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Quasilocal conserved charges and holography

Seungjoon Hyun,* Jaehoon Jeong,† Sang-A Park,‡ and Sang-Heon Yi§

Department of Physics, College of Science, Yonsei University, Seoul 120-749, Korea

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We construct a quasilocal formalism for conserved charges in a theory of gravity in the presence of matter fields that may have slow falloff behaviors at the asymptotic infinity. This construction depends only on equations of motion, and so it is irrespective of ambiguities in the total derivatives of the Lagrangian. By using identically conserved currents, we show that this formalism leads to the same expressions of conserved charges as those in the covariant phase space approach. At the boundary of the asymptotic anti-de Sitter space, we also introduce an identically conserved boundary current that has the same structure as the bulk current and then show that this boundary current gives us the holographic conserved charges identical with those from the boundary stress tensor method. In our quasilocal formalism, we present a general proof that conserved charges from the bulk potential are identical with those from the boundary current. Our results can be regarded as the extension of the existing results on the equivalence of conserved charges by the covariant phase space approach and by the boundary stress tensor method.

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I. INTRODUCTION

The AdS/CFT correspondence has made huge impact on our understanding of strong coupling physics, which is far beyond our usual perturbative approach in field theories. Through this correspondence, the strongly coupled highly quantal regime in the dual field theory side is explored by a classical gravity computation. Many results have been obtained in this route, and various cross-checks have been made for such results verifying the power of the AdS/CFT correspondence. Its successful realization in four-dimensional supersymmetric Yang–Mills theory is still an ongoing productive subject. On the other hand, the quantum gravity is not yet fully understood even under this correspondence, though it may in the future turn out to be a crucial cornerstone of the complete understanding of quantum gravity. However, the lack of its usefulness in the full quantum regime of gravity does not mean that it is powerless in the classical theory of gravity. One such application of the AdS/CFT correspondence to the classical gravity side is the new understanding on conserved charges in a theory of gravity.

In a theory of gravity with diffeomorphism symmetry, it is not so straightforward to define conserved charges. As is well known, the Noether method is insufficient to connect conserved charges and symmetries when those under consideration are local gauge symmetries like diffeomorphisms. There have been various attempts to define conserved charges in gravity, and the final form of such attempts for the asymptotically flat geometry is molded as the so-called Arnowitt-Deser-Misner formula [1,2], which computes total conserved charges at the asymptotic infinity. After failure of many attempts to construct local conserved quantities in gravity, it has been gradually recognized that a local conservation concept like conserved currents has intrinsic ambiguities and denies its complete specification. At most, one may try to construct quasilocal quantities in such a theory. See Ref. [3] for a review on general quasilocal concepts. We use the definition of the term *quasilocal* conserved charge associated with an exact Killing vector as a surface integral in the bulk, not only at the asymptotic boundary, following the spirit given in Refs. [4,5]. One of the important results by the quasilocal construction of conservation law is the understanding of the black hole entropy as a conserved quantity at the Killing horizon [4], which was at first perceived at the level of the analogy with thermodynamics [6] and then confirmed by a semiclassical computation [7].

In contrast to gravity, conserved charges in the dual field theory are rather clear to define and have no ambiguities in their construction. The AdS/CFT correspondence implies that there may be a way to construct quasilocal conserved charges in the bulk gravity side for the asymptotically anti-de Sitter (AdS) space, consistently with the unambiguous field theory side. Indeed, there is a formalism known as the counterterm method or the boundary stress tensor method [8] to obtain holographic conserved charges consistent with the dual field theory. Then, one may ask what is the relation between this holographic approach and the traditional approaches to conserved charges in gravity. This question was answered quite concretely for the asymptotically AdS geometry in Einstein gravity [9–11]. However, the status of this equivalence at the general setup is not so explicit since the boundary stress tensor method depends on the explicit form of Gibbons–Hawking (GH) terms [12,13] and

*sjhyun@yonsei.ac.kr
†j.jeong@yonsei.ac.kr
‡sangapark@yonsei.ac.kr
§shyi@yonsei.ac.kr

counterterms that are not known in general. The boundary stress tensor method is based on Ref. [14] and is basically a kind of the Hamiltonian approach to conserved charges. Because of this nature, this method becomes complicated for a higher derivative theory of gravity. On the other hand, in the bulk gravity side, there are general covariant methods to obtain conserved charges. For instance, the covariant phase space method [4,5,15,16] or Barnich–Brandt–Compère formalism [17–20] can be used in a general covariant theory of gravity. Though there is also a general argument on the consistency of the boundary stress tensor method for the asymptotically AdS geometry with the covariant phase space method [11], it would be much better to have an explicit verification of the equivalence between the conserved charges in the holographic method and those in the bulk covariant one.

To verify the equivalence of boundary and bulk formalisms for the asymptotically AdS geometry, it is useful to recall that there are some modifications on the boundary terms in the Lagrangian in the holographic method, which does not change the bulk equations of motion (EOM). Based on this fact, it is more natural to resort to a covariant formalism for conserved charges that uses the bulk EOM or more accurately the Euler–Lagrange expression. There is one such formalism developed by Abbott, Deser, and Tekin (ADT) [21–24], which has been used successfully for the asymptotic AdS space.

In this paper, we construct a quasilocal formalism for conserved charges in the presence of arbitrary matter fields in the theory of gravity with diffeomorphism symmetry. This construction is based on the EOM and free from any ambiguity in the total derivatives of the Lagrangian, which may be thought of as the extension of the well-known ADT formalism for conserved charges. When the falloff of matter fields is slow enough, the original ADT method needs to be extended since their approach is based on the assumption of the fast falloff of matter fields at the asymptotic infinity so that only metric contribution survives. Here, we give the natural extension of the ADT formalism in the case of the slow falloff of matter fields through the construction of identically conserved currents. It turns out that this quasilocal formalism gives conserved charges that are identical with those from the covariant phase space method. Furthermore, we propose a new holographic method for asymptotically AdS geometry to find the conserved charges at the boundary in the same spirit with the bulk quasilocal formalism. We show that this method gives consistent results with the boundary stress tensor method for holographic conserved charges in Einstein gravity. By using our holographic construction, we confirm the equivalence between conserved charges in the holographic method and those in the bulk covariant one.

This paper is organized as follows. In Sec. II, we construct a quasilocal formalism for conserved charges, based on the Euler–Lagrange expressions, in the presence of arbitrary matter fields, which may be thought of as the extension of the ADT formalism. We introduce the off-shell ADT current and potential and show that the resultant conserved charges are identical with those from the covariant phase space method. In Sec. III, we introduce the identically conserved current at the boundary and show that the corresponding conserved charges are equivalent to those in boundary stress tensor formalism. We also show that the boundary current is equivalent to the bulk ADT potential in appropriate coordinates. These results warrant explicitly the equivalence of the bulk conserved charges with the holographic ones. In Sec. IV, we summarize some generic features for scalar fields. In Sec. IV, we apply our formalism to some interesting examples and explain additional interesting features in our formalism. In the final section, we summarize our results and comment on some future directions.

II. QUASILOCAL FORMALISM AND COVARIANT PHASE SPACE APPROACH

In this section, we extend a quasilocal formalism for conserved charges to a theory of gravity with arbitrary matter fields. We construct the off-shell ADT current and potential and show that the resultant on-shell potential becomes identical with the one from the covariant phase space method. By using these results, we derive the ADT potential straightforwardly for a class of model, which can be used to compute conserved charges.

A. Construction

Let us consider a generic theory of gravity in the presence of arbitrary matter fields denoted collectively as $\psi = (\phi^I, A_\mu, \ldots)$,

$$I[g,\psi] = \frac{1}{16\pi G}\int d^D x\sqrt{-g}\mathcal{L}(g,\psi). \qquad (1)$$

For our convenience, we also denote the metric and matter fields jointly as $\Psi = (g_{\mu\nu}, \phi^I, A_\mu, \ldots)$ in the following. The variation of the action would be taken as

$$\delta I[\Psi] = \frac{1}{16\pi G}\int d^D x[\sqrt{-g}\mathcal{E}_\Psi\delta\Psi + \partial_\mu\Theta^\mu(\delta\Psi)], \qquad (2)$$

where $\mathcal{E}_\Psi = (\mathcal{E}_{\mu\nu}, \mathcal{E}_\psi)$ and Θ^μ denote the Euler–Lagrange expression and the surface term, respectively. We have also adopted the convention such that $\mathcal{E}_\Psi\delta\Psi \equiv \mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}_\psi\delta\psi$.

To introduce the off-shell ADT current and potential in this generic case, we would like to note that there is an off-shell identity in the form of

$$2\zeta_\nu\nabla_\mu\mathcal{E}^{\mu\nu} + \mathcal{E}_\psi \pounds_\zeta\psi = \nabla_\mu(\mathcal{Z}^{\mu\nu}\zeta_\nu), \qquad (3)$$

where $\pounds_\zeta\psi$ denotes the Lie derivative of ψ along the ζ direction. The second rank tensor $\mathcal{Z}^{\mu\nu}$ is a certain function of metric and matter fields of which the specific form will be discussed below. This identity may be thought of as the generalization of the usual Bianchi identity. In fact, one can see that the $\mathcal{Z}^{\mu\nu}$ tensor vanishes when the matter EOM are satisfied by comparing the terms proportional to $\nabla_\mu\zeta_\nu$ in the left- and right-hand sides of Eq. (3). In other words, the $\mathcal{Z}^{\mu\nu}$ tensor should be proportional to a certain combination of the Euler–Lagrange expression, $\mathcal{E}_\psi$, of matter fields. The above off-shell identity can be written in the form of

$$\nabla_\mu(2\mathbf{E}^{\mu\nu}\zeta_\nu) = \mathcal{E}^{\mu\nu}\pounds_\zeta g_{\mu\nu} - \mathcal{E}_\psi\pounds_\zeta\psi = -\mathcal{E}_\Psi\pounds_\zeta\Psi, \tag{4}$$

where $\mathbf{E}^{\mu\nu}$ is defined by

$$\mathbf{E}^{\mu\nu} \equiv \mathcal{E}^{\mu\nu} - \frac{1}{2}\mathcal{Z}^{\mu\nu}. \tag{5}$$

Note that the current $S^\mu_\zeta \equiv 2\mathbf{E}^{\mu\nu}\zeta_\nu$ may be identified with the weakly vanishing Noether current in Refs. [17,18,20].

