

Octet and decuplet masses: A covariant three-body Faddeev calculation

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We determine the baryon octet and decuplet masses as well as their wave functions in a covariant three-body Faddeev approach. We work in an approximation where irreducible three-body forces are neglected. In the two-body interactions we take into account a well-explored rainbow-ladder kernel as well as flavor dependent meson-exchange terms motivated from the underlying quark-gluon interaction. Without flavor dependent forces we find agreement with the experimental spectrum on the 5–10 percent level. Including the flavor dependent terms on an exploratory level delivers a $\Sigma - \Lambda$ splitting with the correct sign although the magnitude is still too small.

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I. INTRODUCTION

Most of the mass of ordinary matter in the observable Universe is generated by the strong interaction. Therefore, it is important to understand the mechanisms of mass generation in QCD, the theory of strong interactions. Of particular theoretical interest is the calculation of the hadron spectrum from QCD and its comparison with the experimentally measured masses. For this reason, great effort is put forth to overcome the technical difficulties of lattice QCD and provide reliable *ab initio* calculations of hadron properties (see, e.g. [1–6] and references therein).

The combination of Dyson-Schwinger equations (DSE) and Bethe-Salpeter equations (BSE) provides, ideally, an alternative approach to first principle QCD. Complementary to the lattice, it provides insight into the underlying mechanism of mass generation in QCD and the details of the interaction mechanisms at work in binding quarks and gluon together into hadrons. In practice, of course, challenges arise. Most prominently for most applications the infinite set of DSE that define a theory must be truncated into something manageable, which in turn induces the necessity of modeling. It is nevertheless remarkable that, even with the simplest truncations, the framework is able to reproduce fairly well many hadron observables (see, e.g. [7,8] for overviews). Despite this fact, the deficiencies of these simple truncations are apparent when the details of the interaction are probed, such as in the calculation of baryon form factors [9–11] or in meson and baryon excited states [12–16]. A major current focus is therefore to improve the approximation of the quark-gluon interaction in the DSE/BSE system [17–25].

The calculation of bound states, especially of baryons, in the DSE/BSE formalism is also technically challenging. A fully covariant three-body calculation of the nucleon [26], delta and omega [27] masses has already been achieved. However, for baryon octet and decuplet states with both

u/d- and *s*-quark content only a simplified two-body framework using the quark-diquark approximation has been considered so far [28]. In this work, we overcome those technical difficulties and unify, for the first time, the treatment of light and strange baryons in the three-body approach. We explore the merits of the simplest truncation of the quark-gluon interaction compatible with Poincaré covariance and chiral symmetry, namely, the rainbow-ladder truncation and provide first steps to include flavor dependent interactions responsible for details such as the $\Sigma - \Lambda$ splitting in the spectrum. The methodology presented here sets the stage for future, more sophisticated, schemes beyond the simple rainbow-ladder framework.

The paper is organized as follows. In Sec. II we briefly summarize the covariant DSE/BSE formalism. A more detailed description of the formalism is given in Appendix A. In Sec. III we discuss the masses of the baryon octet and decuplet in the rainbow-ladder truncation and explore the possible effect of flavor dependent interactions beyond rainbow-ladder. Potential future developments are discussed in the conclusions, Sec. IV.

II. REVIEW OF THE FORMALISM

In the Bethe-Salpeter framework, a baryon is described by the three-body Bethe-Salpeter amplitude $\Gamma_{ABCD}(p_1, p_2, p_3)$, where we use $\{ABC\}$ as generic quark indices for spin, flavor and color indices (e.g. $A \rightarrow \{\alpha, a, r\}$, respectively). The amplitude depends on the three quark momenta $p_{1,2,3}$, which can be expressed in terms of two relative momenta p and q and the total momentum P [see Eq. (A3) in Appendix A]. It is decomposed in a tensor product of a spin-momentum part to be determined and flavor and color parts which are fixed,

$$\Gamma_{ABCD}(p, q, P) = \left(\sum_{\rho} \Psi_{\alpha\beta\gamma\mathcal{I}}^{\rho}(p, q, P) \otimes F_{abcd}^{\rho} \right) \otimes \frac{\epsilon_{rst}}{\sqrt{6}}. \quad (1)$$

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The color term $\epsilon_{rst}/\sqrt{6}$ fixes the baryon to be a color singlet and the flavor terms F_{abcd}^ρ are the quark-model $SU(3)$ -symmetric representations (see Appendix B). The index ρ denotes the representation of the $SU(3)$ group to which the baryon belongs (mixed-symmetric or mixed-antisymmetric representation for the baryon octet and only the symmetric representation for the baryon decuplet).

The spin-momentum part of the Bethe-Salpeter amplitude, $\Psi_{\alpha\beta\gamma\mathcal{I}}^\rho(p, q, P)$, is a tensor with three Dirac indices α, β, γ associated to the valence quarks and a generic index \mathcal{I} whose nature depends on the spin of the resulting bound state. They can in turn be expanded in a covariant basis $\{\tau(p, q; P)\}$,

$$\Psi_{\alpha\beta\gamma\mathcal{I}}^\rho(p, q, P) = f_i^\rho(p^2, q^2, z_0, z_1, z_2) \tau_{\alpha\beta\gamma\mathcal{I}}^i(p, q; P), \quad (2)$$

where the scalar coefficients $\{f\}$ depend on Lorentz scalars $p^2, q^2, z_0 = \widehat{p}_T \cdot \widehat{q}_T, z_1 = \hat{p} \cdot \hat{P}$ and $z_2 = \hat{q} \cdot \hat{P}$ only. The subscript T denotes transverse projection with respect to the total momentum and vectors with hat are normalized. A covariant basis can be obtained using symmetry requirements only; for positive-parity spin-1/2 baryons it contains 64 elements [29,30] whereas for spin-3/2 baryons it contains 128 elements [27]. The task of solving the bound-state equation is now greatly facilitated, as one only needs to solve for the scalar functions f .

