Asymptotic behavior of the Weyl tensor in higher dimensions

Marcello Ortaggio* and Alena Pravdová†
Institute of Mathematics, Academy of Sciences of the Czech Republic,
Žitná 25, 115 67 Prague 1, Czech Republic
(Received 2 August 2014; published 12 November 2014)

We determine the leading order falloff behavior of the Weyl tensor in higher-dimensional Einstein spacetimes (with and without a cosmological constant) as one approaches infinity along a congruence of null geodesics. The null congruence is assumed to "expand" in all directions near infinity (but it is otherwise generic), which includes in particular asymptotically flat spacetimes. In contrast to the well-known four-dimensional peeling property, the falloff rate of various Weyl components depends substantially on the chosen boundary conditions and is also influenced by the presence of a cosmological constant. The leading component is always algebraically special, but in various cases, it can be of type N, III, or II.

DOI: 10.1103/PhysRevD.90.104011 PACS numbers: 04.50.-h, 04.20.Ha, 04.20.-q

I. INTRODUCTION

The study of isolated systems in general relativity is based on the analysis of asymptotic properties of spacetimes. Under certain assumptions, this enables one to define physical quantities such as mass, angular momentum, and energy flux. In particular, properties of gravitational radiation can be determined by considering the spacetime behavior "far away" along a geodesic null congruence.

In four dimensions, the Weyl tensor decay is described by the well-known peeling property, i.e., components of boost weight (b.w.) w fall off as $1/r^{w+3}$ (where $w = \pm 2, \pm 1, 0$, and the $1/r$ term characterizes radiative fields). This result was obtained by coordinate-based approaches that studied Einstein's vacuum equations assuming suitable asymptotic "outgoing radiation" conditions, which were formulated in terms either of the metric coefficients [1,2] or directly of the Weyl tensor [3,4] (see Refs. [5–7] for early results in special cases). From a more geometrical viewpoint, the peeling-off behavior also naturally follows from Penrose's conformal definition of asymptotically simple spacetimes (which also allows for a cosmological constant) [8,9], at least under suitable smoothness conditions on the conformal geometry (see also Ref. [10]).

In an n-dimensional spacetime, the definition of asymptotic flatness at null infinity (along with the "news" tensor and Bondi energy-momentum) using a conformal method turns out to be sound only for even n [11] (see also Ref. [12])—linear gravitational perturbation of the metric tensor typically decays as $r^{-(n/2-1)}$, and the unphysical (conformal) metric is thus not smooth at null infinity if n is odd (see Ref. [13] for further results for even n). In Ref. [14], linear (vacuum) perturbations of Minkowski spacetime were studied in terms of the Weyl tensor, which was found to decay as $r^{-(n/2-1)}$, thus again nonsmoothly in odd dimensions.[1] Reference [14] also pointed out a qualitative difference between $n = 4$ and $n > 4$ in the decay properties of various Weyl components at null infinity and related this to a possible new peeling behavior when $n > 4$. This expectation was indeed confirmed in the full theory in Ref. [15] by studying the Bondi-like metric defined in Refs. [16,17] (also mentioned in Refs. [11,12]) and thus an expansion of the Weyl tensor along the generators of a family of outgoing null hypersurfaces. Not only was the $r^{-(n/2-1)}$ result of Ref. [14] recovered at the leading order, but at higher orders, a new structure of the r-dependence of various Weyl components was also obtained [15]. For odd n, an extra condition on the asymptotic metric coefficients was needed in Ref. [15] (see also Ref. [16]), in relation to the simultaneous appearance of integer and semi-integer powers in the expansions. (Note that the analysis of Ref. [15] includes not only vacuum spacetimes but also possible matter fields that decay "fast enough" at infinity; cf. Ref. [15] for details.)

The present contribution studies the asymptotic behavior of the Weyl tensor in higher-dimensional *Einstein spacetimes* ($R_{ab} = \frac{R}{n} g_{ab}$) under more general boundary conditions, for which a different method seems to be more suitable. The basic idea is still to evaluate the Weyl components in a frame parallelly transported along a congruence of "outgoing" null geodesics, affinely parametrized by r (the congruence is rather "generic" and not assumed to be hypersurface orthogonal—its precise properties will be specified in Sec. II A below). However, on the lines of the classic four-dimensional (4D) work [3], we do not make assumptions on the spacetime metric but work directly with the Weyl tensor, in the framework of the

[1]In the present paper, we discuss the *physical* Weyl tensor only, so here we have accordingly rephrased the results of Ref. [14] (where the *unphysical* Weyl tensor of the conformal spacetime was instead considered).

*ortaggio@math.cas.cz
†pravdova@math.cas.cz

higher-dimensional Newman–Penrose (NP) formalism [18–22] (we follow the notation of the review [22], and we do not repeat here the definitions of all the symbols). This permits a unified study for both even and odd dimensions, and with little extra effort, it also allows for a possible cosmological constant. In the case of asymptotically flat spacetimes, the Bianchi equations naturally give the "$r^{-(n/2-1)}$ result" for the leading Weyl components [see Eq. (2) below], as previously obtained with the methods of Refs. [14,15]. In addition to this special case, a complete pattern of possible falloff behaviors both with (Secs. III A 4, III B, and III C) and without (Secs. IV A 4, IV B, and IV C) a cosmological constant is presented. The precise falloff for a specific spacetime will be determined by a choice of "boundary condition" at null infinity. These are naturally specified by first fixing a bound on the decay rate of b.w. +2 Weyl components Ω_{ij} (which we will assume to be faster than $1/r^2$), as in four dimensions. However, while in four dimensions only the falloff $\Omega_{ij} = O(r^{-5})$ needs to be assumed (and then the standard peeling result follows [3]),[2] for $n > 4$, the r-dependence of the remaining Weyl components will still be partially undetermined, and various possible choices of boundary conditions for lower b.w. components will lead to different falloff behaviors. More specifically, how such numerous cases (and subcases) arise can be better understood by observing that the Weyl components containing arbitrary integration "constants" are Ψ_{ijk} (at order $1/r^n$ or $1/r^3$) and, for $n > 5$, Φ_{ijkl} (at order $1/r^2$). This will be worked out in the paper.[3]

Certain cases of physical interest [including asymptotically (anti-)de Sitter and asymptotically flat spacetimes] arise when we set to zero the terms of order $1/r^3$ in Ψ_{ijk} and $1/r^2$ in Φ_{ijkl}. For $R \neq 0$, we then obtain that necessarily $\Omega_{ij} = O(r^{-1-n})$ (or faster), and the falloff generically is [see (67)]

$$\begin{aligned} \Omega_{ij} &= O(r^{-1-n}),\\ \Psi_{ijk} &= O(r^{-n}),\\ \Phi_{ijkl} &= O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}) \quad (R \neq 0),\\ \Psi'_{ijk} &= O(r^{2-n}),\\ \Omega'_{ij} &= O(r^{3-n}), \end{aligned} \tag{1}$$

where components are ordered by decreasing b.w. Under the same assumptions, more possibilities arise for a vanishing cosmological constant, depending more substantially on the precise falloff prescribed for Ω_{ij}. In particular, if Ω_{ij} falls faster than $1/r^{n/2}$ but not faster than $1/r^{n/2+1}$, we have [cf. (94) and the discussion after it]

$$\begin{aligned} \Omega_{ij} &= O(r^{-\nu}) \qquad \left(\frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right),\\ \Psi_{ijk} &= O(r^{-\nu}),\\ \Phi_{ijkl} &= O(r^{-n/2}), \quad \Phi = O(r^{-\nu}), \quad \Phi^A_{ij} = O(r^{-\nu}) \quad (R = 0),\\ \Psi'_{ijk} &= O(r^{-n/2}),\\ \Omega'_{ij} &= O(r^{1-n/2}). \end{aligned} \tag{2}$$

This includes the behavior found in Ref. [15] for asymptotically flat radiative spacetimes. The radiative term $O(r^{1-n/2})$ in Ω'_{ij} vanishes if $\nu > 1 + \frac{n}{2}$, in which case the falloff is completely different [e.g., it is given by (105) for $\nu > n$, but other cases are also possible; see Sec. IV for details]. On the other hand, if Ω_{ij} falls as $1/r^{n/2}$ or slower, one finds instead the behavior (99) (with $\nu > 3$). Both (1) and (2) are qualitatively different from the corresponding results (69) and (107) for the four-dimensional case [apart from Φ^A_{ij}, (1) with $n = 4$ would look the same as (69), but see comments in the following sections].

More general asymptotia can also be of physical interest, and the corresponding falloff properties are given in the paper. Let us just mention here, for example, that a nonzero term of order $1/r^2$ in Φ_{ijkl} may correspond, e.g., to black holes living in generic Einstein spacetimes (this is manifest in the case of static black holes from the Weyl r-dependence given in Ref. [24]). Although here we restrict to Einstein spacetimes, several results can presumably be easily extended to include matter fields that fall off "sufficiently" fast (cf. Ref. [15]). The method employed here can also be similarly applied to more general contexts such as the coupled Einstein–Maxwell equations, which we leave for future work. We further note that previous results concerning the (exact) r-dependence of the Weyl tensor for algebraically special Einstein spacetimes include Refs. [24–29].

A. Invariance of the results

Once a null direction $\boldsymbol{\ell}$ is chosen, the presented results hold in a generic parallelly transported frame. One may thus wonder if the behavior we find is frame dependent. Similarly as in four dimensions, the answer follows from transformation properties of various Weyl components under null rotations about $\boldsymbol{\ell}$, i.e.,

[2]The Ω_{ij} components of the n-dimensional notation correspond to the NP scalar Ψ_0 in four dimensions.

[3]To be precise, by "arbitrary integration constants," we refer to r-independent quantities that generically may still depend on coordinates different from r. Additionally, (some of) these may be "arbitrary" only at the level of the r-integration of the (asymptotic) NP equations—the remaining "transverse" NP equations would in fact play a role of "constraint equations." This is of course important for a full analysis of the characteristic initial value problem, but it goes beyond the scope of this paper and will not be discussed in the following (for details in four dimensions, see Ref. [4] and, e.g., the review [23]).

$$\hat{\boldsymbol{\ell}} = \boldsymbol{\ell}, \qquad \hat{\boldsymbol{n}} = \boldsymbol{n} + z_i \boldsymbol{m}_i - \frac{1}{2} z_i z^i \boldsymbol{\ell}, \qquad \hat{\boldsymbol{m}}_i = \boldsymbol{m}_i - z_i \boldsymbol{\ell}. \tag{3}$$

Two different parallelly transported frames are related by a transformation (3) (apart from trivial spatial rotations) with the parameters z_i being r independent [20]. Under (3), the change of a Weyl component of a given b.w. w is simply a term linear in components of b.w. *smaller* than w, with coefficients determined by the z_i (see, e.g., Eqs. (2.27)–(2.35) of Ref. [21]). It thus follows, in particular, that at the leading order (when $r \to \infty$) a certain Weyl component will be unchanged *if* all Weyl components of lower boost weight decay faster. This is always the case, for instance, for the b.w. -2 components Ω'_{ij} when the leading-order term is of type N. Therefore, this observation will apply to several of the results of this paper, most notably to the radiative behavior (2) [or (94)], in which case the leading Weyl component can be related to the Bondi flux [15]. By contrast, when leading-order terms are not invariant in the sense just discussed, a transformation (3) can be used to pick up preferred frames, which may simplify certain expressions and be useful for particular applications (see, e.g., Refs. [25,30] in the case of algebraically special spacetimes). This freedom will not be used here since we are interested in the asymptotic behavior in a generic parallelly transported frame.

B. Assumptions and notation

In this paper, we are interested in determining the *leading*-order r-dependence of the Weyl tensor of Einstein spacetimes, while a systematic study of subleading terms and the analysis of asymptotic solutions of the NP equations is left for future work (several results have been already obtained in the case of algebraically special spacetimes [30]). For this reason, we will not need to assume that the NP quantities (Weyl tensor, Ricci rotation coefficients, and derivative operators) admit a series expansion. However, we will assume that for large r the leading terms of those quantities have a powerlike behavior [so that for our purposes the notation $f = O(r^{-\zeta})$ will effectively mean $f \sim r^{-\zeta}$], where the powers will not be restricted to be integer numbers. We will also assume that if $f = O(r^{-\zeta})$ then $\partial_r f = O(r^{-\zeta-1})$ and $\partial_A f = O(r^{-\zeta})$ (where ∂_A denote a derivative with respect to coordinates x^A different from r and that need not be further specified for our purposes). In a few cases it will be useful to consider subleading terms of some expressions [most importantly (10)], and it will be understood that those are also assumed to be powerlike.

