , \rangle$ refers to the natural dual pairing.

We also will need to make use of the following theorem from Paper I, regarding weak solutions. A weak solution of $D\Phi^*(\xi) = f$ is an element $\xi \in \mathcal{N}$ such that

$$\int_\mathcal{M} \xi \cdot D\Phi(h, b, p, f) = \int_\mathcal{M} f \cdot (h, b, p, f), \tag{41}$$

for all $(h, b, p, f) \in \mathcal{G} \times \mathcal{A} \times \mathcal{K} \times \mathcal{E} = T_{(g,A,\pi,\varepsilon)}\mathcal{F}$.

Theorem IV.5. If $\xi \in \mathcal{N}$ is a weak solution of $D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) = (f_1, f_2, f_3, f_4)$, with $(f_1, f_3, f_4) \in L^2_{-5/2} \times W^{1,2}_{-3/2} \times W^{1,2}_{-3/2}$ and $(g, A, \pi, \varepsilon) \in \mathcal{F}$, then $\xi \in W^{2,2}_{-1/2}$ and is indeed a strong solution.

The following theorem from Ref. [2] is stated in reference to a particular operator; however, it is clear from the proof that the theorem applies to a general class of operators. In particular, the theorem could more generally be stated as follows:

Theorem IV.6 [Theorem 3.6 of Ref. [2]]. Let $\Omega \subset \mathcal{M}$ be a connected domain with $E'_R \subset \Omega$ for some R, where E'_R is a connected component of E_R. If $\xi \in W^{2,2}_{-1/2}$ satisfies

$$\mathring{\nabla}^2\xi = b_1\nabla\xi + b_0\xi,$$

with $b_0 \in L^2_{-5/2}$ and $b_1 \in W^{1,2}_{-3/2}$, then $\xi \equiv 0$ in Ω.

From this and Theorem IV.5, we have the following immediate corollary:

Corollary IV.7. Let $(g, A, \pi, \varepsilon) \in \mathcal{F}$. If $\xi \in \mathcal{N}^*$ satisfies $D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) = 0$ on a connected $\Omega \subset \mathcal{M}$ containing some E'_R, then $\xi \equiv 0$ on Ω.

Now we are in a position to prove the main theorem. Below, we use the notation $D\Phi^*_{(g,A,\pi,\varepsilon)}(\xi) = (D\Phi^*_g(\xi), D\Phi^*_A(\xi), D\Phi^*_\pi(\xi), D\Phi^*_\varepsilon(\xi))$ to identify the components of $D\Phi^*$.

Theorem IV.8. Let $(g, A, \pi, \varepsilon) \in \mathcal{C}(s)$, where $s \in L^1$, and suppose there exists a vector field, $\phi \in W^{2,2}_{\mathrm{loc}}$, tangent to Σ with $D\Phi^*_\pi(\phi), D\Phi^*_\varepsilon(\phi) \in W^{1,2}_{-1/2c}(\mathcal{M}_0)$. Further suppose that for all $(h, b, p, f) \in T_{(g,A,\pi,\varepsilon)}\mathcal{C}(s)$,

$$\begin{aligned} Dm_{(g,A,\pi,\varepsilon)}(h,b,p,f) &= \alpha D\mathrm{Ar}_{\Sigma(g,A,\pi,\varepsilon)}(h,b,p,f) + \beta DJ_{\phi(g,A,\pi,\varepsilon)}(h,b,p,f) \\ &\quad + \gamma_\Sigma \cdot DQ_{\Sigma(g,A,\pi,\varepsilon)}(h,b,p,f) - \gamma_\infty \cdot DQ_{\infty(g,A,\pi,\varepsilon)}(h,b,p,f), \end{aligned} \tag{42}$$

where $\alpha, \beta \in \mathbb{R}$ and $\gamma_\Sigma, \gamma_\infty \in \mathfrak{g}$ are constants. Then (g, A, π, ε) is a generalized stationary initial data set. Furthermore, γ is the electric potential, and if Σ is the bifurcation surface of a bifurcate Killing horizon, then $8\pi\alpha$ is the surface gravity and β is the angular velocity.

