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Highly effective action from large N gauge fields

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Recently Schwarz put forward a conjecture that the world-volume action of a probe D3-brane in an $\mathrm{AdS}_5 \times \mathbb{S}^5$ background of type IIB superstring theory can be reinterpreted as the highly effective action (HEA) of four-dimensional $\mathcal{N} = 4$ superconformal field theory on the Coulomb branch. We argue that the HEA can be derived from the noncommutative (NC) field theory representation of the AdS/CFT correspondence and the Seiberg-Witten (SW) map defining a spacetime field redefinition between ordinary and NC gauge fields. It is based only on the well-known facts that the master fields of large N matrices are higher-dimensional NC $U(1)$ gauge fields and the SW map is a local coordinate transformation eliminating $U(1)$ gauge fields known as the Darboux theorem in symplectic geometry.

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I. INTRODUCTION

Recently Schwarz conjectured [1] that the world-volume action of a probe p-brane in a maximally (or $3/4$ maximal) supersymmetric spacetime containing AdS_{p+2} can be reinterpreted as the highly effective action (HEA) of a superconformal field theory in $(p+1)$ dimensions on the Coulomb branch. The HEA is defined by taking a conformal gauge theory on the Coulomb branch and integrating out the massive fields, thereby obtaining an effective action in terms of massless Abelian multiplets only. Then the HEA is conjecturally identified with the world-volume action for a probe p-brane in an $\mathrm{AdS}_{p+2} \times K$ background geometry with N units of flux threading a compact space K. Examples considered in [1] are a D3-brane in $\mathrm{AdS}_5 \times \mathbb{S}^5$, an M2-brane in $\mathrm{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$, a D2-brane in $\mathrm{AdS}_4 \times \mathbb{CP}^3$, and an M5-brane in $\mathrm{AdS}_7 \times \mathbb{S}^4$. This conjecture was driven by a guiding principle [1]: "Take coincidences seriously," with the observation that the probe-brane theory has all of the expected symmetries and dualities. The brane actions fully incorporate the symmetry of the background as an exact global symmetry of the world-volume theory. For example, in the case of a D3-brane in $\mathrm{AdS}_5 \times \mathbb{S}^5$, this symmetry is the superconformal group $PSU(2,2|4)$. In this example, it also includes the $SL(2,\mathbb{Z})$ duality group, which is known to be an exact symmetry of type IIB superstring theory. This conjecture may be further strengthened by showing that the world-volume actions describing probe branes in anti–de Sitter (AdS) space exhibit not only (super)conformal symmetry but also dual (super)conformal symmetry and, taken together, have an infinite-dimensional Yangian-like symmetry.[1] There have also been earlier works [3–7] to note the conformal symmetry of the world-volume theory of a p-brane in an AdS background as well as works [8–11] to emphasize the relationship between probe-brane actions and low-energy effective actions on the Coulomb branch.

In this paper we will argue that the HEA can be derived from the noncommutative (NC) field theory representation of the AdS/CFT correspondence as recently formulated in [12] (see, in particular, Sec. VI). Our argument is based only on the well-known facts that the master fields of large N matrices are higher-dimensional NC $U(1)$ gauge fields [13–16] and the Seiberg-Witten (SW) map [17] defining a spacetime field redefinition between ordinary and NC gauge fields is a local coordinate transformation eliminating $U(1)$ gauge fields via the Darboux theorem in symplectic geometry [16,18–21]. The underlying math for the argument is rather fundamental. For simplicity, let us consider two-dimensional NC space, denoted by $\mathbb{R}^2_\theta$, whose coordinates obey the commutation relation

$$[y^1, y^2] = i\theta, \tag{1.1}$$

where $\theta > 0$ is a constant parameter measuring the noncommutativity of the space $\mathbb{R}^2_\theta$. If we define annihilation and creation operators as

$$a = \frac{y^1 + iy^2}{\sqrt{2\theta}}, \qquad a^\dagger = \frac{y^1 - iy^2}{\sqrt{2\theta}}, \tag{1.2}$$

the NC algebra (1.1) of $\mathbb{R}^2_\theta$ reduces to the Heisenberg algebra of harmonic oscillator, i.e.,

$$[a, a^\dagger] = 1. \tag{1.3}$$

The representation space of the Heisenberg algebra (1.3) is given by the Fock space defined by

$$\mathcal{H} = \{|n\rangle | n \in \mathbb{Z}_{\geq 0}\}, \tag{1.4}$$

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[1] Indeed this problem was addressed by Lipstein and Schwarz [2]. But, unfortunately, this paper was withdrawn due to an error in some equation.

which is orthonormal, i.e., $\langle n|m\rangle = \delta_{n,m}$ and complete, i.e., $\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbf{1}_{\mathcal{H}}$, as is well-known from quantum mechanics.