Some comments are in order:

(i) The explicit form of the $\mathcal{Z}^{\mu\nu}$ tensor may be written as

$$\sqrt{-g}\mathcal{Z}^{\mu\nu}\zeta_\nu = \zeta^\mu\sqrt{-g}\mathcal{L} + \Sigma^\mu(\zeta) - \Theta^\mu(\pounds_\zeta\Psi) + 2\sqrt{-g}\mathcal{E}^{\mu\nu}\zeta_\nu + \partial_\nu U^{\mu\nu},$$

where $U^{\mu\nu} = U^{[\mu\nu]}$ is an arbitrary antisymmetric second rank tensor. However, there are various ambiguities in this expression. We have bypassed these ambiguities by choosing the $\mathcal{Z}^{\mu\nu}$ tensor such that it is proportional to a certain combination of Euler–Lagrange expressions for matter fields, $\mathcal{E}_\psi$.

(ii) In some cases, the $\mathcal{Z}^{\mu\nu}$ tensor turns out to vanish. Let us consider scalar fields specifically. Since the Lie derivative of scalar fields does not contain a derivative of diffeomorphism parameter as $\pounds_\zeta\phi^I = \zeta^\mu\partial_\mu\phi^I$, one cannot obtain terms matching with $\mathcal{Z}^{\mu\nu}\nabla_\mu\zeta_\nu$ as can be inferred from Eq. (3). Therefore, the $\mathcal{Z}^{\mu\nu}$ tensor should vanish generically for scalar fields, though the contribution of the scalar field to $\mathcal{Z}^{\mu\nu}$ may exist indirectly through the interaction with other matter fields. In the case of massless gauge fields, if we use the modified Lie derivative $\pounds'_\zeta$, supplemented by the gauge transformation, one can see that the $\mathcal{Z}^{\mu\nu}$ tensor vanishes. The details will be given in the following.

(iii) In most of the interesting cases, the Lagrangian could be separated as $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_\psi$ for the pure gravity part and the matter field part, respectively. The equations of motion of the metric and matter fields are given by

$$\mathcal{E}_{\mu\nu} = \mathcal{G}_{\mu\nu} - T_{\mu\nu} = 0, \qquad \mathcal{E}_\psi = 0, \tag{6}$$

where $\mathcal{G}_{\mu\nu}$ and $T^{\mu\nu}$ denote the generalized Einstein tensor and the stress tensor of the matter fields, respectively. In these cases, the generalized Einstein tensor $\mathcal{G}_{\mu\nu}$ for the metric field satisfies the Bianchi identity $\nabla_\mu\mathcal{G}^{\mu\nu} = 0$, and the Euler–Lagrange expression of matter fields, $\mathcal{E}_\psi$, satisfies the following off-shell identity, independently:

$$-2\zeta_\nu\nabla_\mu T^{\mu\nu} + \mathcal{E}_\psi\pounds_\zeta\psi = \nabla_\mu(\mathcal{Z}^{\mu\nu}\zeta_\nu). \tag{7}$$

Now, let us recall the form of the off-shell ADT current for a Killing vector, ξ, in the case of pure gravity [25] (see also Ref. [26]):

$$J^\mu_{\rm ADT}(\xi,\delta g) = \delta\mathcal{G}^{\mu\nu}\xi_\nu + \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\mathcal{G}^{\mu\nu}\xi_\nu + \mathcal{G}^{\mu\nu}\delta g_{\nu\rho}\xi^\rho + \frac{1}{2}\xi^\mu\mathcal{G}_{\alpha\beta}\delta g^{\alpha\beta}.$$

As was explained in Ref. [25], this is the natural off-shell extension of the on-shell ADT current, which leads to the on-shell ADT potential in Einstein gravity. One of the essential ingredients in the off-shell conservation of this current is the off-shell identity $\nabla_\mu(\mathcal{G}^{\mu\nu}\xi_\nu) = 0$ for a Killing vector, ξ. By using the off-shell identity given in Eq. (4), one can see that for a Killing vector, ξ, there is an analogous identity even in the presence of arbitrary matters in the form of

$$\nabla_\mu(\mathbf{E}^{\mu\nu}\xi_\nu) = 0. \tag{8}$$

Inspired by this observation, we introduce the off-shell ADT current for a Killing vector, ξ, in the presence of arbitrary matter fields by

$$\mathcal{J}^\mu_{\rm ADT}(\xi,\delta\Psi) = \delta\mathbf{E}^{\mu\nu}\xi_\nu + \frac{1}{2}g^{\alpha\beta}\delta g_{\alpha\beta}\mathbf{E}^{\mu\nu}\xi_\nu + \mathbf{E}^{\mu\nu}\delta g_{\nu\rho}\xi^\rho + \frac{1}{2}\xi^\mu\mathcal{E}_\Psi\delta\Psi, \tag{9}$$

or, more compactly, in the form of

$$\sqrt{-g}\mathcal{J}^\mu_{\rm ADT}(\xi,\delta\Psi) = \delta(\sqrt{-g}\mathbf{E}^{\mu\nu}\xi_\nu) + \frac{1}{2}\sqrt{-g}\xi^\mu\mathcal{E}_\Psi\delta\Psi, \tag{10}$$

where δ denotes the generic variation of Ψ such that $\delta\xi^\mu = 0$. We would like to emphasize that the above construction of the off-shell ADT current, $\mathcal{J}_{\rm ADT}$, depends only on the Euler–Lagrange expressions of metric and matter fields. Using this result, it is straightforward to show the identical conservation of the above off-shell ADT current for a Killing vector, ξ, in the presence of matter fields. The identical conservation of the off-shell ADT current even in the presence of matter fields allows us to introduce the off-shell ADT potential $Q^{\mu\nu}$ as

$$\mathcal{J}^\mu_{\rm ADT} = \nabla_\nu Q^{\mu\nu}_{\rm ADT}. \tag{11}$$

B. Comparison with the covariant phase space approach

We would like to connect the off-shell ADT current in the presence of matter fields to the symplectic current in the covariant phase space approach [15,16]. To this purpose, it is very useful to introduce the off-shell Noether current. For the simplicity of the presentation, let us focus on the action without gravitational Chern–Simons terms in the following. This means that we are considering the case of $\Sigma^\mu = 0$ in Eq. (A2). Recall that the Lagrangian transforms under the diffeomorphism as

$$\delta_\zeta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}\mathcal{E}_\Psi \pounds_\zeta \Psi + \partial_\mu \Theta^\mu(\pounds_\zeta \Psi). \tag{12}$$

Then, one can deduce that the off-shell Noether current in the presence of matter fields can be introduced as

$$J^\mu(\zeta) = 2\sqrt{-g}\mathbf{E}^{\mu\nu}\zeta_\nu + \zeta^\mu\sqrt{-g}\mathcal{L} - \Theta^\mu(\pounds_\zeta g, \pounds_\zeta \psi). \tag{13}$$

To check the identical conservation of this current,[1] we equate two forms of the diffeomorphism variation given in Eqs. (A2) and Eq. (12) and use the off-shell identity given in Eq. (4). From these, one can confirm that $\partial_\mu J^\mu = 0$, identically. Note that the above off-shell Noether current reduces to the on-shell one by using the EOM of metric and matter fields, $\mathbf{E}^{\mu\nu} = 0$. The conservation of the off-shell Noether current J^μ allows us to introduce the off-shell Noether potential $K^{\mu\nu}$ as

$$J^\mu \equiv \partial_\nu K^{\mu\nu}. \tag{14}$$

Now, let us recall that symplectic current in the covariant phase space formalism is introduced as [15]

$$\omega^\mu(\delta_1\Psi, \delta_2\Psi) \equiv \delta_1\Theta^\mu(\delta_2\Psi) - \delta_2\Theta^\mu(\delta_1\Psi). \tag{15}$$

By using the generic variation of the Lagrangian, the Lie derivative of the surface term

$$\pounds_\zeta \Theta^\mu(\delta\Psi) = \zeta^\nu\partial_\nu\Theta^\mu - \Theta^\nu\partial_\nu\zeta^\mu + \Theta^\mu\partial_\nu\zeta^\nu,$$

and the invariance property of the diffeomorphism parameter under a generic variation, $\delta\zeta^\mu = 0$, we have

$$\zeta^\mu\sqrt{-g}\mathcal{E}_\Psi\delta\Psi = \delta(\zeta^\mu\sqrt{-g}\mathcal{L}) - \partial_\nu(2\zeta^{[\mu}\Theta^{\nu]}(\delta\Psi)) - \pounds_\zeta\Theta^\mu(\delta\Psi). \tag{16}$$

By varying Eq. (13) and using Eqs. (10) and (16), we obtain one of our essential results:

$$2\sqrt{-g}\mathcal{J}^\mu_{\rm ADT}(\zeta, \delta\Psi) = \partial_\nu(\delta K^{\mu\nu}(\zeta) - 2\zeta^{[\mu}\Theta^{\nu]}(\delta\Psi)) - \omega^\mu(\pounds_\zeta\Psi, \delta\Psi). \tag{17}$$

We would like to emphasize that this relation holds for any background field configuration and any generic variation, since the conservation of the off-shell ADT current does not require the matter EOM nor the metric EOM. For a Killing vector, ξ, the symplectic current vanishes because $\pounds_\xi\Psi = 0$. As a result, one can see that the off-shell ADT potential for a Killing vector, ξ, is identical with the potential $W^{\mu\nu}$ in the covariant phase space approach [4,5] as

$$2\sqrt{-g}Q^{\mu\nu}_{\rm ADT}(\xi, \delta\Psi) = \delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\Theta^{\nu]}(\delta\Psi) \equiv W^{\mu\nu}(\xi, \delta\Psi). \tag{18}$$

This proves the complete equivalence between the quasilocal formalism and the covariant phase space approach even in the presence of generic matter fields.