Although we are not interested in giving the full set of asymptotic field equations, in some cases it will be useful to display relations among the leading terms of certain Weyl components. For a generic frame Weyl component "f" we thus define the notation

$$f = \frac{f^{(\zeta)}}{r^{\zeta}} + o(r^{-\zeta}), \tag{4}$$

where $f^{(\zeta)}$ does not depend on r [so that we will have, e.g., $\Phi^S_{ij} = \Phi^{S(n-1)}_{ij} r^{1-n} + o(r^{1-n})$, or $\Psi_{ijk} = \Psi^{(3)}_{ijk} r^{-3} + o(r^{-3})$, etc.]. For the Ricci rotation coefficients, we will instead denote r-independent quantities by lowercase latin letters, e.g., $L_{1i} = l_{1i} r^{-1} + o(r^{-1})$, $\overset{i}{M}_{j1} = \overset{i}{m}_{j1} + o(1)$, etc.

Many of the equations will take a more compact form using the rescaled Ricci scalar

$$\tilde{R} = \frac{R}{n(n-1)}. \tag{5}$$

We will be interested in the asymptotic behavior along a geodesic null congruence with an affine parameter r and tangent vector field $\boldsymbol{\ell}$. Calculations will be performed in a frame $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i)$ (with $i, j, k, \ldots = 2, \ldots, n-1$), which is parallelly transported along $\boldsymbol{\ell}$. The above assumptions imply the vanishing of the following Ricci rotation coefficients (cf. Ref. [22] for more details on the notation):

$$\kappa_i = 0 = L_{10}, \qquad \overset{i}{M}_{j0} = 0, \qquad N_{i0} = 0. \tag{6}$$

Directional derivatives along the frame vectors $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i)$ will be denoted, respectively, by D, δ_i, and Δ.

Section II A and the first parts of Secs. III A and IV A are devoted to results on the Ricci rotation coefficients, to preliminary analysis of the Weyl tensor, and to setting up the method. Readers not interested in those details can jump to the summary of the results for the Weyl tensor in Secs. III A 4, III B, and III C ($\tilde{R} \neq 0$) and IV A 4, II B, and II C ($\tilde{R} = 0$). For comparison, four-dimensional results are also reproduced in the various cases and given in (62) and (69) ($\tilde{R} \neq 0$) and (98), (100), and (107) ($\tilde{R} = 0$).

II. BOUNDARY CONDITIONS AND RICCI ROTATION COEFFICIENTS

In this section, we explain our assumptions on the asymptotic behavior of $\boldsymbol{\ell}$ and of the Weyl tensor components of b.w. $+2$ and use those to fix the leading-order behavior of the Ricci rotation coefficients and derivative operators (both for $\tilde{R} \neq 0$ and $\tilde{R} = 0$). It will also follow that subsequent analysis will need to consider three different choices of boundary conditions on the Weyl components of b.w. $+1$, which we will do in later sections.

A. Sachs equation and optical matrix

In the frame $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i)$ (see above), the optical matrix of $\boldsymbol{\ell} = \partial_r$ is given by

$$\rho_{ij} = \ell_{a;b} m^a_{(i)} m^b_{(j)}. \tag{7}$$

From now on, we assume that ρ_{ij} is *asymptotically nonsingular and expanding*; i.e., the leading term of ρ_{ij} (for large r) is a matrix with nonzero determinant and nonzero trace. Roughly speaking, this means that near infinity $\boldsymbol{\ell}$ expands in all spacelike directions at the same speed, which is compatible, in particular, with asymptotically flat spacetimes (as follows from Refs. [15–17]—however, we will see in the following that these assumptions hold also in more general spacetimes).

Next, one needs to specify the speed at which the Weyl tensor tends to zero for $r \to \infty$. In general, we will make only the following rather weak assumption for the falloff for the b.w. +2 components of the Weyl tensor,

$$\Omega_{ij} = O(r^{-\nu}), \quad \nu > 2, \tag{8}$$

although, in most cases of interest, ν will in fact be larger, as we will show (recall that in four dimensions the existence of a smooth null infinity requires $\nu \geq 5$ [3,8–10]).

With the assumptions listed above, the Sachs equation reads $D\rho_{ij} = -\rho_{ik}\rho_{kj} - \Omega_{ij}$ (cf. (11g), Ref. [20]), from which one finds[4]

$$\rho_{ij} = \frac{\delta_{ij}}{r} + o(r^{-1}). \tag{9}$$

In general, it is easy to see from (11g) in Ref. [20] that Ω_{ij} *will affect* ρ_{ij} *at order* $O(r^{-\nu+1})$. At all lower orders, the r-dependence of ρ_{ij} is given by negative integer powers of r, which can be fixed recursively as done (to arbitrary order) in Ref. [30]. Thus, for example, if $\nu > 3$ (which will indeed occur in several cases discussed in the following), one has

$$\rho_{ij} = \frac{\delta_{ij}}{r} + \frac{b_{ij}}{r^2} + o(r^{-2}) \quad (\nu > 3), \tag{10}$$

where the subleading term contains an arbitrary "integration matrix" b_{ij} independent of r. Note that when $\boldsymbol{\ell}$ is twistfree then $b_{[ij]} = 0$ (the reverse is also true if $\boldsymbol{\ell}$ is a Weyl aligned null direction (WAND) [27]).

Since we have now outlined all our assumptions (see also Sec. I), for readers' convenience, let us summarize those before proceeding: (i) the spacetimes in question are Einstein (possibly, Ricci flat); (ii) $\boldsymbol{\ell} = \partial_r$ is a vector field tangent to a congruence of null geodesics, affinely parametrized by r; (iii) a frame $(\boldsymbol{\ell}, \boldsymbol{n}, \boldsymbol{m}_i)$ parallelly transported along $\boldsymbol{\ell}$ is employed [so that (6) holds]; (iv) the optical matrix of $\boldsymbol{\ell}$ is asymptically nonsingular and expanding [as defined by (7) and the following comments]; (v) near infinity (i.e., $r \to \infty$), the frame components of the Weyl tensor, of the Ricci rotations coefficients, and of the derivative operators admit a powerlike behavior at the leading order (in very few cases also at the subleading order, as explained in the text); (vi) the b.w. +2 components of the Weyl tensor fall off as $\Omega_{ij} = O(r^{-\nu})$, with $\nu > 2$ [Eq. (8)]. More specific possible choices of values (or a range of values) of ν will determine various falloff patterns of the remaining Weyl components, as explained in the following sections and summarized in final Tables I and II. We further observe that (again depending on ν) in certain cases it will later be necessary also to specify the falloff of the b.w. +1 components Ψ_{ijk} (see Sec. II C below) and the b.w. 0 components Φ_{ijkl}—all possible cases will be considered, and again we refer to Tables I and II for a summary of those.

[4]Another solution is $\rho_{ij} = O(r^{-\nu+1})$ (for $\nu > 2$), which, however, gives an asymptotically nonexpanding optical matrix (since Ω_{ij} is traceless), contrary to our assumptions.

B. Derivative operators and commutators

Taking r as one of the coordinates, we can write

$$D = \partial_r, \qquad \Delta = U\partial_r + X^A\partial_A, \qquad \delta_i = \omega_i\partial_r + \xi_i^A\partial_A, \tag{11}$$

where $\partial_A = \partial/\partial x^A$ and the x^A represent any set of $(n-1)$ scalar functions such that (r, x^A) is a well-behaved coordinate system (at least locally near infinity, which suffices for our purposes). From the commutators [19]

$$\Delta D - D\Delta = L_{11}D + L_{i1}\delta_i, \tag{12}$$

$$\delta_i D - D\delta_i = L_{1i}D + \rho_{ji}\delta_j, \tag{13}$$

we obtain the differential equations (cf. also Ref. [30])

$$D\omega_i = -L_{1i} - \rho_{ji}\omega_j, \tag{14}$$

$$D\xi_i^A = -\rho_{ji}\xi_j^A, \tag{15}$$

$$DU = -L_{11} - L_{i1}\omega_i, \tag{16}$$

$$DX^A = -L_{i1}\xi_i^A. \tag{17}$$

Using (9), Eq. (15) gives

$$\xi_i^A = O(r^{-1}). \tag{18}$$

Similarly, as mentioned above for ρ_{ij}, Ω_{ij} will affect ξ_i^A at order $O(r^{-\nu+1})$.

To fix the full r-dependence of the derivative operators, we also need to study the behavior of the Ricci rotation coefficients of b.w. 0 and −1. However, the corresponding differential equations will in turn involve also Weyl components of b.w. +1 and 0, respectively, and thus one has to consider the set of the "D"-Ricci identities of b.w. b simultaneously with the "D"-Bianchi identities of b.w. $(b+1)$ (for $b = +1, 0, -1, -2$).

C. Ricci rotation coefficients of b.w. 0 and Weyl components of b.w. +1

We need to study (11b), (11e), and (11n) of Ref. [20]; (B8) of Ref. [18]); and (14) and (17). One starts by assuming a generic behavior for large r for each of the "unknowns" [e.g., $L_{1i} = O(r^\alpha)$, where α need not be specified *a priori*]. By combining conditions coming from all the considered equations, one can constraint such leading terms. For example, from (11b) of Ref. [20], it is easy to see that one can only have either

$$L_{1i} = O(r^{-1}), \qquad \Psi_i = o(r^{-2}) \tag{19}$$

or

$$L_{1i} = O(r^\alpha), \qquad \Psi_i = O(r^{\alpha-1}) \quad (\alpha \neq -1). \tag{20}$$

Working out similar conditions for other quantities from (11n) of Ref. [20], (B8) of Ref. [18], and (14) and requiring compatibility of all such conditions, one concludes that

$$L_{1i} = O(r^{-1}), \quad \overset{i}{M}_{jk} = O(r^{-1}), \quad \omega_i = O(1), \tag{21}$$

where it is understood that for $r \to \infty$ all terms can go to zero faster than indicated, in special cases. However, we will consider only the generic case, in which this does not happen. For the Weyl tensor components of positive b.w., there are three possibilities:

(i) $\Psi_{ijk} = O(r^{-\nu})$, $\Omega_{ij} = O(r^{-\nu})$ $(\nu > 2)$: where $\Psi^{(-\nu)}_{ijk}$ can be expressed in terms of $\Omega^{(-\nu)}_{ij}$ using (B8) of Ref. [18] (except when $\nu = 3, n$). For $\nu > 3$, this case sets the boundary condition $\Psi^{(3)}_{ijk} = 0$, and for $\nu > n$ also $\Psi^{(n)}_{ijk} = 0$. It includes the case in which $\boldsymbol{\ell}$ is a *multiple* WAND (in the formal limit $\nu \to +\infty$) and asymptotically flat radiative spacetimes in higher dimensions (as we will discuss in the following; cf. Ref. [15]).

(ii) $\Psi_{ijk} = O(r^{-n})$, $\Omega_{ij} = o(r^{-n})$: with $(n-3)\Psi^{(n)}_{ijk} = 2\Psi^{(n)}_{[j}\delta_{k]i}$. This case corresponds to the boundary condition $\Psi^{(3)}_{ijk} = 0$, $\Psi^{(n)}_{ijk} \neq 0$. It is compatible with the four-dimensional results of Refs. [3,8,9] (where $\nu = 5$) for $n = 4$.

(iii) $\Psi_{ijk} = O(r^{-3})$, $\Psi_i = o(r^{-3})$, $\Omega_{ij} = O(r^{-\nu})$ $(n > 4, \nu > 3)$: with $\Psi_i = O(r^{-\nu})$ if $3 < \nu \leq 4$ and [using (10)] $\Psi_i = O(r^{-4})$ if $\nu > 4$ (in both cases, the leading term of Ψ_i can be determined by the trace of (B8) in Ref. [18]. This case corresponds to the boundary condition $\Psi^{(3)}_{ijk} \neq 0$. It is not permitted in four dimensions since $\Psi_i = 0 \Leftrightarrow \Psi_{ijk} = 0$ there [22] and cannot be asymptotically flat; cf. Ref. [15].

Only cases ii and iii are permitted if one assumes that asymptotically Ψ_{ijk} goes to zero more slowly than Ω_{ij}.

Furthermore, from (11e) of Ref. [20], we have

$$L_{i1} = O(r^{-1}), \tag{22}$$

which with (17) gives

$$X^A = X^{A0} + O(r^{-1}). \tag{23}$$

When the falloff condition $\nu > 3$ is assumed, thanks to (10), we can strengthen the above results and those of Sec. II B for the derivative operator as follows (assuming that each quantity has a powerlike behavior also at the subleading order):

$$L_{1i} = \frac{l_{1i}}{r} + O(r^{-2}), \qquad L_{i1} = \frac{l_{i1}}{r} + O(r^{-2}),$$
$$\overset{i}{M}_{jk} = \frac{\overset{i}{m}_{jk}}{r} + O(r^{-2}), \tag{24}$$

$$\xi^A_i = \frac{\xi^{A0}_i}{r} + O(r^{-2}), \qquad \omega_i = -l_{1i} + O(r^{-1}) \quad (\nu > 3). \tag{25}$$

This will be useful in the following since many cases of interest have indeed $\nu > 3$. Note that, using null rotations (3), one can always choose a parallelly transported frame such that, e.g., $l_{1i} = 0$ or $l_{i1} = 0$. This may be convenient for particular computations, but for the sake of generality, we will keep our frame unspecified.

