PHYSICAL REVIEW D **90,** 101503(R) (2014)

Chaos removal in $R + qR^2$ gravity: The mixmaster model

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(Received 26 July 2014; revised manuscript received 9 September 2014; published 24 November 2014)

We study the asymptotic dynamics of the mixmaster universe, near the cosmological singularity, considering $f(R)$ gravity up to a quadratic correction in the Ricci scalar R. The analysis is performed in the scalar-tensor framework and adopting Misner-Chitré-like variables to describe the mixmaster universe, whose dynamics resembles asymptotically a billiard ball in a given domain of the half-Poincaré space. The form of the potential well depends on the spatial curvature of the model and on the particular form of the self-interacting scalar field potential. We demonstrate that the potential walls determine an open domain in the configuration region, allowing the point universe to reach the absolute of the considered Lobachevsky space. In other words, we outline the existence of a stable final Kasner regime in the mixmaster evolution, implying chaos removal near the cosmological singularity. The relevance of the present issue relies both on the general nature of the considered dynamics, allowing its direct extension to the Belinski-Khalatnikov-Lifshitz conjecture too, as well as the possibility to regard the considered modified theory of gravity as the first correction to the Einstein-Hilbert action as a Taylor expansion of a generic function $f(R)$ (as soon as a cutoff on the space-time curvature takes place).

DOI: 10.1103/PhysRevD.90.101503 PACS numbers: 04.50.Kd, 04.60.Kz

I. INTRODUCTION

The chaotic dynamics of the mixmaster universe [1–3] is a basic prototype of the local (subhorizon) behavior of the generic cosmological solution [the so-called Belinski-Khalatnikov-Lifshitz (BKL) conjecture [4]]. Investigating the stability of such a chaotic picture with respect to the presence of matter [5–7] and space-time dimensions number [8–10] has seen a great effort over the last four decades and the most significant issue was the proof of chaos removal when a massless scalar field is involved in the dynamics [5]. Such a result is a consequence of the capability manifested by the scalar field kinetic energy of affecting the second (quadratic) Kasner condition, easily restated in the Hamiltonian picture, as shown in [11]. This property of the massless scalar field acquires intriguing perspectives when $f(R)$ modified theory of gravity is considered [12–16]. In fact, these alternative formulations of the gravitational field dynamics can be represented by an equivalent scalar-tensor picture: the scalar degree of freedom associated to the form of the function f is expressed via a self-interacting scalar field, coupled to the ordinary general relativity [17–20]. When implementing this scalar-tensor scheme to the mixmaster universe dynamics, a natural question arises: the kinetic term of the scalar field removes the chaotic behavior, but the presence of a potential term could restore it? Thus we can study, for specific modified theories of gravity, if the mixmaster chaos survives or not, simply characterizing the corresponding scalar field potential. Here we analyze the modified gravity theory corresponding to a quadratic correction in the Ricci scalar to the ordinary Einstein-Hilbert Lagrangian, both because it is the simplest viable deviation from general relativity (apart from a cosmological constant term), as well as the first correction emerging from a Taylor expansion of a $f(R)$ theory for very small values of the space-time Ricci scalar, i.e. for very law curvatures, like we observe today in the Solar System [21]. The quadratic term in the Ricci scalar provides an exponential-like potential term for the self-interacting scalar field, when a scalar-tensor reformulation of the model is considered. This case is particularly appropriate to the analysis we pursue of the mixmaster dynamics in terms of the Misner-Chitré-like variables [7,22–24]. In fact, the kinetic term of the scalar field is on the same footing of the anisotropy term contribution and, for the considered Lagrangian, also the potential term is isomorphic to the spatial curvature of the model; i.e. the total potential term is constituted by equivalent exponential profile. In the asymptotic limit toward the initial singularity the total potential takes the form of four potential walls, whose morphology determines if the configuration domain

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is closed or not. Indeed, we demonstrate how the whole domain, available in principle, is a constant negative curvature space (half-Poincarè space). We first analyze the case of the mixmaster universe in the presence of a massless scalar field, demonstrating the open nature of its configuration space and the implied existence of a stable Kasner regime to the initial singularity. Then, we face a detailed study of the dynamics in the presence of the total potential and the still open structure of the configuration domain. Thus, we demonstrate the nonchaotic nature of the mixmaster universe behavior, as it is described by the scalar-tensor version of the R^2 gravity.

