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Equivalence of the Einstein and Jordan frames

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No experiment can measure an absolute scale: every dimensionful quantity has to be compared to some fixed unit scale in order to be measured, and thus only dimensionless quantities are really physical. The Einstein and Jordan frames are related by a conformal transformation of the metric, which amounts to rescaling all length scales. Since the absolute scale cannot be measured, both frames describe the same physics and are equivalent. In this article we make this explicit by rewriting the action in terms of dimensionless variables, which are invariant under a conformal transformation. For definitiveness, we concentrate on the action of Higgs inflation, but the results can easily be generalized. In addition, we show that the action for $f(R)$ gravity, which includes Starobinsky inflation, can be written in a frame-independent form.

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I. INTRODUCTION

In recent years there has been renewed interest in inflation models with a large nonminimal coupling to gravity, of which Higgs inflation is the prime example [1–3]. Although the predictions of these models fall right in the sweet spot of the Planck Collaboration data [4], they can all go in the dust bin if the polarization signal measured by BICEP is of cosmological origin [5,6]. Even in this case, a (much smaller) nonminimal coupling is still allowed [7–9], and it is thus important to understand its implications.

The nonminimal coupling to gravity entails a coupling between the Ricci scalar and the inflaton field, which mixes the metric and scalar degrees of freedom. This also implies that the effective Planck mass during inflation is field dependent, and thus time dependent, which in turn hampers a physical interpretation of the equations in the Jordan frame. For example, when defining the expansion rate of the Universe, one has to take into account that not only the scale factor is time dependent, but also the measurement unit; when defining an energy-momentum tensor, one has to take into account that gravitational and field energies are mixed, and so on.

The nonminimal coupling can be removed, and the gravity Lagrangian brought into canonical form, by performing a conformal transformation of the metric. Since gravity is now standard and the Planck mass a constant, the Einstein frame equations are easy to interpret—all of the usual textbook intuition applies—but complicated. The scalar field kinetic terms are noncanonical and the potential is nonpolynomial.

Calculations can be done in either frame. The conformal transformation can be seen just as a field redefinition which does not affect the physics. It has been shown that both the classical action and the one-loop corrections are frame independent [10], as is the curvature perturbation [11–15]. Nevertheless, there is still some confusion in the literature and conflicting claims exist [16–23]. Our results are in line with earlier work [24–27].

In this paper we will show explicitly that the Jordan and Einstein frames are equivalent by rewriting the action in terms of dimensionless fields and parameters. A conformal transformation of the metric rescales all length scales, or equivalently all mass scales, in the theory. It is important to note that this does not affect physical quantities, which are dimensionless; no experiment can measure an absolut scale. Hence, if we rewrite the action in terms of physical, dimensionless fields, it is automatically invariant under a conformal transformation: the action obtained describes all frames related by a conformal transformation at once, and thus all the results derived from it apply equally to the Jordan and Einstein frames.

This paper is organized as follows: we start in Sec. II with a brief review of the action of Higgs inflation in the Einstein and Jordan frames, and the conformal transformation which relates them. We then explain our approach to rewriting the Lagrangian in terms of dimensionless variables, applying it to the simple setting of the classical background action only. This is subsequently generalized to the full action, with a generic spacetime metric, more than one field coupled to gravity, and arbitrary kinetic terms. We remark briefly on quantization. In Sec. III we show that our results apply equally to $f(R)$ gravity, and in particular to Starobinsky inflation. Finally, in Sec. IV we give an example and discuss how curvature perturbation ζ can be expressed in dimensionless, Jordan frame, and Einstein frame quantities. We end with some concluding remarks.