At this stage, knowing the r-dependence of the derivative operators at the leading order [Eq. (11) with (18), (21), (23), and (33) or (34)] of course means also knowing the leading-order terms of the spacetime metric (however, to explicitly connect the metric and the Weyl tensor, we would need to study higher-order terms). In the following, we will analyze in detail the above case I (Secs. II D, III A, and IV A). For cases ii and iii, we will only summarize the main results (Secs. III B, III C, IV B, and IV C) without giving intermediate steps since the method to obtain those is essentially the same as for case i.

D. Ricci rotation coefficients of b.w. −1 and Weyl components of b.w. 0: Derivation for case i

The next step consists of the study of (11a), (11j), and (11m) of Ref. [20]; (B5) and (B12) of Ref. [18]; and (16), also using the results of Sec. II C above. It is convenient to start from (11j) from Ref. [20] and (B12) from Ref. [18] (since these do not contain L_{11}, $\overset{i}{M}_{j1}$, and U). Let us first focus on (11j) of Ref. [20] and consider the leading-order behavior of the following quantities:

$$N_{ij} = O(r^\alpha), \qquad \Phi_{ij} = O(r^\beta). \tag{26}$$

By inspecting (11j) of Ref. [20], we arrive at the following possibilities:

(1) For $\tilde{R} \neq 0$:
(a) $\alpha = 1$, $\beta < 0$, with $N_{ij} = -\frac{\tilde{R}}{2}\delta_{ij} r + o(r)$
(b) $\alpha < 1$, $\beta = 0$, with $\Phi_{ij} = -\tilde{R}\delta_{ij} + o(1)$
(c) $\alpha \geq 1$, $\beta = \alpha - 1$
(2) For $\tilde{R} = 0$:
(a) $\alpha = -1$, $\beta < -2$, with $N_{ij} = O(r^{-1})$
(b) $\alpha \geq 1$, $\beta = \alpha - 1$
(c) $\alpha < 1$, $\alpha \neq -1$, $\beta = \alpha - 1$

Let us also define the leading-order behavior of

$$\Phi_{ijkl} = O(r^{\beta_c}). \tag{27}$$

Now, in general, the leading-order term of Eq. (B12) of Ref. [18] can be of order $O(r^{\beta_c - 1})$, $O(r^{\beta - 1})$, $O(r^{\alpha - \nu})$, or $O(r^{-\nu - 1})$, depending on the relative value of the parameters α, β_c, β, and ν (recall that here we are restricting to case i: $\Psi_{ijk} = O(r^{-\nu})$, $\Omega_{ij} = O(r^{-\nu})$). It is easy to see that in the above cases 1b, 1c, and 2b the leading term is either $O(r^{\beta_c - 1})$ or $O(r^{\beta - 1})$ (with possibly $\beta = \beta_c$). However, studying (B12) of Ref. [18] at the leading order reveals that such cases 1b, 1c, and 2b are in fact forbidden, since they all have $\beta \geq 0$. Additionally, it shows that in case 2c one has a stronger restriction $\alpha < -1$ (for $n = 4$, Eq. (B5) of Ref. [18] is also needed). In the permitted cases, we can thus in general conclude

$$N_{ij} = -\frac{\tilde{R}}{2}\delta_{ij} r + o(r) \quad \text{if } \tilde{R} \neq 0, \tag{28}$$

$$N_{ij} = O(r^{-1}) \quad \text{if } \tilde{R} = 0. \tag{29}$$

Note also that in all the permitted cases we have $\beta < 0$. This enables us to use (11a) of Ref. [20] to readily arrive at

$$L_{11} = \tilde{R} r + o(r) \quad \text{if } \tilde{R} \neq 0, \tag{30}$$

$$L_{11} = l_{11} + o(1) \quad \text{if } \tilde{R} = 0, \tag{31}$$

while (11m) of Ref. [20] gives

$$\overset{i}{M}_{j1} = O(1), \tag{32}$$

and (16) leads to

$$U = -\frac{\tilde{R}}{2} r^2 + o(r^2) \quad \text{if } \tilde{R} \neq 0, \tag{33}$$

$$U = -l_{11} r + o(r) \quad \text{if } \tilde{R} = 0. \tag{34}$$

Thanks to the above discussion, we can now study the consequences of (B12) of Ref. [18], as well as those of (B5) of Ref. [18]), more systematically. Clearly, from now on, it will be necessary to distinguish case 1 ($\tilde{R} \neq 0$) from case 2 ($\tilde{R} = 0$).

III. CASE $\tilde{R} \neq 0$

A. Case i: $\Psi_{ijk} = O(r^{-\nu})$, $\Omega_{ij} = O(r^{-\nu})$ ($\nu > 2$)

1. Weyl components of b.w. 0

At the leading order of (B12) of Ref. [18], we can have only (some of) the terms $O(r^{\beta_c - 1})/O(r^{\beta - 1})$, $O(r^{1-\nu})$. (From now on, it will be understood that Φ^S_{ij} and Φ have the same behavior as Φ_{ijkl}, i.e., $\beta = \beta_c$, except when stated otherwise.)

(1) If $1 - \nu > \beta_c - 1$ and $1 - \nu > \beta - 1$, Eq. (B12) of Ref. [18] shows that necessarily $n = 4$, and (B5) of Ref. [18] then gives $\nu = 5$. It also turns out that then $\beta_c = \beta = -4$, so that here we can thus have only

$$\Phi_{ijkl} = O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-4}),$$
$$\Omega_{ij} = O(r^{-5}) \quad (n = 4). \tag{35}$$

(2) In all remaining cases, at least one of the terms $O(r^{\beta_c - 1})$, $O(r^{\beta - 1})$ must appear at the leading order in (B12) of Ref. [18]. Combing this with (B5) of Ref. [18], after some calculations and depending on the value of ν (and of n), one arrives at the following possible behaviors:
(a) $\beta_c = -2$, $\nu = 4$:

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi = o(r^{-2}),$$
$$\Phi^A_{ij} = o(r^{-2}), \qquad \Omega_{ij} = O(r^{-4}) \quad (n > 4), \tag{36}$$

with $\Phi^{S(2)}_{ij} = \frac{\tilde{R}}{2}\Omega^{(4)}_{ij}$. Since in four dimensions $\Phi^S_{ij} \propto \delta_{ij}$, this case is permitted only for $n > 4$.
(b) $\beta_c = -2$, $\nu > 4$: it follows from the last remark that here Φ^S_{ij} becomes subleading. It turns out (by comparing (B5) of Ref. [18] with the trace of (B12) in Ref. [18]) that the ranges $4 < \nu < 5$ and $4 < \nu < 6$ are forbidden, and we can identify three possible subcases, i.e.,

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = o(r^{-3}),$$
$$\Phi^A_{ij} = O(r^{-3}), \qquad \Omega_{ij} = O(r^{-5}) \quad (n > 5), \tag{37}$$

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}),$$
$$\Phi^A_{ij} = o(r^{1-n}), \qquad \Omega_{ij} = O(r^{-n-1}) \quad (n > 5), \tag{38}$$

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{2-\nu}),$$
$$\Phi = o(r^{2-\nu}), \qquad \Phi^A_{ij} = o(r^{2-\nu}),$$
$$\Omega_{ij} = O(r^{-\nu}) \quad (n > 5, \nu \geq 6, \nu \neq n + 1). \tag{39}$$

Here, $n > 5$ since in four and five dimensions one has $\Phi_{ijkl} = 0 \Leftrightarrow \Phi^S_{ij} = 0$ [31]. In (37), the [(anti) symmetric parts of the] trace of (B12) of Ref. [18] [using (10)] give $\Phi^{(2)}_{ijkl} b_{(jl)} = -\frac{\tilde{R}}{2}(n-4)\Omega^{(5)}_{ik}$ and $(n-4)\Phi^{A(3)}_{ij} = \Phi^{(2)}_{ikjl} b_{[kl]}$; moreover, if $\nu > 5$ then necessarily $\nu \geq 6$. In (38) and (39), we have instead $\Phi^{(2)}_{ijkl} b_{(jl)} = 0 = \Phi^{(2)}_{ikjl} b_{[kl]}$. In (38), one finds $(2-n)\Phi^{S(n-1)}_{ij} + \Phi^{(n-1)}\delta_{ij} = \frac{\tilde{R}}{2}(n-4)\Omega^{(n+1)}_{ij}$, and Ω_{ij} can go to zero faster than indicated. In (39), one has $(3-\nu)\Phi^{S(\nu-2)}_{ij} = \frac{\tilde{R}}{2}\Omega^{(\nu)}_{ij}(\nu-5)$ (as obtained from (B5) of Ref. [18]).

(c) $\beta_c = 1-n$: there is a difference between $n > 4$ and $n = 4$, i.e.,

$$\text{if } n > 4: \ \Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = o(r^{1-n}), \qquad \Omega_{ij} = O(r^{-n-1}), \tag{40}$$

$$\text{if } n = 4: \ \Phi_{ijkl} = O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}), \qquad \Omega_{ij} = O(r^{-5}), \tag{41}$$

with $\Phi^{(3)}_{ijkl} = 2\Phi^{(3)}\delta_{j[k}\delta_{l]i}$ for $n = 4$ and $(n-2)(n-3)\Phi^{(n-1)}_{ijkl} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{l]i} - 2(n-3)\tilde{R}(\Omega^{(n+1)}_{j[k}\delta_{l]i} - \Omega^{(n+1)}_{i[k}\delta_{l]j})$ [which implies $(2-n)\Phi^{S(n-1)}_{ij} + \Phi^{(n-1)}\delta_{ij} = \frac{\tilde{R}}{2}(n-4)\Omega^{(n+1)}_{ij}$] for $n > 4$. Note the different behavior of the "magnetic" term Φ^A_{ij}. In both cases, it is understood that Ω_{ij} can go to zero faster (or even vanish identically—for $n = 4$, if $\nu > 5$, then necessarily $\nu \geq 6$]. In (41), both Φ_{ijkl} and Φ^A_{ij} can go to zero faster than indicated. The result of (35) can thus be understood as a subcase of (41)—for this reason, (35) will not be considered anymore in the following.

We have not given explicitly the behavior of Ψ_{ijk} in all the above cases since it always follows from point I of Sec. II C. Note that not all values of ν are permitted. In particular, although we started from the weak assumption $\nu > 2$, in the end, we always have either $\nu = 4$ or $\nu \geq 5$. Thanks to (10), this enables us to specialize (28) to

$$N_{ij} = -\frac{\tilde{R}}{2}\delta_{ij} r + \frac{\tilde{R}}{2} b_{ij} + o(1). \tag{42}$$

Additionally, since in all permitted cases we have $\Phi = o(r^{-2})$ and $\Phi^A_{ij} = o(r^{-2})$ (or faster), Eqs. (30), (33), and (32) can be specialized as

$$L_{11} = \tilde{R} r + l_{11} + O(r^{-1}), \tag{43}$$

$$U = -\frac{\tilde{R}}{2} r^2 - l_{11} r + O(1), \tag{44}$$

$$\overset{i}{M}_{j1} = \overset{i}{m}_{j1} + O(r^{-1}). \tag{45}$$

Using (42) in (B5) of Ref. [18], one is now able to refine all the "o" symbols in Eqs. (36), (38), (39), and (40) [but not in (38)] by appropriate "O" symbols [e.g., $\Phi = o(r^{-2})$ in (36) can be replaced by $\Phi = O(r^{-3})$, etc]. This will be taken into account explicitly in a summary in Sec. III A 4.

2. Ricci rotation coefficients of b.w. −2 and Weyl components of b.w. −1

Let us analyze (11f) of Ref. [20] and (B6), (B9), and (B1) of Ref. [18] in all the possible cases listed above, where we note that always $\nu \geq 4$ (useful for the next comment). First, let us observe from (B9) of Ref. [18] that if Ψ'_{ijk} goes to zero more slowly than Φ_{ij}, then necessarily it goes to zero as $O(r^{-2})$ (or faster). On the other hand, if Ψ'_{ijk} does *not* go to zero more slowly than Φ_{ij}, we also conclude $\Psi'_{ijk} = O(r^{-2})$ (or faster) since $\Phi_{ij} = O(r^{-2})$ (or faster) in all permitted cases. Thus, we always have $\Psi'_{ijk} = O(r^{-2})$ (or faster), which enables one to use (11f) of Ref. [20] [together with the second of (24) and (42)] to arrive at

$$N_{i1} = \frac{\tilde{R}}{2} l_{i1} r + O(1). \tag{46}$$

Thanks to this result, we can now employ (B6) together with (B9) of Ref. [18] and arrive at the following results (where the various points are "numbered" so as to correspond to those of Sec. III A 1). From now on, it will be understood that Ψ'_i has the same behavior as Ψ'_{ijk}, except when stated otherwise:

(a) $\Psi'_{ijk} = O(r^{-2})$:
with $\Psi'^{(2)}_i = -\frac{\tilde{R}}{2}\Psi^{(4)}_i$, and $\Psi'^{(2)}_{ijk}$ can be expressed in terms of $\Omega^{(4)}_{ij}$ and $\Phi^{(2)}_{ijkl}$ using (B6) of Ref. [18] (recall that $\Psi^{(4)}_{ijk}$ and its trace $\Psi^{(4)}_i$ can be expressed in terms of $\Omega^{(4)}_{ij}$, as observed in Sec. II C).

