PHYSICAL REVIEW D **90,** 103501 (2014)

Axion inflation with gauge field production and primordial black holes

Edgar Bugaev* and Peter Klimai†

Institute for Nuclear Research, Russian Academy of Sciences,
60th October Anniversary Prospect 7a, 117312 Moscow, Russia

(Received 15 March 2014; revised manuscript received 28 June 2014; published 5 November 2014)

We study the process of primordial black hole (PBH) formation at the beginning of the radiation era for the cosmological scenario in which the inflaton is a pseudo-Nambu-Goldstone boson (axion) and there is a coupling of the inflaton with some gauge field. In this model inflation is accompanied by the gauge quanta production, and a strong rise of the curvature power spectrum amplitude at small scales (along with non-Gaussianity) is predicted. We show that data on PBH searches can be used for a derivation of essential constraints on the model parameters in such an axion inflation scenario. We compare our numerical results with the similar results published earlier, in the work [A. Linde, S. Mooij, and E. Pajer, Phys. Rev. D 87, 103506 (2013)].

DOI: 10.1103/PhysRevD.90.103501 PACS numbers: 98.80.-k, 04.70.-s

I. INTRODUCTION

It is well known that inflationary models that predict prolonged inflation are very sensitive to Planck-scale physics (see, e.g., the recent reviews in Refs. [1,2]). This sensitivity is especially important for large-field models when one needs to protect the inflationary potential from a possible large effect of an infinite number of higher-dimension operators. Even in supersymmetric models of inflation this protection is not guaranteed, because the supersymmetry is broken by the inflationary background at the Hubble scale.

It was shown very long ago that the simplest and most natural solution of this problem is to assume that the inflaton φ is a pseudo-Nambu-Goldstone boson (PNGB) [3–13], because in this case there is the shift symmetry, $\varphi \to \varphi + \mathrm{const}$, which is broken by instanton effects (or explicitly). In the limit when this symmetry is exact, the potential is flat, and the corrections to slow-roll parameters are under control due to the smallness of the symmetry breaking.

If PNGB is pseudoscalar (e.g., it is an axion), it is natural to assume that there is a coupling of it with some gauge field. This coupling is not forbidden by the shift symmetry and, in general, is phenomenologically favorable (e.g., it can lead to successful reheating). This coupling is essential if the axion decay constant f is not too large [because the interaction term is inversely proportional to f; see Eq. (1) below]. In UV-complete models of axion inflation (e.g., those based on string theory [8]) one has $f \ll M_P$ and, at the same time, a large excursion of the axion field is allowed. The inflationary potential in these models is similar to the potential in large-field models.

The main feature of axion inflation with an inflaton-gauge field coupling is that such a coupling leads to the production of gauge quanta, and—through the inverse decay of these quanta into inflaton perturbations—the rise of non-Gaussianity effects[1] and the violation of scale invariance. In particular, the rather essential formation of primordial black holes (PBHs) becomes possible [19,20].[2]

In the present work we consider a process of PBH formation and PBH constraints for the axion inflation models in which the inflationary expansion is accompanied by the gauge quanta production. Our consideration differs from that carried out in the recent work by Linde *et al.* [20] in two respects. First, we checked the hypothesis that a probability distribution function (PDF) for curvature fluctuations produced in an axion inflation model has the same form as that in χ_n^2 models. Second, for the calculation of the β_{PBH} functions describing the fraction of the Universe's mass in PBHs at their formation time, we use the full machinery of the Press-Schechter [23] formalism rather than the simple integral over the PDF of the curvature field (see, in this connection, Refs. [24,25]).

The plan of the paper is as follows. In the second section we review the main assumptions and formulas of the axion inflation model in which there is a coupling of the inflaton with the gauge field. In the third section, we discuss the choice of a suitable PDF for the ζ field in our scenario. In the fourth section—using the Press-Schechter formalism—we derive the PBH mass spectra needed to obtain the PBH constraints. The last section contains our conclusions. In Appendix A we study the time evolution of the curvature perturbation power spectrum behind the Hubble horizon. In Appendix B we study the shape of the ζ bispectrum in our axion inflation scenario, comparing it with the prediction of the χ^2 model.

[1]Non-Gaussian effects in processes of PBH formation have been studied in several pioneering works [14–18].

[2]Inflation models with pseudo-Nambu-Goldstone fields coexisting with the inflaton, and subsequent PBH production processes, have been considered in Refs. [21,22].

*bugaev@pcbai10.inr.ruhep.ru
†pklimai@gmail.com

II. AXION INFLATION WITH GAUGE FIELD PRODUCTION

A. Outline of the model

We consider the model of axion inflation in which there is a coupling of the pseudoscalar inflaton (axion) to gauge fields of the form

$$\mathcal{L}_{\rm int} = -\frac{\alpha}{4f}\varphi F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength corresponding to some U(1) gauge field A_μ, $\tilde{F}^{\mu\nu} = \eta^{\mu\nu\omega\theta}F_{\omega\theta}/(2\sqrt{-g})$ is the dual strength, f is the axion decay constant, and α is a dimensionless parameter.

It was shown in Ref. [10] that the evolution (rolling) of the inflaton leads to a generation of the field A_μ and to a subsequent amplification (due to tachyonic instability) of its modes. The solutions for the amplified modes are well parametrized by the formula (the index "+" means the circular polarization of quanta)

$$\tilde{A}_+(k,\tau) \cong \frac{1}{\sqrt{2k}}\left(\frac{k}{2\xi aH}\right)^{1/4} \exp\left[\pi\xi - 2\sqrt{\frac{2\xi k}{aH}}\right], \tag{2}$$

where

$$\xi \equiv \frac{\alpha\dot{\varphi}}{2fH}, \tag{3}$$

and $\tau \cong -1/(aH)$. During inflationary expansion the value of ξ changes with time. If ξ is larger than 1, the amplification factor $e^{\pi\xi}$ is essential. The production of gauge field quanta can affect the inflationary process. In general, it prolongs inflation [10] because it sources inflaton perturbations through the inverse decay $\delta A + \delta A \to \delta\varphi$ [26].