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II. EQUIVALENCE OF THE ACTION

The action for Higgs inflation in the Jordan frame is [1–3]

$$S = \int d^4x\sqrt{-g_J}\left[\frac{1}{2}M^2\Omega^2 R[g_J] - \frac{1}{2}\gamma_{Jab}g_J^{\mu\nu}\partial_\mu\phi_J^a\partial_\nu\phi_J^b - V_J(\phi_J)\right],$$

where ϕ_J^a are real scalar fields, ξ is the nonminimal coupling that mixes the metric and scalar degrees of freedom, and

$$\Omega^2 = 1 + \frac{\xi\phi_J^a\phi_{Ja}}{M^2}. \quad (1)$$

Although our focus is on the conformal factor (1) motivated by Higgs inflation, our methods can be generalized to generic conformal factors [28]. The gravitational action can be brought into canonical form via a conformal transformation of the metric

$$g_{E\mu\nu} = \Omega^2 g_{J\mu\nu}. \quad (2)$$

This yields the action in the Einstein frame [29]

$$S = \int d^4x\sqrt{-g_E}\left[\frac{1}{2}M^2 R[g_E] - \frac{1}{2}g_E^{\mu\nu}\gamma_{Eab}\partial_\mu\phi_J^a\partial_\nu\phi_J^b - V_E(\phi_J)\right], \quad (3)$$

with field-space metric

$$\gamma_{Eab} = \frac{1}{\Omega^2}[\gamma_{Jab} + 3\partial_{\phi_J^a}\ln\Omega^2\partial_{\phi_J^b}\ln\Omega^2]. \quad (4)$$

The Einstein frame potential is

$$V_E = \frac{V_J}{\Omega^4}. \quad (5)$$

Note that the fields ϕ_J^a do not change going from one frame to the other. We added the subscript J to denote the fields originally defined in the Jordan frame; the use of this will become clear soon.

From the action we can read off the (reduced) Planck mass in the Jordan and Einstein frames, respectively:

$$m_J = M\Omega, \qquad m_E = M. \quad (6)$$

Although the metric changes under a conformal transformation, distances measured in Planck units are invariant. Indeed, the line element written in Planck units is invariant:

$$m_J^2 ds_J^2 = m_J^2 g_{J\mu\nu}dx^\mu dx^\nu = m_E^2 g_{E\mu\nu}dx^\mu dx^\nu = m_E^2 ds_E^2, \quad (7)$$

where we used (2), (6).

A. Dimensionless action—background

In this subsection we rewrite the classical background action in terms of dimensionless quantities which transform trivially under a conformal transformation: this shows clearly our approach in a simple setting. Then, in the next subsection, we will generalize our results to the full action.

For simplicity, we take the field-space metric in the Jordan frame to be canonical, $\gamma_{Jij} = \delta_{ij}$, and specialize to the case of a single, homogeneous background field $\phi_J^a(x) = \phi_J(t)$. The metric is of the Friedmann-Robertson-Walker form, and (7) can be written

$$m_J^2 ds_J^2 = m_J^2[-N_J^2 dt^2 + a_J^2 dx^2] = m_E^2[-N_E^2 dt^2 + a_E^2 dx^2] = m_E^2 ds_E^2, \quad (8)$$

with N_J, a_J and N_E, a_E the lapse function and scale factor in the Jordan and Einstein frames, respectively. We choose the coordinates to be dimensionless and take N_i, a_i to have dimensions of inverse mass.[1] We define the dimensionless metric functions, denoted by an overbar, via

$$\bar{N}_i = m_i N_i, \qquad \bar{a}_i = m_i a_i, \qquad i = J, E, \quad (9)$$

where m_i is the frame-dependent Plank mass (6). The dimensionless quantities transform trivially under a conformal transformation, e.g., $\bar{N}_J = \bar{N}_E$. To make this explicit we drop the subscript index on the barred quantities and simply write $\bar{N}$, etc. All barred quantities defined below transform trivially; they correspond to the respective quantities expressed in Planck units.

Now let us rewrite the background action in terms of physical dimensionless quantities. We start from the Einstein frame action (3), which can be expressed

$$S = \int d^4x\sqrt{-g_E}m_E^4\left[\frac{1}{2}\frac{R[g_E]}{m_E^2} - \frac{1}{2}\gamma_E\frac{\dot{\phi}_J^2}{N_E^2 m_E^4} - \frac{V_J}{\Omega^4 m_E^4}\right], \quad (10)$$

where $\gamma_E \equiv \gamma_{E\phi\phi}$, with the metric given in (4), and the dot denotes the time derivative $\dot{\phi}_J = \partial_t\phi_J$.