To obtain finite conserved charges of black holes from the above ADT potential, we integrate the infinitesimal form of the potential with respect to parameters $\mathcal{Q}_s$ in the black hole solution, as was adopted in Refs. [16–20,28]. Finally, by assuming that the integral is path independent, the finite conserved charge for a Killing vector can be introduced as

$$Q(\xi) \equiv \frac{1}{8\pi G}\int ds \int d^{D-2}x_{\mu\nu}\sqrt{-g}Q^{\mu\nu}_{\rm ADT} = \frac{1}{16\pi G}\int d^{D-2}x_{\mu\nu}\left(\Delta K^{\mu\nu}(\xi) - 2\xi^{[\mu}\int ds\Theta^{\nu]}(g;\mathcal{Q}_s)\right), \tag{19}$$

where $\Delta K^{\mu\nu}$ denotes the finite difference defined by $\Delta K^{\mu\nu} \equiv K^{\mu\nu}_{\mathcal{Q}} - K^{\mu\nu}_{\mathcal{Q}=0}$ and $d^{D-2}x_{\mu\nu}$ denotes the area element of codimension-2 subspace. This final expression of quasilocal conserved charges is completely identical with the one in the covariant phase space [4,5] and in the Barnich–Brandt–Compère formalism [17,18,20]. This formula can be applied to the computation of the black hole entropy as well as the mass and angular momentum of black holes. From the properties of the Killing vector on a Killing horizon and the rotational Killing vector, one can see that the entropy and the angular momentum of black holes can be computed just by the first term in the above formula.

C. Some models

As an application of our formulation, let us consider the general two derivative Lagrangian of the form

$$I = \frac{1}{16\pi G}\int d^Dx\sqrt{-g}(\mathcal{L}_g + \mathcal{L}_\phi + \mathcal{L}_A), \tag{20}$$

[1]For another direction for the use of off-shell currents, see Ref. [27].

where

$$\mathcal{L}_g = R - 2\Lambda,$$
$$\mathcal{L}_\phi = -\frac{1}{2}G_{IJ}(\phi)\partial_\mu\phi^I\partial^\mu\phi^J - V(\phi),$$
$$\mathcal{L}_A = -\frac{1}{4}\mathcal{N}(\phi)F^{\mu\nu}F_{\mu\nu}. \tag{21}$$

Explicitly, the variation of the Lagrangian is given by

$$\delta(\sqrt{-g}\mathcal{L}) = \sqrt{-g}(\mathcal{E}_{\mu\nu}\delta g^{\mu\nu} + \mathcal{E}^\phi_I\delta\phi^I + \mathcal{E}^\mu_A\delta A_\mu) + \partial_\mu\Theta^\mu, \tag{22}$$

where the Euler–Lagrange expressions for each field are

$$\mathcal{E}_{\mu\nu} \equiv G^\Lambda_{\mu\nu} - T_{\mu\nu}, \qquad \mathcal{E}^\nu_A \equiv \nabla_\mu(\mathcal{N}F^{\mu\nu}),$$
$$\mathcal{E}^\phi_I \equiv G_{IJ}(\phi)(\nabla^2\phi^J + \Gamma^J_{KL}\partial_\mu\phi^K\partial^\mu\phi^L) - \partial_I V(\phi)$$
$$-\frac{1}{4}\partial_I\mathcal{N}F_{\mu\nu}F^{\mu\nu}, \tag{23}$$

and the surface terms are given by

$$\Theta^\mu(\delta g,\delta\phi,\delta A) = \Theta^\mu_g(\delta g) + \Theta^\mu_\phi(\delta\phi) + \Theta^\mu_A(\delta A)$$
$$= \sqrt{-g}[2g^{\alpha[\mu}\nabla^{\beta]}\delta g_{\alpha\beta} - G_{IJ}(\phi)\delta\phi^I\partial^\mu\phi^J$$
$$- \mathcal{N}F^{\mu\nu}\delta A_\nu]. \tag{24}$$

Here, Einstein and bulk stress tensors become

$$G^\Lambda_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu},$$
$$T^\phi_{\mu\nu} = \frac{1}{2}G_{IJ}(\phi)\partial_\mu\phi^I\partial_\nu\phi^J + \frac{1}{2}g_{\mu\nu}\mathcal{L}_\phi,$$
$$T^A_{\mu\nu} = \frac{1}{2}\mathcal{N}F_{\mu\alpha}F^\alpha_\nu + \frac{1}{2}g_{\mu\nu}\mathcal{L}_A.$$

The metric, scalar, and gauge field EOM are given by $\mathcal{E}^{\mu\nu} = 0$, $\mathcal{E}^\phi_I = 0$, and $\mathcal{E}^\mu_A = 0$.

As mentioned earlier, several interesting features appear in the model with vector fields. One may modify the Lie derivative of gauge fields since those fields may be accompanied by a certain gauge transformation. To use the off-shell identity for the gauge field, it is more useful to introduce a modified Lie derivative that is augmented by a certain gauge transformation such that

$$£'_\zeta A_\mu \equiv -F_{\mu\nu}\xi^\nu = £_\zeta A_\mu + \partial_\mu\Lambda, \qquad \Lambda \equiv -\zeta^\alpha A_\alpha.$$

By recalling that gauge fields satisfy a Bianchi identity in the form of

$$\nabla_{[\rho}F_{\mu\nu]} = 0$$

and using this modified Lie derivative, one can show that the $\mathcal{Z}^{\mu\nu}\zeta_\mu$ term is absent in Eq. (3). Surely, this modification is not essential, and the unmodified form can also be used without affecting the final result of conserved charges. For massive gauge fields, one cannot use the modified Lie derivative since there is no gauge invariance. Rather, we should keep the original Lie derivatives

$$£_\zeta A_\mu = -F_{\mu\nu}\zeta^\nu + \partial_\mu(\zeta^\nu A_\nu).$$

In this case, it turns out that the tensor $\mathcal{Z}^{\mu\nu}$ is given in terms of the Euler–Lagrange expression $\mathcal{E}^\mu_A$ of a gauge field, A_μ, as

$$\mathcal{Z}^{\mu\nu} = \mathcal{E}^\mu_A A^\nu. \tag{25}$$

Just as in the massless case, one can see that the final results on the relation between the off-shell ADT potential and the covariant phase space potential should remain the same as Eq. (18).

Now, we obtain the ADT potential in this model by using Eq. (18). Since surface terms are given in Eq. (24), it is sufficient to derive the expression of the Noether potential. By using the off-shell identities in the Noether current and potential,

$$-2\sqrt{-g}T^{\mu\nu}_\phi\zeta_\nu + \zeta^\mu\sqrt{-g}\mathcal{L}_\phi - \Theta^\mu_\phi(£_\zeta\phi) = 0, \tag{26}$$

$$-2\sqrt{-g}T^{\mu\nu}_A\zeta_\nu + \zeta^\mu\sqrt{-g}\mathcal{L}_A - \Theta^\mu_A(£'_\zeta A) = 0, \tag{27}$$

one can see that the Noether potential is given by

$$K^{\mu\nu}(\xi) = 2\sqrt{-g}\nabla^{[\mu}\xi^{\nu]}. \tag{28}$$

This Noether potential as well as the corresponding off-shell Noether current J^μ, even in the presence of matter fields, take the identical forms as those without matter fields. The form of the Noether potential in Eq. (28) explains why there is no apparent contribution of matter fields on the entropy of charged black holes in Einstein gravity, and thus it is simply determined by the area law. As is well known, the Wald's entropy of black holes is captured by the Noether potential only since the contribution of the surface term in $W^{\mu\nu}$, in Eq. (18), vanishes on a Killing horizon. In other words, any contribution of matter fields to the black hole entropy should be indirectly incorporated through the backreaction of the metric due to matter fields.

In this model, the total off-shell ADT potential is given by the sum of the metric, scalar, and gauge field contributions as

$$Q^{\mu\nu}_{\rm ADT}(\xi;\delta\Psi) = Q^{\mu\nu}_{\rm ADT}(\xi;\delta g) + Q^{\mu\nu}_{\rm ADT}(\xi;\delta\phi^I) + Q^{\mu\nu}_{\rm ADT}(\xi;\delta A). \tag{29}$$

By using our relation (18), one can easily show that, for a Killing vector, ξ, the metric contribution to the off-shell ADT potential is given by

$$Q^{\mu\nu}_{\rm ADT}(\xi;\delta g) = -\frac{1}{2}g_{\alpha\beta}\delta g^{\alpha\beta}\nabla^{[\mu}\xi^{\nu]} + \xi^{[\mu}\nabla_\alpha\delta g^{\nu]\alpha} - \xi_\alpha\nabla^{[\mu}\delta g^{\nu]\alpha} - g_{\alpha\beta}\xi^{[\mu}\nabla^{\nu]}\delta g^{\alpha\beta} + \delta g^{\alpha[\mu}\nabla_\alpha\xi^{\nu]}, \qquad (30)$$

and the contributions from the scalar and gauge fields are given by

$$Q^{\mu\nu}_{\rm ADT}(\xi;\delta\phi) = G_{IJ}(\phi)\delta\phi^I\xi^{[\mu}\partial^{\nu]}\phi^J,$$
$$Q^{\mu\nu}_{\rm ADT}(\xi;\delta A) = \mathcal{N}\xi^{[\mu}F^{\nu]\alpha}\delta A_\alpha. \qquad (31)$$

Traditionally, matter contributions through $Q^{\mu\nu}_{\rm ADT}$ to total conserved charges have been ignored by supposing that matter fields fall off fast when they approach the asymptotic infinity. However, one needs to incorporate those with the slow falloff boundary condition, especially in the context of the AdS/CFT correspondence since matter contributions have some dual interpretation.