A crucial, though elementary, fact for our argument is that the NC space $\mathbb{R}^2_\theta$ admits an infinite-dimensional separable Hilbert space (1.4) [22]. Let us apply this elementary fact to dynamical fields defined on $\mathbb{R}^{d-1,1} \times \mathbb{R}^2_\theta$ with local coordinates (x^μ, y^1, y^2) where $\mathbb{R}^{d-1,1} \ni x^\mu$ is a d-dimensional Minkowski spacetime. Consider two arbitrary fields $\hat{\Phi}_1(x,y)$ and $\hat{\Phi}_2(x,y)$ on $\mathbb{R}^{d-1,1} \times \mathbb{R}^2_\theta$. In quantum mechanics physical observables are considered as operators acting on a Hilbert space. Similarly the dynamical variables $\hat{\Phi}_1(x,y)$ and $\hat{\Phi}_2(x,y)$ can be regarded as operators acting on the Hilbert space $\mathcal{H}$, which are elements of the deformed algebra $C^\infty(\mathbb{R}^{d-1,1}) \otimes \mathcal{A}_\theta$. Thus one can represent the operators acting on the Fock space (1.4) as $N \times N$ matrices in $\mathrm{End}(\mathcal{H}) \equiv \mathcal{A}_N$ where $N = \dim(\mathcal{H}) \to \infty$:

$$\begin{aligned}\hat{\Phi}_1(x,y) &= \sum_{n,m=0}^{\infty} |n\rangle\langle n|\hat{\Phi}_1(x,y)|m\rangle\langle m| \coloneqq \sum_{n,m=0}^{\infty} (\Phi_1)_{nm}(x)|n\rangle\langle m|,\\ \hat{\Phi}_2(x,y) &= \sum_{n,m=0}^{\infty} |n\rangle\langle n|\hat{\Phi}_2(x,y)|m\rangle\langle m| \coloneqq \sum_{n,m=0}^{\infty} (\Phi_2)_{nm}(x)|n\rangle\langle m|,\end{aligned} \tag{1.5}$$

where $\Phi_1(x)$ and $\Phi_2(x)$ are $N \times N$ matrices in $C^\infty(\mathbb{R}^{d-1,1}) \otimes \mathcal{A}_N$. Then one gets a natural composition rule for the products

$$\begin{aligned}(\hat{\Phi}_1 \star \hat{\Phi}_2)(x,y) &= \sum_{n,l,m=0}^{\infty} |n\rangle\langle n|\hat{\Phi}_1(x,y)|l\rangle\langle l|\hat{\Phi}_2(x,y)|m\rangle\langle m|\\ &= \sum_{n,l,m=0}^{\infty} (\Phi_1)_{nl}(x)(\Phi_2)_{lm}(x)|n\rangle\langle m|.\end{aligned} \tag{1.6}$$

The above composition rule implies that the ordering in the NC algebra $\mathcal{A}_\theta$ is compatible with the ordering in the matrix algebra $\mathcal{A}_N$, and so it is straightforward to translate multiplications of NC fields in $\mathcal{A}_\theta$ into those of matrices in $\mathcal{A}_N$ using the matrix representation (1.5) without any ordering ambiguity.

It is easy to generalize the matrix representation to $2n$-dimensional NC space $\mathbb{R}^{2n}_\theta$ whose coordinate generators obey the commutation relation

$$[y^a, y^b] = i\theta^{ab}, \qquad a,b = 1,\ldots,2n, \tag{1.7}$$

where the Poisson bivector $\theta = \frac{1}{2}\theta^{ab}\frac{\partial}{\partial y^a} \bigwedge \frac{\partial}{\partial y^b}$ is assumed to be invertible and so $B \equiv \theta^{-1}$ defines a symplectic structure on $\mathbb{R}^{2n}$. Consider a $D = (d+2n)$-dimensional NC space $\mathbb{R}^{d-1,1} \times \mathbb{R}^{2n}_\theta$ with coordinates $Y^M = (x^\mu, y^a)$, $M = 0,1,\ldots,D-1, \mu = 0,1,\ldots,d-1$. The star product for smooth functions $\hat{f}(Y), \hat{g}(Y) \in C^\infty(\mathbb{R}^{D-1,1})$ is defined by

$$(\hat{f} \star \hat{g})(Y) = e^{\frac{i}{2}\theta^{ab}\frac{\partial}{\partial y^a}\otimes\frac{\partial}{\partial z^b}} \hat{f}(x,y)\hat{g}(x,z)|_{y=z}. \tag{1.8}$$