Proof.—We assume that Eq. (42) holds at some fixed point $\tilde{G} = (\tilde{g}, \tilde{A}, \tilde{\pi}, \tilde{\varepsilon}) \in \mathcal{F}$. Then we fix ξ_{ref} such that it satisfies the following boundary and asymptotic conditions:

(i) ξ^μ_∞ corresponds to a future-pointing unit vector at spatial infinity in the spacetime that is proportional to $\mathbb{P}^\mu$;

(ii) ξ^a_{ref} is constant at infinity and on Σ, with values $\xi^a_\infty = \gamma^a_\infty$ and $\xi^a_\Sigma = \gamma^a_\infty$;

(iii) ξ^0_{ref} vanishes on Σ;

(iv) $\xi^i_{\mathrm{ref}} = -\beta\phi^i$ on Σ; and

(v) $\partial_i(\xi^0_{\mathrm{ref}})\tilde{n}^i = 16\pi\alpha$ on Σ.

We use $\tilde{n}$ to denote the unit normal with respect to $\tilde{g}$, pointing towards infinity in M_0. Note that the condition on ξ^μ_∞ implies $\xi^\mu_\infty \mathbb{P}_\mu = m$, and the conditions on α, β and γ ensure that they correspond to the appropriate physical quantities in the statement of the theorem.

Now for some $\xi \in W^{2,2}_{\xi_{\mathrm{ref}}}$, we define

$$\tilde{f}(G) := \mathcal{H}^\xi(G) - 16\pi\alpha \mathrm{Ar}_\Sigma(G), \tag{43}$$

where $G = (g, A, \pi, \varepsilon) \in \mathcal{F}$. We again let $K(G) = \Phi(G) - s$, and note that for all constrained variations, $(h, b, p, f) \in \ker(DK_{\tilde{G}}) = T_{\tilde{G}}\mathcal{C}(s)$, we have [see Eq. (27)]

$$\begin{aligned}\frac{1}{16\pi} D\mathcal{H}^\xi_{\tilde{G}}(h,b,p,f) &= \xi_\infty \cdot D\mathbb{P}_{\tilde{G}}(h,b,p,f) + D\tilde{J}^\xi_{\tilde{G}}(h,b,p,f) - \xi^a_\Sigma DQ_{\Sigma\tilde{G}a}(h,b,p,f)\\ &= Dm_{\tilde{G}}(h,b,p,f) - \beta DJ_{\phi\tilde{G}}(h,b,p,f)\\ &\quad - \gamma_\Sigma \cdot DQ_{\Sigma\tilde{G}}(h,b,p,f) + \gamma_\infty \cdot DQ_{\infty\tilde{G}}(h,b,p,f).\end{aligned}$$

By hypothesis (42), we have $D\tilde{f}_{\tilde{G}}(h,b,p,f) = 0$ for all $(h,b,p,f) \in \ker(DK_{\tilde{G}})$. It follows from Theorem IV.4, that there exists $\lambda \in \mathcal{N}$ such that

$$D\tilde{f}_{\tilde{G}} = \langle D\Phi_{\tilde{G}}, \lambda \rangle; \tag{44}$$

that is,

$$D\tilde{f}_{\tilde{G}}(h,b,p,f) = \int_{\mathcal{M}} D\Phi_{\tilde{G}}(h,b,p,f) \cdot \lambda, \tag{45}$$

for all $(h,b,p,f) \in T_{\tilde{G}}\mathcal{F}$. However, from Proposition IV.3, we have

$$\begin{aligned}D\tilde{f}_{\tilde{G}}(h,b,p,f) = &-\oint_\Sigma (\mathring{\nabla}^j(\xi^0)h_{ij} - \mathrm{tr}_g h \mathring{\nabla}_i(\xi^0))\sqrt{g}dS^i\\ &- \int_{\mathcal{M}_0} D\Phi^*(\xi) \cdot (h,b,p,f)\\ &- 16\pi\alpha D\mathrm{Ar}_{\Sigma(\tilde{G})}(h,b,p,f).\end{aligned} \tag{46}$$