(b) For the three subcases we find, respectively,

$$\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}), \tag{47}$$

$$\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{1-n}), \tag{48}$$

$$\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{2-\nu}), \tag{49}$$

with $\Psi'^{(2)}_{ijk} = -\Phi^{(2)}_{isjk} l_{s1}$ and where the behavior of Ψ'_i has been obtained using (B1) of Ref. [18].

(c)

$$\text{if } n > 4: \quad \Psi'_{ijk} = O(r^{1-n}), \tag{50}$$

$$\text{if } n = 4: \quad \Psi'_{ijk} = O(r^{-2}). \tag{51}$$

3. Weyl components of b.w. −2

To conclude, let us study (B4) of Ref. [18]. It will be also useful to use (B13) of Ref. [18], for which the trace immediately tells us that the terms containing Ω'_{ij} cannot be leading over all the remaining terms in that equation (when $n > 4$). Bearing this in mind, in the various cases listed above, (B4) of Ref. [18] leads to:

(a) $\Omega'_{ij} = O(1)$,
with $\Omega'^{(0)}_{ij} = (\frac{\tilde{R}}{2})^2 \Omega^{(4)}_{ij}$. (One can arrive at the same result also using (B13) of Ref. [18].)

(b) In the first case [Eq. (37)], we find

$$\Omega'_{ij} = O(r^{-1}) \quad (\text{case (37)}), \tag{52}$$

with $\Omega'^{(1)}_{ij} = -(\frac{\tilde{R}}{2})^2 \Omega^{(5)}_{ij}$, and for the second and third cases [Eqs. (38) and (39)]

$$\Omega'_{ij} = O(r^{-2}) \quad [\text{cases (38) and (39)}]. \tag{53}$$

The different behavior in case (37) stems from (B13) of Ref. [18] using the fact that $\Phi^{(2)}_{ijkl} b_{(jl)} \neq 0$ when $\nu = 5$. In case (39), one has $\Omega'^{(2)}_{ij} = \Phi^{(2)}_{isjk} l_{s1} l_{k1} + (\frac{\tilde{R}}{2})^2 \Omega^{(6)}_{ij}$ (recall that $\nu \geq 6$; cf. Sec. III A 1). For case (38), one has simply $\Omega'^{(2)}_{ij} = \Phi^{(2)}_{isjk} l_{s1} l_{k1}$.

(c)

$$\text{if } n > 4: \quad \Omega'_{ij} = O(r^{3-n}), \tag{54}$$

$$\text{if } n = 4: \quad \Omega'_{ij} = O(r^{-1}), \tag{55}$$

where $\Omega'^{(n-3)}_{ij} = (\frac{\tilde{R}}{2})^2 \Omega^{(n+1)}_{ij}$ for $n > 4$. (One can arrive at the same result also using (B13) of Ref. [18].)

It is clear that if $n > 4$ and $\boldsymbol{\ell}$ is a WAND (possible in cases c and b above) the falloff of Ω'_{ij} will be faster since $\Omega_{ij} = 0$ (in agreement with the results of Ref. [30] for multiple WANDs).

4. Summary of case i

In all cases given here, we have

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu \geq 4), \qquad \Psi_{ijk} = O(r^{-\nu}). \tag{56}$$

These two equations will not be repeated every time below, where we will give only possible further restrictions on ν. See also Secs. III A 1–III A 3 for relations among the leading-order terms of various boost weight.

(a) Here, $n > 4$, and

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi = O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}) \quad (n > 4, \nu = 4), \qquad \Psi'_{ijk} = O(r^{-2}), \qquad \Omega'_{ij} = O(1). \tag{57}$$

The leading term at infinity is of order r^0, and it is of type N. At order $1/r^2$, the type becomes II(ad). This case does not seem of great physical interest since the frame components Ω'_{ij} do not decay near infinity. In particular, it cannot describe asymptotically anti-de Sitter spacetimes according to the definition of Ref. [32] [this applies also to cases below and in Secs. III B and III C having $\Phi_{ijkl} = O(r^{-2})$ and/or $\Psi_{ijk} = O(r^{-3})$]. Here, $\boldsymbol{\ell}$ cannot be a WAND.

(b) Here, $n > 5$, and we have three subcases. Generically [case (37)], we have

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = o(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}) \quad (n > 5, \nu = 5 \quad \text{or} \quad \nu \geq 6), \qquad \Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}), \qquad \Omega'_{ij} = O(r^{-1}), \tag{58}$$

where, however, if $\nu \geq 6$, then $\Phi^S_{ij} = O(r^{-4})$ and $\Omega'_{ij} = O(r^{-2})$. The leading term is thus of type N for $\nu = 5$ and of type II(abd) for $\nu \geq 6$. As a special subcase, here $\boldsymbol{\ell}$ can be a multiple WAND; cf. the results of Ref. [30].

If $\Phi^{(2)}_{ijkl} b_{[jl]} = 0$, this becomes either

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}) \quad (n > 5, \nu \geq n+1), \qquad \Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{1-n}), \qquad \Omega'_{ij} = O(r^{-2}), \tag{59}$$

which describes, in particular, the falloff along a multiple WAND in Robinson–Trautman Einstein spacetimes [24] (such as static Einstein black holes) or (if $6 \leq \nu < n+1$, or $\nu > n+1$ but with $\Phi^{S(n-1)}_{ij} = 0$)

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{2-\nu}), \qquad \Phi = O(r^{1-\nu}), \qquad \Phi^A_{ij} = O(r^{1-\nu}) \quad (n > 5, \nu \geq 6, \nu \neq n+1), \qquad \Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{2-\nu}), \qquad \Omega'_{ij} = O(r^{-2}). \tag{60}$$

The leading term is of type II(abd) in both of the above two cases.

(c) This possibility arises when $\Phi^{(2)}_{ijkl} = 0$ and includes the four-dimensional case. For $n > 4$, we have

$$\Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}) \quad (n > 4, \nu \geq n+1),$$
$$\Psi'_{ijk} = O(r^{1-n}), \qquad \Omega'_{ij} = O(r^{3-n}). \tag{61}$$

The leading term at infinity is of order $1/r^{n-3}$ (provided $\Omega^{(n+1)}_{ij} \neq 0$) and it is of type N. At order $1/r^{n-1}$, the type becomes II(cd) [II(bcd) if $\Omega^{(n+1)}_{ij} = 0$]. In special cases, ℓ can be a multiple WAND. This case thus includes the behavior of algebraically special spacetimes along a nondegenerate geodesic multiple WAND under the assumption $\Phi^{(2)}_{ijkl} = 0$, for which, however, $\Omega'_{ij} = O(r^{1-n})$ [30] [the r-dependence at the leading order has been worked out explicitly also for concrete examples such as Kerr–Schild–(A)dS geometries (with a nondegenerate Kerr–Schild vector) [29], including rotating (A)dS black holes, and for Robinson–Trautman spacetimes with (A)dS asymptotics [24], such as the Schwarzschild–Tangherlini (A)dS black hole].

For $n = 4$, one has instead [recall that (35) is a subcase of (41)]

$$\Phi_{ijkl} = O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}) \quad (n = 4, \nu \geq 5),$$
$$\Psi'_{ijk} = O(r^{-2}), \qquad \Omega'_{ij} = O(r^{-1}). \tag{62}$$

This is a special subcase of the standard four-dimensional peeling (69).

B. Case ii: $\Psi_{ijk} = O(r^{-n})$, $\Omega_{ij} = o(r^{-n})$

The behavior of the Ricci rotation coefficients and derivative operators is the same as in case i, and it will not be repeated here [in particular, (28), (30), (32), (33), and (46) still apply].

1. Case $\beta_c = -2$, $n > 5$

All the following cases can occur only for $n > 5$. In general, one has

$$\begin{aligned}
&\Omega_{ij} = o(r^{-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{63}$$

with $\Phi^{(2)}_{ijkl} b_{(jl)} = 0$, $(n-4)\Phi^{A(3)}_{ij} = \Phi^{(2)}_{ikjl} b_{[kl]}$, and $\Psi'^{(2)}_{ijk} = -\Phi^{(2)}_{isjk} l_{s1}$. Here, ℓ can be a single WAND, in special cases. For $\Psi^{(n)}_{ijk} = 0$, this reduces to (58) (with $\nu > n$).

If $\Phi^{(2)}_{ikjl} b_{[kl]} = 0$ (but $\Phi^{(2)}_{ikjl} \neq 0$), we have the subcase

$$\begin{aligned}
&\Omega_{ij} = O(r^{-1-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{2-n}),\\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{64}$$

with $\Psi'^{(n-2)}_i = \frac{\tilde{R}}{2}\Psi^{(n)}_i$. Ω_{ij} can go to zero faster than indicated.

If, additionally, $\Phi^{S(n-1)}_{ij} = 0$, we have, depending on the range of ν, either

$$\begin{aligned}
&\Omega_{ij} = O(r^{-\nu}) \quad (n < \nu < 2+n, \nu \neq n+1),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{2-\nu}),\\
&\Phi = o(r^{2-\nu}), \qquad \Phi^A_{ij} = o(r^{2-\nu}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{2-n}),\\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{65}$$

where the precise power of r for both Φ and Φ^A_{ij} is given by $\max\{1-\nu, -n\}$, or

$$\begin{aligned}
&\Omega_{ij} = O(r^{-2-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi_i{}' = O(r^{2-n}),\\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{66}$$

where Ω_{ij} can go to zero faster than indicated.

In all of the above cases, the leading term is of type II(abd).

2. Case $\beta_c < -2$, $n > 4$

If $\Phi^{(2)}_{ijkl} = 0$, then (64) reduces to

$$\begin{aligned}
&\Omega_{ij} = O(r^{-1-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{2-n}),\\
&\Omega'_{ij} = O(r^{3-n}),
\end{aligned} \tag{67}$$

with $\Omega'^{(n-3)}_{ij} = (\frac{\tilde{R}}{2})^2 \Omega^{(n+1)}_{ij}$ and $(n-2)(n-3)\Phi^{(n-1)}_{ijkl} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{m]i} - 2(n-3)\tilde{R}(\Omega^{(n+1)}_{j[k}\delta_{m]i} - \Omega^{(n+1)}_{i[k}\delta_{m]j})$. The leading term is type N. If $\Psi^{(n)}_{ijk} = 0$, this reduces to (61) with

$\nu = n + 1$. Although the above falloff looks very similar to the standard 4D peeling (69), an important difference for $n > 4$ is that $\Omega'^{(n-3)}_{ij} \neq 0$ implies that $\boldsymbol{\ell}$ is not a WAND.

If $\Phi^{(n-1)} = 0 = \Omega^{(n+1)}_{ij}$, this becomes

$$
\begin{aligned}
\Omega_{ij} &= O(r^{-2-n}),\\
\Psi_{ijk} &= O(r^{-n}),\\
\Phi_{ijkl} &= O(r^{-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
\Psi'_{ijk} &= O(r^{2-n}),\\
\Omega'_{ij} &= O(r^{2-n}).
\end{aligned}
\tag{68}
$$

Here, the leading term is of type III. Ω_{ij} can go to zero faster than indicated.

In both of the above cases, we have $(n-3)\Psi'^{(n-2)}_{ijk} = \tilde{R}\Psi^{(n)}_{[j}\delta_{k]i}$.

3. Case $n = 4$

In four dimensions, we recover the standard asymptotic behavior [9,10], i.e.,

$$
\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu \geq 5),\\
\Psi_{ijk} &= O(r^{-4}),\\
\Phi_{ijkl} &= O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
\Psi'_{ijk} &= O(r^{-2}),\\
\Omega'_{ij} &= O(r^{-1}).
\end{aligned}
\tag{69}
$$

In our study, the condition $\nu \geq 5$ followed by analyzing the Ricci and Bianchi equations (where we initially only assumed $\nu > 2$), thanks to $\tilde{R} \neq 0$. Additionally, we observe that if $\nu > 5$ then necessarily $\nu \geq 6$. For $\Psi^{(4)}_{ijk} = 0$, this case reduces to (62).

C. Case iii: $\Psi_{ijk} = O(r^{-3})$, $\Omega_{ij} = o(r^{-3})$ $(n > 4)$

Again the behavior of the Ricci rotation coefficients and derivative operators is the same as in case i.[5]

[5]To arrive at (46) in the present case, one needs to use also (10) and (42) and thus to observe that, although (B9) of Ref. [18] gives $\Psi'_{ijk} = O(r^{-1})$, from its trace one gets $\Psi_i{}' = O(r^{-2})$ [see also (70)–(72) below].