Therefore, to formulate a gauge theory on $\mathbb{R}^{d-1,1} \times \mathbb{R}^{2n}_\theta$, it is necessary to dictate the gauge covariance under the NC star product (1.8). The covariant field strength of NC $U(1)$ gauge fields $\hat{A}_M(Y) = (\hat{A}_\mu, \hat{A}_a)(x,y)$ is then given by

$$\hat{F}_{MN}(Y) = \partial_M \hat{A}_N(Y) - \partial_N \hat{A}_M(Y) - i[\hat{A}_M, \hat{A}_N]_\star(Y). \tag{1.9}$$

Using the matrix representation (1.5), one can show [13–16] that the $D = (d+2n)$-dimensional NC $U(1)$ gauge theory is exactly mapped to the d-dimensional $U(N \to \infty)$ Yang-Mills theory,

$$S = -\frac{1}{4G_{YM}^2}\int d^D Y (\hat{F}_{MN} - B_{MN})^2 \tag{1.10}$$

$$= -\frac{1}{g_{YM}^2}\int d^d x \mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\Phi_a D^\mu\Phi^a - \frac{1}{4}[\Phi_a, \Phi_b]^2\right), \tag{1.11}$$

where $G_{YM}^2 = (2\pi)^n |\mathrm{Pf}\theta| g_{YM}^2$ and

$$B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}.$$

We refer more details to Sec. 6.1 of Ref. [12].

We emphasize that the equivalence between the D-dimensional NC $U(1)$ gauge theory (1.10) and d-dimensional $U(N\to\infty)$ Yang-Mills theory (1.11) is an exact mathematical identity, not a dimensional reduction, and has been known long ago, for example, in [13,14]. A remarkable point is that the resulting matrix models or large N gauge theories described by the action (1.11) arise as a nonperturbative formulation of string/M theories. For instance, we get the Ishibashi-Kawai-Kitazawa-Tsuchiya matrix model for $d=0$ [23], the Banks-Fischler-Shenker-Susskind matrix quantum mechanics for $d=1$ [24], and the matrix string theory for $d=2$ [25]. The most interesting case arises for $d=4$ and $n=3$, which suggests an engrossing duality that the ten-dimensional NC $U(1)$ gauge theory on $\mathbb{R}^{3,1}\times\mathbb{R}_\theta^6$ is equivalent to the bosonic action of four-dimensional $\mathcal{N}=4$ supersymmetric $U(N)$ Yang-Mills theory, which is the large N gauge theory of the AdS/CFT duality [3,26,27]. According to the large N duality or gauge/gravity duality, the large N matrix model (1.11) is dual to a higher-dimensional gravity or string theory. Hence it should not be surprising that the D-dimensional NC $U(1)$ gauge theory should describe a theory of gravity (or a string theory) in D dimensions. Nevertheless the possibility that gravity can emerge from NC $U(1)$ gauge fields has been largely ignored until recently. But the emergent gravity picture based on NC $U(1)$ gauge theory [12,16,28] debunks that this coincidence did not arise by some fortuity. Here we want to take an advantage following the advice of Schwarz [1]: "Take coincidences seriously."

In this paper, we will seriously take the equivalence between the D-dimensional NC $U(1)$ gauge theory (1.10) and the d-dimensional $U(N\to\infty)$ Yang-Mills theory (1.11) to derive the HEA conjectured in [1]. It is to be hoped that we also clarify why the emergent gravity from NC gauge fields is actually the manifestation of the gauge/gravity duality or large N duality in string/M theories. We think that the emergent gravity from NC gauge fields opens a lucid avenue to understand the gauge/gravity duality such as the AdS/CFT correspondence. While the large N duality is still a conjectural duality and its understanding is far from being complete to identify an underlying first principle for the duality, it is possible [12,16,28] to reasonably identify the first principle for the emergent gravity from NC $U(1)$ gauge fields and to derive in a systematic way gravitational variables from gauge theory quantities. Moreover, it can be shown [12] that the four-dimensional $\mathcal{N}=4$ supersymmetric $U(N)$ Yang-Mills theory is equivalent to the ten-dimensional $\mathcal{N}=1$ supersymmetric NC $U(1)$ gauge theory on $\mathbb{R}^{3,1}\times\mathbb{R}_\theta^6$ if we consider the Moyal-Heisenberg vacuum (1.7) which is a consistent solution of the former—the $\mathcal{N}=4$ super Yang-Mills theory. Here is a foothold for our departure.