II. $f(R)$ GRAVITY

The $f(R)$ theories of gravity are a direct generalization of the Einstein-Hilbert Lagrangian consisting in a replacement of the Ricci scalar R by a general function $f(R)$ [13–15,25]:

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}f(R), \tag{1}$$

where g is the determinant of the metric.[1] The introduction of the additional degree of freedom, related to the presence of the $f(R)$ term, can be translated into a dynamics of a self-interacting scalar field coupled with the Einstein-Hilbert action, the so-called *scalar-tensor framework*. In this approach, a new auxiliary field χ is introduced to get the following equivalent version of the action (1):

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}[f(\chi) - f'(\chi)(R-\chi)]. \tag{2}$$

The variation of the action (2) with respect to χ provides $f''(\chi)(R-\chi) = 0$, implying $\chi = R$ if $f''(\chi) \neq 0$. By a redefinition of the auxiliary field χ in the form $\varphi = f'(\chi)$ the action becomes

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}[\varphi R - \chi(\varphi)\varphi + f(\chi(\varphi))]. \tag{3}$$

It is now possible to perform a conformal transformation on the metric $g_{\mu\nu} \to \tilde{g_{\mu\nu}} = \varphi g_{\mu\nu}$ and a scalar field redefinition $\varphi \equiv f'(R) \to \phi = \sqrt{\frac{3}{16\pi}}\ln f'(R)$ in order to obtain

$$S = \int d^4x\sqrt{-\tilde{g}}\left[\frac{\tilde{R}}{16\pi} - \frac{1}{2}\partial^\alpha\phi\partial_\alpha\phi - U(\phi)\right], \tag{4}$$

where the potential term $U(\phi)$ has the form

$$U(\phi) = \frac{Rf'(R) - f(R)}{16\pi(f'(R))^2}. \tag{5}$$

[1]We use the $(-,+,+,+)$ signature of the metric and the geometric unit system $(c = G = \hbar = 1)$.

For small values of the Ricci scalar, the first-order correction to the Einstein-Hilbert Lagrangian is represented by a quadratic correction, i.e.

$$f(R) = R + qR^2. \tag{6}$$

By this choice, the potential term (5) takes the form

$$U(\phi) = \frac{1}{64\pi q}\left(1 - 2\exp^{-4\sqrt{\frac{\pi}{3}}\phi} + \exp^{-8\sqrt{\frac{\pi}{3}}\phi}\right). \tag{7}$$

This is the effective potential that emerges in the so-called Starobinsky-inflation model [21]. Such a model ensures a "slow-rolling" face and it is an inflationary model passing the latest inflation constraint [26].

III. THE MIXMASTER MODEL DYNAMICS

Following the standard representation of the Bianchi IX model [27] in the Misner variables [2,3] the Einstein-Hilbert action takes the form

$$S_g = \int dt\left[p_\alpha\dot{\alpha} + p_+\dot{\beta_+} + p_-\dot{\beta_-} - \frac{Ne^{-3\alpha}}{24\pi}\mathcal{H}_{\text{IX}}\right], \tag{8}$$

where the dynamics of the model implies the super-Hamiltonian constraint

$$\mathcal{H}_{\text{IX}} \equiv -p_\alpha^2 + p_+^2 + p_-^2 + 12\pi^2e^{4\alpha}V_{\text{IX}}(\beta_\pm) = 0. \tag{9}$$

Here α expresses the isotropic component of the Universe (i.e. the volume of the Universe) and the initial singularity is reached for $\alpha \to -\infty$, while the traceless matrix $\beta_{ab} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$ accounts for the anisotropy of this model. Furthermore, $p_\alpha, p_\pm$ are the conjugated momenta to $\alpha, \beta_\pm$, respectively, and $V_{\text{IX}}(\beta_\pm)$ is the potential term depending only on $\beta_\pm$, corresponding to the spatial curvature. If we execute an Arnowitt-Deser-Misner reduction of the dynamics [28], the Bianchi IX model resembles the behavior of a two-dimensional particle, evolving with respect to the timelike variable α in the β_+, β_- plane. In other words, the system dynamics is summarized by the time-dependent Hamiltonian H_{IX}:

$$-p_\alpha = H_{\text{IX}} \equiv \sqrt{p_+^2 + p_-^2 + 12\pi^2e^{4\alpha}V_{\text{IX}}(\beta_\pm)}. \tag{10}$$