The amplitudes Ψ are, in general, solutions of a three-body BSE (see Fig. 1), for which one needs to specify the three-particle and two-particle irreducible kernels, $K^{(3)}$ and $K^{(2)}$, respectively. In the Faddeev approximation the three-body irreducible interactions are neglected, and we refer to the simplified BSE as the Faddeev equation. Central elements of the Faddeev equation are the full quark propagator S (omitting now Dirac indices),

$$S^{-1}(p) = A(p^2)(i\not{p} + B(p^2)/A(p^2)), \quad (3)$$

with vector and scalar dressing functions $A(p^2)$ and $B(p^2)$. The ratio $M(p^2) = B(p^2)/A(p^2)$ is a renormalization group invariant and describes the running of the quark mass with momentum. The dressing functions are obtained as solutions of the quark DSE,

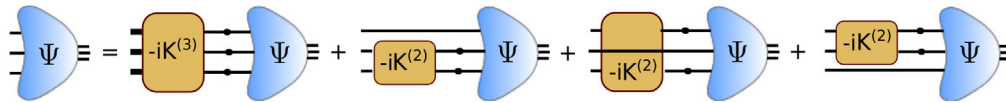


FIG. 1 (color online). Diagrammatic representation of the three-body Bethe-Salpeter equation with amplitude Ψ . Full quark propagators are denoted by straight lines with black dots. The interaction between the quarks are encoded in the three-body and two-body kernels $K^{(3)}$ and $K^{(2)}$.

$$S^{-1}(p) = S_0^{-1}(p) + Z_{1f} g^2 C_F \int_q \gamma^\mu D_{\mu\nu}(p-q) \Gamma^\nu(p, q) S(q), \quad (4)$$

which also contains the full quark-gluon vertex Γ^ν and the full gluon propagator $D_{\mu\nu}$; S_0 is the (renormalized) bare propagator with inverse

$$S_0^{-1}(p) = Z_2(i\not{p} + m), \quad (5)$$

where m is the bare quark mass and Z_{1f} and Z_2 are renormalization constants. The renormalized strong coupling is denoted by g and $\int_q = \int \frac{d^4q}{(2\pi)^4}$ abbreviates a four-dimensional integral in momentum space supplemented with a translationally invariant regularization scheme.

Under certain symmetry requirements, the practical solution of the Faddeev equation can be greatly simplified by relating the three two-body interaction diagrams in Fig. 1. As shown in Appendix A, taking the flavor part of the Faddeev amplitudes as representations of the $SU(3)$ group induces a specific transformation rule for the spin-momentum part under interchange of its valence-quark indices. In this case the Faddeev equation for the coefficients f reduces to

$$\begin{aligned} f_i^\rho(p^2, q^2, z_0, z_1, z_2) &= C \mathcal{F}_1^{\rho\rho':\lambda} H_1^{ij} g_j^{\rho'\lambda}(p'^2, q'^2, z'_0, z'_1, z'_2) \\ &+ C \mathcal{F}_2^{\rho\rho':\lambda} H_2^{ij} g_j^{\rho'\lambda}(p''^2, q''^2, z''_0, z''_1, z''_2) \\ &+ C \mathcal{F}_3^{\rho\rho':\lambda} g_i^{\rho'\lambda}(p^2, q^2, z_0, z_1, z_2), \end{aligned} \quad (6)$$

with the color factor C , the flavor matrices \mathcal{F} , the rotation matrices H and other symbols defined in Appendix A. Here, the index λ runs over all elements in a given flavor state (e.g. $\lambda = 1, 2$ for the mixed-symmetric flavor wave function of the proton and $\lambda = 1, 3$ for the mixed-antisymmetric one; see Appendix B). The simplification is that one needs to solve for one of the diagrams only,

$$\begin{aligned} g_i^{\rho,\lambda}(p^2, q^2, z_0, z_1, z_2) &= \int_k \text{Tr} \left[\bar{\tau}_{\beta\alpha\mathcal{I}\gamma}^i(p, q, P) K_{\alpha\alpha',\beta\beta'}(p, q, k) \delta_{\gamma\gamma'} S_{\alpha'\alpha''}^{(\lambda_1)}(k_1) \right. \\ &\times S_{\beta'\beta''}^{(\lambda_2)}(k_2) \tau_{\alpha''\beta''\gamma''\mathcal{I}}^j(p(3), q(3), P) \left. \right] \\ &\times f_j^\rho(p_{(3)}^2, q_{(3)}^2, z_0^{(3)}, z_1^{(3)}, z_2^{(3)}), \end{aligned} \quad (7)$$

where now the quark at the position ℓ in each term of the flavor wave function is denoted by the superindex λ_ℓ . The internal relative momenta $p_{(3)}$, $q_{(3)}$ [and analogously for $z_0^{(3)}$, $z_1^{(3)}$ and $z_2^{(3)}$] are obtained from (A3) upon substitution of the quark momenta by the internal counterparts $k_1 = p_1 - k$ and $k_2 = p_2 + k$. The conjugate of the covariant basis $\bar{\tau}$ has been defined in [27,29] and it is assumed that the basis $\{\tau\}$ is orthonormal.