The paper is organized as follows. In Sec. II we review the result in Ref. [12] showing that the four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch defined by the NC space (1.7) is equivalent to the ten-dimensional $\mathcal{N}=1$ supersymmetric NC $U(1)$ gauge theory. In Sec. III we consider the ten-dimensional $\mathcal{N}=1$ NC $U(1)$ super Yang-Mills theory (2.8) as a nontrivial leading approximation of the supersymmetric completion of the NC Dirac-Born-Infeld (DBI) action. The supersymmetric completion is postponed to Sec. V. In Sec. IV, we identify a commutative DBI action that is mapped to the NC one by the exact SW map defining a spacetime field redefinition between ordinary and NC gauge fields [17]. It is observed that the spacetime geometry dual to four-dimensional large N matrices or ten-dimensional NC $U(1)$ gauge fields is simply derived from the Darboux transformation eliminating $U(1)$ gauge fields whose statement is known as the Darboux theorem in symplectic geometry. We also identify a possible candidate giving rise to $\mathrm{AdS}_5\times\mathbb{S}^5$ geometry. It is shown and will also be checked in Appendix A that the duality between NC $U(1)$ gauge fields and gravitational fields is the SW map between commutative and NC $U(1)$ gauge fields. See Eq. (4.20). We thus argue that the emergent gravity from NC gauge fields is the manifestation of the gauge/gravity duality or large N duality in string/M theories [12]. In Sec. V, we derive the world-volume action of a probe D3-brane in $\mathrm{AdS}_5\times\mathbb{S}^5$ geometry from the DBI action of ten-dimensional NC $U(1)$ gauge fields, which was obtained from the four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch. We consider a supersymmetric D9-brane with the local κ symmetry [29–34] to yield the supersymmetric version of DBI actions. We finally identify the supersymmetric world-volume action of a probe D3-brane in $\mathrm{AdS}_5\times\mathbb{S}^5$ geometry with the HEA conjectured by Schwarz [1]. Our approach sheds light on why $N=1$ (i.e., Abelian gauge group) is the proper choice for the HEA that was elusive in the original conjecture (see the discussion in Sec. 5 of Ref. [1]). In Sec. VI, we discuss why the emergent gravity from NC gauge fields provides a lucid avenue to understand the gauge/gravity duality such as the AdS/CFT correspondence [3,26,27]. We conclude the paper with a few speculative remarks. In Appendix A, we demonstrate how to determine $2n$-dimensional Kähler metrics from $U(1)$ gauge fields by solving the identities (4.14) and (4.15) between DBI actions, which are underlying equations for our argument. In particular, we show that Calabi-Yau n-folds for $n=2$ and 3 arise from symplectic $U(1)$ instantons in four and six dimensions, respectively.

II. NC $U(1)$ GAUGE FIELDS FROM LARGE N MATRICES

The AdS/CFT correspondence [3,26,27] implies that a wide variety of quantum field theories provide a nonperturbative realization of quantum gravity. In the AdS/CFT duality, the dynamical variables are large N matrices, and so gravitational physics at a fundamental level is described by NC operators. We argued in [12] that the AdS/CFT correspondence is a particular case of emergent gravity from NC U(1) gauge fields. An underlying argumentation is to realize the equivalence between the actions (1.10) and (1.11) in a reverse way by observing that the Moyal-Heisenberg vacuum (1.7) is a consistent vacuum solution of the $\mathcal{N}=4$ super Yang-Mills theory.

It is easy to understand an underlying logic, and so we recapitulate only the essential points deferring to [12] on a detailed description. The action of four-dimensional $\mathcal{N}=4$ super Yang-Mills theory is given by [35]

$$S=\int d^4x\mathrm{Tr}\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}D_\mu\Phi_a D^\mu\Phi_a+\frac{g^2}{4}[\Phi_a,\Phi_b]^2+i\bar{\lambda}_i\bar{\sigma}^\mu D_\mu\lambda^i+\frac{ig}{2}\bar{\Sigma}^a_{ij}\lambda^i[\Phi_a,\lambda^j]-\frac{ig}{2}\Sigma^{a,ij}\bar{\lambda}_i[\Phi_a,\bar{\lambda}_j]\right\}. \quad (2.1)$$

Consider a vacuum configuration defined by

$$\langle\Phi_a\rangle_{\mathrm{vac}}=p_a,\qquad \langle A_\mu\rangle_{\mathrm{vac}}=0,\qquad \langle\lambda^i\rangle_{\mathrm{vac}}=0. \quad (2.2)$$