1. Case $\beta_c = -2$

Here, in general, one has $(n \geq 5)$

$$
\begin{aligned}
\Omega_{ij} &= O(r^{-4}),\\
\Psi_{ijk} &= O(r^{-3}), \qquad \Psi_i = O(r^{-4}),\\
\Phi_{ijkl} &= O(r^{-2}), \qquad \Phi = O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
\Psi'_{ijk} &= O(r^{-1}), \qquad \Psi'_i = O(r^{-2}),\\
\Omega'_{ij} &= O(1),
\end{aligned}
\tag{70}
$$

with $\Phi^{S(2)}_{ij} = \frac{\tilde{R}}{2}\Omega^{(4)}_{ij}$, $\Psi'^{(1)}_{ijk} = \frac{\tilde{R}}{2}\Psi^{(3)}_{ijk}$, $\Psi'^{(2)}_i$ can be expressed in terms of $\Omega^{(4)}_{ij}$ and $\Psi^{(3)}_{ijk}$ thanks to (B6) of Ref. [18], $\Omega'^{(0)}_{ij} = (\frac{\tilde{R}}{2})^2\Omega^{(4)}_{ij}$, and

$$
\begin{aligned}
(n-4)\Phi^{A(3)}_{ki} &= \Phi^{(2)}_{klij}b_{[lj]} + \xi^{A0}_j\Psi^{(3)}_{[ki]j,A} + 2l_{1j}\Psi^{(3)}_{[ki]j}\\
&\quad + \Psi^{(3)}_{[ki]l}\overset{l}{m}_{jj} + \Psi^{(3)}_{jl[k}\overset{l}{m}_{i]j} + \tilde{R}\Omega^{(4)}_{j[k}b_{i]j}.
\end{aligned}
$$

The leading term is type N. In the limit $\Psi^{(3)}_{ijk} = 0$, this reduces to case (57).

If Ω_{ij} has a faster falloff, one finds for $n > 5$ (as in Sec. III A the range $4 < \nu < 5$ is forbidden by imposing (B5) and (B12) of Ref. [18]; see Sec. III C 2 for the case $n = 5$)

$$
\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu \geq 5),\\
\Psi_{ijk} &= O(r^{-3}), \qquad \Psi_i = O(r^{-4}),\\
\Phi_{ijkl} &= O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}),\\
\Phi &= O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-3}), \quad (n > 5)\\
\Psi'_{ijk} &= O(r^{-1}), \qquad \Psi'_i = O(r^{-2}),\\
\Omega'_{ij} &= O(r^{-1}),
\end{aligned}
\tag{71}
$$

where from (B5) of Ref. [18] $\Phi^{S(3)}_{ij} = -\Psi^{(3)}_{(ij)l}l_{l1}$, from (B12) of Ref. [18]

$$
\begin{aligned}
(n-4)\Phi^{A(3)}_{ki} &= \Phi^{(2)}_{klij}b_{[lj]} + \xi^{A0}_j\Psi^{(3)}_{[ki]j,A} + 2l_{1j}\Psi^{(3)}_{[ki]j}\\
&\quad + \Psi^{(3)}_{[ki]l}\overset{l}{m}_{jj} + \Psi^{(3)}_{jl[k}\overset{l}{m}_{i]j},\\
-\Phi^{(2)}_{klij}b_{(lj)} &= \xi^{A0}_j\Psi^{(3)}_{(ki)j,A} + [(n-6)l_{j1} + 2l_{1j}]\Psi^{(3)}_{(ki)j}\\
&\quad + \Psi^{(3)}_{(ki)l}\overset{l}{m}_{jj} + (2\Psi^{(3)}_{l(k|j} + \Psi^{(3)}_{jl(k|})\overset{l}{m}_{|i)j}\\
&\quad + \frac{\tilde{R}}{2}(n-4)\Omega^{(5)}_{ik},
\end{aligned}
$$

and $\Omega'^{(1)}_{ij}$ can be expressed (using the trace of (B13) of Ref. [18]) in terms of $\Omega^{(5)}_{ik}$ and $\Psi^{(3)}_{ijk}$.

TABLE I. Falloff behavior of the Weyl tensor in the presence of a cosmological constant ($\tilde{R} \neq 0$). We list here in a compact way the cases summarized in Secs. III A 4, III B, III C. Recall that the cases i, ii and iii differ by the falloff of the component Ψ_{ijk}. Whenever there is just one power of r in the column for Φ^S_{ij} and Φ (the 5th column), it means that these two quantities have the same falloff (the same holds for Ψ_{ijk}, Ψ_i and Ψ'_{ijk}, Ψ'_i–the 3rd and the 7th column, respectively), while when the column is empty it means that both Φ^S_{ij} and Φ have same falloff as Φ_{ijkl}. It is always understood that $n > 4$ except when we explicitly indicate $n = 4$ (last but one column). The shortcuts RT and KS stands for Robinson–Trautman and Kerr–Schild spacetimes, respectively (last column).

Case	Ω_{ij}	Ψ_{ijk}, Ψ_i	Φ_{ijkl}	Φ^S_{ij}, Φ	Φ^A_{ij}	Ψ'_{ijk}, Ψ'_i	Ω'_{ij}	Restrictions	Comments
i a	r^{-4}	r^{-4}	r^{-2}	r^{-2}, r^{-3}	r^{-3}	r^{-2}	$O(1)$	$\nu = 4$	ℓ not a WAND
i b	r^{-5}	r^{-5}	r^{-2}	$o(r^{-3})$	r^{-3}	r^{-2}, r^{-3}	r^{-1}	$n > 5$, $\nu = 5$	ℓ not a WAND
	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	r^{-4}	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n > 5$, $\nu \geq 6$	
	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	r^{1-n}	r^{-n}	r^{-2}, r^{1-n}	r^{-2}	$n > 5$, $\nu \geq n+1$	includes RT
	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	$r^{2-\nu}$, $r^{1-\nu}$	$r^{1-\nu}$	r^{-2}, $r^{2-\nu}$	r^{-2}	$n > 5$, $\nu \geq 6$, $\nu \neq n+1$	
i c	r^{-n-1}	r^{-n-1}	r^{1-n}		r^{-n}	r^{1-n}	r^{3-n}	$\nu = n+1$	ℓ not a WAND
	$r^{-\nu}$	$r^{-\nu}$	r^{1-n}		r^{-n}	r^{1-n}	$o(r^{3-n})$	$\nu > n+1$	includes KS (A)dS
	$r^{-\nu}$	$r^{-\nu}$	r^{-3}		r^{-3}	r^{-2}	r^{-1}	$n = 4$, $\nu \geq 5$	
ii	$o(r^{-n})$	r^{-n}	r^{-2}	r^{-4}	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n > 5$	
	r^{-n-1}	r^{-n}	r^{-2}	r^{1-n}	r^{-n}	r^{-2}, r^{2-n}	r^{-2}	$n > 5$, $\nu \geq n+1$	
	$r^{-\nu}$	r^{-n}	r^{-2}	$r^{2-\nu}$, $o(r^{2-\nu})$	$o(r^{2-\nu})$	r^{-2}, r^{2-n}	r^{-2}	$5 < n < \nu < n+2$, $\nu \neq n+1$	ℓ not a WAND
	r^{-n-2}	r^{-n}	r^{-2}	r^{-n}	r^{-n}	r^{-2}, r^{2-n}	r^{-2}	$n > 5$, $\nu \geq n+2$	
	r^{-n-1}	r^{-n}	r^{1-n}	r^{1-n}	r^{-n}	r^{2-n}	r^{3-n}	$\nu = n+1$	ℓ not a WAND
	r^{-n-2}	r^{-n}	r^{-n}	r^{-n}	r^{-n}	r^{2-n}	r^{2-n}	$\nu \geq n+2$	
	$r^{-\nu}$	r^{-4}	r^{-3}	r^{-3}	r^{-3}	r^{-2}	r^{-1}	$n = 4$, $\nu \geq 5$	
iii	r^{-4}	r^{-3}, r^{-4}	r^{-2}	r^{-2}, r^{-3}	r^{-3}	r^{-1}, r^{-2}	$O(1)$	$\nu = 4$	ℓ not a WAND
	$r^{-\nu}$	r^{-3}, r^{-4}	r^{-2}	r^{-3}, r^{-4}	r^{-3}	r^{-1}, r^{-2}	r^{-1}	$n > 5$, $\nu \geq 5$	
	$r^{-\nu}$	r^{-3}, r^{-4}	r^{-3}	r^{-3}	r^{-3}	r^{-1}, r^{-2}	r^{-1}	$n \geq 5$, $\nu \geq 5$	

The leading term is of type III(a), and ℓ can be a single WAND. If $\Psi^{(3)}_{ijk} = 0$, this reduces to (58) for $5 \leq \nu \leq n$ and to (63) for $\nu > n$.

2. Case $\beta_c < -2$

For $n = 5$, or for $n > 5$ with $\Phi^{(2)}_{ijkl} = 0$, instead of (71), one has

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu \geq 5),$$
$$\Psi_{ijk} = O(r^{-3}), \qquad \Psi_i = O(r^{-4}),$$
$$\Phi_{ijkl} = O(r^{-3}), \qquad \Phi = O(r^{-3}),$$
$$\Phi^A_{ij} = O(r^{-3}), \quad (n \geq 5) \Psi'_{ijk} = O(r^{-1}),$$
$$\Psi'_i = O(r^{-2}), \quad \Omega'_{ij} = O(r^{-1}), \tag{72}$$

where $\Phi^{(3)}_{ijkl}$ can be expressed in terms of $\Omega^{(5)}_{ij}$ and $\Psi^{(3)}_{ijk}$ using (B12) of Ref. [18] (or (B13) of Ref. [18]). The leading term is of type III(a). Again, $\Psi'^{(1)}_{ijk} = \frac{\tilde{R}}{2} \Psi^{(3)}_{ijk}$.

All of the above results for the case $\tilde{R} \neq 0$ are summarized in Table I.

IV. Case $\tilde{R} = 0$

A. Case i: $\Psi_{ijk} = O(r^{-\nu})$, $\Omega_{ij} = O(r^{-\nu})$ ($\nu > 2$)

1. Weyl components of b.w. 0

In this case, at the leading order of (B12) of Ref. [18] we can have only (some of) the terms $O(r^{\beta_c - 1})$, $O(r^{\beta - 1})$, $O(r^{-1-\nu})$. The same is true for the antisymmetric part of (B5) of Ref. [18], while the leading-order terms of the symmetric part of (B5) of Ref. [18] can only be $O(r^{\beta_c - 1})$, $O(r^{\beta - 1})$, and $O(r^{-\nu})$. Here, we are mainly interested in studying the case when the leading terms of (B12) of Ref. [18] are $O(r^{\beta_c - 1})$ or $O(r^{\beta - 1})$, i.e., $\beta_c > -\nu$ or $\beta > -\nu$. (In all the remaining cases, the asymptotic behavior of b.w. zero components can be represented by $\Phi_{ijkl} = O(r^{-\nu})$, $\Phi^A_{ij} = O(r^{-\nu})$, and $\Omega_{ij} = O(r^{-\nu})$, with $\nu > 2$. The behavior of higher b.w. components is given in Sec. IV A 5 below.)

By combining (B12) and (B5) of Ref. [18], we arrive at the following possibilities, also depending on the value of ν and of n:

(A) $\beta_c = -2$: there are several possibilities, i.e.,

A1:

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = o(r^{-2}),$$
$$\Phi^A_{ij} = o(r^{-2}),$$
$$\Omega_{ij} = O(r^{-\nu}) \quad (n > 5, 2 < \nu \leq 3). \tag{73}$$

A2:

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}),$$
$$\Phi = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-3}),$$
$$\Omega_{ij} = O(r^{-\nu}) \qquad (n > 5, 3 < \nu < 4). \tag{74}$$

The (anti)symmetric parts of the trace of (B12) of Ref. [18] [using (10)] give $(n-4)\Phi^{A(3)}_{ij} = \Phi^{(2)}_{ikjl}b_{[kl]}$ and $(n-6)\Phi^{S(3)}_{ki} = \Phi^{(2)}_{klij}b_{(lj)}$. In the special case, $\Phi^{(2)}_{ikjl}b_{[kl]} = 0$, and thus Φ^A_{ij} goes to zero faster, namely, $\Phi^A_{ij} = O(r^{-\nu})$.

A3:

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}), \qquad \Phi = O(r^{-4}),$$
$$\Phi^A_{ij} = O(r^{-3}), \qquad \Omega_{ij} = O(r^{-4}) \qquad (n > 5). \tag{75}$$

As above, $(n-4)\Phi^{A(3)}_{ij} = \Phi^{(2)}_{ikjl}b_{[kl]}$, and $(n-6)\Phi^{S(3)}_{ki} = \Phi^{(2)}_{klij}b_{(lj)}$, but here with the latter, (B5) of Ref. [18] further gives $\Phi^{(2)}_{klij}b_{(lj)} = -(n-6)(l_{11}\Omega^{(4)}_{ki} + \frac{1}{2}X^{A0}\Omega^{(4)}_{ki,A} + \Omega^{(4)}_{s(k}\overset{s}{m}_{i)1})$. Here, Ω_{ij} can go to zero faster than indicated, i.e., $\Omega_{ij} = O(r^{-\nu})$ with $\nu > 4$, but in that case, clearly also Φ^S_{ij} does [namely, $\Phi^S_{ij} = O(r^{1-\nu})$ for $4 < \nu < 5$, and $\Phi^S_{ij} = O(r^{-4})$ for $\nu \geq 5$—in particular, for $\nu > 5$, the symmetric part of (B5) of Ref. [18] gives $\Phi^{S(4)}_{ij}$ in terms of $\Phi^{A(3)}_{ij}$].