So far we have not specified the two-body kernel $K_{\alpha\alpha',\beta\beta'}$ and, in fact, the simplification (6), (7) applies for any kernel provided that $K_{\alpha\alpha',\beta\beta'} = K_{\beta\beta',\alpha\alpha'}$. However, the kernel contains all possible two-body irreducible interactions among two quarks and for any practical implementation of Eq. (7) one must truncate it. The two-body kernel is related to the integration kernel in the quark DSE via the requirements of chiral symmetry expressed by axial vector and vector Ward-Takahashi identities [31–33], leading to massless pions in the chiral limit from the pseudoscalar-meson BSE. Using crossing symmetry, the quark-antiquark kernel is then related to the quark-quark kernel that appears in the Faddeev equation. With these restrictions, if the full gluon propagator and full quark-gluon vertex in (4) are truncated to their tree-level part, the corresponding two-body kernel is a single-gluon exchange with a tree-level vector-vector coupling to quarks. All nonperturbative effects from both the gluon and the vertex are encoded in an effective coupling α_{eff} which has to be modeled. This is the rainbow-ladder truncation of the DSE/BSE system. In a series of works [24,25], this truncation has been extended to include effects from the four-quark Green's function in the quark-gluon interaction parametrized in terms of (off shell) meson exchange. In the DSE and Bethe-Salpeter kernel, these effects can be represented by one-pion exchange between the quarks while still maintaining the pseudo-Goldstone nature of the pion; see [24] for details. In the following we specify the rainbow-ladder gluon and pion exchange parts of the resulting interaction and then discuss our results.

A. Effective coupling of one-gluon exchange

For the effective interaction in the rainbow-ladder truncation we use the Maris-Tandy model [34,35] which has been employed frequently in hadron studies within the rainbow-ladder BSE/DSE framework. Despite its simplicity it performs very well when it comes to purely phenomenological calculations of ground-state meson and baryon properties in certain channels; see e.g. [13] of a corresponding discussion. It combines the relevant parts in the quark self-energy according to

$$Z_{1f}C_F \frac{g^2}{4\pi} D_{\mu\nu}(k)\Gamma^\nu(p, q) = Z_2^2 C_F T_{\mu\nu}^k \frac{\alpha_{\text{eff}}(k^2)}{k^2} \gamma^\nu, \quad (8)$$

such that the resulting kernel in the baryon Faddeev equation is given by

$$K_{\alpha\alpha'\beta\beta'}^{RL}(k) = -4\pi C_F Z_2^2 \frac{\alpha_{\text{eff}}(k^2)}{k^2} T_{\mu\nu}(k) \gamma_{\alpha\alpha'}^\mu \gamma_{\beta\beta'}^\nu, \quad (9)$$

with gluon momentum $k = p - q$, transverse projector $T_{\mu\nu}^k$ and the effective running coupling α_{eff} given by

$$\alpha_{\text{eff}}(k^2) = \pi\eta^7 \left(\frac{k^2}{\Lambda^2}\right)^2 e^{-\eta^2 \frac{k^2}{\Lambda^2}} + \frac{2\pi\gamma_m(1 - e^{-k^2/\Lambda_f^2})}{\ln[e^2 - 1 + (1 + k^2/\Lambda_{\text{QCD}}^2)^2]}. \quad (10)$$

This interaction reproduces the one-loop QCD behavior of the quark propagator at large momenta and features a Gaussian distribution of interaction strength in the intermediate momentum region that provides for dynamical chiral symmetry breaking. The scale $\Lambda_f = 1$ GeV is introduced for technical reasons and has no impact on the results. Therefore, the interaction strength is characterized by an energy scale Λ and a dimensionless parameter η that controls the width of the interaction. For the anomalous dimension we use $\gamma_m = 12/(11N_c - 2N_f) = 12/25$, corresponding to $N_f = 4$ flavors and $N_c = 3$ colors. For the QCD scale $\Lambda_{\text{QCD}} = 0.234$ GeV.

The scale $\Lambda = 0.72$ GeV is adjusted to reproduce the experimental pion decay constant from the truncated pion BSE. This as well as several other pseudoscalar ground-state observables turn out to be insensitive to the value of η in the range of values of η between 1.6 and 2.0; see, e.g. [9,36,37]). Moreover, the current-quark masses are fixed to reproduce the physical pion (for the u/d quarks) and kaon (for the s quark) masses. The corresponding values are $m_{u/d}(\mu^2) = 3.7$ MeV and $m_s(\mu^2) = 85$ MeV. The renormalization scale is chosen to be $\mu^2 = (19 \text{ GeV})^2$.

B. Flavor dependent part of the interaction

In Refs. [24,25] the explicit construction, the chiral properties and some consequences for meson and baryon spectra of pion contributions to the quark-quark interaction have been explored. Since all technical details of this beyond rainbow-ladder framework have been discussed at several places already, we refrain from repeating the details here and merely state the resulting kernel for the exchange of a pion between two quark lines,

$$K_{\alpha\alpha'\beta\beta'}^{\text{pion}}(l_1, l_2, l_3, l_4; P) = \frac{1}{2} [\Gamma_\pi^j]_{\alpha\alpha'} \left(\frac{l_1 + l_2}{2}; P\right) [Z_2 \tau^j \gamma_5]_{\beta\beta'} D_\pi(P) + \frac{1}{2} [Z_2 \tau^j \gamma_5]_{\alpha\alpha'} [\Gamma_\pi^j]_{\beta\beta'} \left(\frac{l_3 + l_4}{2}; P\right) D_\pi(P). \quad (11)$$

Here $l_{1..4}$ are the incoming and outgoing quark momenta, $\Gamma_\pi^j(p, P)$ is the pion Bethe-Salpeter amplitude, with p being the relative momentum and P the total momentum, and D_π is the on shell pion propagator,

$$D_\pi(P) = \frac{1}{M_\pi^2 + P^2}. \quad (12)$$

Including the pion exchange kernel in the Faddeev equation, the resulting total kernel is then given by the sum of the rainbow-ladder part and the pion exchange part,

$$K_{\alpha\alpha'\beta\beta'} = K_{\alpha\alpha'\beta\beta'}^{\text{RL}} + K_{\alpha\alpha'\beta\beta'}^{\text{pion}}. \quad (13)$$

The same is true for the quark DSE; see Refs. [24,25] for details.