Assume that the vacuum expectation value (vev) $p_a\in\mathcal{A}_N(N\to\infty)$ satisfies the Moyal-Heisenberg algebra

$$[p_a,p_b]=-iB_{ab}I_{N\times N}. \quad (2.3)$$

Of course, the commutation relation (2.3) is meaningful only when we take the limit $N\to\infty$. It is obvious that the vacuum configuration (2.2) in this limit is definitely a solution of the theory. We emphasize that the vev (2.2) of adjoint scalar fields does not break four-dimensional Lorentz symmetry. Actually the vacuum algebra (2.3) refers to NC space $\mathbb{R}^6_\theta$ if we define $p_a\equiv B_{ab}y^b$ and $B\equiv\theta^{-1}$. Now fluctuations of large N matrices around the vacuum (2.2) are parametrized by

$$\hat{D}_\mu(x,y)=\partial_\mu-i\hat{A}_\mu(x,y),$$
$$\hat{D}_a(x,y)\equiv-i\hat{\Phi}_a(x,y)=-i(p_a+\hat{A}_a(x,y)), \quad (2.4)$$

$$\hat{\Psi}(x,y)=\begin{pmatrix}P_+\hat{\lambda}^i\\ P_-\tilde{\hat{\bar{\lambda}}}_i\end{pmatrix}(x,y), \quad (2.5)$$

where we assumed that fluctuations also depend on vacuum moduli y^a. Note that, if we apply the matrix representation (1.5) to the fluctuations in Eqs. (2.4) and (2.5) again, we recover the original large N gauge fields in the action (2.1). Therefore let us introduce ten-dimensional coordinates $Y^M=(x^\mu,y^a), M=0,1,\ldots,9$ and ten-dimensional connections defined by

$$\hat{D}_M(Y)=\partial_M-i\hat{A}_M(x,y)=(\hat{D}_\mu,\hat{D}_a)(x,y), \quad (2.6)$$

whose field strength is given by

$$\hat{F}_{MN}(Y)=i[\hat{D}_M,\hat{D}_N]_\star=\partial_M\hat{A}_N-\partial_N\hat{A}_M-i[\hat{A}_M,\hat{A}_N]_\star. \quad (2.7)$$

Thus the correspondence between the NC $\star$ algebra $\mathcal{A}_\theta$ and the matrix algebra $\mathcal{A}_N=\mathrm{End}(\mathcal{H})$ under the Moyal-Heisenberg vacuum (2.3) implies that the master fields of large N matrices are higher-dimensional NC $U(1)$ gauge fields. In the end large N matrices in the $\mathcal{N}=4$ vector multiplet on $\mathbb{R}^{3,1}$ are mapped to NC gauge fields and their superpartners in the $\mathcal{N}=1$ vector multiplet on $\mathbb{R}^{3,1}\times\mathbb{R}^6_\theta$ where $\mathbb{R}^6_\theta$ is an extra NC space whose coordinate generators $y^a\in\mathcal{A}_\theta$ obey the commutation relation (1.7).

Using the ordering (1.6) for $U(N)$ and NC $U(1)$ gauge fields, it is straightforward to organize the four-dimensional $\mathcal{N}=4$ $U(N)$ super Yang-Mills theory (2.1) into the ten-dimensional $\mathcal{N}=1$ NC $U(1)$ super Yang-Mills theory with the action [12]

$$S=\int d^{10}Y\left\{-\frac{1}{4G^2_{YM}}(\hat{F}_{MN}-B_{MN})^2+\frac{i}{2}\bar{\hat{\Psi}}\Gamma^M\hat{D}_M\hat{\Psi}\right\}, \quad (2.8)$$

where B fields take the same form as Eq. (1.10). Now the fermion $\hat{\Psi}(Y)$ is a ten-dimensional gaugino, the superpartner of the ten-dimensional NC $U(1)$ gauge field $\hat{A}_M(y)$, which is the Majorana-Weyl spinor of $SO(9,1)$. The action (2.8) is invariant under $\mathcal{N}=1$ supersymmetry transformations given by

$$\delta\hat{A}_M=i\bar{\alpha}\Gamma_M\hat{\Psi},$$
$$\delta\hat{\Psi}=\frac{1}{2}(\hat{F}_{MN}-B_{MN})\Gamma^{MN}\alpha. \quad (2.9)$$

It should be remarked that the relationship between the four-dimensional $U(N)$ super Yang-Mills theory (2.1) and ten-dimensional NC $U(1)$ super Yang-Mills theory (2.8) is not a dimensional reduction, but they are exactly equivalent to each other. Therefore any quantity in lower-dimensional $U(N)$ gauge theory can be transformed into an object in higher-dimensional NC $U(1)$ gauge theory using the compatible ordering (1.6) [12].