All of the separate terms in the action (10) and also the measure are written as dimensionless combinations. We are now going to rewrite these terms in a form that makes explicit that they are all separately invariant under a conformal transformation. Consider first the measure: we define the invariant combination

[1]We could just as well have used the more standard convention with a, N dimensionless and the coordinate's dimensions of inverse mass, but in that case attention should be paid in defining the Hubble parameter, which should be taken as $H = \partial_t \ln \Delta x$ (i.e., the rate of change of a physical coordinate's distance rather than of the unphysical scale factor $H = \partial_t \ln a$).

$$\sqrt{-\bar{g}} = \sqrt{-g_i} m_i^4, \quad i = J, E. \tag{11}$$

It can be checked explicitly that it is invariant under a conformal transformation $\sqrt{-\bar{g}} = \sqrt{-g_E} m_E^4 = N_E a_E^3 m_E^4 = N_J a_J^3 m_J^4 = \sqrt{-g_J} m_J^4$. The dimensionless potential can likewise be defined

$$\bar{V} = \frac{V_i}{m_i^4}, \tag{12}$$

where we used (5). Note that if $V_J = \lambda \phi_J^4$, this implies the scaling

$$\phi_E = \frac{\phi_J}{\Omega}. \tag{13}$$

This is, in general, the case: dimensionful variables scale with a factor Ω under a conformal transformation. As a consequence, physical observables which are dimensionless ratios remain invariant. Care should be taken, though, when defining dimensionless quantities involving time derivatives, such as the Hubble constant. The reason is that in the Jordan frame not only is the quantity itself time dependent, but also the measurement stick [e.g., when expressed in Planck units, it is important to take into account that the Jordan frame Planck mass (6) is itself time dependent]. On the background the Ricci scalar is $R_i = 6(2H_i^2 + H_i')$, with $i = J, E$. To write this in physical quantities we define the dimensionless Hubble constant

$$\bar{H} = \frac{\bar{a}'}{\bar{a}} = \frac{1}{\bar{a}} \frac{\partial_t \bar{a}}{\bar{N}} = \frac{1}{a_i m_i} \frac{1}{m_i N_i} \partial_t (a_i m_i), \tag{14}$$

with $i = E, J$ for Einstein and Jordan frame quantities, respectively, and the prime derivative is defined as $\bar{\phi}_i' = \frac{1}{N_i} \dot{\bar{\phi}}_i$. This dimensionless Hubble constant transforms trivially under a conformal transformation. Likewise we define

$$\bar{H}' = \frac{1}{\bar{N}} \partial_t \bar{H}, \tag{15}$$

so we can write the dimensionless Ricci scalar (on the background) as

$$\bar{R} = 6(2\bar{H}^2 + \bar{H}'). \tag{16}$$

To formulate the kinetic term in invariant form it is convenient to first express the Einstein frame quantities in terms of the rescaled field ϕ_E using (13). Then, in a next step, it is straightforward to introduce the invariant and dimensionless field, defined in the usual way:

$$\bar{\phi} = \frac{\phi_i}{m_i}, \qquad \bar{\phi}' = \frac{1}{\bar{N}} \partial_t \bar{\phi}. \tag{17}$$

Before going further, let us reformulate Ω^2 in terms of the Einstein frame field[2]:

$$\Omega^2 = 1 + \frac{\xi \phi_J^2}{m_E^2} = 1 + \xi \bar{\phi}^2 \Omega^2$$
$$\Rightarrow \Omega^2 = \frac{1}{1 - \xi \phi_E^2 / m_E^2} = \frac{1}{1 - \xi \bar{\phi}^2}. \tag{19}$$

The derivatives of the Einstein and Jordan frame fields $\phi_J = \Omega \phi_E$ are related via

$$\begin{aligned} \frac{1}{N_E} \dot{\phi}_J &= \Omega \left(\phi_E' + \phi_E \frac{\Omega'}{\Omega} \right) \\ &= \frac{\Omega}{1 - \xi \phi_E^2 / m_E^2} \frac{1}{N_E} \dot{\phi}_E \\ &= \Omega^3 \frac{1}{N_E} \dot{\phi}_E. \end{aligned} \tag{20}$$