If $\Phi^{(2)}_{ikjl}b_{[kl]} = 0$, we obtain the following two subcases, depending on whether $\nu \neq n$ or $\nu = n$.

A4:

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-\nu}), \qquad \Phi = O(r^{-\nu}),$$
$$\Phi^A_{ij} = O(r^{-\nu}), \qquad \Omega_{ij} = O(r^{-\nu})$$
$$(n > 5, \nu \geq 4, \nu \neq n), \tag{76}$$

with $\Phi^{(2)}_{ikjl}b_{[kl]} = 0$ and $(n-6)\Phi^{S(3)}_{ki} = \Phi^{(2)}_{klij}b_{(lj)}$ (if $\nu = 4$) or $\Phi^{(2)}_{klij}b_{(lj)} = 0$ (if $\nu > 4$). For $\nu > n$, this can be seen as a subcase of (77) with $\Phi^{(n-1)} = 0$.

A5:

$$\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}),$$
$$\Phi^A_{ij} = O(r^{-n}), \qquad \Omega_{ij} = O(r^{-n}) \quad (n > 5), \tag{77}$$

with $\Phi^{(2)}_{ikjl}b_{[kl]} = 0$ and $\Phi^{(2)}_{klij}b_{(lj)} = 0$. Ω_{ij} can go to zero faster than indicated, with no effect on the falloff of Φ^S_{ij}. If $\nu > n$, then $(2-n)\Phi^{S(n-1)}_{ij} + \Phi^{(n-1)}\delta_{ij} = 0$.

(B) $\beta_c = -n/2$:

$$\Phi_{ijkl} = O(r^{-n/2}), \qquad \Phi = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-\nu}),$$
$$\Omega_{ij} = O(r^{-\nu}) \quad \left(n > 4, \frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right), \tag{78}$$

with $(n-4)\Phi^{(n/2)}_{ijkl} = 4(\Phi^{S(n/2)}_{i[l}\delta_{k]j} - \Phi^{S(n/2)}_{j[l}\delta_{k]i})$. Note that here Ω_{ij} cannot become $o(r^{-n/2-1})$ as long as $\Phi_{ijkl} = O(r^{-n/2})$. In the special case $\nu = 1 + n/2$, from (B5) of Ref. [18], we obtain $(n-2)\Phi^{S(n/2)}_{ij} = -2X^{A0}\Omega^{(n/2+1)}_{ij,A} - (n-2)l_{11}\Omega^{(n/2+1)}_{ij} - 4\Omega^{(n/2+1)}_{s(j}\overset{s}{m}_{i)1}$, while for $\frac{n}{2} < \nu < 1 + \frac{n}{2}$ we have $X^{A0}\Omega^{(\nu)}_{ij,A} + (\nu - 2)l_{11}\Omega^{(\nu)}_{ij} + 2\Omega^{(\nu)}_{s(j}\overset{s}{m}_{i)1} = 0$.

(C) $\beta_c = 1 - n$: similarly as in Sec. III A, one has to distinguish between the cases $n > 4$ and $n = 4$, i.e.,

$$\text{if } n > 4: \quad \Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = o(r^{1-n}),$$
$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > n-1), \tag{79}$$

$$\text{if } n = 4: \quad \Phi_{ijkl} = O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}),$$
$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 3), \tag{80}$$

with (for $n \geq 4$) $(n-2)(n-3)\Phi^{(n-1)}_{ijkl} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{l]i}$ and $(2-n)\Phi^{S(n-1)}_{ij} + \Phi^{(n-1)}\delta_{ij} = 0$. In (79), we have $\Phi^A_{ij} = O(r^{-\nu})$ for $n - 1 < \nu < n$ and $\Phi^A_{ij} = O(r^{-n})$ for $\nu \geq n$.

Again, see point I of Sec. II C for the behavior of Ψ_{ijk} in all the above cases. As shown above, in all cases except (73), we have $\nu > 3$, which enables us [thanks to (10)] to specialize (29) to

$$N_{ij} = \frac{n_{ij}}{r} + O(r^{-2}) \quad [\text{except for } (73)]. \tag{81}$$

Similarly as for $\tilde{R} \neq 0$ (cf. Sec. III A 1), since in all permitted cases one has $\Phi = o(r^{-2})$ and $\Phi^A_{ij} = o(r^{-2})$, for L_{11}, U, and $\overset{i}{M}_{j1}$, one obtains the refined equations that follow by setting $\tilde{R} = 0$ in (43), (44), and (45) [in contrast to (81), this applies also when $2 < \nu \leq 3$].

2. Ricci rotation coefficients of b.w. −2 and Weyl components of b.w. −1

Let us analyze (11f) of Ref. [20] and (B6), (B9), and (B1) of Ref. [18] in all of the possible cases listed above. Similarly as in Sec. III A 2, it is easy to conclude from (B9) of Ref. [18] that we always have $\Psi'_{ijk} = O(r^{-2})$ (or faster; see more details below), which enables one to use (11f) of Ref. [20] to obtain

$$N_{i1} = O(1). \tag{82}$$

Using (B9), (B6), and (B1) of Ref. [18], one arrives at the following results (the numbering corresponds to that of Sec. IV A 1):

(A) For the five subcases, we find, respectively,
A1: $\Psi'_{ijk} = O(r^{-2})$,
A2: $\Psi'_{ijk} = O(r^{-2})$, $\Psi'_i = O(r^{-3})$,
A3: $\Psi'_{ijk} = O(r^{-2})$, $\Psi'_i = O(r^{-3})$,
A4: $\Psi'_{ijk} = O(r^{-2})$, $\Psi'_i = O(r^{1-\nu})$,
A5: $\Psi'_{ijk} = O(r^{-2})$, $\Psi'_i = O(r^{1-n})$.

In all cases except A1, we have $\Psi'^{(2)}_{ijk} = -\Phi^{(2)}_{isjk} l_{s1}$ (in case A1, if $\nu = 3$, then (B6) of Ref. [18] gives $\Psi'^{(2)}_{ijk}$ in terms of $\Omega^{(3)}_{ij}$, $\Psi^{(3)}_{ijk}$, and $\Phi^{(2)}_{isjk}$).

(B) We have $\Psi'_{ijk} = O(r^{-n/2})$ for any $n \geq 6$ and for $n = 5$ provided $3 < \nu \leq \frac{7}{2}$ [in both cases, (B9) of Ref. [18] enables one to express $\Psi'^{(n/2)}_{ijk}$ in terms of $\Phi^{S(n/2)}_{ij}$]. If, instead, $n = 5$ and $\frac{5}{2} < \nu \leq 3$, we have $\Psi'_{ijk} = O(r^{-2})$.

(C)

$$\text{if } n > 4: \quad \Psi'_{ijk} = O(r^{1-n}), \tag{83}$$

$$\text{if } n = 4: \quad \Psi'_{ijk} = O(r^{-2}). \tag{84}$$

For $n > 4$, (B9) of Ref. [18] gives $(n-3)\Psi'^{(n-1)}_{ijk} = 2\Psi'^{(n-1)}_{[j}\delta_{k]i}$, with $(n-2)\Psi'^{(n-1)}_{i} = -(n-1)\Phi^{(n-1)} l_{1i} - \xi^{A0}_i \Phi^{(n-1)}_{,A}$.

3. Weyl components of b.w. −2

Using (B4) and (B14) of Ref. [18], we arrive at

(A) For the five subcases, we find, respectively,

A1: $\Omega'_{ij} = O(r^{\sigma})$, with $-2 \leq \sigma < -1$ [the precise value of σ depends on the values taken by ν and β—recall (26)].

A1–A5: $\Omega'_{ij} = O(r^{-2})$, with $\Omega'^{(2)}_{ij} = -3 l_{11}\Phi^{S(3)}_{ij} - X^{A0}\Phi^{S(3)}_{ij,A} - 2\Phi^{S(3)}_{s(j}\overset{s}{m}_{i)1} - \Psi'^{(2)}_{(ij)k} l_{k1}$ (note that in some of these cases $\Phi^{S(3)}_{ij} = 0$).

(B) In all cases ($n \geq 5$), we have

$$\Omega'_{ij} = O(r^{1-n/2}), \tag{85}$$

with $(n-4)\Omega'^{(n/2-1)}_{ij} = -n l_{11}\Phi^{S(n/2)}_{ij} - 2X^{A0}\Phi^{S(n/2)}_{ij,A} - 4\Phi^{S(n/2)}_{s(j}\overset{s}{m}_{i)1}$. In the special case $\nu = 1 + n/2$, this can be written in terms of $\Omega^{(n/2+1)}_{ij}$ using the form of $\Phi^{S(n/2)}_{ij}$ given in the above Sec. IVA 1.

(C)

$$\text{if } n > 4: \quad \Omega'_{ij} = o(r^{2-n}), \tag{86}$$

$$\text{if } n = 4: \quad \Omega'_{ij} = O(r^{-1}). \tag{87}$$

To obtain the above behavior, in the $n > 4$ case, it is also necessary to recall that at the leading order $\Phi^S_{ij} \propto \delta_{ij}$ (cf. Sec. IVA 1).

4. Summary of case i

In all cases given here, we have

$$\Omega_{ij} = O(r^{-\nu}) \quad (\nu > 2),$$
$$\Psi_{ijk} = O(r^{-\nu}). \tag{88}$$

This will not be repeated every time below, where we will give only possible further restrictions on ν. See also Secs. IVA 1–IVA 3 for relations among the leading-order terms of various boost weight.

(A) Here, we have $n > 5$ and the following possible behaviors (cf. Sec. IVA 1 for a few further special subcases):

A1:

$$\begin{aligned} &\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = o(r^{-2}),\\ &\Phi^A_{ij} = o(r^{-2}) \quad (n > 5, 2 < \nu \leq 3),\\ &\Psi'_{ijk} = O(r^{-2}),\\ &\Omega'_{ij} = O(r^{\sigma}) \quad (-2 \leq \sigma < -1). \end{aligned} \tag{89}$$

A2:

$$\begin{aligned} &\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}),\\ &\Phi = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-3})\\ &\quad (n > 5, 3 < \nu < 4),\\ &\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\ &\Omega'_{ij} = O(r^{-2}). \end{aligned} \tag{90}$$

A3:

$$\begin{aligned} &\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}),\\ &\Phi = O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-3}) \quad (n > 5, \nu \geq 4),\\ &\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\ &\Omega'_{ij} = O(r^{-2}), \end{aligned} \tag{91}$$

with the further restrictions $\Phi^S_{ij} = O(r^{1-\nu})$ for $4 \leq \nu < 5$ and $\Phi^S_{ij} = O(r^{-4})$ for $\nu \geq 5$.

A4:

$$\begin{aligned} &\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-\nu}),\\ &\Phi = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-\nu})\\ &\quad (n > 5, \nu \geq 4, \nu \neq n),\\ &\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{1-\nu}),\\ &\Omega'_{ij} = O(r^{-2}). \end{aligned} \tag{92}$$

A5:

$$\begin{aligned} &\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}),\\ &\Phi^A_{ij} = O(r^{-n}) \quad (n > 5, \nu \geq n),\\ &\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{1-n}),\\ &\Omega'_{ij} = O(r^{-2}). \end{aligned} \tag{93}$$

None of the above five cases can describe asymptotically flat spacetimes; cf. Ref. [15]. In cases

A2–A5, the leading term at infinity falls off as $1/r^2$, and it is of type II(abd). In cases A3–A5, ℓ can be a multiple WAND; cf. also the results of Ref. [30]. Examples in case A5 are Robinson–Trautman Ricci-flat spacetimes [24].

(B) For any $n > 5$, we have

$$\Phi_{ijkl} = O(r^{-n/2}), \qquad \Phi = O(r^{-\nu}),$$
$$\Phi^A_{ij} = O(r^{-\nu}) \quad \left(n > 5, \frac{n}{2} < \nu \leq 1 + \frac{n}{2}\right),$$
$$\Psi'_{ijk} = O(r^{-n/2}), \qquad \Omega'_{ij} = O(r^{1-n/2}). \tag{94}$$

Note that here ℓ cannot be a WAND. The leading term at infinity falls off as $1/r^{n/2-1}$, and it is of type N. At order $1/r^{n/2}$, the type becomes II(acd) (as follows from Sec. IVA 1).

For $n = 5$, the same behavior applies if $3 < \nu \leq \frac{7}{2}$, while $\Psi'_{ijk} = O(r^{-2})$ if $\frac{5}{2} < \nu \leq 3$ (the other terms being unchanged).