We wish to emphasize some general points related to this interaction. First, we stress that the pion in Eq. (11) is not an elementary field. The pion propagator together with the Bethe-Salpeter amplitudes,

$$\Gamma_\pi^j(p; P) = \tau^j \gamma_5 [E_\pi(p; P) - i\not{P}F_\pi(p; P) - i\not{p}G_\pi(p; P) - [\not{P}, \not{p}]H_\pi(p; P)], \quad (14)$$

describe the propagation of a quark antiquark bound state and its coupling to two quark lines. The Bethe-Salpeter amplitudes are determined by the pion BSE with the corresponding interaction kernel, i.e. one-gluon and pion exchange. Thus the coupling strength of the pion to the quark is not a separate model input, but in principle follows from a self-consistent calculation. In practice, one may resort to suitable approximations as detailed below. In general, however, the combined interaction of one-gluon plus pion exchange is derived and motivated from the structure of the QCD Dyson-Schwinger equations [24] and not some extra element added “on top of QCD.”

III. RESULTS

A. Spectrum for one-gluon exchange

We start our investigation using the one-gluon exchange part of the rainbow-ladder kernel, Eq. (9), only. This part is flavor blind, and as a consequence the bound state masses are determined by their quark content only. That is, although the symmetry properties of the flavor part of the Faddeev amplitude are important to relate all three diagrams in the Faddeev equation, this flavor part plays no further role in the calculation of the spectrum in the present truncation. As a consequence, in the baryon octet, the Λ and Σ baryons are degenerate. Moreover, since we use the same current-quark mass for the u/d quarks, all members in the same isospin multiplet are degenerate.

Table I shows the calculated masses of the three representatives of the baryon octet and compares them to the experimental values. As an estimate of the model uncertainty, we calculate for a fixed value of the parameter Λ and for a range of values from $\eta = 1.6$ to $\eta = 2.0$ as discussed above; we give the central value of this band together with the range of results indicated in brackets. The result for the nucleon is not new and was presented for

TABLE I. Positive-parity baryon octet masses (in GeV) at the physical point from the rainbow-ladder truncated Faddeev equation. We give the central value of the bands corresponding to a variation of η between $1.6 \leq \eta \leq 2.0$ with the half-width of the bands added in brackets. We compare also with experimental values [38].

$1/2^+$	N	Σ	Λ	Ξ
Faddeev	0.930 (3)	1.073 (1)	1.073 (1)	1.235 (5)
Experiment	0.938	1.189	1.116	1.315
Relative difference	< 1%	10%	4%	6%

the first time in [26]. It is in excellent agreement with the experimental value. To fully appreciate that, one must recall that not only the truncation and model used are among the simplest one can define, but also that the model parameters were fit in the pseudoscalar meson sector. Still, the agreement in the baryon sector is remarkable. For baryons with one strange quark the mass obtained is lower than the experimental value for both the Σ multiplet and the Λ singlet. However, the error is reasonably small, namely $\sim 10\%$ if compared with the Σ and $\sim 4\%$ if compared with the Λ . The same qualitative behavior appears for the Ξ baryons (or uss state), where we find a $\sim 6\%$ deviation as compared to experiment.

In Table II we show the results of the corresponding representatives in the baryon decuplet. The states with equal quark content, namely the Δ and the Ω were already given in [27].¹ The agreement with experiment in these cases is also remarkable. For the newly calculated spin-3/2 states, Σ^* (uus) and Ξ^* (uss), we observe the same qualitative behavior as for their octet counterparts. The resulting masses are below the experimental values, with an error of only $\sim 4\%$ in both cases.

Despite these relatively small differences between our results and the experimental values a clear deficiency of the rainbow-ladder truncation is manifest: there is no mechanism to generate a mass splitting between the Λ and Σ baryons. In order to improve this situation a flavor dependent interaction is necessary for a more complete description of the octet and decuplet baryons. In this respect one can think of two different categories of flavor dependent two-body kernels. The first one would have a quark-mass dependence manifest already in the effective coupling of the rainbow-ladder kernel reflecting the underlying mass dependence of the quark line going through the quark-gluon vertex. We test this heuristically in the next section through a minimal modification of the effective interaction. The second possibility would be a kernel with an explicit nondiagonal flavor part which, in particular, would

¹The difference between the results presented here, especially for the Δ and the Ω , and those shown in previous calculations is due to an improvement in the numerics. In particular we have managed to use more angular points in our numerical integration.

TABLE II. Positive-parity baryon decuplet masses (in GeV) at the physical point from the rainbow-ladder truncated Faddeev equation. We give the central value of the bands corresponding to a variation of η between $1.6 \leq \eta \leq 2.0$ with the half-width of the bands added in brackets. We compare also with experimental values [38].

$3/2^+$	Δ	Σ^*	Ξ^*	Ω
Faddeev	1.21 (2)	1.33 (2)	1.47 (3)	1.65 (4)
Experiment	1.232 (1)	1.385 (2)	1.533 (2)	1.672
Relative difference	2%	4%	4%	1%

distinguish between the Λ and Σ baryons. An exploratory study in this direction is performed in Sec. III C.

B. Exploration of a quark-mass dependent interaction

In this section we investigate, in a very simplistic manner, the possible effects of a mass dependence in the model used as the effective interaction in a rainbow-ladder truncation. In a recent calculation [39] the full quark-gluon vertex has been solved using its DSE under certain truncations. There it is observed that the dressing accompanying the tree-level component shows a significant quark-mass dependence, becoming weaker as the quark mass increases.