The dimensionless expression for the kinetic terms in (10) can now be rewritten

$$\begin{aligned} \frac{1}{2} \frac{\gamma_E \dot{\phi}_J^2}{N_E^2 m_E^4} &= \frac{1}{2\Omega^2 N_E^2 m_E^4} (1 + 6 m_E^2 (\partial_{\phi_J} \Omega)^2) \dot{\phi}_J^2 \\ &= \frac{\Omega^4}{2} \left(1 + 6 \frac{\xi^2 \phi_E^2}{m_E^2} \right) \frac{\dot{\phi}_E^2}{N_E^2 m_E^4} \\ &= \frac{\Omega^4}{2} (1 + 6\xi^2 \bar{\phi}) \bar{\phi}'^2 \\ &\equiv \frac{1}{2} \bar{S}(\bar{\phi}) \bar{\phi}'^2. \end{aligned} \tag{21}$$

In the second expression we used the explicit form of the Einstein frame metric (4), and in the third we used the relations between the fields in the two frames (13), (20). Finally, in the last two expressions we introduced the frame-invariant fields (17). $\bar{S}(\bar{\phi})$ is the dimensionless and frame-invariant field-space metric, which is a function of the dimensionless field $\bar{\phi}$.[3]

Now we have all of the expressions (11,12,16,21) needed to write the action in terms of dimensionless quantities; this action reads

[2]The dimensionless field is bounded from above,

$$\xi \bar{\phi}^2 = \frac{\xi \phi_J^2}{(m_{\rm p}^2 + \xi \phi_J^2)} < 1, \tag{18}$$

as the denominator is always larger than the numerator. It follows that Ω^2 in (19) is always positive definite.

[3]$\bar{S}$ is not directly related to either γ_E [because the fields in (3) are still the Jordan frame fields] or γ_J (when writing out the $\bar{S}$ and $\bar{R}$ terms in Jordan frame quantities, both contribute to the Jordan field kinetic terms γ_J).

$$S = \int d^4x\sqrt{-\bar{g}}\left[\frac{1}{2}\bar{R}(\bar{a},\bar{N}) - \frac{1}{2}\bar{S}(\bar{\phi})\bar{\phi}'^2 - \bar{V}(\bar{\phi})\right]. \quad (22)$$

Making a conformal transformation leaves the action invariant; therefore, the latter describes all frames related by a conformal transformation. In fact all relevant equations and expressions can be derived from this action. If at some point it is desired to express them in frame-dependent quantities, it can easily be done by substituting in the explicit definitions of the barred quantities. This way it can be checked that (22) indeed returns to the Jordan frame action when the Planck mass and variables proper to that frame are substituted in.

We have expressed all quantities in Planck units. This is a very convenient choice for calculations in cosmology and, moreover, the results can readily be compared with experiments. Of course, the choice of units is not unique. The reason (22) takes the form of the Einstein frame action is precisely because in that frame the Planck mass (our reference mass) is constant.

B. Dimensionless action—full action

In the previous section we showed our idea at work in a very simple but significant example. Now we want to extend the results to the full action (not just the background) and allow for several nonminimally coupled scalar fields.

The metric and scalar fields are thus taken as both time and space dependent; we keep the Jordan frame field metric γ_J and potential V_J.

The approach is the same as before: start with the Einstein frame action (3) and write it in terms of the Einstein frame fields $\phi_J^a = \Omega\phi_E^a$; in the next step, go to dimensionless and frame-invariant variables by dividing everything with the proper powers of the Planck mass. Care should be taken for quantities that involve derivatives: the derivatives should always act on dimensionless and frame-invariant quantities themselves; this properly takes into account that the Planck mass is spacetime dependent in the Jordan frame.

The only nontrivial step is to relate the derivatives of the Jordan and Einstein frame fields and rewrite the kinetic terms. It is convenient to start expressing Ω in terms of the Einstein frame fields:

$$\Omega^2 = 1 + \frac{\xi}{m_E^2}\phi_J^a\phi_{Ja} = 1 + \frac{\xi}{m_E^2}\phi_E^a\phi_{Ea}\Omega^2$$
$$\Rightarrow \Omega^2 = \left(1 - \frac{\xi}{m_E^2}\phi_E^a\phi_{Ea}\right)^{-1} = (1 - \xi\bar{\phi}^a\bar{\phi}_a)^{-1}. \quad (23)$$