The tachyonic amplification of gauge-field modes leads to a characteristic evolution of the power spectrum of primordial curvature perturbations. The production of gauge quanta causes a strong increase in the spectrum amplitude. To put constraints on this increase from PBHs one must study the behavior of the ξ parameter as a function of time during inflationary expansion. The cosmological evolution equations for the inflaton with extra contributions from the gauge field are [10]

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = \frac{\alpha}{f}\langle\vec{E}\cdot\vec{B}\rangle, \tag{4}$$

$$3H^2M_P^2 = \frac{1}{2}\dot{\varphi}^2 + V + \frac{1}{2}\langle\vec{E}^2 + \vec{B}^2\rangle. \tag{5}$$

Here,

$$\vec{B} \equiv \frac{1}{a^2}\vec{\nabla}\times\vec{A}, \qquad \vec{E} \equiv -\frac{1}{a^2}\vec{A}'. \tag{6}$$

The connection of $\langle\vec{E}\cdot\vec{B}\rangle$ and $\langle\vec{E}^2 + \vec{B}^2\rangle$ with ξ is given by [10]

$$\langle\vec{E}\cdot\vec{B}\rangle \approx -2.4\times 10^{-4}\frac{H^4}{\xi^4}e^{2\pi\xi},$$
$$\frac{1}{2}\langle\vec{E}^2 + \vec{B}^2\rangle \approx 1.4\times 10^{-4}\frac{H^4}{\xi^3}e^{2\pi\xi}. \tag{7}$$

For a calculation of the curvature power spectrum one needs the evolution equation for the inflaton field perturbation, $\delta\varphi$. To derive this equation one must take the backreaction effects into account [10,27]. The approximate accounting of these effects leads to the (operator) equation [10,27]

$$\delta\ddot{\varphi} + 3\beta H\delta\dot{\varphi} - \frac{\nabla^2}{a^2}\delta\varphi + V''\delta\varphi = \frac{\alpha}{f}[\vec{E}\cdot\vec{B} - \langle\vec{E}\cdot\vec{B}\rangle], \tag{8}$$

where β is defined by the expression

$$\beta \equiv 1 - 2\pi\xi\frac{\alpha}{f}\frac{\langle\vec{E}\cdot\vec{B}\rangle}{3H\dot{\varphi}}. \tag{9}$$

Equations (4) and (5) are solved numerically, giving the solutions $\varphi(t)$ and $H(t)$ with initial conditions for $\varphi(0)$ and $H(0)$, where $t = 0$ corresponds, in our case, to the moment when cosmic microwave background (CMB) scales exit the horizon. As a byproduct one obtains the function $\xi(t)$.

B. Axion potential

A typical axion inflationary potential that is exploited in natural inflation models [3,4] is given by the formula

$$V(\varphi) = \Lambda^4\left[1 - \cos\left(\frac{\varphi}{f}\right)\right]. \tag{10}$$

In UV-complete models of axion inflation, the axion action is shift symmetric, i.e., the shift symmetry $\varphi \to \varphi + \text{const}$ is broken only nonperturbatively. In particular, in closed string models with type IIB Calabi-Yau orientifold compactifications, such axions are available (see, e.g., the review paper [28]). The inflationary potential in such models is periodic, due to instanton effects, but it is flat enough to drive inflation only in the case when the axion decay constant is larger than M_P. It is well known, however, that it is difficult to obtain such large values of f in UV-complete theories [29,30]. So, the potential of a single axion, Eq. (10), cannot provide the large-field inflation with long slow-roll evolution and a large value of the field excursion.

There are several groups of models in which large-field inflation is possible with sub-Planckian axion decay constants: "Racetrack inflation" models [31], N-flation models [6], assisted inflation models [32,33], and axion monodromy inflation models [8,9,11–13]. The latter approach looks very promising and we have used it in the present paper for numerical calculations. In particular, it was shown in Ref. [8] that, in type IIB string theory, the presence of space-filling D_p-branes wrapping some two-cycles of the compact internal space leads to a breaking of the shift symmetry and to the monodromy phenomenon: the potential energy for the axion arising from integrating two-form fields over these two-cycles is not periodic and increases with an increase of the axion field. As a result, one has an additional component of the axion potential,

$$V(\varphi) = V_{\rm sr}(\varphi) + V_{\rm inst}(\varphi). \tag{11}$$

Here, the subscript "sr" means slow-roll, and "inst" means instanton. In the concrete model [8], with the C_2 axion and $NS5$-brane wrapping Σ_2 (see Ref. [28] for notations), the potential $V_{\rm sr}$ is given by the expression

$$V_{\rm sr}(\varphi) = \frac{\epsilon}{g_s^2(2\pi)^5\alpha'^2}\sqrt{L^4 + g_s^2\frac{\varphi^2}{f^2}}. \tag{12}$$

Here, L is the dimensionless modulus (L^2 is the size of the two-cycle Σ_2), g_s is the string coupling constant, $1/(2\pi\alpha')$ is the string tension, and ϵ is the warp factor [8]. At large values of φ/f one has the linear potential

$$V_{\rm sr}(\varphi) \approx \mu^3\varphi. \tag{13}$$

A different realization of the monodromy idea (which is not based on string theory) was suggested in Refs. [11,12]. In these works, the axion potential was generated by a modification of the action that introduced the coupling of the axion to a 4-form. This new coupling leads to a spontaneous breaking of the shift symmetry and to the appearance (in the simplest case) of the quadratic axion potential, just like in the original chaotic inflation scenario [34].

In this work we will consider both cases: the axion inflation with the quadratic potential

$$V(\varphi) = \frac{m^2\varphi^2}{2} \tag{14}$$

(PBH constraints for the axion inflationary model with such a potential were considered in Ref. [20]), and the inflation with the linear potential given by Eq. (13). We assume that effects from the presence of $V_{\rm inst}$ are subdominant, and hence we neglect this term.

Using the expressions for the axion potentials, Eqs. (4) and (5) can be solved. The initial conditions for $t = 0$—corresponding to the moment of time when the scale with the comoving wave number $k = k_* = 0.002\ {\rm Mpc}^{-1}$ enters the horizon—are

$$\varphi(t=0) = \varphi_0, \qquad \dot\varphi(t=0) = -\frac{V'(\varphi_0)}{3H_0},$$
$$H(t=0) = H_0 = \frac{1}{M_P\sqrt{3}}V(\varphi_0)^{1/2}. \tag{15}$$

The constant m (or μ) is fixed by the requirement that the curvature perturbation power spectrum $\mathcal{P}_\zeta$ reaches the observed value [35] at cosmological scales, $\mathcal{P}_\zeta(k_*) \approx 2.4\times10^{-9}$. For the linear potential (13), we obtained $\mu \approx 6.3\times10^{-4}M_P$ and the following set of initial conditions: $\varphi_0 \approx 10.6M_P$, $|\dot\varphi_0| \approx 2.8\times10^{-6}M_P^2$, $H_0 \approx 2.9\times10^{-5}M_P$. For quadratic potential (14), we have $m = 6.8\times10^{-6}M_P$ and $\varphi_0 \approx 15M_P$, $|\dot\varphi_0| \approx 5.6\times10^{-6}M_P^2$, $H_0 \approx 4.2\times10^{-5}M_P$.