In order to mimic this behavior, we allow the strength parameter Λ of the one-gluon exchange part of the interaction, Eq. (10), to depend on the quark flavor. We proceed as follows: for u/d quarks we use the same value $\Lambda = 0.72$ GeV as in the previous section; for the s quark we weaken the interaction taking $\Lambda = 0.67$ GeV instead in line with the results of [39]. In order to reproduce correctly the kaon mass we change the s -quark current mass to $m_s(19 \text{ GeV}) = 91.5$ MeV. The resulting values for the kaon mass and kaon decay constant are 496 and 106 MeV, respectively. Also, with this choice of parameters, the mass of a fictitious $s\bar{s}$ pseudoscalar meson increases by about ~ 30 MeV. In the present case we also allow the parameter η to vary between 1.6 and 2.0 to estimate the model dependence.

The resulting strange-baryon masses for both the octet and decuplet are presented in Table III. Contrary to what might be expected, the masses of all states remain virtually unchanged as compared to the results presented in Sec. III A above. The reason for this is a compensation mechanism: since the same interaction model is used in both the Faddeev equation and the quark DSE, a weaker binding in the former is accompanied by less dynamical quark generation from the latter. For the quark this can be seen in Fig. 2 where we plot the mass function for both the original and the modified effective interactions. The decrease in quark mass together with the weaker binding in the Faddeev kernel compensate each other.

Within a rainbow-ladder truncation, a quark-mass dependent effective interaction is the only new feature that

TABLE III. Positive-parity strange-baryon octet (left) and decuplet (right) masses (in GeV) at the physical point from the modified (quark-mass dependent) effective interaction. We give the central value of the bands corresponding to a variation of η between $1.6 \leq \eta \leq 2.0$ with the half-width of the bands added in brackets.

Σ/Λ	Ξ
1.07 (1)	1.22 (1)
Σ^*	Ξ^*
1.32 (2)	1.47 (2)
Ω	1.64 (4)

one could incorporate in the framework. We have seen that, for baryons, this doesn't improve the comparison with experiment. This is an indication that important corrections must rather come from missing structures in our interaction kernel. This is explored in the next section.

C. Exploration of a flavor nondiagonal interaction

In the rainbow-ladder truncation used so far, as well as in any other truncation with a flavor independent kernel, the flavor matrices \mathcal{F} appearing in (6) and defined in (A11) are the same for any state in the same $SU(3)$ multiplet. In particular, this means that the spin-1/2 Λ and Σ baryons, with equal quark content but different symmetry properties of their flavor amplitudes, will have the same mass and other properties. If, instead, the interaction kernel features a flavor dependence via the mixing of quark legs, the flavor matrices will be different for different states. Such an interaction kernel has been specified in Sec. II B. Note that the pion couples to the quark lines according to its flavor content encoded in the Pauli matrices τ . These generate the flavor mixing. In particular, this kernel generates the following flavor matrices for the lambda:

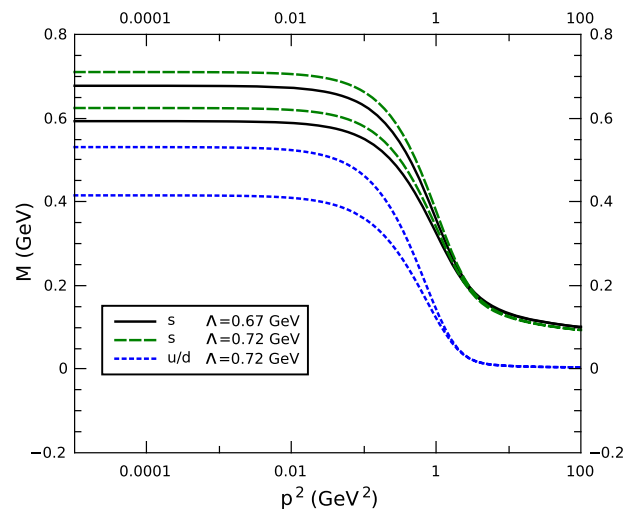


FIG. 2 (color online). Quark mass function as function of the squared momentum.

$$\mathcal{F}_1 = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{\sqrt{3}}{2} & 0 \end{pmatrix},$$

$$\mathcal{F}_3 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (15)$$

and for the sigma

$$\mathcal{F}_1 = \begin{pmatrix} 0 & \frac{-1}{2\sqrt{3}} \\ 0 & \frac{-1}{6} \end{pmatrix}, \quad \mathcal{F}_2 = \begin{pmatrix} 0 & \frac{1}{2\sqrt{3}} \\ 0 & \frac{-1}{6} \end{pmatrix},$$

$$\mathcal{F}_3 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}. \quad (16)$$

A self-consistent calculation, where the quark DSE, the pion BSE and the Faddeev equation are all solved with the inclusion of the kernel (11), was presented in [25] for the case of the nucleon and Δ baryons. In the case of the baryon octet and decuplet, a self-consistent calculation would imply solving the DSE/BSE system with an extension of the kernel (11) in which the exchange of any member of the pseudoscalar meson octet is allowed. This is beyond the scope of the present work. However, we can illustrate the effect of these types of interactions with the pion exchange only, i.e. Eq. (11), in a minimal approximation where we replace the pion amplitude Γ_π , by the exact solution of its leading amplitude in the chiral limit $\Gamma_\pi(p, P) = \gamma_5 B(p^2)/f_\pi$. Here f_π is the pion decay constant and $B(p^2)$ the quark dressing function (3). Following [25] we use a higher interaction parameter $\Lambda = 0.84$, but we maintain here the aforementioned quark masses. The resulting Λ and Σ masses are shown in Table IV. Although this simplified interaction generates an exceedingly small splitting,² it has the correct sign and therefore serves to illustrate what kind of interaction kernels must be included in order to reproduce correctly the baryon octet spectrum.