III. QUASILOCAL FORMALISM AND BOUNDARY STRESS TENSOR METHOD

In this section, we introduce the boundary off-shell current according to the spirit of our bulk construction and compare conserved charges by this current with those from the bulk off-shell ADT potential. In the context of the AdS/CFT correspondence, there is another way to obtain conserved charges from the renormalized boundary stress tensor. We show that the construction of our boundary current is a kind of the reformulation of the conventional boundary stress tensor method along our bulk construction. Furthermore, we show that conserved charges by our boundary current or from the boundary stress tensor method match completely with those from the bulk ADT formalism.

A. Boundary off-shell current

For the construction of the boundary current in the asymptotic AdS space, let us recall that Arnowitt-Deser-Misner decomposition along the radial direction can be taken as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = N^2dr^2 + \gamma_{ij}(r,x)(dx^i + N^i dr)(dx^j + N^j dr), \qquad (32)$$

where $i, j = 0, 1, \ldots, D-2$. In the following, we denote the space-time dimension of the dual field theory as $d \equiv D-1$. To obtain conserved charges from the holographic renormalization perspective [29–34], one may consider the renormalized action that includes the GH boundary term I_{GB} and the counterterm I_{ct} as

$$I_r[g,\psi] = I[g,\psi] + I_{GH}[\gamma] + I_{ct}[\gamma,\psi],$$

where the GH boundary and counterterms are defined on a hypersurface and depend on the boundary values of γ and ψ there. The on-shell valued renormalized action I^{on}_r would be the functional of the boundary value (γ,ψ) at the boundary $\mathcal{B}$. The generic variation of the on-shell renormalized action is given by[2]

$$\delta I^{on}_r[\gamma,\psi] = \frac{1}{16\pi G}\int_{\mathcal{B}} d^dx\sqrt{-\gamma}[T^{ij}_B\delta\gamma_{ij} + \Pi_\psi\delta\psi], \qquad (33)$$

where the boundary stress tensor, up to the radial rescaling, T^{ij}_B, is identified with the stress tensor of dual conformal field theory (CFT) according to the AdS/CFT correspondence and the renormalized momentum Π_ψ of the matter field ψ corresponds to the vacuum expectation value of the operator dual to the matter field.

One can construct the identically conserved boundary current $\mathcal{J}^i_B$ from the on-shell renormalized action. We begin with the identity, analogous to the bulk one given in Eq. (7),

$$-2\zeta_j\nabla_iT^{ij}_B + \Pi_\psi\pounds_\zeta\psi = \nabla_i(\mathcal{Z}^{ij}_B\zeta_j), \qquad (34)$$

where ζ denotes an arbitrary boundary diffeomorphism paramter and the $\mathcal{Z}^{ij}_B$ tensor is a certain combination of Π_ψ. This identity follows from the boundary diffeomorphism invariance. Just as in the bulk case, the scalar field contribution to the $\mathcal{Z}^{ij}_B$ tensor vanishes generically, and the vector field contribution to the $\mathcal{Z}^{ij}_B$ tensor turns out to be given by $\Pi^i_AA^j$. Then, one can introduce the boundary ADT-like current for a boundary Killing vector, ξ_B, as

$$\mathcal{J}^i_B(\xi_B) \equiv -\delta\mathbf{T}^{ij}_B\xi^B_j - \frac{1}{2}\gamma^{kl}\delta\gamma_{kl}\mathbf{T}^{ij}_B\xi^B_j - \mathbf{T}^{ij}_B\delta\gamma_{jk}\xi^k_B + \frac{1}{2}\xi^i_B(T^{kl}_B\delta\gamma_{kl} + \Pi_\psi\delta\psi), \qquad (35)$$

where

$$\mathbf{T}^{ij}_B \equiv T^{ij}_B + \frac{1}{2}\mathcal{Z}^{ij}_B. \qquad (36)$$

By using $\delta\xi^i_B = 0$, this boundary current can be written more compactly as

$$\sqrt{-\gamma}\mathcal{J}^i_B(\xi_B) = -\delta(\sqrt{-\gamma}\mathbf{T}^{ij}_B\xi^B_j) + \frac{1}{2}\sqrt{-\gamma}\xi^i_B(T^{kl}_B\delta\gamma_{kl} + \Pi_\psi\delta\psi). \qquad (37)$$

The above boundary current takes the analogous form of the bulk off-shell ADT current given in Eq. (10) except for the absence of a generalized Einstein tensor. This is natural since the boundary metric field is nondynamical. By using the fact that

[2]Our convention for the boundary stress tensor T^{ij}_B is such that it denotes only the finite part after holographic renormalization and thus corresponds to $\pi^{ij}_{(d)}$ in Ref. [9]. And so does the matter part Π_ψ.

$$\nabla_i(\mathbf{T}_B^{ij}\xi_j^B) = 0, \tag{38}$$

for the boundary Killing vector, ξ_B, one can show that the corresponding current, $\mathcal{J}_B^i$, is also conserved identically for a generic variation such that $\delta\xi_B^i = 0$. Note that one may regard $\mathcal{J}_B^i$ as a 1-form on the solution parameter space. To introduce boundary conserved charges, we integrate the 1-form boundary current in the same manner as in the bulk case. Therefore, the boundary conserved charges are given by

$$Q_B(\xi_B) = \frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i \int ds\sqrt{-\gamma}\mathcal{J}_B^i(\xi_B), \tag{39}$$

where we integrate over the path parametrized by s in the parameter space in the given solution.[3]

B. Equivalence with the boundary stress tensor method

We would like to uncover the relation between (linearized) conserved charges obtained from the boundary current introduced in the previous section and those from the conventional boundary stress tensor method in Refs. [8–10]. As alluded to earlier, we perform the variation δ along the one-parameter path in the solution space. As will be explained through examples, the one-parameter path in the solution space corresponds to the choice of a representative in the conformal class at the boundary with a restricted diffeomorphism preserving the gauge choice.

When the contribution from the second term of the boundary current $\mathcal{J}_B^i$ in Eq. (37) is absent, the boundary current reduces to

$$\sqrt{-\gamma}\mathcal{J}_B^i = -\delta(\sqrt{-\gamma}\mathbf{T}_B^{ij}\xi_j^B). \tag{40}$$

By using the conventional expression of holographic charges in the form of

$$\hat{Q}_B(\xi_B) = -\frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i\sqrt{-\gamma}\mathbf{T}_B^{ij}\xi_j^B, \tag{41}$$

the expression of finite conserved charges for the Killing vector ξ_B from the boundary ADT formalism can be obtained as

$$Q_B(\xi_B) = \hat{Q}_B(\xi_B) - \hat{Q}_B^{\mathrm{AdS}}(\xi_B). \tag{42}$$

This verifies the equivalence, up to the AdS vacuum value, between the boundary quasilocal ADT formalism and the conventional boundary stress tensor method.

To see the meaning of the second term in Eq. (37), let us focus on the specific model introduced in (20). In this model, we would like to consider the relation between the allowed boundary condition on the asymptotic AdS space and the absence of the contribution from the second term of the boundary current $\mathcal{J}_B^i$. As was discussed in Ref. [9] in the context of the well-posedness of the variational problem, the boundary condition allowed in the asymptotic AdS space needs to be relaxed as

$$\delta\gamma_{ij} = 2\gamma_{ij}\delta\sigma, \qquad \delta A_i = 0, \qquad \delta\phi^I = (\Delta_I - d)\phi^I\delta\sigma, \tag{43}$$

where Δ_I is the conformal dimension of a dual operator to a scalar field ϕ^I. This boundary condition shows us that the second term in Eq. (37) is nothing but the conformal anomaly $\mathcal{A}$ in the boundary field theory. Explicitly, the second term becomes

$$T_B^{kl}\delta\gamma_{kl} + \Pi_\psi\delta\psi = \left[2T_{Bi}^i + \sum_I(\Delta_I - d)\Pi_{\phi^I}\phi^I\right]\delta\sigma \equiv \mathcal{A}\delta\sigma. \tag{44}$$

There is no conformal anomaly in the dual field theory of the even-dimensional AdS geometry. On the other hand, in odd-dimensional AdS geometry, the dual CFT has a conformal anomaly. We consider the boundary conditions of metric and matter fields satisfying $\int\delta\sigma\mathcal{A} = 0$, which holds in all our examples. This leads to the absence of the contribution from the second term in the boundary current in Eq. (37).

Because of the absence of the scalar field contribution to $\mathcal{Z}_B^{ij}$, we have

$$\mathbf{T}_B^{ij} = T_B^{ij} + \frac{1}{2}\Pi_A^i A^j, \tag{45}$$

and we can see that Eq. (42), up to the AdS vacuum value, gives us the identical expression of conserved holographic charges with the one in the conventional boundary stress tensor method (see Eq. (4.28) in Ref. [9]).

C. Equivalence with the bulk ADT potential

In this section, we would like to show that the boundary current and the bulk potential lead to the same conserved charges. One may recall that the holographic renormalization process introduces new boundary terms in the given Lagrangian with the on-shell condition. These new boundary terms do not affect the bulk EOM, and thus the construction of the bulk current given in Eq. (9), which depends only on the bulk Euler–Lagrange expressions, is valid and so can be used without any modification. The effect of the new boundary terms comes in through the modifications of the Noether potential $K^{\mu\nu}$ and the surface term Θ^μ given in Eq. (18).