In the second line, $\xi\bar{\phi}^a\bar{\phi}_a < 1$ always (see footnote [2]). Now we can proceed as follows:

$$\begin{aligned}\nabla^\mu\phi_J^a &= \phi_E^a\nabla^\mu\Omega + \Omega\nabla^\mu\phi_E^a \\ &= \Omega\left(\delta_c^a + \frac{\xi}{m_E^2}\Omega^2\phi_E^a\phi_{Ec}\right)\nabla^\mu\phi_E^c \\ &\equiv \mathcal{M}_c^a\nabla^\mu\phi_E^c. \end{aligned} \quad (24)$$

Finally, the field-space metric tensor for the Einstein frame fields can be calculated:

$$S_{ab} = \gamma_{Ecd}\mathcal{M}_a^c\mathcal{M}_b^d \quad (25)$$

$$\begin{aligned} &= \Omega^{-2}\left(\gamma_{Jcd} + 6\frac{\xi}{m_E^2}\phi_{Ec}\phi_{Ed}\right)\Omega^2\left(\delta_a^c + \Omega^2\frac{\xi}{m_E^2}\phi_E^c\phi_{Ea}\right) \\ &\quad\times\left(\delta_b^d + \Omega^2\frac{\xi}{m_E^2}\phi_E^d\phi_{Eb}\right) \\ &= \gamma_{Jab} + \frac{\xi}{m_E^2}\Omega^2(\gamma_{Jad}\phi_E^d\phi_{Eb} + \gamma_{Jbd}\phi_E^d\phi_{Ea}) \\ &\quad + \Omega^4\phi_{Ea}\phi_{Eb}\left(\gamma_{Jcd}\phi_E^c\phi_E^d\frac{\xi^2}{m_E^4} + 6\frac{\xi}{m_E^2}\right). \end{aligned} \quad (26)$$

As it should, in the single field limit $\phi_E^a = \phi_E$ and for a trivial field metric $\gamma_{Jab} = \delta_{ab}$, the expression reduces, after some simplifications, to the field-space metric found in the previous section (21).

Now it is clear how to pass to the dimensionless action and frame invariant action. We define the frame-independent fields

$$\bar{\phi}^a = \frac{\phi_{ia}}{m_i}, \qquad \bar{g}_{\mu\nu} = g_{i\mu\nu}m_i^2, \quad (27)$$

with $i = J, E$. All other quantities are constructed from these:

$$\bar{V} = \frac{V_i}{m_i^4}, \quad (28)$$

$$\bar{\Gamma}^\sigma_{\alpha\beta} = \frac{1}{2}\bar{g}^{\sigma\rho}(\partial_\alpha\bar{g}_{\beta\sigma} + \partial_\beta\bar{g}_{\alpha\sigma} - \partial_\sigma\bar{g}_{\alpha\beta}), \quad (29)$$

$$\bar{R} = \bar{g}^{\alpha\beta}(\partial_\sigma\bar{\Gamma}^\sigma_{\alpha\beta} - \partial_\beta\bar{\Gamma}^\sigma_{\alpha\sigma} + \bar{\Gamma}^\sigma_{\alpha\beta}\bar{\Gamma}^\rho_{\sigma\rho} - \bar{\Gamma}^\rho_{\alpha\sigma}\bar{\Gamma}^\sigma_{\beta\rho}), \quad (30)$$

$$\begin{aligned}\bar{S}_{ab} &= \gamma_{Jab} + \xi\Omega^2(\gamma_{Jad}\bar{\phi}^d\bar{\phi}_b + \gamma_{Jbd}\bar{\phi}^d\bar{\phi}_a) \\ &\quad + \Omega^4\bar{\phi}_a\bar{\phi}_b(\gamma_{Jcd}\bar{\phi}^c\bar{\phi}^d\xi^2 + 6\xi). \end{aligned} \quad (31)$$

Choosing $i = E$ and substituting in the Einstein frame action (3) finally gives the action in explicitly frame-independent and dimensionless form:

$$S = \int d^4x\sqrt{-\bar{g}}\left(\frac{1}{2}\bar{R} - \frac{1}{2}\bar{S}_{ab}\bar{g}_{\mu\nu}\nabla^\mu\bar{\phi}^a\nabla^\nu\bar{\phi}^b - \bar{V}\right). \quad (32)$$

This is our main result: the action is written in a frame-invariant form, so all equations derived from it apply equally to all actions related by a conformal transformation; moreover, the results can readily be related to experiments which measure only dimensionless quantities. In practice, we can simply take the usual Einstein frame results, set the Planck mass to unity $m_E = 1$, and put a bar on all quantities: this gives the frame-invariant equations.