Looking at the form of the potential term $V_{\text{IX}}(\beta_\pm)$, it is possible, taking into account the three leading terms, to parametrize it as an infinitely steep potential well [3]. This way, the point universe lives inside the triangular region of the configuration space where the potential term is negligible; such a region is individuated when the following three conditions hold:

$$\frac{1}{3} + \frac{\beta_+ + \sqrt{3}\beta_-}{3\alpha} > 0,$$
$$\frac{1}{3} + \frac{\beta_+ - \sqrt{3}\beta_-}{3\alpha} > 0,$$
$$\frac{1}{3} - \frac{2\beta_+}{3\alpha} > 0. \tag{11}$$

The presence of the "time" variable α in the relations (11) causes the outside motion of the potential walls and the corresponding time dependence of the domain allowed to the point-universe motion. Such a dependence can be removed in the framework of the *Misner-Cithrè* variables τ, ζ, θ [24,27] as standing in the *Poincaré half-plane*:

$$\alpha - \alpha_0 = -e^{\tau}\frac{1 + u + u^2 + v^2}{\sqrt{3}v},$$
$$\beta_+ = e^{\tau}\frac{-1 + 2u + 2u^2 + 2v^2}{2\sqrt{3}v},$$
$$\beta_- = e^{\tau}\frac{-1 - 2u}{2v}, \tag{12}$$

where $-\infty < \tau < \infty$, $-\infty < u < +\infty$, and $0 < v < +\infty$. In this scheme the role of the Hamiltonian time is assigned to τ and the singularity is approached for $\tau \to \infty$. The transformations (12) permit one to rewrite the conditions (11) as independent of the variable τ and thus the domain within which the particle lives is fixed with respect to the time variable. Making use of the transformations (12), the Hamiltonian (10) in the free-potential case rewrites as

$$-p_\tau = H_{\rm IX} \equiv \sqrt{v^2(p_u^2 + p_v^2)}, \tag{13}$$

and the point universe lives in the u, v plane inside the region individuated when the following three conditions hold:

$$\frac{-u}{1 + u + u^2 + v^2} > 0,$$
$$\frac{1 + u}{1 + u + u^2 + v^2} > 0,$$
$$\frac{u(u+1) + v^2}{1 + u + u^2 + v^2} > 0. \tag{14}$$

As shown by [23], the asymptotic evolution towards the singularity is covariantly chaotic because it is isomorphic to a billiard on the Lobachevsky plane. This demonstration is based on three points: (i) the Jacobi metric in the u, v plane has a negative constant curvature; (ii) the Lyapunov exponent, defined as in [29], is greater than zero; (iii) the configuration space is (dynamically) compact. The occurrence of these three properties ensures that the geodesic trajectories cover the whole configuration space.

IV. MIXMASTER UNIVERSE IN THE R^2 GRAVITY

Now we analyze the case of the gravitational Lagrangian (6) when the Bianchi IX model is considered. As a starting point we consider the modified gravity model (4) in terms of the variables $\alpha, \beta_+, \beta_-, \phi$. Following the same procedure of the previous section we get the generalized reduced Hamiltonian $-p_\alpha = H$ of the form

$$H \equiv \sqrt{p_+^2 + p_-^2 + p_\phi^2 + 12\pi^2 e^{4\alpha} V_{\rm IX} + 4e^{6\alpha} U}. \tag{15}$$