As such, this not a new result. Flavor dependent interaction terms have been discussed in great detail in the context of quark-model calculations; see e.g. [40–45] and Refs. therein. The novelty of our approach lies in the fact that our framework is entirely based on QCD. As already discussed above, the flavor dependent interaction terms are not introduced by adding additional fundamental meson fields to the theory but appear self-consistently as a result of dynamical chiral symmetry breaking and the formation of colorless bound states that in turn may contribute to QCD correlation functions. We believe this is important conceptual progress as compared to previous approaches.

²To ensure that this is not a purely numerical effect, we have run the calculation enhancing the effect of (11) by an artificial large factor. We confirmed that the splitting is indeed an effect of the flavor-mixing kernel.

TABLE IV. Lambda and Sigma baryon masses (in GeV) at the physical point from the flavor dependent kernel (11), using B/f_π for the full pion amplitude. We give the central value of the bands corresponding to a variation of η between $1.6 \leq \eta \leq 2.0$ with the half-width of the bands added in brackets.

Λ	Σ
1.161 (7)	1.164 (9)

IV. SUMMARY

We presented a fully covariant three-body calculation of the octet and decuplet baryon spectrum in the rainbow-ladder truncated DSE/BSE approach. We showed that a simple rainbow-ladder-like interaction is sufficient to reproduce the mass spectrum of the octet and the decuplet on a level better than ten percent. However, when it comes to the fine details, the deficiencies of this type of interaction become apparent. In particular, it was necessary to include flavor nondiagonal interaction terms to induce a splitting between the Λ and Σ baryons of the octet and decuplet. These terms have been motivated and derived from the underlying quark-gluon interaction in previous work [24,25]. In principle, although with much increased numerical effort, it is possible to include these interaction terms in a much more complete manner. This will be the subject of future work.

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APPENDIX A: FADDEEV EQUATION WITH DIFFERENT FLAVORS

The method described in this appendix was first introduced in [9] for the nucleon Faddeev equation and applied in [27] for the delta and omega. Here, we review its main aspects making explicit the possibility of having quarks with different masses in the problem.

Being a representation of a three-quark Green's function, the three-body Bethe-Salpeter amplitude (1) must be antisymmetric in the quark indices $\{A, B, C\}$. The color part ϵ_{rst} is already antisymmetric in order to restrict the state to be a color singlet, which implies that the product of the flavor and spin-momentum parts must be symmetric. Assuming flavor- $SU(3)$ symmetry (the same symmetry arguments would hold, of course, if we take the flavor symmetry group larger) one constructs the usual octet and decuplet representations for baryons (see, e.g. [46] and Appendix B). There are two flavor octets, one symmetric

and one antisymmetric with respect to the first two quark indices. The physical octet is a quantum superposition of these two. There is only one flavor decuplet, whose states are symmetric in all their indices. That is, each baryon is associated to a flavor wave function F_{abc}^ρ , with $\rho = 1, 2$ for the antisymmetric and symmetric octet representations, respectively, and $\rho = 1$ for the decuplet representations. We will be interested here in the quark permutations $(123) \rightarrow (231)$ and $(123) \rightarrow (312)$, under which the

product of spin-momentum and flavor parts transforms symmetrically,

$$\begin{aligned}\Psi_{\alpha\beta\gamma\mathcal{I}}^\rho(p_1, p_2, p_3)F_{abcd}^\rho &= \Psi_{\beta\gamma\alpha\mathcal{I}}^\rho(p_2, p_3, p_1)F_{bcad}^{\rho'}, \\ \Psi_{\alpha\beta\gamma\mathcal{I}}^\rho(p_1, p_2, p_3)F_{abcd}^\rho &= \Psi_{\gamma\alpha\beta\mathcal{I}}^\rho(p_3, p_1, p_2)F_{cabd}^{\rho'}.\end{aligned}\quad (\text{A1})$$

In general, the Faddeev equation reads

$$\begin{aligned}\Gamma_{ABC\mathcal{I}}(p, q, P) &= \int_k [\mathbf{K}_{BB',CC'}(k_2, \tilde{k}_3; k)\delta_{AA''}\mathbf{S}_{B'B''}(k_2)\mathbf{S}_{C'C''}(\tilde{k}_3)\Gamma_{A''B''C''\mathcal{I}}(p^{(1)}, q^{(1)}, P)] \\ &+ \int_k [\mathbf{K}_{AA',CC'}(k_3, \tilde{k}_1; k)\delta_{BB''}\mathbf{S}_{A'A''}(\tilde{k}_1)\mathbf{S}_{C'C''}(k_3)\Gamma_{A''B''C''\mathcal{I}}(p^{(2)}, q^{(2)}, P)] \\ &+ \int_k [\mathbf{K}_{AA',BB'}(k_1, \tilde{k}_2; k)\delta_{CC''}\mathbf{S}_{A'A''}(k_1)\mathbf{S}_{B'B''}(\tilde{k}_2)\Gamma_{A''B''C''\mathcal{I}}(p^{(3)}, q^{(3)}, P)],\end{aligned}\quad (\text{A2})$$

with

$$\begin{aligned}p &= (1 - \zeta)p_3 - \zeta(p_1 + p_2), & p_1 &= -q - \frac{p}{2} + \frac{1 - \zeta}{2}P, \\ q &= \frac{p_2 - p_1}{2}, & p_2 &= q - \frac{p}{2} + \frac{1 - \zeta}{2}P, \\ P &= p_1 + p_2 + p_3, & p_3 &= p + \zeta P,\end{aligned}\quad (\text{A3})$$

and $\zeta = 1/3$ is a momentum partitioning parameter. The internal quark propagators depend on the internal quark momenta $k_i = p_i - k$ and $\tilde{k}_i = p_i + k$, with k being the exchanged momentum. The internal relative momenta, for each of the three terms in the Faddeev equation, are

$$\begin{aligned}p^{(1)} &= p + k, & p^{(2)} &= p - k, & p^{(3)} &= p, \\ q^{(1)} &= q - k/2, & q^{(2)} &= q - k/2, & q^{(3)} &= q + k.\end{aligned}\quad (\text{A4})$$