The coherent condensate (2.2) is described by vev's of adjoint scalar fields. Thus we will call the vacuum (2.2) a

"Coulomb branch" although $[\Phi_a, \Phi_b]|_{\text{vac}} \neq 0$.[2] However, note that $[\Phi_a, \Phi_b]|_{\text{vac}} = -iB_{ab}I_{N\times N}$ take values in a center of the gauge group $U(N)$, which may be identified with the unbroken $U(1)$ gauge group. Hence the Coulombic vacuum (2.2) is compatible with the usual definition of the Coulomb branch. We also remark that the conformal symmetry of four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory is spontaneously broken by the vev (2.2) of scalar fields because it introduces a NC scale $|\theta| \equiv l_{\text{NC}}^2$. But it needs not be specified because the theories with different θ's are SW equivalent [17]. These are also a typical feature of the Coulomb branch.

Under a Coulomb branch described by the coherent condensate (2.2), large N matrices in $\mathcal{N} = 4$ supersymmetric gauge theory can be regarded as a linear representation of operators acting on a separable Hilbert space $\mathcal{H}$ that is the Fock space of the Moyal-Heisenberg vacuum (2.3). Therefore an important point is that a large N matrix $\Phi(x)$ on four-dimensional spacetime $\mathbb{R}^{3,1}$ in the limit $N \to \infty$ on the Coulomb branch (2.2) can be represented by its master field $\hat{\Phi}(x, y)$, which is a higher-dimensional NC $U(1)$ gauge field or its superpartner. Since the large N gauge theory (2.1) on the Coulomb branch (2.2) is mathematically equivalent to the NC $U(1)$ gauge theory described by the action (2.8), it should be possible to isomorphically map the ten-dimensional NC $U(1)$ super Yang-Mills theory to a ten-dimensional type IIB supergravity according to the AdS/CFT correspondence [3,26,27]. Indeed, the emergent gravity from NC $U(1)$ gauge fields provides the first principle to found the large N duality or gauge/gravity duality in a systematic way [12,16,28].

III. COMMUTATIVE AND NC D-BRANES

The world-volume action for a Dp-brane can be viewed as a $(p+1)$-dimensional nonlinear sigma model with a target space M where the embedding functions $X^M(\sigma)$ define a map $X\colon W \to M$ from the $(p+1)$-dimensional world volume W with coordinates $\sigma^\alpha(\alpha = 0, 1, \ldots, p)$ to the target space M with coordinates $X^M(M = 0, 1, \ldots, 9)$. This embedding induces a world-volume metric

$$h_{\alpha\beta} = g_{MN}(X)\partial_\alpha X^M \partial_\beta X^N. \tag{3.1}$$

The D-brane action in general contains a dilaton coupling $e^{-\phi}$ where ϕ is the ten-dimensional dilaton field. Then the string coupling constant is defined by $g_s = e^{\langle\phi\rangle}$ where the vev $\langle\phi\rangle$ at hand is assumed to be constant. The world volume also carries $U(1)$ gauge fields $A_\alpha(\sigma)$ with field strength

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{3.2}$$

Recall that the DBI action is a nonlinear generalization of electrodynamics with self-interactions of $U(1)$ gauge fields and reproduces the usual Maxwell theory at quadratic order. In string theory a generalization of this action appears in the context of Dp-branes. Open strings ending on the Dp-brane couple directly to closed string background fields (g_{MN}, B_{MN}, ϕ) in the bulk. A low-energy effective field theory deduced from the open string dynamics on a single D-brane is obtained by integrating out all the massive modes, keeping only massless fields that are slowly varying at the string scale $\kappa \equiv 2\pi\alpha'$. The DBI action describes the dynamics of $U(1)$ gauge fields on a D-brane world volume in the approximation of slowly varying fields, $\sqrt{\kappa}|\frac{\partial F}{F}| \ll 1$, in the sense keeping field strengths (without restriction on their size) but not their derivatives. The resulting DBI action on a Dp-brane is given by

$$S_1 = -T_{\text{D}p}\int_W d^{p+1}\sigma\sqrt{-\det(h+\kappa\mathcal{F})} + \mathcal{O}(\sqrt{\kappa}\partial F, \ldots), \tag{3.3}$$

where

$$\mathcal{F} \equiv B + F \tag{3.4}$$

is the total $U(1)$ field strength and the Dp-brane tension is given by

$$T_{\text{D}p} = \frac{2\pi}{g_s(2\pi\kappa)^{\frac{p+1}{2}}}. \tag{3.5}$$

In general, the DBI action (3.3) contains derivative corrections $\mathcal{O}(\sqrt{\kappa}\partial F, \ldots)$. However, we will ignore possible terms involving higher derivatives of fields since we are mostly interested in the approximation that world-volume fields are slowly varying. We will also consider the probe-brane approximation ignoring the backreaction of the brane on the geometry and the other background fields. The world-volume theory of a D-brane is given as the sum of two terms $S = S_1 + S_2$. The first term S_1 is given by the DBI action (3.3), and the second term S_2 is the form of the Wess-Zumino–type given by