In Fig. 1 we show the results of our numerical calculations, i.e., the dependence of ξ on N (the number of

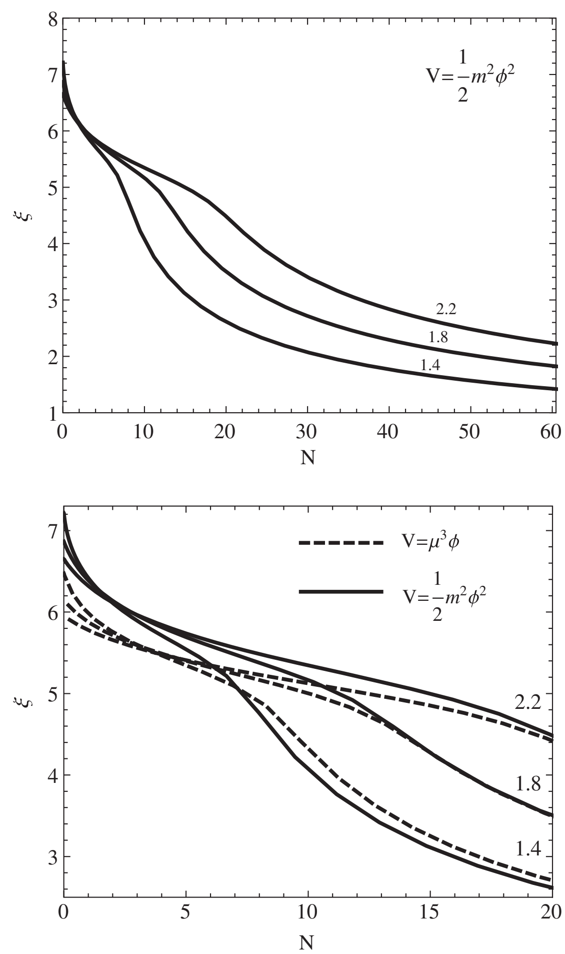

FIG. 1. The value of $\xi(N)$ for different values of ξ at CMB scales and different choices of the model potential. Solid curves are for the case of the potential (14); dashed curves are for the potential (13). The curves are labeled with the value of $\xi(N_{\rm CMB})$.

e-folds before the end of inflation) for different values of ξ at CMB scales.

To close this subsection, one should note that axion monodromy inflation with potentials given by Eqs. (13) and (14) predicts rather large values for the tensor-to-scalar ratio: $r = 0.07$ for the linear potential, and $r = 0.14$ for the quadratic one. The latter value is not excluded by the Planck [35] and BICEP2 [36] data.

C. Curvature power spectrum

In the limit of very small backreaction one has $\beta \to 1$. In this limit, the solution of Eq. (8) is [26,37]

$$\mathcal{P}_\zeta(k) = \mathcal{P}_{\zeta,\mathrm{sr}}(k)(1 + \mathcal{P}_{\zeta,\mathrm{sr}}(k) f_2(\xi) e^{4\pi\xi}), \tag{16}$$

$$\mathcal{P}_{\zeta,\mathrm{sr}}(k) = \left(\frac{H^2}{2\pi\dot{\varphi}}\right)^2. \tag{17}$$

The function $f_2(\xi)$ is defined in Refs. [26,37].

Near the horizon crossing one has an approximate solution of Eq. (8) (everywhere below we omit the contribution of the vacuum part, i.e., the solution of the homogenous equation):

$$\delta\varphi \approx \frac{\alpha}{f} \frac{(\vec{E}\cdot\vec{B} - \langle\vec{E}\cdot\vec{B}\rangle)}{3\beta H^2}, \tag{18}$$

and, correspondingly, one has for the curvature (see Appendix A)

$$\zeta \approx -\frac{\alpha}{f} \frac{(\vec{E}\cdot\vec{B} - \langle\vec{E}\cdot\vec{B}\rangle)}{3\beta H\dot{\varphi}}. \tag{19}$$

The variance of the curvature power spectrum is [20]

$$\langle\zeta(x)^2\rangle = \frac{H^2}{\dot{\varphi}^2}\langle\delta\varphi^2\rangle \approx \frac{\alpha^2}{f^2}\frac{\langle\vec{E}\cdot\vec{B}\rangle^2}{(3\beta H\dot{\varphi})^2}. \tag{20}$$

From this equation, in the limit of large backreaction (when $\beta \gg 1$), the simple approximate formula for the power spectrum is obtained as [10,20,27]

$$\mathcal{P}_\zeta(k) \approx \langle\zeta(x)^2\rangle = \frac{1}{(2\pi\xi)^2}. \tag{21}$$

Some examples of the power-spectrum solutions are shown in Fig. 2. For the calculations we used the approximate formula (20) which takes backreaction into account (at the latest stages of inflation the backreaction effects are quite essential). We then added the contribution of the vacuum part, which is dominant at small values of k. The connection of the comoving wave number k with N is given by

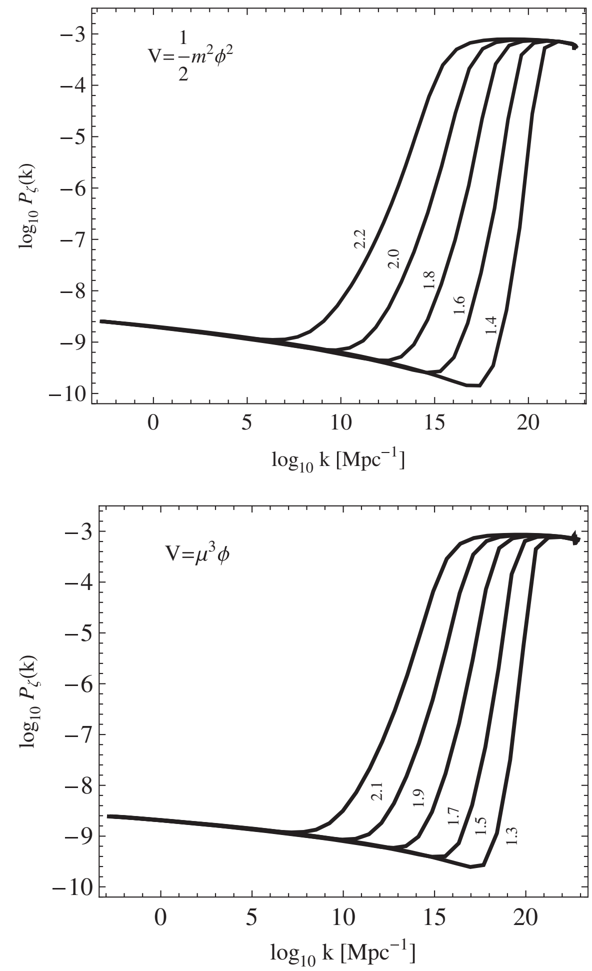

FIG. 2. The curvature perturbation power spectrum $\mathcal{P}_\zeta(k)$ calculated for different values of ξ_{CMB}, for two shapes of the inflaton potential. The curves are labeled with the value of $\xi(N_{\mathrm{CMB}})$.

$$k = a_e H(N) e^{-N}, \tag{22}$$

where a_e is the scale factor at the end of inflation.

III. PDFs AND NON-GAUSSIANITY

For a derivation of the PBH constraints we need an expression for the PDF of the ζ field. Evidently, this is a technical problem in the non-Gaussian case because, for in order to calculate the PDF one must know (in principle) all of the cumulants (moments) contributing to its series expansion.