In Eq. (15), we rescaled the zero point of $\alpha \to \alpha - \alpha_0$, so that the spatial metric factor $e^{3\alpha} \to \frac{1}{(6\pi)} e^{3\alpha}$, and a redefinition of the scalar field amplitude $\phi \to \sqrt{2}(6\pi)\phi$ is considered too. A natural parametrization, in the Misner-Cithrè-Poincaré half-plane scheme, that reduces to the relations (12) if the scalar field is turned off reads as follows:

$$\alpha - \alpha_0 = -e^{\tau}\frac{1 + u + u^2 + v^2}{\sqrt{3}v},$$
$$\beta_+ = e^{\tau}\frac{-1 + 2u + 2u^2 + 2v^2}{2\sqrt{3}v},$$
$$\beta_- = e^{\tau}\frac{-1 - 2u}{2v}\cos\delta,$$
$$\phi = e^{\tau}\frac{-1 - 2u}{2v}\sin\delta, \tag{16}$$

where $-\infty < \tau < \infty$, $-\infty < u < +\infty$, $0 < v < +\infty$ and $0 < \delta < 2\pi$. In this new system of variables the reduced Hamiltonian takes the form

$$-p_\tau = H \equiv \sqrt{v^2\left[p_u^2 + p_v^2 + 4\frac{p_\delta^2}{(1+2u)^2}\right] + e^{2\tau}\mathcal{V}}. \tag{17}$$

The introduction of the degree of freedom related to the scalar field implies that the point universe lives inside a three-dimensional domain in the configuration space u, v, δ determined by the potential term:

$$\begin{aligned} e^{2\tau}\mathcal{V} &= e^{2\tau}[12\pi^2 e^{-4e^{\tau}\xi(u,v)} V_{\rm IX}(u,v,\delta,\tau) \\ &\quad + 4e^{-6e^{\tau}\xi(u,v)} U(u,v,\delta,\tau)] \\ &= 12\pi^2 e^{2\tau}\Big(e^{-\frac{12e^{\tau}}{\sqrt{3}v}(u+u^2+v^2)} + e^{-\frac{6e^{\tau}}{\sqrt{3}v}(1+(1+2u)\cos\delta)} \\ &\quad + e^{-\frac{6e^{\tau}}{\sqrt{3}v}(1-(1+2u)\cos\delta)}\Big) + \frac{e^{2\tau}}{8\pi q}\Big(e^{-\frac{12e^{\tau}}{\sqrt{3}v}(1+u+u^2+v^2)} \\ &\quad - 2e^{-\frac{6e^{\tau}}{\sqrt{3}v}(1+u+u^2+v^2-2\sqrt{2\pi^3}(1+2u)\sin\delta)} \\ &\quad + e^{-\frac{6e^{\tau}}{\sqrt{3}v}(1+u+u^2+v^2-4\sqrt{2\pi^3}(1+2u)\sin\delta)}\Big), \end{aligned} \tag{18}$$

where $\xi(u,v) = \frac{1+u+u^2+v^2}{\sqrt{3}v}$. Because of the exponential forms of the terms in Eq. (18), when the singularity is approached ($\tau \to \infty$) the point universe is confined to live inside a three-dimensional domain defined as the region where all the exponents of the six terms are simultaneously greater than zero. Looking at Eq. (18), the potential term $\mathcal{V}$ behaves as an infinitely steep potential well as in the Poincarè variables (14). So for the evolution of the point universe it is possible to neglect the potential everywhere in a suitable domain. As a first step we study the case in the absence of all the potential terms ($\mathcal{V} = 0$); i.e. we deal with the Hamiltonian problem:

$$H = v\sqrt{p_u^2 + p_v^2 + 4\frac{p_\delta^2}{(1+2u)^2}}. \tag{19}$$

The Hamiltonian equations for this potential-free system (Bianchi I model with the massless scalar field) are

$$\dot{u} = \frac{\partial H}{\partial p_u} = \frac{v^2}{\epsilon}p_u, \qquad \dot{p_u} = -\frac{\partial H}{\partial u} = \frac{8v^2}{\epsilon}\frac{p_\delta^2}{(1+2u)^3},$$
$$\dot{v} = \frac{\partial H}{\partial p_v} = \frac{v^2}{\epsilon}p_v, \qquad \dot{p_v} = -\frac{\partial H}{\partial v} = -\frac{\epsilon}{v},$$
$$\dot{\delta} = \frac{\partial H}{\partial p_\delta} = \frac{4v^2}{\epsilon}\frac{p_\delta}{(1+2u)^2}, \qquad \dot{p_\delta} = -\frac{\partial H}{\partial \delta} = 0. \tag{20}$$