[3] As a working hypothesis, we assume that the 1-form boundary current is independent of path. This assumption holds in all the examples given in the following sections.

For definiteness, it is convenient to use the, so-called, Fefferman–Graham (FG) coordinates for an asymptotically AdS space [35], which is given in the form of

$$ds^2 = d\eta^2 + \gamma_{ij}dx^i dx^j. \quad (46)$$

In the following, we take the radius of asymptotic AdS space as unity and the cosmological constant $\Lambda = -\frac{d(d-1)}{2}$. In these coordinates, the boundary is located at η_0, which will be sent to the infinity in the end. The radial expansion of the metric and the matter fields is generically taken as

$$\gamma_{ij} = e^{2\eta}[\gamma^{(0)}_{ij} + \mathcal{O}(e^{-\eta})],$$
$$\psi = e^{-(d_\psi-\Delta_\psi)\eta}[\psi_{(0)} + \mathcal{O}(e^{-\eta})], \quad (47)$$

where Δ_ψ is the conformal dimension of the operator dual to ψ and d_ψ is given by $d_\psi = d - p$ for the rank p tensor field ψ. The boundary metric $\gamma^{(0)}_{ij}$ represents the background geometry of the dual CFT according to the AdS/CFT dictionary. Formally, the GH boundary term and counterterm are taken by

$$I_{GH}[\gamma] = \frac{1}{8\pi G}\int d^d x\sqrt{-\gamma}L_{GH}(\gamma),$$
$$I_{ct}[\gamma,\psi] = \frac{1}{16\pi G}\int d^d x\sqrt{-\gamma}L_{ct}(\gamma,\psi), \quad (48)$$

which make the renormalized action finite in the limit $\eta_0 \to \infty$.

The modification in boundary terms can be succinctly captured by the introduction of a modified surface term $\tilde{\Theta}^\eta$ as

$$\tilde{\Theta}^\eta(\delta\Psi) = \Theta^\eta(\delta\Psi) + \delta(2\sqrt{-\gamma}L_{GH}) + \delta(\sqrt{-\gamma}L_{ct})$$
$$= \sqrt{-\gamma}(T^{ij}_B\delta\gamma_{ij} + \Pi_\psi\delta\psi), \quad (49)$$

where the second line equality comes from Eq. (33). This expression tells us that $\tilde{\Theta}^\eta \sim \mathcal{O}(1)$ in the radial expansion. Correspondingly, the modified Noether current $\tilde{J}^\eta$ for a diffeomorphism parameter. ζ. becomes

$$\tilde{J}^\eta = \partial_i\tilde{K}^{\eta i}(\zeta) = \zeta^\eta\sqrt{-\gamma}\mathcal{L}^{on}_r - \tilde{\Theta}^\eta(\pounds_\zeta\Psi), \quad (50)$$

where we have used the on-shell condition on the background fields in Eq. (13). Here, one may also note that the on-shell renormalized Lagrangian $\sqrt{-\gamma}\mathcal{L}^{on}_r$ is related to the so-called A-type trace anomaly [29,36].

Just as in Einstein gravity [9], the asymptotic behavior of general diffeomorphism parameter ζ is given by

$$\zeta^\eta \sim \mathcal{O}(e^{-d\eta}), \qquad \zeta^i \sim \mathcal{O}(1), \quad (51)$$

in order to preserve the asymptotic gauge choice and the renormalized action. This asymptotic behavior in the diffeomorphism parameter ζ allows us to discard the first term in the right-hand side of Eq. (50) when we approach the boundary. In the following, we keep only the relevant boundary values of parameters such that a bulk Killing vector, ξ^i, is replaced by its boundary value, ξ^i_B. For the diffeomorphism variation $\pounds_\zeta\Psi$, the modified surface term $\tilde{\Theta}^\eta$ is given by

$$\tilde{\Theta}^\eta(\pounds_\zeta\Psi) = \sqrt{-\gamma}(2T^{ij}_B\nabla_i\zeta_j + \Pi_\psi\pounds_\zeta\psi) = \partial_i(2\sqrt{-\gamma}\mathbf{T}^{ij}_B\zeta_j), \quad (52)$$

where we have used the identity given in Eq. (34). By using this result, one can see that the Noether potential $\tilde{K}^{\eta i}$ becomes

$$\tilde{K}^{\eta i} = -2\sqrt{-\gamma}\mathbf{T}^{ij}_B\zeta_j + \partial_j(\sqrt{-\gamma}\mathcal{U}^{ij}_B), \quad (53)$$

where $\mathcal{U}^{ij}_B$ is an arbitrary antisymmetric second rank tensor. Since we are interested in conserved charges, the total derivative term $\partial_j(\sqrt{-\gamma}\mathcal{U}^{ij}_B)$ is irrelevant and can be discarded for simplicity. As a result, the relation between the ADT and Noether potentials in Eq. (18) for a Killing vector, ξ, becomes

$$2\sqrt{-g}Q^{\eta i}_{\rm ADT}|_{\eta\to\infty} = -\delta(2\sqrt{-\gamma}\mathbf{T}^{ij}_B\xi^B_j)$$
$$+ \sqrt{-\gamma}\xi^i_B(T^{kl}_B\delta\gamma_{kl} + \Pi_\psi\delta\psi)$$
$$\equiv 2\sqrt{-\gamma}\mathcal{J}^i_B. \quad (54)$$

That is to say the leading parts of the bulk ADT potential and the boundary current are identical when we go to the asymptotic infinity.[4] This proves the equivalence of conserved charges by the bulk potential, Q, and those by the boundary current, Q_B:

$$Q(\xi) = \frac{1}{8\pi G}\int_{\mathcal{B}} d^{D-2}x_{\eta i}\int ds\sqrt{-g}Q^{\eta i}_{\rm ADT}$$
$$= \frac{1}{8\pi G}\int_{\partial\mathcal{B}} d^{d-1}x_i\int ds\sqrt{-\gamma}\mathcal{J}^i_B = Q_B(\xi_B). \quad (55)$$

Our results extend, to a general theory of gravity, the equivalence statement given for a specific model in Ref. [9] and are completely consistent with the rather formal argument on such equivalence given in Ref. [11]. We would like to emphasize that the matching between the ADT potential and the boundary current is valid only at the boundary, while the bulk ADT potential in the quasilocal

[4]The holographic charges from boundary stress tensor method are defined by the first term only. In Einstein gravity, it was shown in Ref. [9] that the holographic charges are identical with those from the covariant phase space formalism when conformal anomaly is absent. Our modification of the holographic charges, in which the second term is naturally incorporated, maintain the equivalence between the holographic and bulk charges.

sense could be applied even to the deep interior region like the black hole horizon.

IV. GENERALITIES FOR SCALAR FIELDS

In this section, we introduce the radial expansion of the metric and matter fields and explain some properties related to the computation of conserved charges. We also explain how to construct the boundary stress tensor. For simplicity, we consider only a scalar field in the matter sector with the action given in Eq. (22). The boundary metric is taken to be flat as $\gamma^{(0)}_{ij} = \eta_{ij}$. In pure Einstein gravity, the conformal anomaly of the dual field theory is absent as a consequence of the flat boundary metric. And thus logarithmic terms do not appear in the metric, and the radial expansion of the on-shell metric, in the FG coordinates, is generically given by

$$\gamma_{ij} = e^{2\eta}(\eta_{ij} + e^{-d\eta}\gamma^{(d)}_{ij} + \cdots). \tag{56}$$

It is well known that the leading-order term, $e^{-d\eta}\gamma^{(d)}_{ij}$, gives the well-defined, finite, total conserved charges, like the mass and angular momentum of black holes.

A. Radial expansion

We assume the scalar field depends only on the radial coordinate η. In general, the leading order in the radial expansion of the scalar field is given by $\phi \sim e^{-(d-\Delta_\pm)\eta}\phi_\pm$, where ϕ_+ and ϕ_- correspond to the leading-order terms of the non-normalizable and normalizable modes, respectively, and $\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2}$. The mass of the scalar field has unitary bound or the Breitenlohner–Freedman (BF) bound [37], $m^2 = m^2_{BF} = -\frac{d^2}{4}$, in which the exponents degenerate as $\Delta_+ = \Delta_- = \frac{d}{2}$. In this case, the scalar field includes the logarithmic mode behaving as $\phi \sim \eta e^{-\frac{d}{2}\eta}\phi_{\log}$. We consider the BF-saturated case first.

1. Class I: $m^2 = m^2_{BF} = -\frac{d^2}{4}$

We can apply our formalism to the case with the logarithmic mode, which was studied in Refs. [38,39] by using the Hamiltonian formalism. For simplicity, we consider the case in which the leading-order term in the radial expansion starts at the order $e^{-\frac{d}{2}\eta}$ and take the radial expansion as

$$\phi = e^{-\frac{d}{2}\eta}(\phi_{(0)} + \cdots). \tag{57}$$

The corresponding radial expansion of the metric solution takes the same form given in Eq. (56).