As $\partial_i(\xi^0)\tilde{n}^i = 16\pi\alpha$ on Σ, the first and last terms cancel exactly [see Eq. (38)], leaving

$$D\tilde{f}_{\tilde{G}}(h,b,p,f) = -\int_{\mathcal{M}_0} (h,b,p,f) \cdot D\Phi^*_G(\xi); \tag{47}$$

that is,

$$-\int_{\mathcal{M}_0} (h,b,p,f) \cdot D\Phi^*_G(\xi) = \int_{\mathcal{M}} D\Phi_{\tilde{G}}(h,b,p,f) \cdot \lambda, \tag{48}$$

for all $(h,b,p,f) \in T_{(\tilde{G})}\mathcal{F}$.

Since the first integral in Eq. (48) is over $\mathcal{M}_0$, rather than $\mathcal{M}$, Theorem IV.5 does not directly apply. Instead we extend $D\Phi^*_{\tilde{G}}(\xi)$ by zero, noting that the hypotheses on $D\Phi^*_{\tilde{G}}(\phi)$ ensure that we can do this without losing regularity.

We define the function

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4) := \begin{cases} -D\Phi^*_{\tilde{G}}(\xi) & \text{on } \mathcal{M}_0,\\ 0 & \text{otherwise.}\end{cases} \tag{49}$$

We then have

$$\int_{\mathcal{M}} \psi \cdot (h,b,p,f) = \int_{\mathcal{M}} D\Phi_{\tilde{G}}(h,b,p,f) \cdot \lambda \tag{50}$$

for all $(h,b,p,f) \in T_{\tilde{G}}\mathcal{F}$. It is straightforward to check that $\psi_1 \in L^2_{-5/2}(\mathcal{M})$ and $\psi_3, \psi_4 \in W^{1,2}_{-3/2}(\mathcal{M})$ (see Lemma 6.5 of Ref. [10] for details), and therefore Theorem IV.5 gives $\lambda \in W^{2,2}_{-1/2}(\mathcal{M})$ and $D\Phi^*_{\tilde{G}}(\lambda) = \psi$ in the strong sense. It then follows that $D\Phi^*_{\tilde{G}}(\tilde{\xi}) = 0$ on $\mathcal{M}_0$, where $\tilde{\xi} := \xi + \lambda$ is the generalized stationary Killing vector. □

Note that we have $D\Phi^*_{\tilde{G}}(\lambda) = 0$ on $\mathcal{M}\backslash\mathcal{M}_0$, so Corollary IV.7 implies $\lambda = 0$ on $\mathcal{M}\backslash\mathcal{M}_0$. It then follows that $\tilde{\xi} = \xi = -\beta\phi$ on Σ, and in particular we have that $\tilde{\xi} + \beta\phi^i$ vanishes on Σ. It is interesting to note that while we do not assume that Σ is a horizon in the above theorem, the conclusion that $\tilde{\xi}^\mu + \beta\phi^i$ vanishes on Σ gives us the following corollary.

Corollary IV.9. If the hypotheses of Theorem IV.8 hold and (g, A, π, ε) is axially symmetric with axial Killing field, ϕ, then Σ is the bifurcation surface of a bifurcate Killing horizon, where $8\pi\alpha$ is the surface gravity and β is the angular velocity.

Proof.—This is an immediate consequence of the fact that if a Killing field vanishes on a spacelike 2-surface then that surface is the bifurcation surface of a bifurcate

Killing horizon (see, for example, Chapter 5 of Ref. [21]). □

Remark IV.10. By virtue of the fact that $D\Phi^*(\xi) = 0$ for a Killing vector, ξ, we do indeed have $D\Phi^*_\pi(\phi)$, $D\Phi^*_\varepsilon(\phi) \in W^{1,2}_{-1/2c}(\mathcal{M}_0)$ when ϕ is the axial Killing vector.

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