It is possible to demonstrate, as we approach the singularity, that H is a constant of motion with respect to the "time" variable τ, following [23]. Thus, we perform the substitution $H \simeq \epsilon = \text{const}$ inside Eqs. (20). It is now possible, by following the Jacobi procedure [30] and using the equations of motion (20), to write down the line element for the three-dimensional Jacobi metric in terms of the configuration variables, i.e.

$$ds^2 = \frac{\epsilon}{v^2}\left[du^2 + dv^2 + \frac{(1+2u)^2}{4}d\delta^2\right]. \tag{21}$$

By a direct calculation we see that this metric has a negative constant curvature (the associated Ricci scalar is $R = -\frac{6}{\epsilon}$) and then the point universe moves over a negatively curved three-dimensional space. Furthermore, we can find two singular values for the metric in correspondence to $u = -\frac{1}{2}$, $v = 0$. This feature allows us to restrict the domain of the configuration space in which we will study the trajectories of the point universe to the fundamental one identified by the inequalities $-\frac{1}{2} < u < +\infty$, $0 < v < +\infty$, and $0 < \delta < 2\pi$. Indeed there is no way for the point-universe trajectories to cross over the two planes $u = -\frac{1}{2}$, $v = 0$ (each choice of the Lobachevsky "half-space" is equivalent with respect to the other one). The intermediate step toward the general case of the potential (18), corresponding to the ordinary mixmaster model in the presence of a massless scalar field, takes place when we retain only the exponential terms due to the spatial curvature, namely $\mathcal{V} \simeq 12\pi^2 e^{-4e^\tau \xi(u,v)} V_{\rm IX}(u,v,\delta,\tau)$. Then, the point universe lives in the region where are simultaneously satisfied the three following conditions:

$$1 + (1+2u)\cos\delta > 0,$$
$$1 - (1+2u)\cos\delta > 0,$$
$$u(u+1) + v^2 > 0. \tag{22}$$

We now implement a numerical integration of the system (20) in order to analyze the behavior of the trajectories in the potential-free region and then use this result for interpreting the effect of the scalar curvature. As we can see in Fig. 1 an opening of the domain emerges due to the presence of the scalar field and it is possible to individuate two families of trajectories: those ones corresponding to a point universe that bounces against the walls and turn back inside the domain (the black ones) and those corresponding to a particle that approach the so-called "absolute" [31] (the red ones), for values $v \to 0, \infty$, with no other bounces until the singularity. The presence of the trajectories of the second family shows the removal of the oscillatory behavior of the mixmaster model coupled with a massless scalar field [4,11]. Let us see what happens if we consider the complete potential term (18). This time the restrictions on the dynamics imply that the particle is confined inside a region where all six exponential terms in Eq. (18) are simultaneously greater than zero. We can immediately remove one of the six conditions because the first exponent related to the potential of the scalar field $1 + u + u^2 + v^2$ is always greater than zero for any values of u, v taking in consideration. Thus, the five conditions that identify the domain are

$$1 + (1+2u)\cos\delta > 0,$$
$$1 - (1+2u)\cos\delta > 0,$$
$$u(u+1) + v^2 > 0,$$
$$1 + u + u^2 + v^2 - 2\sqrt{2\pi^3}(1+2u)\sin\delta > 0,$$
$$1 + u + u^2 + v^2 - 4\sqrt{2\pi^3}(1+2u)\sin\delta > 0. \tag{23}$$