In our case, the simplest assumption that we can use in this concrete model is the following [20]: the ζ field is distributed as a square of some Gaussian field χ,

$$\zeta = \chi^2 - \langle\chi^2\rangle, \tag{23}$$

keeping in mind that the non-Gaussianity of fluctuations $\delta\varphi$ [described by, e.g., Eq. (8)] arises solely from the fact that the particular solution of this equation is bilinear in the field A_μ (the latter is assumed to be Gaussian).

If Eq. (23) holds (in this case, we have the so-called χ^2 model), the PDF of ζ is given by (see, e.g., Refs. [38,39])

$$p_\zeta(\zeta) = \frac{1}{\sqrt{\zeta + \langle\chi^2\rangle}} p_\chi\left(\sqrt{\zeta + \langle\chi^2\rangle}\right), \tag{24}$$

$$p_\chi(\chi) = \frac{1}{\sigma_\chi\sqrt{2\pi}} e^{-\frac{\chi^2}{2\sigma_\chi^2}}, \qquad \sigma_\chi^2 \equiv \langle\chi^2\rangle. \tag{25}$$

The variance and skewness of the ζ field are, respectively,

$$\langle\zeta^2\rangle = 2\langle\chi^2\rangle^2, \qquad \langle\zeta^3\rangle = 8\langle\chi^2\rangle^3, \tag{26}$$

so that the first nontrivial reduced cumulant is

$$D_3 = \frac{\langle\zeta^3\rangle}{\langle\zeta^2\rangle^{3/2}} = \sqrt{8}. \tag{27}$$

More generally, one can use the χ_n^2 model, in which the ζ field is written as a sum of n squares of Gaussian fields,

$$\zeta = \sum_{i=1}^{n} \chi_i^2 - n\langle\chi_i^2\rangle. \tag{28}$$

In this case, the PDF of ζ is [40,41]

$$p_\nu(\nu) = \frac{\left(1 + \nu\sqrt{\frac{2}{n}}\right)^{\frac{n}{2}-1}}{\left(\frac{2}{n}\right)^{\frac{n-1}{2}}\Gamma\left(\frac{n}{2}\right)} \exp\left(-\frac{n}{2}\left(1 + \sqrt{\frac{2}{n}}\nu\right)\right), \tag{29}$$

$$\nu \equiv \frac{\zeta}{\sqrt{\langle\zeta^2\rangle}}, \qquad p_\nu(\nu)d\nu = p_\zeta(\zeta)d\zeta. \tag{30}$$

The cumulants of the χ_n^2 distribution are given by the simple formula

$$D_m = (m-1)!\left(\frac{2}{n}\right)^{\frac{m}{2}-1}. \tag{31}$$

It is tempting to assume that the best choice in our case is the χ_2^2 model, i.e., $n = 2$, in accordance with the fact that the photon has two polarizations. The expression for the corresponding PDF follows from Eq. (29),

$$p_\nu(\nu) = e^{-(1+\nu)}, \tag{32}$$

and the PDF for the ζ field is

$$p_\zeta(\zeta) = \frac{1}{\sqrt{\langle\zeta^2\rangle}} p_\nu(\nu), \tag{33}$$

with the properties

$$\int_{-\sqrt{\langle\zeta^2\rangle}}^{\infty} \zeta p_\zeta(\zeta)d\zeta = 0,$$
$$\int_{-\sqrt{\langle\zeta^2\rangle}}^{\infty} p_\zeta(\zeta)d\zeta = 1,$$
$$\int_{-\sqrt{\langle\zeta^2\rangle}}^{\infty} \zeta^2 p_\zeta(\zeta)d\zeta = \langle\zeta^2\rangle. \tag{34}$$

If a PDF of the ζ field is known, one can calculate not only the reduced cumulants D_m but also the shapes of ζ polyspectra (e.g., the shapes of ζ bispectra). From the other side, some of these functions can be calculated in our inflation model directly, without using the PDF. In particular, the reduced cumulant D_3 is given by the simple relation [20] (in the region where the backreaction is large)

$$D_3 = \frac{\langle\zeta^3\rangle}{\langle\zeta^2\rangle^{3/2}} \cong \frac{1/(4\pi^3\xi^3)}{(1/(2\pi\xi)^2)^{3/2}} = 2. \tag{35}$$

This value coincides with the D_3 following from Eq. (31) for $n = 2$ [compare it with the D_3 value given by Eq. (27)]. So, the choice of the χ_2^2 model as a description of the PDF seems to be appropriate.

Some results of the ζ bispectrum calculations in our axion inflation model and a comparison with corresponding χ^2-model predictions are given in Appendix B.

IV. PBH CONSTRAINTS

For calculations of PBH constraints we need the PDF for the *smoothed* ζ field, ζ_R (R is the smoothing radius). We assume, using the arguments of Refs. [41–43] (see also Ref. [39]), that PDF of the smoothed ζ field can be expressed in the form

$$p_{\zeta,R}(\zeta_R) = \frac{1}{\sqrt{\langle\zeta_R^2\rangle}} \tilde{p}_{\tilde{\nu}}(\tilde{\nu}), \qquad \tilde{\nu} = \frac{\zeta_R}{\sqrt{\langle\zeta_R^2\rangle}}. \tag{36}$$

Besides, we assume (following conclusions of Refs. [43,44]) that cumulants of the PDF are approximately equal in smoothing and nonsmoothing cases,

$$D_{m,R} \approx D_m. \tag{37}$$

It follows from Eq. (37) that the PDF of the smoothed ζ field can be written as [39]

$$p_{\zeta,R}(\zeta_R) = \frac{1}{\sqrt{\langle\zeta_R^2\rangle}} p_{\tilde{\nu}}(\tilde{\nu}), \tag{38}$$

where $p_{\tilde{\nu}}(\tilde{\nu})$ is given by Eq. (29) with $n = 1$ for the χ^2 model and $n = 2$ for the χ_2^2 model, with the substitution $\nu \to \tilde{\nu}$. In this approximation the effects of the smoothing come only through the variance $\sqrt{\langle\zeta_R^2\rangle}$, while the shape of the PDF is the same as in the nonsmoothing case. The variance of ζ_R is given by the formula

$$\langle \zeta_R^2 \rangle = \int_0^\infty \tilde{W}^2(kR)\mathcal{P}_\zeta(k)\frac{dk}{k}, \tag{39}$$

where $\tilde{W}(kR)$ is a Fourier transform of the window function [45], and we use a Gaussian one, $\tilde{W}^2(kR) = e^{-k^2R^2}$.