The kernel contains spin-momentum, flavor and color parts as well,

$$\mathbf{K}_{AA',BB'}(p_1, p_2; p) = K_{\alpha\alpha',\beta\beta'}(p_1, p_2; p)k_{aa'bb'}^F k_{rr',ss'}^C. \quad (\text{A5})$$

Moreover, we write the flavor wave functions as a sum of several terms,

$$\begin{aligned}\left(p_1 = -q - \frac{p}{2} + \frac{P}{3}\right) &\equiv \left(p'_3 = p' + \frac{P}{3}\right) \\ \left(k_2 = q - \frac{p}{2} + \frac{P}{3} - k\right) &\equiv \left(k'_1 = -q' - \frac{p'}{2} + \frac{P}{3} - k\right) \Rightarrow p' = -q - \frac{p}{2}, & q' &= -\frac{q}{2} + \frac{3p}{4}, \\ \left(\tilde{k}_3 = p + \frac{P}{3} + k\right) &\equiv \left(\tilde{k}'_2 = q' - \frac{p'}{2} + \frac{P}{3} + k\right),\end{aligned}\quad (\text{A8})$$

and the kinematics of the second term is the same as that of the third term but evaluated at the transformed momenta $\{p'', q''\}$,

$$F_{abcd}^\rho = \sum_\lambda^{d_F} F_{abcd}^{\rho,\lambda}, \quad (\text{A6})$$

where d_F is the number of such terms [for example, the antisymmetric component of the Ξ^0 is $(uss - sus)/\sqrt{2}$ and therefore $d_F = 2$]. The action of the propagators on the Faddeev amplitudes in (A2) can therefore be written as a sum of d_F terms, for instance

$$\begin{aligned}\mathbf{S}_{AA'}(p_1)\mathbf{S}_{BB'}(p_2)\Gamma_{A'B'CI}(p_1, p_2, p_3) \\ = \sum_\rho \sum_i^{d_F} S_{\alpha\alpha'}^{(\lambda_1)}(p_1)S_{\beta\beta'}^{(\lambda_2)}(p_2)\Psi_{\alpha\beta\gamma\mathcal{I}}^\rho(p_1, p_2, p_3)F_{abc}^{\rho,i},\end{aligned}\quad (\text{A7})$$

where, for example, for the first term in the antisymmetric component of the Ξ^0 given above, $S^{(\lambda_1)}$ and $S^{(\lambda_2)}$ would be the propagators for u and s quarks, respectively.

We wish to make use of (A1) to relate the first two terms in (A2) to the third one, which has simpler kinematics and is hence easier to calculate. First, it is necessary to realize that if the third term in the equation is evaluated for relative momenta $\{p, q\}$ then, considering the permutation of momenta in (A1), the kinematics of the first term is the same as that of the third term but evaluated at the transformed momenta $\{p', q'\}$,

$$\begin{aligned}
 \left(\tilde{k}_1 = -q - \frac{p}{2} + \frac{p}{3} + k \right) &\equiv \left(\tilde{k}_2'' = q'' - \frac{p''}{2} + \frac{p}{3} + k \right) \\
 \left(p_2 = q - \frac{p}{2} + \frac{p}{3} \right) &\equiv \left(p_3'' = p'' + \frac{p}{3} \right) \quad \Rightarrow \quad p'' = q - \frac{p}{2}, \quad q'' = -\frac{q}{2} - \frac{3p}{4}. \\
 \left(k_3 = p + \frac{p}{3} - k \right) &\equiv \left(k_1'' = -q'' - \frac{p''}{2} + \frac{p}{3} - k \right).
 \end{aligned} \tag{A9}$$

Putting in all the elements defined above and permuting the indices in (A2) as in (A1), and after renaming dummy indices conveniently, Eq. (A2) now becomes

$$\begin{aligned}
 \Psi_{\alpha\beta\gamma\mathcal{I}}^{\rho}(p, q, P) &= C\mathcal{F}_1^{\rho\rho'\lambda} \int_k [K_{\beta\beta'\gamma\gamma'}(k_1', \tilde{k}_2'; k) \delta_{\alpha\alpha''} S_{\beta'\beta''}^{(\lambda_2)}(k_1') S_{\gamma'\gamma''}^{(\lambda_3)}(\tilde{k}_2') \Psi_{\beta''\gamma''\alpha''\mathcal{I}}^{\rho'}(p^{(3)}, q^{(3)}, P)] \\
 &+ C\mathcal{F}_2^{\rho\rho'\lambda} \int_k [K_{\alpha\alpha'\gamma\gamma'}(k_1'', \tilde{k}_2''; k) \delta_{\beta\beta''} S_{\gamma'\gamma''}^{(\lambda_1)}(\tilde{k}_1'') S_{\alpha'\alpha''}^{(\lambda_3)}(k_2'') \Psi_{\gamma''\alpha''\beta''\mathcal{I}}^{\rho'}(p''^{(3)}, q''^{(3)}, P)] \\
 &+ C\mathcal{F}_3^{\rho\rho'\lambda} \int_k [K_{\alpha\alpha'\beta\beta'}(k_1, \tilde{k}_2; k) \delta_{\gamma\gamma''} S_{\alpha'\alpha''}^{(\lambda_1)}(k_1) S_{\beta'\beta''}^{(\lambda_2)}(\tilde{k}_2) \Psi_{\alpha''\beta''\gamma''\mathcal{I}}^{\rho'}(p^{(3)}, q^{(3)}, P)],
 \end{aligned} \tag{A10}$$