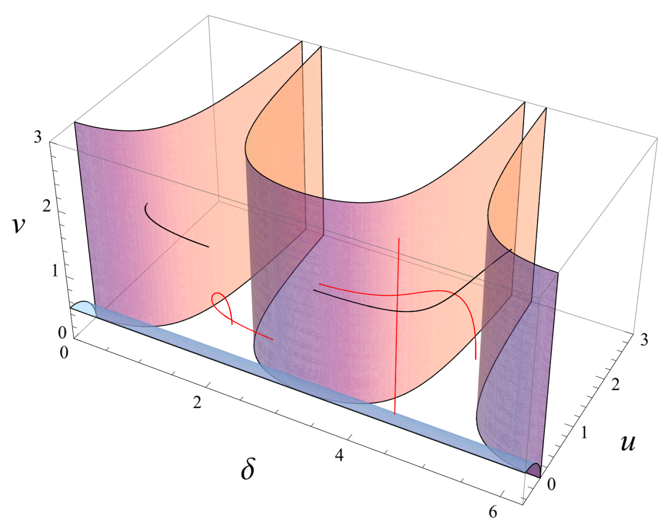

FIG. 1 (color online). The black lines represent the trajectories associated to a point universe that bounce against the walls. Instead, the red lines describe the point universe which directly approaches the singularity.

We observe that the last of the conditions above naturally implies the validity of the fourth one too. Thus, we indeed deal with four potential walls only. As we can see in Fig. 2, taking into account also the potential term $U(u, v, \delta, \tau)$ implies that the available configuration space for the point universe is clearly reduced with respect to the case $U = 0$ (see Fig. 1). However, trajectories yet exist (the red lines in Fig. 2) corresponding to a point universe that is able to reach the absolute for $v \to 0, \infty$. For this reason we can firmly conclude that a quadratic correction in the Ricci scalar to the Einstein-Hilbert action, that in the scalar-tensor theory is equivalent to the dynamics of a self-interacting scalar field [with potential terms of the form (7)], is able to remove the never-ending bounces of the point universe against the walls. As a result of the bounces against the infinite potential walls (which can be described by a reflection rule [22,32]), sooner or later the point universe reaches a trajectory connected with the absolute. It is worth noting that the analysis above is referred to the choice $q > 0$, in which case the sign of the scalar field potential is the same one of the scalar curvature. This choice is forced by the request that the additional scalar mode, associated to the quadratic modification, be a real (nontachyonic) massive one, accordingly to the original Starobinsky approach in [21] and demonstrated also in [33]. However, in the case $q < 0$, the scalar field potential would not contribute an infinite positive wall, but an infinite depression. Since in the region of zero potential, the point universe has always positive "energy," we can easily conclude that such a case overlaps the nonchaotic potential-free one. We now observe that, in correspondence to the configuration region $v = 0, \infty$ and $\delta = \frac{3\pi}{2}$, the scalar field acquires negative diverging values and its potential terms manifests a diverging behavior. Such a profile of the scalar field is typical of a Bianchi I solution near the singularity [5] and the diverging character of the potential term means that general relativity can not be asymptotically recovered. Rigorously speaking, the present result on the chaos structure applies to a quadratic correction in the Ricci scalar only, because it is the first terms of the Taylor expansion of the function $f(R)$ working nearby the singularity. Nonetheless, our analysis has a general validity, as soon as we take into account a physical cutoff at the Planck time, where classical theory starts to fail and a quantum treatment is required. In fact, the Planckian cutoff would remove the ϕ and $U(\phi)$ divergences, allowing the Taylor expansion for $q \lesssim \frac{(ct_{\rm cut})^2}{l_P^2}$, where $t_{\rm cut}$ is the cutoff time and l_P the Planck length. Since $t_{\rm cut} > \frac{l_P}{c}$, we deal with the (nonsevere) restriction $q \lesssim 1$ for preserving the general nature of our result. This estimation follows requiring $R > qR^2$ and remembering that for the case of a Kasner solution, in the presence of a potential-free scalar field, the Ricci scalar behaves as $R \sim \frac{1}{t_s^2}$, where t_s is the synchronous time. We stress that qualitatively, a similar argument is at the ground of the nonchaotic nature of the Bianchi IX loop quantum dynamics in the semiclassical limit [34]. However, the field $\phi(t_s)$ admits, both for $v \to 0, \infty$ and $\delta = \pi/2$, trajectories implying its positive divergence. For such behaviors, corresponding to an open region in the initial condition, the potential $U(\phi)$ approaches a constant value and ϕ is effectively massless. It is just the existence of these diverging profiles at the ground of the chaos removal in the present model. The massless nature of the potential along specific trajectories is a good criterion for determining the chaotic properties of the mixmaster universe in a specific nonexpanded $f(R)$ model. In fact, the behavior of the free scalar field reads $\phi_f(t_s) \propto \ln t_s$ and the corresponding kinetic energy density stands as $1/t_s^2$. Then, for a given $f(R)$ model, fixing the potential $U(\phi)$, the chaos removal is ensured by the validity of the condition $\lim_{t_s \to 0} U(\phi_f(t_s)) t_s^2 = 0$. Clearly, the nonchaoticity is ensured if such a limit holds for a nonzero measure set of trajectories.