C. Quantization

The discussion in the previous section was fully classical: we showed that the classical action can be written in manifestly frame-invariant form. But someone might still be worried that quantization introduces a frame dependence; however, it is clear that if the quantization prescription is formulated in terms of the frame-independent barred quantities, no such issues arise. We can thus use the standard quantization procedures, applied to the action (32).

III. $f(R)$ GRAVITY

In this section we show that theories of $f(R)$ gravity, or equivalently scalar-tensor gravity with the Brans-Dicke parameter $\omega_{\mathrm{BD}} = 0$, can also be written in a frame-independent way. The key here is to realize that this class of theories can be rewritten as a scalar theory with a nonminimal coupling to gravity [30–34]; then the frame-invariant approach of the previous subsection can be applied.

Consider the action of $f(R)$ gravity:

$$S = \frac{M^2}{2}\int \mathrm{d}^4x\sqrt{-g_J}f(R), \tag{33}$$

whose function $f(R)$ begins with the Einstein-Hilbert term. Starobinsky inflation is a specific example, with $f(R) = R + \alpha R^2$ [35,36]. Introducing an auxiliary scalar field A_J, the action can be rewritten [34]

$$S = \frac{M^2}{2}\int \mathrm{d}^4x\sqrt{-g_J}[A_J R - V_J(A_J)]. \tag{34}$$

Applying the equations of motion for the scalar $R = \partial_A V$, substituting in the action, one retrieves the original $f(R)$ action (33), provided that f and V are related by a Legendre transformation,

$$f(R) = RA_J - V_J(A_J). \tag{35}$$

Now the action (34) is exactly of the form of the Jordan frame action (1) for a single field, if we identify

$$\gamma_{Jab} = 0, \qquad A_J = \Omega^2(\phi_J). \tag{36}$$

One can make a conformal transformation (2) to go to the Einstein frame. Using the results of the previous section (29–31), the action can be written in explicitly dimensionless and frame-invariant form.

IV. AN EXAMPLE: EQUIVALENCE OF THE CURVATURE AND ISOCURVATURE PERTURBATIONS

In the literature there are calculations of the curvature and isocurvature perturbations in both the Einstein and Jordan frames. It was shown that the curvature perturbation is frame independent [11–15], but a frame dependence was claimed for the isocurvature perturbations [22,23]. Applying the results of the previous section, we can express the perturbations spectrum in terms of the barred fields, which manifestly shows their frame independence.

The frame-independent perturbations can be rewritten in either Einstein or Jordan frame quantities, using the definitions of the barred quantities; the complicated relations between the two frames show how easy it is to make mistakes when comparing results in different frames, when they are not written in physical, dimensionless quantities.

To write the perturbation spectrum, we have to perturb the field and metric to first order:

$$\bar{\Phi}^a(t,x) = \bar{\phi}^a(t) + \bar{\varphi}^a(t,x), \quad \begin{aligned} \bar{g}_{00} &= -\bar{N}^2(1+2\bar{n}), \\ \bar{g}_{i0} &= \bar{g}_{0i} = 2\bar{a}\bar{N}\bar{n}_i, \\ \bar{g}_{ij} &= \bar{a}^2(1-2\bar{\psi})\delta_{ij} + \bar{F}_{ij}. \end{aligned} \tag{37}$$

With this, we can express the gauge invariant scalar curvature perturbation as

$$\zeta = -\bar{\psi} - \bar{H}\frac{\delta\bar{\rho}}{\dot{\bar{\rho}}}. \tag{38}$$