If we take for b.w. +2 components $\nu = 1 + \frac{n}{2}$ and additionally *assume* that

$$\Omega_{ij} = \frac{\Omega^{(n/2+1)}_{ij}}{r^{n/2+1}} + \frac{\Omega^{(n/2+2)}_{ij}}{r^{n/2+2}} + o(r^{-n/2-2}), \tag{95}$$

then (B4) with (B5) of Ref. [18] show that the subleading term of Ω'_{ij} is of order $O(r^{-n/2})$, which with (94) implies the following peeling-off behavior:

$$C_{abcd} = \frac{N_{abcd}}{r^{n/2-1}} + \frac{II_{abcd}}{r^{n/2}} + o(r^{-n/2}) \quad (n \geq 5). \tag{96}$$

This result is in agreement with the conclusions of Ref. [15] for asymptotically flat spacetimes (and extends it to asymptotics along twisting null geodesics). However, to obtain higher-order terms, one would need to make further assumptions on how Ω_{ij} can be expanded, which goes beyond the analysis of the present paper (however, recall that it is precisely at a higher order in (96) that Ref. [15] found a qualitative difference between five and higher dimensions). In five dimensions, a permitted behavior more general than (96) is described in Sec. IV C 2 below (it does not appear here because it belongs to case iii).

In view of Ref. [15], we conclude that the above behavior (94) includes *radiative spacetimes* that are asymptotically flat in the Bondi definition [16,17] (which is equivalent [15] to the conformal definition [11,12] in even dimensions).

If one takes $\nu > 1 + \frac{n}{2}$ in (94), this reduces to (99) if $1 + \frac{n}{2} < \nu \leq n - 1$, to (97) if $n - 1 < \nu \leq n$, and to (105) if $\nu > n$.

(C) For $n > 4$, the falloff is

$$\Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = o(r^{1-n})$$
$$(n > 4, \nu > n - 1),$$
$$\Psi'_{ijk} = O(r^{1-n}), \qquad \Omega'_{ij} = o(r^{2-n}), \tag{97}$$

with $\Phi^A_{ij} = O(r^{-\nu})$ for $n - 1 < \nu < n$ and $\Phi^A_{ij} = O(r^{-n})$ for $\nu \geq n$. Here, ℓ can become a multiple WAND; cf. Refs. [27,30]. This behavior is compatible with the results of Ref. [15] for asymptotically flat spacetimes, in the case of *vanishing radiation*. In particular, it includes asymptotically flat spacetimes for which ℓ is a multiple WAND [27,30], such as Ricci-flat Robinson–Trautman spacetimes [24] (e.g., Schwarzschild–Tangherlini black holes) and Kerr–Schild spacetimes [26] with a nondegenerate Kerr–Schild vector[6] (e.g., Myers–Perry black holes).

For $n = 4$, we have instead

$$\Phi_{ijkl} = O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}) \quad (n = 4, \nu > 3),$$
$$\Psi'_{ijk} = O(r^{-2}), \qquad \Omega'_{ij} = O(r^{-1}), \tag{98}$$

where the leading $1/r$ term is of type N. However, this is not the "standard" four-dimensional peeling behavior, which would require the stronger condition $\nu = 5$ [3]. Generalized peeling properties under asymptotic conditions weaker than those of Ref. [3] have been already studied in four dimensions, e.g., in Refs. [34–37]. We note that the assumption made in this paper that leading-order terms of Weyl components are powerlike is in fact generically too restrictive in those cases (for example, for $\nu = 4$, the natural framework to consider is that of polyhomogenous expansions [37]). Similar comments will apply to (100) below.

5. Special subcase $\beta_c = \beta = -\nu$

In addition, there is the case $\beta_c = \beta = -\nu$ (briefly mentioned in Sec. IVA 1 above but not explicitly studied in Secs. IVA 2 and IVA 3), for which one easily arrives for $n > 4$ at [note that (82) still applies here]

[6]For these, one finds $\Omega'_{ij} = O(r^{1-n})$. Note that in order to explicitly verify this using the general expressions given in Ref. [26] one should recall to enforce the vacuum equation $R_{11} = 0$; cf. Ref. [33]. The same comment applies to the (A)dS Kerr–Schild spacetimes [29] mentioned in Sec. III A 4.

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-\nu}), \quad (n > 4)\\
&\Psi'_{ijk} = O(r^{-2}) \quad \text{if } 2 < \nu \le 3,\\
&\Psi'_{ijk} = O(r^{-\nu}) \quad \text{if } \nu > 3,\\
&\Omega'_{ij} = o(r^{1-\nu}) \quad \text{if } \nu \neq \frac{n}{2},\\
&\Omega'_{ij} = O(r^{1-n/2}) \quad \text{if } \nu = \frac{n}{2},
\end{aligned} \tag{99}$$

with $X^{A0}\Omega^{(\nu)}_{ij,A} + (\nu-2)l_{11}\Omega^{(\nu)}_{ij} + 2\Omega^{(\nu)}_{s(j}\overset{s}{M}_{i)1} = 0$. ℓ cannot be a WAND. The above conditions on Ω'_{ij} have been obtained by using (B4) and the trace of (B13) of Ref. [18].

For $n = 4$, one finds instead

$$\begin{aligned}
&\Phi_{ijkl} = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-\nu}), \quad (n = 4, \nu > 2)\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Omega'_{ij} = O(r^{-1}),
\end{aligned} \tag{100}$$

which is asymptotically of type N. For $\nu > 4$, this is a subcase of (107) having $\Phi^{(3)}_{ijkl} = 0$, $\Phi^{A(3)}_{ij} = 0$ and $\Psi^{(4)}_{ijk} = 0$.

B. Case ii: $\Psi_{ijk} = O(r^{-n})$, $\Omega_{ij} = o(r^{-n})$

The behavior of the Ricci rotation coefficients and derivative operators is the same as in case (i) [in particular, 34) ,(32) ,(31) ,(29)), and (82) still apply].

1. Case $\beta_c = -2$, $n > 5$

All the following cases can occur only for $n > 5$:

$$\begin{aligned}
&\Omega_{ij} = o(r^{-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-4}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{-3}), \qquad \Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{101}$$

with $(n-4)\Phi^{A(3)}_{ij} = \Phi^{(2)}_{ikjl}b_{[kl]}$ and $\Phi^{(2)}_{ikjl}b_{(kl)} = 0$. Here, ℓ can be a single WAND. For $\Psi^{(n)}_{ijk} = 0$, this case reduces to (91) (with $\nu > n$).

If $\Phi^{(2)}_{ikjl}b_{[kl]} = 0$ (in particular, if ℓ is twistfree), the following subcase arises:

$$\begin{aligned}
&\Omega_{ij} = o(r^{-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi_i{}' = O(r^{1-n}),\\
&\Omega'_{ij} = O(r^{-2}),
\end{aligned} \tag{102}$$

with $(2-n)\Phi^{S(n-1)}_{ij} + \Phi^{(n-1)}\delta_{ij} = 0$.

As a further "subcase", if $\Phi^{S(n-1)}_{ij} = 0$, we obtain, depending on the value of ν,

$$\begin{aligned}
&\Omega_{ij} = O(r^{-\nu}) \quad (n < \nu \le n+1),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{1-\nu}),\\
&\Phi = O(r^{-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi_i{}' = O(r^{1-\nu}),\\
&\Omega'_{ij} = O(r^{-2})
\end{aligned} \tag{103}$$

or

$$\begin{aligned}
&\Omega_{ij} = O(r^{-\nu}) \quad (\nu > n+1),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{-2}), \qquad \Psi'_i = O(r^{1-n}),\\
&\Omega'_{ij} = O(r^{-2}).
\end{aligned} \tag{104}$$

In all the above cases, $\Psi'^{(2)}_{ijk} = -\Phi^{(2)}_{isjk}l_{s1}$ and $\Omega'^{(2)}_{ij} = -\Psi'^{(2)}_{(ij)k}l_{k1} = \Phi^{(2)}_{isjk}l_{s1}l_{k1}$. The asymptotically leading term is of type II(abd), but it reduces to type D(abd) if a particular frame with $l_{i1} = 0$ is employed; cf. the comments at the end of Sec. II C. The terms $\Phi_{ijkl} = O(r^{-2})$ violate the asymptotically flat conditions [15].

2. Case $\beta_c < -2$, $n > 4$

If $\Phi^{(2)}_{ijkl} = 0$, one is left with

$$\begin{aligned}
&\Omega_{ij} = o(r^{-n}),\\
&\Psi_{ijk} = O(r^{-n}),\\
&\Phi_{ijkl} = O(r^{1-n}), \qquad \Phi^A_{ij} = O(r^{-n}),\\
&\Psi'_{ijk} = O(r^{1-n}), \qquad \Psi_i{}' = O(r^{1-n}),\\
&\Omega'_{ij} = o(r^{2-n}),
\end{aligned} \tag{105}$$

with $(n-2)(n-3)\Phi^{(n-1)}_{ijkl} = 4\Phi^{(n-1)}\delta_{j[k}\delta_{m]i}$, $(n-3)\Psi^{(n-1)}_{ijk} = 2\Psi^{(n-1)}_{[j}\delta_{k]i}$, $(n-2)\Psi^{(n-1)}_i = -(n-1)\Phi^{(n-1)}l_{i1}$, and where ℓ can be a single WAND. This behavior is compatible with the results of Ref. [15] for asymptotically flat spacetimes, in the case of vanishing radiation. For $\Psi^{(n)}_{ijk} = 0$, this case reduces to (97) (with $\nu > n$).

If $\Phi^{(n-1)} = 0$, this reduces to

$$\begin{aligned}
\Omega_{ij} &= o(r^{-n}),\\
\Psi_{ijk} &= O(r^{-n}),\\
\Phi_{ijkl} &= O(r^{-n}), \qquad \Phi^A_{ij} = O(r^{-1-n}),\\
\Psi'_{ijk} &= O(r^{-n}),\\
\Omega'_{ij} &= O(r^{1-n}). \qquad (106)
\end{aligned}$$

The asymptotically leading term is of type N.

3. Case $n = 4$

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu > 4),\\
\Psi_{ijk} &= O(r^{-4}),\\
\Phi_{ijkl} &= O(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
\Psi'_{ijk} &= O(r^{-2}),\\
\Omega'_{ij} &= O(r^{-1}). \qquad (107)
\end{aligned}$$

The above behavior agrees with the well-known results of Ref. [3] (where it was assumed $\nu = 5$). For $\Psi^{(4)}_{ijk} = 0$, this case reduces to (98) (with $\nu > 4$). See Ref. [4] for results also at the subleading order.

C. Case iii: $\Psi_{ijk} = O(r^{-3})$, $\Omega_{ij} = o(r^{-3})$ $(n > 4)$

Again, the behavior of the Ricci rotation coefficients and derivative operators is the same as in case i.

1. Case $n > 5$

In more than five dimensions, we generically have $\beta_c = -2$, giving rise to

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu > 3),\\
\Psi_{ijk} &= O(r^{-3}), \qquad \Psi_i = o(r^{-3}),\\
\Phi_{ijkl} &= O(r^{-2}), \qquad \Phi^S_{ij} = O(r^{-3}),\\
\Phi &= o(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
\Psi'_{ijk} &= O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\
\Omega'_{ij} &= O(r^{-2}), \qquad (108)
\end{aligned}$$

where $\Psi_i = O(r^{-\nu})$, $\Phi = O(r^{-\nu})$ for $3 < \nu \leq 4$, while $\Psi_i = O(r^{-4})$, $\Phi = O(r^{-4})$ for $\nu > 4$ and

$$\begin{aligned}
(n-4)\Phi^{A(3)}_{ki} &= \Phi^{(2)}_{klij} b_{[lj]} + \xi^{A0}_j \Psi^{(3)}_{[ki]j,A} + 2l_{1j}\Psi^{(3)}_{[ki]j} + \Psi^{(3)}_{[ki]l}\stackrel{l}{M}_{jj} + \Psi^{(3)}_{jl[k}\stackrel{l}{M}_{i]j},\\
(n-6)\Phi^{S(3)}_{ki} &= \Phi^{(2)}_{klij} b_{(lj)} + \xi^{A0}_j \Psi^{(3)}_{(ki)j,A} + 2l_{1j}\Psi^{(3)}_{(ki)j} + \Psi^{(3)}_{(ki)l}\stackrel{l}{M}_{jj} + (2\Psi^{(3)}_{l(k|j} + \Psi^{(3)}_{jl(k|})\stackrel{l}{M}_{|i)j}.
\end{aligned}$$

Here, $\boldsymbol{\ell}$ can be a single WAND, and the asymptotically leading term is of type II(abd). For $\Psi^{(3)}_{ijk} = 0$, this case reduces for $3 < \nu < 4$ to (90) (with $\nu > n$), for $4 \leq \nu \leq n$ to (91), and for $\nu > n$ to (101).

A subcase with $\Phi^{(2)}_{ijkl} = 0$ is also possible, giving

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad (\nu > 3),\\
\Psi_{ijk} &= O(r^{-3}), \qquad \Psi_i = o(r^{-3}),\\
\Phi_{ijkl} &= O(r^{-3}), \qquad \Phi = o(r^{-3}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
\Psi'_{ijk} &= O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\
\Omega'_{ij} &= O(r^{-2}), \qquad (109)
\end{aligned}$$

with the same behavior as above for Ψ_i and Φ. In this case, the leading term at infinity is of type III(a).