One can show that the energy-density fraction of the Universe contained in PBHs which form near the time of formation $t = t_f$ [at this time the horizon mass is equal to $M_h(t_f) = M_h^f$] is given by the integral [39,46]

$$\Omega_{\rm PBH}(M_h^f) \approx \frac{1}{\rho_i}\left(\frac{M_h^f}{M_i}\right)^{1/2}\int n_{\rm BH}(M_{\rm BH})M_{\rm BH}^2 d\ln M_{\rm BH} \approx \frac{(M_h^f)^{5/2}}{\rho_i M_i^{1/2}} n_{\rm BH}(M_{\rm BH})|_{M_{\rm BH}\approx f_h M_h^f}. \tag{40}$$

Here, $n_{\rm BH}(M_{\rm BH})$ is the PBH mass spectrum, ρ_i and M_i are, respectively, the energy density and horizon mass at the beginning of the radiation era (if the reheating is fast, it coincides with an end of inflation), and f_h is a constant [equal to $(1/3)^{1/2}$] which connects the value of PBH mass forming at the moment t_f with the horizon mass at that moment (see, e.g., Ref. [24]). The PBH mass spectrum in the Press-Schechter [23] formalism is proportional to the derivative $\partial P/\partial R$, where P is the integral over the ζ PDF [46],

$$P(R) = \int_{\zeta^c}^{\infty} p_\zeta d\zeta. \tag{41}$$

Approximately, one has [39,46]

$$\Omega_{\rm PBH}(M_h^f) \approx \beta_{\rm PBH}(M_h^f), \tag{42}$$

where $\beta_{\rm PBH}$ is, by definition, the fraction of the Universe's mass in PBHs at their formation time,

$$\beta_{\rm PBH}(M_h^f) \equiv \frac{\rho_{\rm PBH}(t_f)}{\rho(t_f)}. \tag{43}$$

Now, utilizing Eqs. (40) and (42), one can use the experimental limits on the value of $\beta_{\rm PBH}$ [47,48] to constrain parameters of models used for PBH production predictions. The PBH mass spectrum needed for a derivation of $\Omega_{\rm PBH}$ in Eq. (40) depends on the amplitude of the curvature power spectrum $\mathcal{P}_\zeta$ (see Refs. [38,39,46] for details).

The results of the PBH mass spectra calculation for the considered model are given in Fig. 3 for several values of the parameter $\xi_{\rm CMB} \equiv \xi(N_{\rm CMB})$ and for two choices of the parameter ζ_c, which is a model-dependent PBH formation

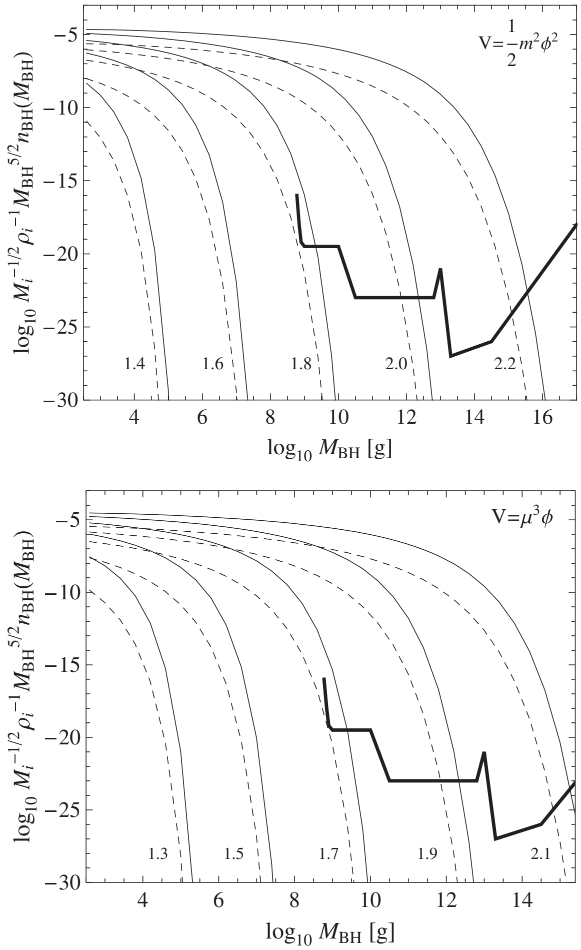

FIG. 3. Primordial black hole mass spectra corresponding to curvature perturbation power spectra shown in Fig. 2. Solid curves are for the case $\zeta_c = 0.75$, while dashed curves are for $\zeta_c = 1$. The curves are labeled with the value of $\xi(N_{\rm CMB})$. The thick lines schematically show the existing constraints on the PBH abundance [47,48].

threshold (see, e.g., Ref. [46]). For a calculation of the ζ PDF entering Eq. (41), we have used the χ_2^2 model.

The PBH mass value (as a function of N) in our model is given by the formula

$$M_{\rm BH} = \frac{f_h M_{\rm eq} k_{\rm eq}^2}{a_e^2}\frac{e^{2N}}{H(N)^2}, \tag{44}$$

where $H(N)$ is the Hubble constant during inflation at the epoch determined by the value of N, a_e is the scale factor at the end of inflation, and $M_{\rm eq}$ and $k_{\rm eq}$ are the horizon mass and wave number corresponding to the moment of matter-radiation equality. The result of the calculation using Eq. (44) is shown in Fig. 4, together with the result of the calculation using the more simple formula suggested in Ref. [20] (namely, $M_{\rm BH} = 10e^{2N}$ g). It is seen that the curves start at almost the same value at $N = 0$. The

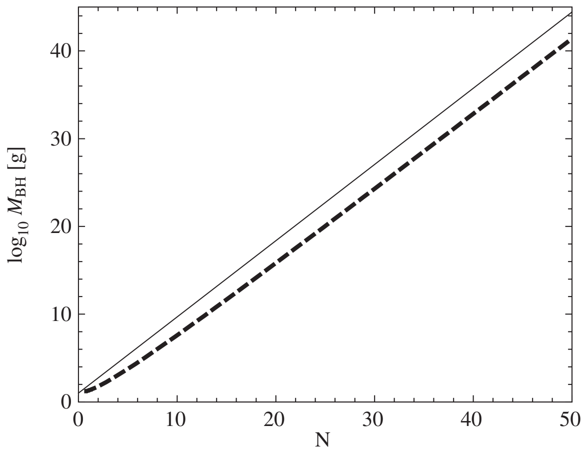

FIG. 4. Primordial black hole mass $M_{\rm BH}$ that is produced, depending on the number N of inflation e-folds. The dashed line is the calculation using our formulas [Eq. (44)], while the solid line is obtained using the formula $M_{\rm BH} = 10e^{2N}$ g of Ref. [20].

difference at larger N is due to the fact that Eq. (44) takes into account the dependence of H on N.

V. RESULTS AND DISCUSSION

The main results of the paper are shown in Figs. 2 and 3. Figure 2 illustrates the fact that—due to the tachyonic instability of the gauge field—an amplitude of the curvature power spectrum is very large (up to 10^{-3}) at small scales, $k \sim (10^{15}\text{–}10^{20})$ Mpc^{-1}, for a broad range of $\xi_{\rm CMB}$ values. Figure 3 shows the PBH mass spectra for definite values of the parameter $\xi_{\rm CMB}$. On the vertical axis of Fig. 3 the combination $M_i^{-1/2}\rho_i^{-1}M_{\rm BH}^{5/2}n_{\rm BH}(M_{\rm BH})$ is shown; just this combination is approximately equal to $\beta_{\rm PBH}$, as it follows from Eq. (40). We compare these spectra with PBH data [47,48], in which we consider only data for $M_{\rm BH} > 10^9$ g as the most reliable ones. For such a comparison we drew in Fig. 3 the zigzag line representing, schematically, the well-known $\beta_{\rm PBH}$-constraint summary curve (see Fig. 9 in Ref. [48]). If some of our curves cross this zigzag line, the corresponding ξ value is, according to our logic, forbidden. Finally, we obtain the constraint on the value of $\xi_{\rm CMB}$, for the quadratic potential (14),

$$\xi_{\rm CMB} < 1.8. \tag{45}$$

This constraint can be compared with the corresponding result of Ref. [20], $\xi_{\rm CMB} < 1.5$. In terms of the constants α and f, the limit (45) corresponds to $\alpha/f < 26M_P^{-1}$.