[2]The usual Coulomb branch is defined by $[\Phi_a, \Phi_b]|_{\text{vac}} = 0$ and so $\langle\Phi_a\rangle_{\text{vac}} = \text{diag}(\alpha_{a_1}, \ldots, \alpha_{a_N})$. In this case the gauge group $U(N)$ or $SU(N+1)$ is broken to $U(1)^N$. But we remark that the HEA is conjectured to correspond to the choice, $N = 1$ [1] while the probe brane approximation requires $N \to \infty$. Therefore the conventional choice of vacuum finds difficulty in explaining why $N = 1$ (i.e., Abelian gauge group) is the proper choice for the HEA. We emphasize that the Coulomb branch as the NC space (2.2) is a key origin of emergent gravity and is completely consistent with the HEA because it requires the $N \to \infty$ limit and preserves only the $U(1)$ gauge group. Hence our approach sheds light on why HEA preserves only the $U(1)$ gauge symmetry in spite of $N \to \infty$, which was elusive in the original conjecture as discussed in Sec. 5 of Ref. [1].

$$S_2 = \int_W C_{RR} \wedge e^{\kappa \mathcal{F}}, \tag{3.6}$$

where the coupling to background Ramond-Ramond (RR) n-form gauge fields is collected in the formal sum

$$C_{RR} = \bigoplus_{n=0}^{10} C_n. \tag{3.7}$$

The coupling S_2 is a characteristic feature of D-branes that they carry an RR charge [36] and support the world-volume gauge fields (3.2).

Some important remarks are in order. The DBI action (3.3) respects several local gauge symmetries. It has $(p+1)$-dimensional general coordinate invariance since the integrand transforms as a scalar density in $\mathrm{Diff}(W)$. It also admits the so-called Λ symmetry,

$$(B, A) \mapsto (B - d\Lambda, A + \Lambda), \tag{3.8}$$

where the two-form $B \equiv X^*(B_{\mathrm{bulk}})$ is the pullback of target space B field B_{bulk} to the world volume W and the gauge parameter Λ is a one-form in $\Gamma(T^*W)$. Let (W, B) be a symplectic manifold. The symplectic structure B is a nondegenerate, closed two-form, i.e., $dB = 0$, and so it can locally be written as $B = d\xi$ by the Poincaré lemma. The B-field transformation (3.8) can then be understood as a shift of the canonical one-form, $\xi \to \xi - \Lambda$. An important point for us is that the symplectic structure defines a bundle isomorphism $B\colon TW \to T^*W$ by $X \mapsto \Lambda = -\iota_X B$. Thus the B-field transformation (3.8) is equivalent to $(B, A) \mapsto (B + \mathcal{L}_X B, A - \iota_X B)$ where $\mathcal{L}_X = d\iota_X + \iota_X d$ is the Lie derivative with respect to the vector field X. Since vector fields are infinitesimal generators of local coordinate transformations, in other words, Lie algebra generators of $\mathrm{Diff}(W)$, the B-field transformation (3.8) can be identified with a coordinate transformation generated by a vector field $X \in \Gamma(TW)$. Consequently the Λ symmetry (3.8) can be considered on par with diffeomorphisms [12,16]. Moreover, it is well known [29–34] that the D-brane world-volume theory has a local fermionic symmetry called "κ symmetry" if fermion coordinates $\psi^\alpha (\alpha = 1, \ldots, 32)$ are included in the target spacetime with supercoordinates $Z^{\mathbf{M}} = (X^M, \psi^\alpha)$. See a recent review [37] for brane effective actions with the κ symmetry. In sum, the world-volume theory of a supersymmetric D-brane admits the following local gauge symmetries: (I) $\mathrm{Diff}(W)$, (II) Λ symmetry, and (III) κ symmetry.