Now, let us perform a linearized analysis to see the backreaction of the metric to the scalar field. By taking into account the leading-order behavior of the scalar field, it is sufficient to take the scalar potential up to quadratic order as

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \cdots. \tag{58}$$

The linearized EOM of our specific model become

$$h''_{ij} + (d-4)h'_{ij} + (4-2d)h_{ij} - e^{2\eta}\eta_{ij}(h'' + dh') = 0, \tag{59}$$

$$(d-1)h' - \frac{d^2}{4}e^{-d\eta}\phi^2_{(0)} = 0, \qquad h \equiv e^{-2\eta}\eta^{ij}h_{ij} \tag{60}$$

$$\varphi'' + d\varphi' - m^2\varphi = 0, \tag{61}$$

where primes denote derivatives with respect to η and $\gamma_{ij} \equiv e^{2\eta}\eta_{ij} + h_{ij}$ and $\phi \equiv e^{-\frac{d}{2}\eta}\phi_{(0)} + \varphi$. Since the leading-order contribution of the scalar field to the metric starts from the order $e^{-d\eta}$, the linear analysis is sufficient to compute conserved charges. From Eq. (60), the leading-order coefficient $\gamma^{(d)}_{ij}$ in the metric satisfies the trace relation,

$$\eta^{ij}\gamma^{(d)}_{ij} = -\frac{d}{4(d-1)}\phi^2_{(0)}. \tag{62}$$

The form of the coefficients $\gamma^{(d)}_{ij}$ would be further specified by the metric ansatz of the solution. As in the case of pure Einstein gravity, these coefficients can be used to determine the conserved charges.

2. Class II: $m^2 > m^2_{BF} = -\frac{d^2}{4}$

In this class, we consider the case with $\Delta_\phi = \Delta_+$, and then the radial expansion of the scalar field solution is given in the form of

$$\phi = e^{-(d-\Delta_\phi)\eta}(\phi_{(0)} + e^{-2(d-\Delta_\phi)\eta}\phi_{(2)} + e^{-4(d-\Delta_\phi)\eta}\phi_{(4)} + \cdots), \tag{63}$$

for the even scalar potential for which the generic expansion is given by

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 + \cdots. \tag{64}$$

If $\Delta_\phi \geq d$, the presence of this non-normalizable mode changes the asymptotic AdS structure. Henceforth, we restrict ourselves to the case $\Delta_\phi < d$, which corresponds to $m^2 < 0$. The corresponding metric solution has the radial expansion,

$$\gamma_{ij} = e^{2\eta}[\eta_{ij} + e^{-2(d-\Delta_\phi)\eta}\gamma^{(2d-2\Delta_\phi)}_{ij} + \cdots + e^{-d\eta}\gamma^{(d)}_{ij} + \cdots], \tag{65}$$

where the leading-order term in the expansion of the metric is given by

$$\gamma_{ij}^{(2d-2\Delta_\phi)} = -\frac{\phi_{(0)}^2}{4(d-1)}\eta_{ij}. \tag{66}$$

The slower falloff terms than $e^{-d\eta}\gamma_{ij}^{(d)}$ may give divergent contributions to conserved charges. However, such divergencies should be automatically taken care of, and finite values emerge since our bulk formalism, by using the one-parameter path in the solution space, gives identical results with those from the boundary stress tensor formalism. One may note that conserved charges are generically determined by $\gamma_{ij}^{(d)}$. Since the contribution of the scalar source to the metric starts, at least, from the ϕ^2 term, we need to know all the coefficients up to the order $e^{-(2\Delta_\phi-d)\eta}$ in the expansion of the full solution of the scalar field. This will be clearly shown through the explicit computation of conserved charges in specific examples in Sec. V.

B. Counterterms and boundary stress tensor

In this section, we present the generic forms of the GH and counterterms in the model (20). By using these forms, we give the resultant form of the boundary stress tensor and the renormalized momentum of the scalar field.

First of all, the GH term for the Einstein gravity is given by

$$L_{\rm GH} = K(\gamma), \tag{67}$$

where $K(\gamma)$ is the extrinsic curvature scalar at the boundary. The counterterms $L_{ct}(\gamma,\phi)$ consist of two parts,

$$L_{ct} = 2K_{ct}(\gamma) + \Phi_{ct}(\phi), \tag{68}$$

where the first term is the counterterm for the pure gravity and the second one is the one for the scalar field. The counterterms for the pure gravity part are given by [8,40–42]

$$K_{ct}(\gamma) = -(d-1) - \frac{1}{2(d-2)}R_B - \frac{1}{2(d-4)(d-2)^2} \times \left(R_{ij}^B R_B^{ij} - \frac{d}{4(d-1)}R_B^2\right) + \cdots, \tag{69}$$

where R_{ij}^B and R_B are the intrinsic Ricci tensor and scalar at the boundary, respectively. The counterterms for the scalar field ϕ are chosen as the polynomial of the scalar field as

$$\Phi_{ct}(\phi) = \alpha_1\phi^2 + \alpha_2\phi^4 + \cdots, \tag{70}$$

where α_k are determined to cancel the divergences in the renormalized action at the boundary.

It follows that the boundary stress tensor consists of two parts,

$$T_B^{ij} = T_g^{ij} + T_\phi^{ij}, \tag{71}$$

where T_g^{ij} and T_ϕ^{ij} come from the metric and scalar fields, respectively. They are given by

$$T_g^{ij} = K\gamma^{ij} - K^{ij} - (d-1)\gamma^{ij} + \frac{1}{(d-2)}\left(R_B^{ij} - \frac{1}{2}R_B\gamma^{ij}\right) + \cdots, \tag{72}$$

$$T_\phi^{ij} = \frac{\gamma^{ij}}{2}(\alpha_1\phi^2 + \alpha_2\phi^4 + \cdots). \tag{73}$$

One may note that the contribution of the scalar field to the boundary stress tensor comes only from the counterterm action, and the concrete expression of T_B depends on the form of the counterterm action. One may also note that in this case

$$\mathbf{T}_B^{ij} = T_B^{ij},$$

since we are considering a scalar field only. The renormalized momentum of the scalar field at the boundary is given by

$$\sqrt{-\gamma}\Pi_\phi = \sqrt{-\gamma}[-\partial_\eta\phi + 2\alpha_1\phi + 4\alpha_2\phi^3 + \cdots]. \tag{74}$$

In class I, it is sufficient to take $\alpha_1 = -\frac{d}{4}$, $\alpha_2 = \cdots = 0$, and then it turns out that $\Pi_\phi = 0$ generically.

V. APPLICATION TO VARIOUS BLACK HOLES

In this section, we apply our quasilocal formalism to some specific examples. In particular, we compute the total conserved charges from both bulk and boundary constructions. We support the general proof of the equivalence on total charges in the bulk and boundary constructions through explicit computations. All the examples we have presented in this section correspond to the specific cases such that the one-parameter path in the solution space is taken as $\delta_s\gamma_{ij}^{(0)} = 0$. In our bulk construction, we compute each contribution from the metric and matter sectors to conserved charges, by using Eqs. (30) and (31). We find each contribution to conserved charges matches with the corresponding one in our boundary construction. Specifically, we reproduce the mass and angular momentum of AdS black holes in various dimensions and explain additional salient features in our formalism through explicit examples.

A. Three-dimensional black holes

In three-dimensional gravity, we have various analytic black hole solutions that allow us to apply our formalism concretely. Specifically, we consider the three-dimensional, AdS black hole space with scalar hair.

1. Class I: $m^2 = m^2_{\mathrm{BF}} = -1$

By solving the linearized EOM, we obtain the most general solution of the metric as

$$\gamma^{(2)}_{ij} = \begin{pmatrix} C_1 + \frac{1}{4}\phi^2_{(0)} & -C_2 \\ -C_2 & C_1 - \frac{1}{4}\phi^2_{(0)} \end{pmatrix}, \tag{75}$$

where C_1 and C_2 are arbitrary parameters that turn out to be proportional to the mass and the angular momentum, respectively, of AdS black holes with scalar hair. To compute the mass and angular momentum of these black holes in the bulk quasilocal formalism, we take the timelike and rotational Killing vectors as $\xi_T = \frac{\partial}{\partial t}$ and $\xi_R = \frac{\partial}{\partial \theta}$ and take the relevant path in the solution space parametrized by C_1, C_2, and $\phi_{(0)}$.

The ADT potentials in Eqs. (30) and (31) for the timelike Killing vector $\xi^i_T = (1, 0)$ are computed as

$$\sqrt{-g}Q^{\eta i}_{\mathrm{ADT}}(\xi_T; \delta g)|_{\eta\to\infty} = \left(\delta C_1 - \frac{1}{2}\phi_{(0)}\delta\phi_{(0)}, \delta C_2\right), \tag{76}$$

$$\sqrt{-g}Q^{\eta i}_{\mathrm{ADT}}(\xi_T; \delta\phi)|_{\eta\to\infty} = \left(\frac{1}{2}\phi_{(0)}\delta\phi_{(0)}, 0\right). \tag{77}$$

By using Eq. (19) with the convention $dx_{\eta t} = \frac{1}{2\sqrt{-g}}\epsilon_{\eta t\theta} d\theta = d\theta$, we obtain

$$M^g_{\mathrm{ADT}} = \frac{1}{4G}\left(C_1 - \frac{1}{4}\phi^2_{(0)}\right), \quad M^\phi_{\mathrm{ADT}} = \frac{1}{16G}\phi^2_{(0)}. \tag{78}$$

Therefore, the total mass of these black holes is given by

$$M_{\mathrm{ADT}} \equiv M^g_{\mathrm{ADT}} + M^\phi_{\mathrm{ADT}} = \frac{1}{4G}C_1. \tag{79}$$

The ADT potentials for the rotational Killing vector $\xi^i_R = (0, 1)$ are computed as

$$\sqrt{-g}Q^{\eta i}_{\mathrm{ADT}}(\xi_R; \delta g)|_{\eta\to\infty} = \left(-\delta C_2, -\delta C_1 - \frac{1}{2}\phi_{(0)}\delta\phi_{(0)}\right), \tag{80}$$

$$\sqrt{-g}Q^{\eta i}_{\mathrm{ADT}}(\xi_R; \delta g)|_{\eta\to\infty} = (0, 0). \tag{81}$$

Therefore, the scalar contribution to the angular momentum is absent, and the total angular momentum of these black holes is given by

$$J_{\mathrm{ADT}} \equiv J^g_{\mathrm{ADT}} + J^\phi_{\mathrm{ADT}} = \frac{1}{4G}C_2. \tag{82}$$

Now, we present the boundary stress tensor explicitly and confirm the equivalence relation (54) between the bulk ADT potential and the boundary current. After a bit of computation, one obtains the boundary stress tensor as

$$(\mathbf{T}_g)^i{}_j = \begin{pmatrix} -C_1 + \frac{1}{4}\phi^2_{(0)} & -C_2 \\ -C_2 & C_1 + \frac{1}{4}\phi^2_{(0)} \end{pmatrix},$$
$$(\mathbf{T}_\phi)^i{}_j = \begin{pmatrix} -\frac{1}{4}\phi^2_{(0)} & 0 \\ 0 & -\frac{1}{4}\phi^2_{(0)} \end{pmatrix}. \tag{83}$$

It is straightforward to confirm the equivalence relation (54) for Killing vectors ξ_T and ξ_R. One may note that the equivalence relation holds for the metric and matter parts separately.