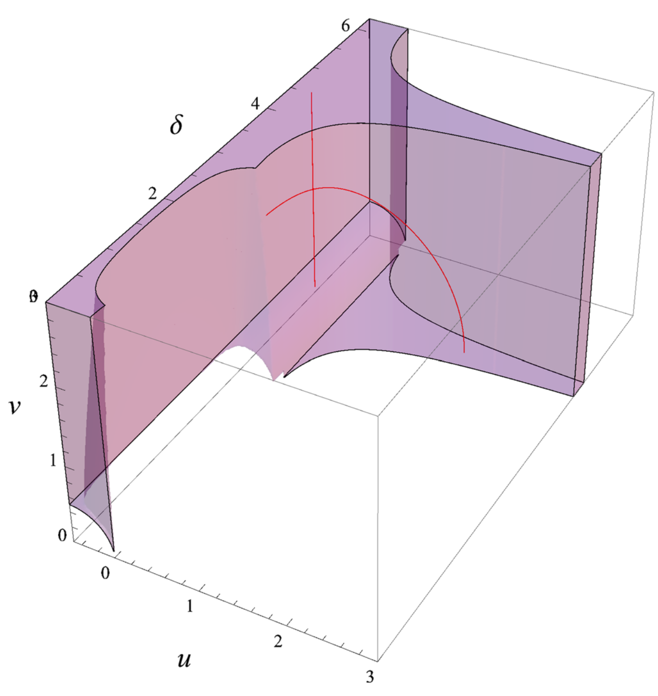

FIG. 2 (color online). The point universe lives inside the region marked by the walls, where the conditions (23) are verified. We also sketch the trajectories reaching the absolute.

V. CONCLUSIONS

The analysis above demonstrated how including a quadratic correction in the Ricci scalar to the Einstein-Hilbert Lagrangian of the gravitational field gives a deep insight on the nature of the mixmaster singularity: the evolution of the scale factors is no longer chaotic and a stable Kasner regime emerges as the final approach to the singular point.

The relevance of this result is in its generality with respect to the behavior of the cosmological gravitational field. In fact, on one hand, the result we derived in the homogeneous cosmological setting can be naturally extended to a generic inhomogeneous universe, simply following the line of investigation discussed in [4,5].

The basic statement, at the ground of the BKL conjecture, is the space point decoupling in the asymptotic dynamics toward the cosmological singularity. Such a dynamical property of a generic inhomogeneous cosmological model allows one to reduce the behavior of a subhorizon spatial region [4,35] to the prototype offered by the homogeneous mixmaster universe. We are actually stating that the time derivatives of the dynamical variables asymptotically dominate their spatial gradients, limiting the presence of the spatial coordinates in the Einstein equation to a pure parametrical role. We are speaking of a conjecture because the chaotic features of the pointlike dynamics induce a corresponding stochastic behavior of the spatial dependence and the statement above requires a nontrivial treatment for its proof. Nonetheless a valuable estimation of the spatial gradient behavior, when the space-time takes the morphology of a foam, is provided in [36]. When a scalar field is present the situation is even more simple, because, after a certain number of iterations of the BKL map, in each space point, a stable Kasner regime takes place [37] and the validity of the solution is rigorously determined [38]. Thus, we can extend our result to a generic inhomogeneous cosmological model simply considering the dynamical variables as space-time functions $u = u(\tau, x^i)$, $v = v(\tau, x^i)$ and $\delta = \delta(\tau, x^i)$, which, in each space point, live in a half-Poincarè space and are governed by an independent and morphologically equivalent dynamics. On the other hand, the extension of general relativity we considered here is the most simple and natural one, widely studied in the literature in view of its implications on the primordial Universe features. Since the classical evolution is expected to be predictive up to a finite value of the Universe volume, i.e. up to a given amplitude of the space-time curvature, for sufficiently small coupling constant q values, the present model can be considered as the quadratic Taylor expansion of a generic $f(R)$ theory and we can then guess that the nonchaotic feature is a very general dynamical property, at least within the classical domain of validity of the $f(R)$ theory. In this sense we traced very general and reliable properties of the cosmological gravitational field in modified theories of gravity of significant impact on the so-called billiard representation of the generic primordial Universe [8,22,32].