where the contraction of the color parts of the kernel and those of the Faddeev amplitudes (before the permutation of indices) gives a global factor C and, similarly, the contraction of the corresponding flavor parts leads to the flavor matrices

$$\begin{aligned}
 \mathcal{F}_1^{\rho\rho'\lambda} &= F_{bac}^{\dagger}{}^{\rho} k_{bb'cc'}^F F_{b'c'a'}^{\rho'\lambda}, \\
 \mathcal{F}_2^{\rho\rho'\lambda} &= F_{bac}^{\dagger}{}^{\rho} k_{aa'cc'}^F F_{c'a'b}^{\rho'\lambda}, \\
 \mathcal{F}_3^{\rho\rho'\lambda} &= F_{bac}^{\dagger}{}^{\rho} k_{aa'bb'}^F F_{a'b'c}^{\rho'\lambda}.
 \end{aligned} \tag{A11}$$

If we denote the result of the integral in the third line of the last equation as $[\Psi_{\lambda_1\lambda_2\lambda_3}^{(3)}]_{\alpha\beta\gamma\mathcal{I}}^{\rho}(p, q, P)$, it is clear that if the kernel is such that $K_{\alpha\alpha'\beta\beta'}(k_1, k_2, k) = K_{\beta\beta'\alpha\alpha'}(k_1, k_2, k)$ then we have

$$\begin{aligned}
 \Psi_{\alpha\beta\gamma\mathcal{I}}^{\rho}(p, q, P) &= C\mathcal{F}_1^{\rho\rho'\lambda} [\Psi_{\lambda_2\lambda_3}^{(3)}]_{\beta\gamma\alpha\mathcal{I}}^{\rho'}(p', q', P) \\
 &+ C\mathcal{F}_2^{\rho\rho'\lambda} [\Psi_{\lambda_1\lambda_3}^{(3)}]_{\gamma\alpha\beta\mathcal{I}}^{\rho'}(p'', q'', P) \\
 &+ C\mathcal{F}_3^{\rho\rho'\lambda} [\Psi_{\lambda_1\lambda_2}^{(3)}]_{\alpha\beta\gamma\mathcal{I}}^{\rho'}(p, q, P).
 \end{aligned} \tag{A12}$$

Therefore, the problem has been reduced to the calculation of only one of the diagrams in the Faddeev equation for all possible combinations of pairs of quarks. If we rewrite (A12) in terms of the coefficients f of the expansion of the Faddeev amplitudes in a covariant basis (2) we finally obtain Eq. (6) with

$$H_1^{ij} = [\tilde{\tau}_{\beta\alpha\mathcal{I}\gamma}^i(p, q, P) \tau_{\beta\gamma\alpha\mathcal{I}}^j(p', q', P)], \tag{A13}$$

$$H_2^{ij} = [\tilde{\tau}_{\beta\alpha\mathcal{I}\gamma}^i(p, q, P) \tau_{\gamma\alpha\beta\mathcal{I}}^j(p'', q'', P)]. \tag{A14}$$

APPENDIX B: FLAVOR AMPLITUDES

For convenience, we reproduce here the usual quark-model flavor amplitudes of baryons F_{abc} . The octet is

composed of a superposition of states which are symmetric (S) or antisymmetric (A) under the exchange of the first two indices. In terms of quarks these combinations are shown in Tables V and VI.

TABLE V. Baryon octet flavor amplitudes.

State	S	A
p	$\frac{1}{\sqrt{2}}(udu - duu)$	$\frac{1}{\sqrt{6}}(2uud - udu - duu)$
n	$\frac{1}{\sqrt{2}}(udd - dud)$	$\frac{1}{\sqrt{6}}(udd + dud - 2ddu)$
Σ^+	$\frac{1}{\sqrt{2}}(usu - suu)$	$\frac{1}{\sqrt{6}}(2uus - usu - suu)$
Σ^0	$\frac{1}{2}(usd + dsu - sud - sdu)$	$\frac{1}{\sqrt{12}}(2uds + 2dus - usd - dsu - sud - sdu)$
Σ^-	$\frac{1}{\sqrt{2}}(dsd - sdd)$	$\frac{1}{\sqrt{6}}(2dds - dsd - sdd)$
Ξ^0	$\frac{1}{\sqrt{2}}(uss - sus)$	$\frac{1}{\sqrt{6}}(uss + sus - 2ssu)$
Ξ^-	$\frac{1}{\sqrt{2}}(dss - sds)$	$\frac{1}{\sqrt{6}}(dss + sds - 2ssd)$
Λ^0	$\frac{1}{\sqrt{12}}(2uds - 2dus + dsu - dsu + usd - sud)$	$\frac{1}{2}(usd + sud - dsu - sdu)$

TABLE VI. Baryon decuplet flavor amplitudes.

State	S
Δ^{++}	uuu
Δ^+	$\frac{1}{\sqrt{3}}(uud + udu + duu)$
Δ^0	$\frac{1}{\sqrt{3}}(udd + dud + ddu)$
Δ^-	ddd
Σ^{*+}	$\frac{1}{\sqrt{3}}(uus + usu + suu)$
Σ^{*0}	$\frac{1}{\sqrt{6}}(uds + usd + dus + dsu + sud + sdu)$
Σ^{*-}	$\frac{1}{\sqrt{3}}(dds + dsd + sdd)$
Ξ^{*0}	$\frac{1}{\sqrt{3}}(uss + sus + ssu)$
Ξ^{*-}	$\frac{1}{\sqrt{3}}(dss + sds + ssd)$
Ω^-	sss

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