Now, we present some known black hole solutions which belong to this class:

(i) Banados-Teitelboim-Zanelli black hole solutions [43,44]:

$$ds^2 = -\frac{(r^2 - r_-^2)(r^2 - r_+^2)}{r^2}dt^2 + \frac{r^2}{(r^2 - r_-^2)(r^2 - r_+^2)}dr^2 + r^2\left(d\theta - \frac{r_- r_+}{r^2}dt\right)^2. \tag{84}$$

These are solutions in pure gravity with a cosmological constant or solutions without scalar hair, $\phi_{(0)} = 0$. After transforming to FG coordinates, one can read off

$$C_1 = \frac{r_-^2 + r_+^2}{2}, \qquad C_2 = r_- r_+, \tag{85}$$

which reproduce the well-known expressions of the total mass and angular momentum of Banados-Teitelboim-Zanelli black holes,

$$M = \frac{r_-^2 + r_+^2}{8G}, \qquad J = \frac{r_- r_+}{4G}. \tag{86}$$

(ii) The extremal rotating black holes with scalar hair [45–47]:

$$ds^2 = r^2\left[-1 + \frac{\mu_0}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right]dt^2 + \frac{1}{r^2}\left[1 + \frac{\mu_0 - \frac{1}{2}\phi^2_{(0)}}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right]dr^2 + r^2\left[d\theta - \left(\frac{\mu_0}{2r^2} + \mathcal{O}\left(\frac{1}{r^3}\right)\right)dt\right]^2, \tag{87}$$

$$\phi(r) = \frac{\phi_{(0)}}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{88}$$

These are solutions corresponding to the case $C_1 = C_2 = \frac{\mu_0}{2}$. The total mass and angular momentum of these black holes are computed as

$$M = J = \frac{\mu_0}{8G}, \tag{89}$$

which satisfy the extremality condition.

2. Class II: $-1 < m^2 < 0$

In this class, we apply our formalism to those solutions given in Refs. [39,48]. The scalar potential with a cosmological constant is taken as

$$V(\phi) - 2 = -2\left[\cosh^6\left(\frac{\phi}{4}\right) + \nu \sinh^6\left(\frac{\phi}{4}\right)\right]. \tag{90}$$

The radial expansion, Eq. (63), of the scalar field in FG coordinates becomes

$$\phi = e^{-\frac{1}{2}\eta}\left(\phi_{(0)} + \frac{1}{48}\phi_{(0)}^3 e^{-\eta} + \cdots\right), \tag{91}$$

while the coefficients in the radial expansion of the metric solution up to the $e^{-2\eta}$ order are given by

$$\gamma_{ij}^{(1)} = -\frac{1}{4}\phi_{(0)}^2\eta_{ij}, \qquad \gamma_{ij}^{(2)} = \frac{3}{128}\phi_{(0)}^4\left[\eta_{ij} + \frac{(1+\nu)}{4}\delta_{ij}\right]. \tag{92}$$

The ADT potentials for the timelike Killing vector ξ_T^i are computed as

$$\sqrt{-g}Q_{\rm ADT}^{\eta i}(\xi_T;\delta g)|_{\eta\to\infty} = \left[-\frac{1}{4}e^{\eta}\phi_{(0)}\delta\phi_{(0)} + \frac{1}{32}\phi_{(0)}^3\delta\phi_{(0)} + \frac{3(1+\nu)}{128}\phi_{(0)}^3\delta\phi_{(0)}\right]\xi_T^i, \tag{93}$$

$$\sqrt{-g}Q_{\rm ADT}^{\eta i}(\xi_T;\delta\phi)|_{\eta\to\infty} = \left[\frac{1}{4}e^{\eta}\phi_{(0)}\delta\phi_{(0)} - \frac{1}{32}\phi_{(0)}^3\delta\phi_{(0)}\right]\xi_T^i. \tag{94}$$

By using Eq. (19), we obtain the total mass of black holes:

$$M_{\rm ADT} \equiv M_{\rm ADT}^g + M_{\rm ADT}^\phi = \frac{1}{4G}\frac{3(1+\nu)}{512}\phi_{(0)}^4. \tag{95}$$

The ADT potentials for the rotational Killing vector ξ_R^i become

$$\sqrt{-g}Q_{\rm ADT}^{\eta i}(\xi_R;\delta g)|_{\eta\to\infty} = \left[-\frac{1}{4}e^{\eta}\phi_{(0)}\delta\phi_{(0)} + \frac{1}{32}\phi_{(0)}^3\delta\phi_{(0)} - \frac{3(1+\nu)}{128}\phi_{(0)}^3\delta\phi_{(0)}\right]\xi_R^i, \tag{96}$$

$$\sqrt{-g}Q_{\rm ADT}^{\eta i}(\xi_R;\delta\phi)|_{\eta\to\infty} = \left[\frac{1}{4}e^{\eta}\phi_{(0)}\delta\phi_{(0)} - \frac{1}{32}\phi_{(0)}^3\delta\phi_{(0)}\right]\xi_R^i. \tag{97}$$

Therefore, it turns out that the total angular momentum vanishes,

$$J_{\rm ADT} \equiv J_{\rm ADT}^g + J_{\rm ADT}^\phi = 0. \tag{98}$$

Now, we turn to the boundary formalism. In this case, we choose counterterms of the scalar field as

$$\Phi_{ct} = -\frac{1}{4}\phi^2 - \frac{1}{96}\phi^4. \tag{99}$$

By using this form of counterterms, one can see that

$$\sqrt{-\gamma}(T_G)^i{}_j = \left[\frac{1}{8}e^{\eta}\phi_{(0)}^2 - \frac{3}{128}\phi_{(0)}^4 + \frac{3(1+\nu)}{512}\phi_{(0)}^4\right]\delta_j^i - \frac{3(1+\nu)}{256}\phi_{(0)}^4\delta^{it}\delta_{jt}, \tag{100}$$

$$\sqrt{-\gamma}(T_\phi)^i{}_j = -\left[\frac{1}{8}e^{\eta}\phi_{(0)}^2 - \frac{3}{128}\phi_{(0)}^4\right]\delta_j^i, \tag{101}$$

$$\sqrt{-\gamma}\Pi_\phi = 0. \tag{102}$$

Once again, it is straightforward to confirm the equivalence relation (54) for Killing vectors ξ_T and ξ_R. As a result, the identical expression for the mass and angular momentum can be obtained through the boundary stress tensor method as well. Furthermore, one can see that each leading divergent term in $Q_{\rm ADT}(\delta g)$ and $Q_{\rm ADT}(\delta\phi)$ matches with the corresponding one in $\delta(\sqrt{-\gamma}T_G)$ and $\delta(\sqrt{-\gamma}T_\phi)$, respectively. It is amusing to note that each ADT potential $Q_{\rm ADT}^{\eta i}(\xi)$ is proportional to the corresponding Killing vector ξ, which is not clear *a priori* from the bulk formalism. This seems natural from the equivalence relation since the boundary stress tensor $(T_B)_j^i$ for the static black holes becomes diagonal.

B. General d-dimensional static black holes

In general d dimensions, we focus on planar static black holes with scalar hair in class I. The relevant coefficient in the radial expansion of the metric is given by

$$\gamma^{(d)}_{ij} = \left[C - \frac{1}{4(d-1)}\phi^2_{(0)}\right]\eta_{ij} + dC\delta_{it}\delta_{jt}, \quad (103)$$

where C is an arbitrary constant. By using the expression of the quasilocal ADT potential given in Eq. (30), one can see that

$$\sqrt{-g}Q^{\eta t}_{\rm ADT}(\xi_T;\delta g)|_{\eta\to\infty} = -\frac{d}{4}\phi_{(0)}\delta\phi_{(0)} + \frac{d(d-1)}{2}\delta C, \quad (104)$$

$$\sqrt{-g}Q^{\eta t}_{\rm ADT}(\xi_T;\delta\phi)|_{\eta\to\infty} = \frac{d}{4}\phi_{(0)}\delta\phi_{(0)}. \quad (105)$$

The full expression of counterterms for the metric field in general d dimensions is not known explicitly even in Einstein gravity. Yet one may still ignore their contributions to the boundary stress tensor except for the boundary cosmological constant if the boundary metric is taken flat, $\gamma^{(0)}_{ij} = \eta_{ij}$. With this assumption, the boundary stress tensor is given by

$$\sqrt{-\gamma}(T_g)^{tj}\xi^T_j = \frac{d}{8}\phi^2_{(0)} - \frac{d(d-1)}{2}C, \quad (106)$$

$$\sqrt{-\gamma}(T_\phi)^{tj}\xi^T_j = -\frac{d}{8}\phi^2_{(0)}, \quad (107)$$

$$\sqrt{-\gamma}\Pi_\phi = 0. \quad (108)$$

Once again, we confirm our general results given in Eq. (54).