[1] V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970).
[2] C. W. Misner, Phys. Rev. Lett. **22**, 1071 (1969).
[3] C. W. Misner, Phys. Rev. **186**, 1319 (1969).
[4] V. A. Belinski, I. M. Khalatnikov, and E. M. Lifshitz, Adv. Phys. **31**, 639 (1982).
[5] V. A. Belinskii and I. M. Khalatnikov, Sov. Phys. JETP **36**, 591 (1973).
[6] G. Montani, M. V. Battisti, R. Benini, and G. Imponente, Int. J. Mod. Phys. A **23**, 2353 (2008).
[7] G. Montani, M. V. Battisti, G. Imponente, and R. Benini, *Primordial Cosmology* (World Scientific, Singapore, 2011).
[8] J. Demaret, M. Henneaux, and P. Spindel, Phys. Lett. **164B**, 27 (1985).
[9] Y. Elskens and M. Henneaux, Nucl. Phys. **B290**, 111 (1987).
[10] N. Deruelle and P. Spindel, Classical Quantum Gravity **7**, 1599 (1990).
[11] B. K. Berger, Phys. Rev. D **61**, 023508 (1999).
[12] S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002).
[13] S. Capozziello and M. De Laurentis, Phys. Rep. **509**, 167 (2011).
[14] S. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011).
[15] S. Nojiri and S. D. Odintsov, Int. J. Geom. Methods Mod. Phys. **04**, 115 (2007).
[16] J. Q. Guo, D. Wang, and A. V. Frolov, Phys. Rev. D **90**, 024017 (2014).
[17] J. D. Barrow and S. Cotsakis, Phys. Lett. B **232**, 172 (1989).
[18] J. D. Barrow and H. Sirousse-Zia, Phys. Rev. D **39**, 2187 (1989).
[19] J. D. Barrow and S. Cotsakis, Phys. Lett. B **214**, 515 (1988).
[20] N. Deruelle, Nucl. Phys. **B327** 253 (1989).
[21] A. A. Starobinsky, Phys. Lett. **91B**, 99 (1980).
[22] A. A. Kirillov and G. Montani, Pis'ma Zh. Eksp. Teor. Fiz. **66**, 449 (1997).
[23] G. P. Imponente and G. Montani, Phys. Rev. D **63**, 103501 (2001).
[24] D. M. Chitré, Ph.D. thesis, University of Maryland, 1972.
[25] S. Nojiri and S. D. Odintsov, Tomsk State Pedag. Univ. Bull. **N8**, 7 (2011).
[26] P. A. R. Ade *et al.* (Planck Collaboration), arXiv:1303.5082.
[27] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[28] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **117**, 1595 (1960).
[29] Ya. B. Pesin, Russ. Math. Surv. **32**, 55 (1977).
[30] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1989).
[31] A. A. Kirillov and G. Montani, Phys. Rev. D **56**, 6225 (1997).
[32] T. Damour and O. M. Lecian, Phys. Rev. D **83**, 044038 (2011).
[33] S. Capozziello and S. Vignolo, Classical Quantum Gravity **26**, 175013 (2009).
[34] M. Bojowald, G. Date, and G. M. Hossain, Classical Quantum Gravity **21**, 3541 (2004).
[35] G. Montani, Classical Quantum Gravity, **12**, 2505 (1995).
[36] A. A. Kirillov, Sov. Phys. JETP **76**, 355 (1993).
[37] A. A. Kirillov and A. A. Kochnev, Pis'ma Zh. Eksp. Teor. Fiz. **46**, 345 (1987).
[38] E. M. Lifshitz and I. M. Khalatnikov, Adv. Phys. **12**, 185 (1963).