We performed a similar analysis for the case of the linear potential (13), and in this case the constraint on $\xi_{\rm CMB}$ turns out to be stronger,

$$\xi_{\rm CMB} < 1.7, \tag{46}$$

corresponding to $\alpha/f < 36M_P^{-1}$.

For a derivation of these results, we used the assumption that the ζ field has a χ_2^2 distribution. For a comparison, we also performed the same calculations for a simple χ^2 model (with one degree of freedom) and obtained the following PBH limits on the parameters: for the quadratic potential $\xi_{\rm CMB} < 1.75$, and for the linear potential $\xi_{\rm CMB} < 1.65$. Luckily, the constraints weakly depend on the choice of PDF ($n = 1$ or $n = 2$).

In conclusion, one should note that PBH constraints are stronger than those from CMB scales [2] and forthcoming constraints from gravity-wave experiments [49].

ACKNOWLEDGMENTS

The work of P. K. was partially supported by the grant of the President of RF number SS-3110.2014.2.

APPENDIX A: CURVATURE POWER SPECTRUM BEHIND THE HUBBLE HORIZON

It is well known that, in general, the curvature perturbation amplitude ζ does not stay constant in time after its scale exits the horizon during inflation. This is the case even in the standard single-field inflation model if, in particular, slow-roll is temporarily violated in the process of the inflationary expansion [50–52]. It was shown in Refs. [51,52] (see also Ref. [53]) that in such models the modes can have a very complicated evolution and can be strongly amplified on super-horizon scales. As a result of such amplification, in particular, the perturbation amplitudes at horizon reentry can differ rather strongly from amplitudes at the time of the exit.

In this appendix we derive the curvature perturbation power spectrum, closely following Ref. [10]. Two main differences are: (i) the authors of Ref. [10] assumed that α is very large ($\sim 10^2$ or larger), and (ii) they considered the case of the cosine potential [given by Eq. (10)]. In contrast with this, we consider the case when $f/\alpha \ll M_P$, $\alpha \sim 1$ and our potentials are nonperiodic. Nevertheless, we show in this appendix that the resulting spectrum formula in our case is just the same as that in Ref. [10] if we limit ourselves to considering the small scales, which exit the horizon at the final stages of inflationary expansion. Only these scales are of interest to us because we are studying the PBH production processes.

Equation (8), which takes into account the backreaction effects, can be simplified using the slow-roll approximation in the background equation (4). We assume that the slow-roll regime is mainly supported by the dissipation into gauge-field modes, i.e.,

$$3H\dot{\varphi} \ll V', \tag{A1}$$

$$V' \cong \frac{\alpha}{f}\langle \vec{E}\cdot\vec{B}\rangle. \tag{A2}$$

The inequality (A1) holds if f/α is small compared to M_P. Using the definition of ξ [Eq. (3)] and the approximate relation $3H^2M_P^2 \cong V$, one can rewrite Eq. (A1) in the form

$$2\xi \cdot \frac{f}{\alpha} \cdot \frac{V}{V'} \ll 1. \tag{A3}$$

For the quadratic potential, $V = \frac{1}{2}m^2\varphi^2$, one obtains from Eq. (A3) that

$$\xi \cdot \frac{f}{\alpha}\varphi \ll 1, \tag{A4}$$

and, for the linear potential, $V = \mu^3\varphi$,

$$2\xi \cdot \frac{f}{\alpha}\varphi \ll 1. \tag{A5}$$

We are interested in the final stage of inflation when small scales exit the horizon ($N \sim 10$). During this stage $\xi \sim 5$ (see Fig. 1) and $\varphi \sim M_P$ [20,27]. Substituting our limiting values of f/α (see Sec. V) into Eqs. (A4) and (A5), one can see that the inequalities (A4) and (A5) really hold.

To obtain the approximate equation (A2) one must show that the term $\ddot{\varphi}$ in Eq. (4) is small in comparison to V'. The proof of this is easily performed in complete analogy with the proof of Eq. (A1). Now, using Eq. (A2) and the relation following from Eq. (9),

$$\beta = 1 - \pi\langle\vec{E}\cdot\vec{B}\rangle\frac{\alpha^2}{3H^2f^2}, \tag{A6}$$

one can rewrite Eq. (8) in the following form [changing the time variable $\tau \cong -1/(aH)$ and going over into k space) [10]:

$$\delta\varphi''(\vec{k}) - \frac{2}{\tau}\left(1 + \frac{\pi\alpha V'}{2fH^2}\right)\delta\varphi'(\vec{k}) + \left(k^2 + \frac{V''}{H^2\tau^2}\right)\delta\varphi(\vec{k}) = -\frac{\alpha}{f}a^2\mathcal{J}(\tau,\vec{k}), \tag{A7}$$

$$\mathcal{J}(\tau,\vec{k}) = \int \frac{d^3x}{(2\pi)^{3/2}}e^{-i\vec{k}\vec{x}}[\vec{E}\cdot\vec{B} - \langle\vec{E}\cdot\vec{B}\rangle]. \tag{A8}$$

We can treat V'/H^2 and V''/H^2 as adiabatically evolving parameters, as well as H and ξ [e.g., for the quadratic potential, $V'/H^2 \sim (V'/V)M_P^2 \sim M_P^2/\varphi$, $V''/H^2 \sim M_P^2/\varphi^2$], because $\Delta\varphi \ll \varphi$ when $\Delta t \sim H^{-1}$ [20,27]. Due to this, we neglect their time dependencies during the essential part of the inflationary evolution of each mode. In this case the homogenous equation (A7) [i.e., Eq. (A7) with $\mathcal{J}(\tau,\vec{k}) = 0$] can be written in the form

$$\tau^2\delta\varphi'' + b\tau\delta\varphi' + (c\tau^2 + d)\delta\varphi = 0, \tag{A9}$$

$$b = -\frac{\pi\alpha V'}{fH^2} - 2, \qquad c = k^2, \qquad d = \frac{V''}{H^2}. \tag{A10}$$