The total mass of these black holes is obtained as

$$M = M^g + M^\phi = \frac{d(d-1)}{16\pi G}V_{d-1}C, \quad (109)$$

where V_{d-1} denotes the volume of the $(d-1)$-dimensional planar space. In class II, it is straightforward to apply our formalism to the known analytic solutions, for instance, those given in Ref. [49].

VI. CONCLUSION

In this paper, we have constructed a quasilocal formalism for conserved charges in a general theory of gravity with diffeomorphism symmetry in the presence of arbitrary matter fields. This construction can be regarded as the full- fledged extension of the covariant formalism developed by Abott, Deser, and Tekin, which depends on the Euler–Lagrange expressions only. While the original ADT formulation incorporates the metric fields only at the asymptotic infinity, our construction incorporates the contribution of slow falloff matter fields and can be applied even in the interior region in the sense of quasilocal conserved charges.

We have shown that our formalism or the full-fledged extension of the ADT formalism at the quasilocal level gives us completely identical results on potentials as those from the covariant phase space approach. In fact, the equivalence of potentials in both formalisms is proven at the off-shell level. Technically, we have adopted a one-parameter path in the solution space in order to obtain finite conserved charges from the off-shell expression.

For the asymptotically (locally) AdS space, we have also introduced identically conserved boundary currents in the same spirit as in the bulk case and obtained the corresponding conserved charges. We have shown that these charges have the same expression as those from the conventional holographic approach known as the boundary stress tensor method. Furthermore, we have proved that the bulk formalism on conserved charges leads to the same results as the boundary one by showing that the bulk off-shell ADT potential reduces to the boundary current when we approach the asymptotic infinity. In all, we have shown that our quasilocal formalism can be matched completely with the previously well-known methods. As a byproduct of these matchings, we have verified in a general theory of gravity that conserved charges by the covariant phase approach should be identical with those by the holographic method. This result can be regarded as the extension of the proof on the equivalence of conserved charges in Einstein gravity from the covariant phase space formalism and those from the boundary stress tensor method.

As an application of our formalism, we have considered some examples in order to show some details in our formalism concretely. The necessity of the matter contribution to conserved charges is manifest in these examples. Through the linear analysis, some additional features on matchings between the quasilocal ADT potential and the boundary stress tensor have been explained.

Our matchings among various approaches to conserved charges clarify some equivocal aspects in each formulation on conserved charges. For instance, the consistency of conserved charges with the first law of black hole thermodynamics is not so manifest in the holographic approach, while the finiteness of the ADT potential for the asymptotically AdS geometry is not manifest in the ADT formalism. On the other hand, the consistency of conserved charges with the first law of the black hole thermodynamics is usually taken as the property in the covariant phase space, and the finiteness of conserved charges is manifest, by construction, in the holographic approach. All such equivocal aspects disappear since conserved charges are matched through our construction.

One may note that the second term in Eq. (10) plays essential roles to define conserved charge consistent with known results. The analogous term in the boundary formalism is the second one in Eq. (37), which is not revealed in literature on the boundary stress tensor formalism. By presuming that the conformal anomaly is invariant along the path, we argue that there is no contribution from the second term in Eq. (37), which corresponds to the known results. Indeed, there is no contribution from the second term in all the examples we have presented in this paper. It is amusing to speculate the case in which the conformal anomaly is not invariant along the path in the solution space. In that case, the second term in Eq. (37) would be essential, and our expression of holographic conserved charges would be an improvement over the known one.

We would like to give some comments on the further extension of our formalism. As mentioned in the previous sections, our bulk quasilocal construction can be applied even to the case in which a bulk Lagrangian contains nonmanifestly covariant terms like gravitational Chern–Simons terms. Though explicit steps are not presented in the presence of nonmanifestly covariant terms in the bulk Lagrangian, it would be straightforward to match our final expressions with those in the covariant phase space approach by modifying it to accommodate such terms [50–53]. The equivalence with holographic methods would also hold in the presence of such terms. Though the equivalence between conserved charges from the bulk and boundary formalisms is shown by adopting FG coordinates, it is expected to hold in other coordinates. It would be interesting to prove this in general. In this paper, we have focused on exact Killing vectors. It would also be straightforward to extend our construction to asymptotic Killing vectors by following steps worked out in Refs. [54]. It would be an interesting direction to extend our equivalence between the bulk and boundary constructions to geometries that are not asymptotically (locally) AdS space.

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APPENDIX A: DERIVATION OF THE OFF-SHELL IDENTITY

To obtain the off-shell identity given in Eq. (3), let us note that the diffeomorphism variation $\delta_\zeta\Psi = \pounds_\zeta\Psi$ leads to

$$\begin{aligned}\delta_\zeta(\sqrt{-g}\mathcal{L}) &= \sqrt{-g}[-\mathcal{E}^{\mu\nu}\pounds_\zeta g_{\mu\nu} + \mathcal{E}_\psi\pounds_\zeta\psi] + \partial_\mu\Theta^\mu(\pounds_\zeta\Psi)\\ &= \sqrt{-g}[2\zeta_\nu\nabla_\mu\mathcal{E}^{\mu\nu} + \mathcal{E}_\psi\pounds_\zeta\psi]\\ &\quad + \partial_\mu(\Theta^\mu(\pounds_\zeta\Psi) - 2\sqrt{-g}\mathcal{E}^{\mu\nu}\zeta_\nu),\end{aligned}\tag{A1}$$

where we have used $\pounds_\zeta g_{\mu\nu} = 2\nabla_{(\mu}\zeta_{\nu)}$ and performed the integration by parts on the first term. Alternatively, since the diffeomorphism is the symmetry of the given action, the diffeomorphism variation of the Lagrangian can be written as a total derivative in the form of

$$\delta_\zeta(\sqrt{-g}\mathcal{L}) = \partial_\mu(\zeta^\mu\sqrt{-g}\mathcal{L} + \Sigma^\mu(\zeta)),\tag{A2}$$

where Σ^μ denotes an additional surface term that exists for nonmanifestly covariant terms like gravitational Chern–Simons terms. By equating the above two forms of diffeomorphism variation, one can see that

$$\begin{aligned}&\sqrt{-g}[2\zeta_\nu\nabla_\mu\mathcal{E}^{\mu\nu} + \mathcal{E}_\psi\pounds_\zeta\psi]\\ &\quad = \partial_\mu(\zeta^\mu\sqrt{-g}\mathcal{L} + \Sigma^\mu(\zeta) - \Theta^\mu(\pounds_\zeta\Psi) + 2\sqrt{-g}\mathcal{E}^{\mu\nu}\zeta_\nu).\end{aligned}\tag{A3}$$

Since the left-hand side of Eq. (A3) is composed only of ζ and $\nabla\zeta$ terms for an arbitrary function ζ, one can deduce that the right-hand side should be taken in the form of

$$\begin{aligned}\text{r.h.s} &= \sqrt{-g}\nabla_\mu(\mathcal{Y}^{\mu\nu}\zeta_\nu + \mathcal{Y}^{[\mu\nu]\rho}\nabla_\nu\zeta_\rho)\\ &= \sqrt{-g}\nabla_\mu(\mathcal{Y}^{\mu\nu}\zeta_\nu - \nabla_\nu\mathcal{Y}^{[\mu\nu]\rho}\zeta_\rho),\end{aligned}$$

where we have used $\nabla_\mu\nabla_\nu(\mathcal{Y}^{[\mu\nu]\rho}\zeta_\rho) = 0$. As a result, the off-shell identity follows.

APPENDIX B: FORMULAS FOR THE CONSERVATION OF CURRENTS

In this Appendix, we show some formulas that are used for the derivation of the conservation of off-shell currents. One may note that the generic double variations of the bulk action can be written as

$$\delta_2\delta_1 I[\Psi] = \frac{1}{16\pi G}\int d^D x[\delta_2(\sqrt{-g}\mathcal{E}_\Psi\delta_1\Psi) + \partial_\mu\delta_2\Theta^\mu(\delta_1\Psi)].\tag{B1}$$

By using the fact that the antisymmetrization of double variations of the action vanish, $(\delta_1\delta_2 - \delta_2\delta_1)I = 0$, and taking one of the variations as a diffeomorphism variation, one can see that

$$\begin{aligned}0 = \frac{1}{16\pi G}\int d^D x[&\delta_\zeta(\sqrt{-g}\mathcal{E}_\Psi\delta\Psi) - \delta(\sqrt{-g}\mathcal{E}_\Psi\delta_\zeta\Psi)\\ &- \partial_\mu\omega^\mu(\delta\Psi, \delta_\zeta\Psi)].\end{aligned}\tag{B2}$$

Since $\delta_\xi\Psi = 0$ and $\omega^\mu(\delta\Psi, \delta_\xi\Psi) = 0$ for a Killing vector, ξ, it is straightforward to obtain the following formula:

$$\delta_\xi(\sqrt{-g}\mathcal{E}_\Psi\delta\Psi) = \partial_\mu(\xi^\mu\sqrt{-g}\mathcal{E}_\Psi\delta\Psi) = 0. \qquad \text{(B3)}$$

Combining this formula with Eq. (8), one can check the identical conservation of $\mathcal{J}^\mu_{\rm ADT}$.

By applying the same argument to the on-shell renormalized action given in Eq. (33), one can obtain

$$\partial_i[\xi^i_B\sqrt{-\gamma}(T^{kl}_B\delta\gamma_{kl} + \Pi_\psi\delta\psi)] = 0, \qquad \text{(B4)}$$

which is used to show the identical conservation of the boundary current $\mathcal{J}^i_B$.

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