Neither of the above behaviors can represent asymptotically flat spacetimes since the falloff of the Weyl tensor is too slow [15].

2. Case $n = 5$

In five dimensions, we generically have

$$\begin{aligned}
\Omega_{ij} &= O(r^{-\nu}) \quad \left(3 < \nu \leq \frac{7}{2}\right),\\
\Psi_{ijk} &= O(r^{-3}), \qquad \Psi_i = O(r^{-\nu}),\\
\Phi_{ijkl} &= O(r^{-5/2}), \qquad \Phi = O(r^{-\nu}), \qquad \Phi^A_{ij} = O(r^{-3}),\\
\Psi'_{ijk} &= O(r^{-2}), \qquad \Psi'_i = O(r^{-3}),\\
\Omega'_{ij} &= O(r^{-3/2}), \qquad (110)
\end{aligned}$$

with $\Phi^{A(3)}_{ki} = \xi^{A0}_j \Psi^{(3)}_{[ki]j,A} + 2l_{1j}\Psi^{(3)}_{[ki]j} + \Psi^{(3)}_{[ki]l}\stackrel{l}{M}_{jj} + \Psi^{(3)}_{jl[k}\stackrel{l}{M}_{i]j}$, $\Psi'^{(2)}_{ijk}$ can be expressed in terms of $\Psi^{(3)}_{ijk}$ using (B6) of Ref. [18], and $\Omega'^{(3/2)}_{ij} = -5l_{11}\Phi^{S(5/2)}_{ij} - 2X^{A0}\Phi^{S(5/2)}_{ij,A} - 4\Phi^{S(5/2)}_{s(j}\stackrel{s}{M}_{i)1}$. If $\nu = 7/2$, this can be rewritten using $3\Phi^{S(5/2)}_{ij} = -2X^{A0}\Omega^{(7/2)}_{ij,A} - 3l_{11}\Omega^{(7/2)}_{ij} - 4\Omega^{(7/2)}_{s(j}\stackrel{s}{M}_{i)1}$. Recalling the comments following (94), one finds that the same behavior (110) holds in fact for the full range $\frac{5}{2} < \nu \leq \frac{7}{2}$

TABLE II. Falloff behavior of the Weyl tensor for Ricci-flat spacetimes ($\tilde{R} = 0$), listing the cases summarized in Secs. IV A 4, IV B, IV C. The conventions explained in Table I apply also here. Note that, for brevity, we have not included here the very special subcase of Sec. IV A 5.

Case	Ω_{ij}	Ψ_{ijk}, Ψ_i	Φ_{ijkl}	Φ^S_{ij}, Φ	Φ^A_{ij}	Ψ'_{ijk}, Ψ'_i	Ω'_{ij}	Restrictions	Comments
i A1	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	$o(r^{-2})$	$o(r^{-2})$	r^{-2}	r^{σ}	$n>5,\ 2<\nu\leq 3,\ -2\leq\sigma<-1$	ℓ not a WAND
i A2	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	r^{-3}, $r^{-\nu}$	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5,\ 3<\nu<4$	ℓ not a WAND
i A3	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	$r^{1-\nu}$, r^{-4}	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5,\ 4\leq\nu<5$	ℓ not a WAND
	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	r^{-4}	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5,\ \nu\geq 5$	
i A4	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	$r^{1-\nu}$, $r^{-\nu}$	$r^{-\nu}$	r^{-2}, $r^{1-\nu}$	r^{-2}	$n>5,\ \nu\geq 4,\ \nu\neq n$	
i A5	$r^{-\nu}$	$r^{-\nu}$	r^{-2}	r^{1-n}	r^{-n}	r^{-2}, r^{1-n}	r^{-2}	$n>5,\ \nu\geq n$	includes RT
i B	$r^{-\nu}$	$r^{-\nu}$	$r^{-n/2}$	$r^{-n/2}$, $r^{-\nu}$	$r^{-\nu}$	$r^{-n/2}$	$r^{1-n/2}$	$n>5$ and $n/2<\nu\leq n/2+1$	radiation, ℓ not a WAND
								or $n=5$ and $3<\nu\leq 7/2$	ℓ not a WAND
	$r^{-\nu}$	$r^{-\nu}$	$r^{-5/2}$	$r^{-5/2}$, $r^{-\nu}$	$r^{-\nu}$	r^{-2}	$r^{-3/2}$	$n=5$ and $5/2<\nu\leq 3$	ℓ not a WAND
i C	$r^{-\nu}$	$r^{-\nu}$	r^{1-n}		$o(r^{1-n})$	r^{1-n}	$o(r^{2-n})$	$\nu>n-1$	includes RT, KS
	$r^{-\nu}$	$r^{-\nu}$	r^{-3}		r^{-3}	r^{-2}	r^{-1}	$n=4,\ \nu>3$	
ii	$o(r^{-n})$	r^{-n}	r^{-2}	r^{-4}	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5$	
	$o(r^{-n})$	r^{-n}	r^{-2}	r^{1-n}	r^{-n}	r^{-2}, r^{1-n}	r^{-2}	$n>5$	
	$r^{-\nu}$	r^{-n}	r^{-2}	$r^{1-\nu}$, r^{-n}	r^{-n}	r^{-2}, $r^{1-\nu}$	r^{-2}	$5<n<\nu\leq n+1$	ℓ not a WAND
	$r^{-\nu}$	r^{-n}	r^{-2}	r^{-n}	r^{-n}	r^{-2}, r^{1-n}	r^{-2}	$n>5,\ \nu>n+1$	
	$o(r^{-n})$	r^{-n}	r^{1-n}		r^{-n}	r^{1-n}	$o(r^{2-n})$		
	$o(r^{-n})$	r^{-n}	r^{-n}		r^{-n-1}	r^{-n}	r^{1-n}		
	$r^{-\nu}$	r^{-4}	r^{-3}		r^{-3}	r^{-2}	r^{-1}	$n=4,\ \nu>4$	
iii	$r^{-\nu}$	r^{-3}, $r^{-\nu}$	r^{-2}	r^{-3}, $r^{-\nu}$	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5,\ 3<\nu\leq 4$	ℓ not a WAND
	$r^{-\nu}$	r^{-3}, r^{-4}	r^{-2}	r^{-3}, r^{-4}	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5,\ \nu>4$	
	$r^{-\nu}$	r^{-3}, $o(r^{-3})$	r^{-3}	r^{-3}, $o(r^{-3})$	r^{-3}	r^{-2}, r^{-3}	r^{-2}	$n>5$	
								or $n=5$ and $\nu>7/2$	
	$r^{-\nu}$	r^{-3}, $r^{-\nu}$	$r^{-5/2}$	$r^{-5/2}$, $r^{-\nu}$	r^{-3}	r^{-2}, r^{-3}	$r^{-3/2}$	$n=5,\ 5/2<\nu\leq 7/2$	ℓ not a WAND

(unless $\Psi^{(3)}_{ijk} = 0$). In all cases here, ℓ cannot be a WAND, and the asymptotically leading term is of type N.

Note an important difference with the behavior (94) with $n = 5$: after the leading type N term, the subleading term in (110) is of type III(a) [it was of type II(acd) in (94)]. If we assume for Ω_{ij} a falloff as in (95), this shows that the subleading term of Ω'_{ij} is of order $O(r^{-2})$, thus leading to the qualitatively different peeling-off behavior

$$C_{abcd} = \frac{N_{abcd}}{r^{3/2}} + \frac{III_{abcd}}{r^2} + o(r^{-2}) \quad (n = 5). \qquad (111)$$

However, according to Ref. [15], this behavior is not permitted in asymptotically flat spacetimes. For the latter, one thus concludes that $\Psi^{(3)}_{ijk} = 0$ [in which case (110) reduces to (94) with $n = 5$] is a necessary boundary condition in five dimensions. This is perhaps not surprising since $\Psi^{(3)}_{ijk} = 0$ already in four dimensions [where $\Psi_{ijk} = O(r^{-4})$ [3]; cf. also (107) above].

If $\nu > 7/2$, the asymptotic behavior is described by (109) (in which cases ℓ can be a single WAND).

All the above results for the case $\tilde{R} = 0$ are summarized in Table II.

ACKNOWLEDGMENTS

The authors acknowledge support from research plan RVO: 67985840 and research Grant No. GAČR 13-10042S.

[1] H. Bondi, M. G. J. van der Burg, and A. W. K Metzner, Proc. R. Soc. A **269**, 21 (1962).
[2] R. K. Sachs, Proc. R. Soc. A **270**, 103 (1962).
[3] E. T. Newman and R. Penrose, J. Math. Phys. (N.Y.) **3**, 566 (1962)see also E. Newman and R. Penrose, J. Math. Phys. (N.Y.) **4**, 998(E) (1963).

[4] E. T. Newman and T. W. J. Unti, J. Math. Phys. (N.Y.) **3**, 891 (1962).
[5] I. Robinson and A. Trautman, Phys. Rev. Lett. **4**, 431 (1960).
[6] R. K. Sachs, Z. Phys. **157**, 462 (1960).
[7] R. Sachs, Proc. R. Soc. A **264**, 309 (1961).
[8] R. Penrose, Phys. Rev. Lett. **10**, 66 (1963).
[9] R. Penrose, Proc. R. Soc. A **284**, 159 (1965).
[10] R. Penrose and W. Rindler, *Spinors and Space-Time*, Vol. 2 (Cambridge University Press, Cambridge, England, 1986).
[11] S. Hollands and A. Ishibashi, J. Math. Phys. (N.Y.) **46**, 022503 (2005).
[12] A. Ishibashi, Classical Quantum Gravity **25**, 165004 (2008).
[13] Y. Choquet-Bruhat, P. T. Chruściel, and J. Loizelet, Classical Quantum Gravity **23**, 7383 (2006).
[14] S. Hollands and R. M. Wald, Classical Quantum Gravity **21**, 5139 (2004).
[15] M. Godazgar and H. S. Reall, Phys. Rev. D **85**, 084021 (2012).
[16] K. Tanabe, N. Tanahashi, and T. Shiromizu, J. Math. Phys. (N.Y.) **51**, 062502 (2010).
[17] K. Tanabe, S. Kinoshita, and T. Shiromizu, Phys. Rev. D **84**, 044055 (2011).
[18] V. Pravda, A. Pravdová, A. Coley, and R. Milson, Classical Quantum Gravity **21**, 2873 (2004); see also V. Pravda, A. Pravdová, A. Coley, and R. Milson, Classical Quantum Gravity **24**, 1691 (2007) (Corrigendum).
[19] A. Coley, R. Milson, V. Pravda, and A. Pravdová, Classical Quantum Gravity **21**, 5519 (2004).
[20] M. Ortaggio, V. Pravda, and A. Pravdová, Classical Quantum Gravity **24**,1657 (2007).
[21] M. Durkee, V. Pravda, A. Pravdová, and H. S. Reall, Classical Quantum Gravity **27**, 215010 (2010).
[22] M. Ortaggio, V. Pravda, and A. Pravdová, Classical Quantum Gravity **30**, 013001 (2013).
[23] E. T. Newman and K. P. Tod, *General Relativity and Gravitation: One Hundred Years After the Birth of Albert Einstein*, Vol. 2, edited by A. Held (Plenum, New York, 1980), p. 1.
[24] J. Podolský and M. Ortaggio, Classical Quantum Gravity **23**, 5785 (2006).
[25] A. Pravdová and V. Pravda, Classical Quantum Gravity **25**, 235008 (2008).
[26] M. Ortaggio, V. Pravda, and A. Pravdová, Classical Quantum Gravity **26**, 025008 (2009).
[27] M. Ortaggio, V. Pravda, and A. Pravdová, Phys. Rev. D **80**, 084041 (2009).
[28] M. Ortaggio, V. Pravda, and A. Pravdová, Phys. Rev. D **82**, 064043 (2010).
[29] T. Málek and V. Pravda, Classical Quantum Gravity **28**, 125011 (2011).
[30] M. Ortaggio, V. Pravda, and A. Pravdová (unpublished).
[31] V. Pravda, A. Pravdová, and M. Ortaggio, Classical Quantum Gravity **24**, 4407 (2007).
[32] A. Ashtekar and S. Das, Classical Quantum Gravity **17**, L17 (2000).
[33] M. Kršsák, Diploma thesis, Charles University in Prague, 2009.
[34] W. E. Couch and R. J. Torrence, J. Math. Phys. (N.Y.) **13**, 69 (1972).
[35] D. Christodoulou and S. Klainerman, *The Global Nonlinear Stability of the Minkowski Space* (Princeton University, Princeton, NJ, 1993).
[36] P. T. Chruściel, M. A. H. MacCallum, and D. B. Singleton, Phil. Trans. R. Soc. A **350**, 113 (1995).
[37] J. A. Valiente Kroon, Classical Quantum Gravity **15**, 2479 (1998).