Note that we have not put a bar over ζ because it is both invariant and dimensionless. The energy density appearing in this equation is defined in the usual way:

$$\begin{aligned} \bar{\rho} &= -\bar{g}^{0\nu}\bar{T}_{\nu 0} \\ &= \bar{g}^{0\nu}\frac{2}{\sqrt{\bar{g}}}\frac{\delta\bar{S}_M}{\delta\bar{g}^{\mu\nu}}\delta_0^\mu \\ &= \bar{S}_{ab}\left[\frac{1}{2}\bar{g}^{\alpha\beta}\partial_\alpha\bar{\Phi}^a\partial_\nu\bar{\Phi}^b - \bar{g}^{0\nu}\partial_\nu\bar{\Phi}^a\partial_0\bar{\Phi}^b\right] + \bar{V}. \end{aligned} \tag{39}$$

In the presence of isocurvature perturbations, the curvature perturbation is nonconserved,

$$\zeta' = -\frac{\bar{H}}{\bar{\rho} + p}\delta\bar{p}_{\mathrm{nad}}, \tag{40}$$

with, as before, $\zeta' = \dot{\zeta}/\bar{N}$ the dimensionless time derivative and $\bar{p}_{\mathrm{nad}}$ the nonadiabatic pressure:

$$\delta\bar{p}_{\rm nad} = \delta\bar{p} - \frac{\dot{\bar{p}}}{\dot{\bar{\rho}}}\delta\bar{\rho}. \tag{41}$$

A. Invariance of ζ

In this subsection we write the curvature perturbation in Jordan and Einstein frame variables and show how these are related.

First we have to find the transformation between $\bar{\psi} = \psi_E$[4] and ψ_J. We can express the invariant metric in either the Jordan or the Einstein frame variables $\bar{g}_{\mu\nu} = M^2\bar{g}_{E\mu\nu} = M^2\Omega^2\bar{g}_{J\mu\nu}$. Expanding this relation to first order then gives

$$\begin{aligned} a_E^2(1-2\psi_E) &= a_J^2\Omega^2(1-2\psi_J) \\ &= a_J^2\left(1+\frac{\xi}{M^2}\phi_J^a\phi_{Ja} + 2\frac{\xi}{M^2}\phi_J^a\varphi_{Ja}\right)(1-2\psi_J) \\ &\simeq a_J^2\Omega_{(0)}^2\left(1-2\psi_J+\frac{2\frac{\xi}{M^2}\phi_J^a\varphi_{Ja}}{\Omega_{(0)}^2}\right), \end{aligned} \tag{42}$$

up to second order in perturbation. Further, we defined $\Omega_{(0)}^2 = 1 + \frac{\xi}{M^2}\phi_J^a\phi_{Ja}$. It follows that

$$\psi_E = \psi_J - \frac{1}{\Omega_{(0)}^2}\frac{\xi}{M^2}\phi_J^a\varphi_{Ja} = \psi_J - \frac{1}{2\Omega_{(0)}^2}\partial_a(\Omega_{(0)}^2)\varphi_J^a. \tag{43}$$

Using the slow-roll approximation we can write

$$\begin{aligned} \dot{\bar{\rho}} &\simeq \partial_t\left[\frac{1}{2\bar{N}^2}\bar{S}_{(0)ab}\dot{\bar{\phi}}^a\dot{\bar{\phi}}^b + \dot{\bar{V}}_{(0)}\right] \simeq \dot{\bar{V}}^{(0)} = 4\lambda\bar{\phi}^2\bar{\phi}^a\dot{\bar{\phi}}_a, \\ \delta\bar{\rho} &\simeq \delta\bar{V} = 4\lambda\bar{\phi}^2\bar{\phi}^a\bar{\varphi}_a, \end{aligned} \tag{44}$$

where in the last line we have approximated $\bar{\rho}$ with its background value, using the expansion expressed in (37); to simplify the notation we let $\phi^2 = \phi^b\phi^a\delta_{ab}$. Now we relate the ratio $\delta\bar{\rho}/\dot{\bar{\rho}}$ in the two frames:

$$\begin{aligned} \frac{\delta\rho_E}{\dot{\rho}_E} &= \frac{\delta(\lambda(\phi_J^2)^2)\Omega^2 - \lambda(\phi_J^2)^2\delta\Omega^2}{\lambda\Omega^2\partial_t(\phi_J^2)^2 - \lambda(\phi^2)^2\partial_t\Omega^2} \\ &= \frac{4\lambda(1+\frac{\xi}{M^2}\phi_J^2)\phi_J^2\phi_J^a\varphi_{Ja} - 2\lambda(\phi_J^2)^2\frac{\xi}{M^2}\phi_J^2\phi_J^a\varphi_{Ja}}{4\lambda(1+\frac{\xi}{M^2}\phi_J^2)\phi_J^2\phi_J^a\dot{\phi}_{Ja} - 2\lambda(\phi_J^2)^2\frac{\xi}{M^2}\phi_J^2\phi_J^a\dot{\phi}_{Ja}} \\ &= \frac{2\lambda\frac{1+\Omega^2}{\Omega^2}\phi_J^2\phi_J^a\varphi_{Ja}}{2\lambda\frac{1+\Omega^2}{\Omega^2}\phi_J^2\phi_J^a\dot{\phi}_{Ja}} = \frac{\phi_J^a\varphi_{Ja}}{\phi_J^a\dot{\phi}_{Ja}} = \frac{\delta\rho_J}{\dot{\rho}_J}. \end{aligned} \tag{45}$$

So, finally,

$$\begin{aligned} \zeta &= -\psi_E - \frac{\dot{a}_E}{a_E}\frac{\delta\rho_E}{\dot{\rho}_E} \\ &= -\psi_J + \frac{1}{2\Omega_{(0)}^2}\partial_a(\Omega_{(0)}^2)\varphi_J^a - \frac{1}{\Omega a_J}(\Omega\dot{a}_J + \dot{\Omega}a_J)\frac{\phi_J^a\varphi_{Ja}}{\phi_J^a\dot{\phi}_{Ja}} \\ &= -\psi_J - \frac{\dot{a}_J}{a_J}\frac{\delta\rho_J}{\dot{\rho}_J} + \frac{1}{2\Omega_{(0)}^2}\partial_a(\Omega_{(0)}^2)\varphi_J^a - \frac{\dot{\Omega}}{\Omega}\frac{\phi_J^a\varphi_{Ja}}{\phi_J^a\dot{\phi}_{Ja}} \\ &= \zeta. \end{aligned} \tag{46}$$

V. CONCLUSIONS

Higgs inflation has attracted considerable interest over recent years. The key ingredient of the model is a nonminimal coupling of the Higgs field to the Ricci tensor. Unfortunately, this brings along with it the issue of frames. The freedom to carry out the desired calculation in a given frame without worrying about possible implications raised from the choice of the frame itself is very important. With this in mind, in this paper we have demonstrated the equivalence between the Jordan and Einstein frames, and more generally between every frame related to these by a conformal transformation.

The equivalence of the various frames is not immediately obvious: directly transforming quantities calculated in one frame to another can lead (and has led) to mismatching results. Seen from a physics perspective, these frame-dependent results make no sense. The key point is that the conformal transformation that relates the Einstein and Jordan frames rescales all length scales. Since the absolute scale cannot be measured, both frames describe the same physics and must be equivalent. It is thus important to realize that when applying a conformal transformation, not only does the spacetime change, but the unit of measure is also modified. Therefore it is not surprising that when comparing quantities between two frames without changing the measuring reference accordingly, one obtains different results.

To avoid the unpleasant consequences of not keeping track of all scale changes, we have introduced the concept of dimensionless variables: in particular, we have chosen to express all dimensionful quantities in terms of Planck units. When transforming between frames, all mass scales including the Planck mass scale in the same way; the dimensionless ratios—our dimensional variables—remain, however, the same. Rewriting the Lagrangian terms of these dimensionless quantities provides a manifestly frame-invariant formulation of the theory, which can be directly related to what is actually measured in experiments. Moreover, formulating the action and all equations derived from it in terms of dimensionless variables is very convenient because, in any moment, it is possible to choose a frame and immediately convert the desired quantities into their frame-specific counterparts by simply substituting in the expressions for variables and for the Plank mass proper of

[4] Clearly $\bar{\psi} = \psi_E$ as ψ_E is dimensionless.

that frame. In the last section we have given an example of this mechanism, showing how it works for the gauge invariant curvature perturbation ζ.

Our results are applicable to $f(R)$ gravity and, in particular, Starobinsky inflation. Introducing an auxiliary field, these types of actions can be written as a Jordan frame Lagrangian with a nonminimal coupling to gravity. These can, in turn, be expressed in our dimensionless, physical quantities.

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