The solution of this equation is expressed through the cylindrical functions (see, e.g., Ref. [54]):

$$\delta\varphi = \tau^{\frac{1-b}{2}}Z_\nu(k\tau), \qquad \nu = \frac{1}{2}\sqrt{(1-b)^2 - 4d}, \tag{A11}$$

$$Z_\nu(k\tau) = C_1J_\nu(k\tau) + C_2N_\nu(k\tau), \tag{A12}$$

$$N_\nu(k\tau) = \frac{J_\nu(k\tau)\cos(\pi\nu) - J_{-\nu}(k\tau)}{\sin(\pi\nu)}. \tag{A13}$$

Using the estimates given above, one can check that $|b| \gg 1$, $d \ll |b|$, so

$$\nu \approx \frac{1}{2}(1-b)\sqrt{1 - \frac{4d}{b^2}} \approx \frac{1-b}{2} - \frac{d}{1-b}. \tag{A14}$$

We are interested in the power spectrum at $k \ll aH$, i.e., at $k|\tau| \ll 1$, so one can use the approximation

$$J_\nu(x) \approx \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)}. \tag{A15}$$

The solution of the full equation (A7) is obtained by the variation-of-constants method (or by the method of Green functions, which is technically the same) and is given by the integration over the source function $\mathcal{J}(\tau,\vec{k})$. Finally, using the approximation (A15) one obtains

$$\delta\varphi \sim -\frac{\alpha}{f}\int_{-\infty}^{\tau} d\tau'\tau'\left\{\left(\frac{\tau}{\tau'}\right)^{\nu+\frac{1}{2}-\frac{b}{2}} - \left(\frac{\tau}{\tau'}\right)^{-\nu+\frac{1}{2}-\frac{b}{2}}\right\} \times a^2(\tau')\mathcal{J}(\tau',\vec{k}). \tag{A16}$$

Since $|\tau| < |\tau'|$ one can neglect the first term in the brackets, because $\nu + \frac{1}{2} - \frac{b}{2} \approx 1 - b \gg 1$, $-\nu + \frac{1}{2} - \frac{b}{2} \approx \frac{d}{1-b} \ll 1$. Using Eq. (A14), this leads to

$$\delta\varphi \sim \frac{\alpha}{f}\int_{-\infty}^{\tau} d\tau'\tau'\left(\frac{\tau}{\tau'}\right)^{\frac{d}{|b|}}, \tag{A17}$$

$$\frac{d}{|b|} \approx \frac{V''f}{\pi\alpha V'} \ll 1. \tag{A18}$$

Using this expression and the relation $\zeta = H(\delta\varphi/\dot{\varphi})$, a formula for the curvature perturbation power spectrum is obtained straightforwardly [10], with the result

$$\mathcal{P}_\zeta \approx \frac{10^{-2}}{\xi^2}\left(\frac{\xi k}{aH}\right)^{\frac{2d}{|b|}}, \qquad k \ll aH. \tag{A19}$$

We see from this formula that the power spectrum at super-horizon scales has no amplification; on the contrary, it decreases with time when the scale moves away from the horizon. Due to a small value of $d/|b|$ the time dependence is rather mild. Further, we see from Eq. (A19) that in the limit of small $d/|b|$ (which corresponds to the limit of large backreaction) the curvature spectrum is almost scale invariant in a region of small scales, in accordance with the results shown in Fig. 2. We come to the conclusion that our estimates of the spectrum amplitude based on the approximate solution of Eq. (8) [given by Eq. (19)] are reliable.

APPENDIX B: THE SHAPE OF THE ζ BISPECTRUM

The bispectrum of the non-Gaussian ζ field is defined by the expression

$$\langle\zeta(\mathbf{k_1})\zeta(\mathbf{k_2})\zeta(\mathbf{k_3})\rangle = (2\pi)^3\delta(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3})B(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}). \tag{B1}$$

If $\zeta = \chi^2 - \langle\chi^2\rangle$, the formula for B is [55]

$$B(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) = \frac{8}{3}\left[\int \frac{d^3k'}{(2\pi)^3} P_G(|\mathbf{k_1}-\mathbf{k}'|)P_G(|\mathbf{k_2}+\mathbf{k}'|)P_G(\mathbf{k}') + 2\,\text{perm}\right], \tag{B2}$$

where $P_G(k)$ is the curvature power spectrum of the Gaussian χ field, $P_G(k) \sim k^n$. The shape S of the bispectrum, which is defined by the formula

$$S(k_1,k_2,k_3) = (k_1k_2k_3)^2 B(k_1,k_2,k_3), \tag{B3}$$

has a characteristic "squeezed" form, shown in Fig. 5 (upper panel; $n = -2.9$).

The bispectrum in our axion inflation model is calculated using the formula

$$B(k_1,k_2,k_3) = \frac{3}{10}\mathcal{P}^3_{\zeta,\text{sr}} e^{6\pi\xi} \frac{k_1^3+k_2^3+k_3^3}{k_1^3k_2^3k_3^3} f_3\left(\xi, \frac{k_2}{k_1}, \frac{k_3}{k_1}\right). \tag{B4}$$

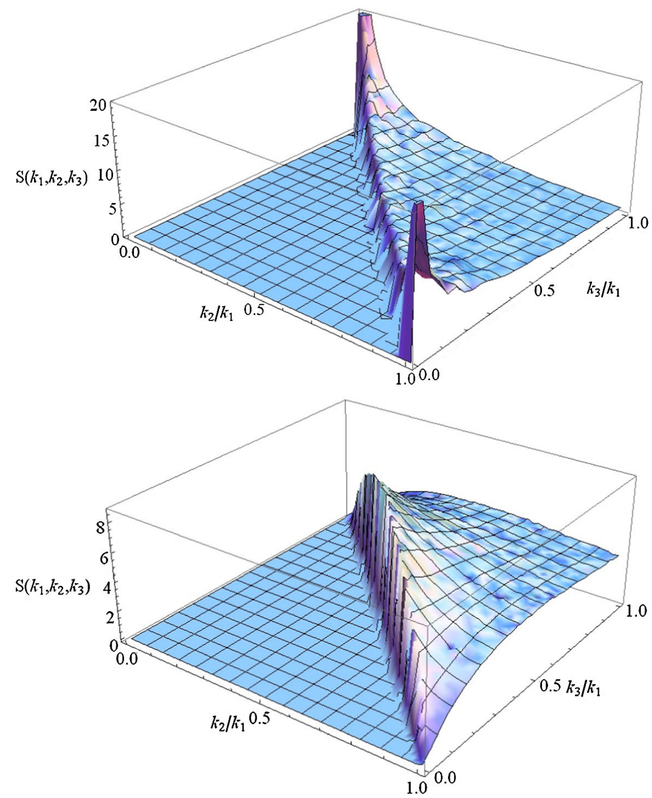

FIG. 5 (color online). Shape functions $S(k_1,k_2,k_3)$ (arbitrarily normalized) for the χ^2 model (upper panel) and the axion inflation model (lower panel).

Here, the function f_3 is defined in Refs. [37,56]. An example of the calculation of the corresponding shape function (for $\xi = 6$) is shown in Fig. 5 (lower panel).

Comparing the two shape functions, one can see that the shape function of our model differs rather strongly from the typical equilateral shape function (see, e.g., Ref. [57] for examples of equilateral shapes). At the same time, there is some similarity with the χ^2 model prediction, namely, in both figures there is some concentration of points along the diagonal line.

[1] D. Baumann, arXiv:0907.5424.
[2] E. Pajer and M. Peloso, Classical Quantum Gravity **30**, 214002 (2013).
[3] K. Freese, J. A. Frieman, and A. V. Olinto, Phys. Rev. Lett. **65**, 3233 (1990).
[4] F. C. Adams, J. R. Bond, K. Freese, J. A. Frieman, and A. V. Olinto, Phys. Rev. D **47**, 426 (1993).
[5] J. E. Kim, H. P. Nilles, and M. Peloso, J. Cosmol. Astropart. Phys. 01 (2005) 005.
[6] S. Dimopoulos, S. Kachru, J. McGreevy, and J. G. Wacker, J. Cosmol. Astropart. Phys. 08 (2008) 003.
[7] R. Easther and L. McAllister, J. Cosmol. Astropart. Phys. 05 (2006) 018.
[8] L. McAllister, E. Silverstein, and A. Westphal, Phys. Rev. D **82**, 046003 (2010).
[9] R. Flauger, L. McAllister, E. Pajer, A. Westphal, and G. Xu, J. Cosmol. Astropart. Phys. 06 (2010) 009.
[10] M. M. Anber and L. Sorbo, Phys. Rev. D **81**, 043534 (2010).

[11] N. Kaloper and L. Sorbo, Phys. Rev. Lett. **102**, 121301 (2009).
[12] N. Kaloper, A. Lawrence, and L. Sorbo, J. Cosmol. Astropart. Phys. 03 (2011) 023.
[13] M. Berg, E. Pajer, and S. Sjors, Phys. Rev. D **81**, 103535 (2010).
[14] J. S. Bullock and J. R. Primack, Phys. Rev. D **55**, 7423 (1997).
[15] P. Ivanov, Phys. Rev. D **57**, 7145 (1998).
[16] J. Yokoyama, Phys. Rev. D **58**, 083510 (1998).
[17] P. Pina Avelino, Phys. Rev. D **72**, 124004 (2005).
[18] R. Saito, J. Yokoyama, and R. Nagata, J. Cosmol. Astropart. Phys. 06 (2008) 024.
[19] C.-M. Lin and K.-W. Ng, Phys. Lett. B **718**, 1181 (2013).
[20] A. Linde, S. Mooij, and E. Pajer, Phys. Rev. D **87**, 103506 (2013).
[21] S. G. Rubin, A. S. Sakharov, and M. Yu. Khlopov, J. Exp. Theor. Phys. **92**, 921 (2001); **92**, 921 (2001).
[22] M. Yu. Khlopov, S. G. Rubin, and A. S. Sakharov, Astropart. Phys. **23**, 265 (2005).
[23] W. H. Press and P. Schechter, Astrophys. J. **187**, 425 (1974).
[24] E. Bugaev and P. Klimai, Phys. Rev. D **79**, 103511 (2009).
[25] E. Bugaev and P. Klimai, Phys. Rev. D **83**, 083521 (2011).
[26] N. Barnaby and M. Peloso, Phys. Rev. Lett. **106**, 181301 (2011).
[27] N. Barnaby, E. Pajer, and M. Peloso, Phys. Rev. D **85**, 023525 (2012).
[28] M. Cicoli and F. Quevedo, Classical Quantum Gravity **28**, 204001 (2011).
[29] T. Banks, M. Dine, P. J. Fox, and E. Gorbatov, J. Cosmol. Astropart. Phys. 06 (2003) 001.
[30] N. Arkani-Hamed, L. Motl, A. Nicolis, and C. Vafa, J. High Energy Phys. 06 (2007) 060.
[31] J. J. Blanco-Pillado, C. P. Burgess, J. M. Cline, C. Escoda, M. Gomez-Reino, R. Kallosh, A. D. Linde, and F. Quevedo, J. High Energy Phys. 11 (2004) 063.
[32] A. R. Liddle, A. Mazumdar, and F. E. Schunck, Phys. Rev. D **58**, 061301 (1998).
[33] E. J. Copeland, A. Mazumdar, and N. J. Nunes, Phys. Rev. D **60**, 083506 (1999).
[34] A. D. Linde, Phys. Lett. B **129**, 177 (1983).
[35] P. A. R. Ade *et al.* (Planck Collaboration), arXiv:1303.5082.
[36] P. A. R. Ade *et al.* (BICEP2 Collaboration), Phys. Rev. Lett. **112**, 241101 (2014).
[37] N. Barnaby, R. Namba, and M. Peloso, J. Cosmol. Astropart. Phys. 04 (2011) 009.
[38] E. V. Bugaev and P. A. Klimai, Int. J. Mod. Phys. D **22**, 1350034 (2013).
[39] E. Bugaev and P. Klimai, Phys. Rev. D **88**, 023521 (2013).
[40] T. Matsubara and J. Yokoyama, Astrophys. J. **463**, 409 (1996).
[41] K. Koyama, J. Soda, and A. Taruya, Mon. Not. R. Astron. Soc. **310**, 1111 (1999).
[42] P. J. E. Peebles, Astrophys. J. **510**, 531 (1999).
[43] M. J. White, Mon. Not. R. Astron. Soc. **310**, 511 (1999).
[44] N. Seto, Astrophys. J. **553**, 488 (2001).
[45] D. H. Lyth and A. R. Liddle, *The primordial density perturbation* (Cambridge University Press, Cambridge, England, 2009).
[46] E. Bugaev and P. Klimai, Phys. Rev. D **85**, 103504 (2012).
[47] K. Kohri and J. Yokoyama, Phys. Rev. D **61**, 023501 (2000).
[48] B. J. Carr, K. Kohri, Y. Sendouda, and J. Yokoyama, Phys. Rev. D **81**, 104019 (2010).
[49] S. G. Crowder, R. Namba, V. Mandic, S. Mukohyama, and M. Peloso, Phys. Lett. B **726**, 66 (2013).
[50] O. Seto, J. Yokoyama, and H. Kodama, Phys. Rev. D **61**, 103504 (2000).
[51] S. M. Leach and A. R. Liddle, Phys. Rev. D **63**, 043508 (2001).
[52] S. M. Leach, M. Sasaki, D. Wands, and A. R. Liddle, Phys. Rev. D **64**, 023512 (2001).
[53] E. Bugaev and P. Klimai, Phys. Rev. D **78**, 063515 (2008).
[54] E. Kamke, *Handbook on Ordinary Differential Equations* (Nauka, Moscow, 1971).
[55] M. LoVerde, E. Nelson, and S. Shandera, J. Cosmol. Astropart. Phys. 06 (2013) 024.
[56] P. D. Meerburg and E. Pajer, J. Cosmol. Astropart. Phys. 02 (2013) 017.
[57] Y. Wang, Commun. Theor. Phys. **62**, 109 (2014).