We can use the general coordinate invariance of the action $S = S_1 + S_2$ to eliminate unphysical degrees of freedom. We choose a static gauge so that $X^M = (x^\mu(\sigma), \phi^a(\sigma)) = (\delta^\mu_\alpha \sigma^\alpha, \phi^a(x))$ where $\mu = 0, \ldots, p$ and $a = p+1, \ldots, 9$. The $(9-p)$ coordinates $\phi^a(x)$ will be identified as the world-volume scalar fields of the Dp-brane. In this gauge the metric (3.1) becomes

$$h_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu \phi^a \partial_\nu \phi^a, \tag{3.9}$$

where we assumed $g_{MN}(X) = \eta_{MN}$ for the target spacetime. Now we focus on a D9-brane for which there are no world-volume scalar fields, i.e., $\phi^a = 0$ and so $h_{MN} = g_{MN}$. Suppose that the D9-brane supports the two-form B field with $\mathrm{rank}(B) = 6$. In this case it is convenient to split the world-volume coordinates $X^M = \sigma^M$ in the static gauge into two parts, $X^M = (x^\mu, z^a), \mu = 0, 1, 2, 3, a = 1, \ldots, 6$, so that $B = \frac{1}{2} B_{ab} dz^a \wedge dz^b$. Then the total field strength (3.4) takes the form

$$\mathcal{F}_{MN} = \begin{pmatrix} F_{\mu\nu} & F_{\mu a} \\ F_{a\mu} & B_{ab} + F_{ab} \end{pmatrix}. \tag{3.10}$$

It is well known [17] that the open string gives rise to the NC geometry when the two-form B field is present on a D-brane world volume. The D-brane dynamics in the static gauge is then described by $U(1)$ gauge fields on a NC spacetime with coordinates $Y^M = (x^\mu, y^a)$ obeying the commutation relation (1.7). The resulting DBI action on the NC D9-brane is given by

$$\hat{S}_1 = -T_9 \int d^{10} Y \sqrt{-\det(G + \kappa(\hat{F} + \Phi))} + \mathcal{O}(\sqrt{\kappa} \hat{D} \hat{F}, \ldots), \tag{3.11}$$

where the NC $U(1)$ field strength $\hat{F}_{MN}(Y)$ is given by Eq. (1.9) and the NC D9-brane tension is

$$T_9 = \frac{2\pi}{G_s (2\pi\kappa)^5}. \tag{3.12}$$

The open string moduli (G, Φ, G_s) in the NC description (3.11) are related to the closed string moduli (g, B, g_s) in the commutative description (3.3) by [17]

$$\frac{1}{g + \kappa B} = \frac{1}{G + \kappa \Phi} + \frac{\theta}{\kappa}, \tag{3.13}$$

$$G_s = g_s \sqrt{\frac{\det(G + \kappa\Phi)}{\det(g + \kappa B)}} = g_s \left(\frac{\det G}{\det g}\right)^{\frac{1}{4}}, \tag{3.14}$$

where the two-form Φ parametrizes some freedom in the description of commutative and NC gauge theories. It is worthwhile to remark that the NC DBI action (3.11) can be obtained by applying the (exact) SW map to the commutative one (3.3) [20,38,39], as will be shown later. Similarly the Wess-Zumino–type term $\hat{S}_2$ for the NC D9-brane can be obtained from the RR couplings in Eq. (3.6) for a commutative D9-brane by considering the (exact) SW map [20,40].

Let us expand the NC DBI action (3.11) in powers of κ. First, note that

$$\sqrt{-\det(G+\kappa(\hat{F}+\Phi))} = \sqrt{-\det G}\sqrt{\det(1+\kappa M)}$$
$$= \sqrt{-\det G}\left(1-\frac{\kappa^2}{4}\mathrm{Tr}M^2-\frac{\kappa^4}{8}\mathrm{Tr}M^4+\frac{\kappa^4}{32}(\mathrm{Tr}M^2)^2+\cdots\right), \tag{3.15}$$

where

$$M_N{}^Q \equiv (\hat{F}+\Phi)_{NP}G^{PQ}, \tag{3.16}$$

and so $\mathrm{Tr}M = 0$. At nontrivial leading orders, we find

$$\hat{S}_1 = -T_9\int d^{10}Y\sqrt{-\det G} - \frac{1}{4G_{YM}^2}\int d^{10}Y\sqrt{-\det G}G^{MP}G^{NQ}(\hat{F}+\Phi)_{MN}(\hat{F}+\Phi)_{PQ}+\mathcal{O}(\kappa^4), \tag{3.17}$$

where the ten-dimensional Yang-Mills coupling constant is given by

$$G_{YM}^2 = (\kappa^2T_9)^{-1} = (2\pi)^4\kappa^3 G_s. \tag{3.18}$$

In our case at hand, the open string metric can be set to be flat, i.e., $G_{MN}=\eta_{MN}$. The first term of $\hat{S}_1$ is a vacuum energy due to the D-brane tension that will be canceled against a contribution from $\hat{S}_2$ [1,3]. The second term is precisely equal to the bosonic part of the action (2.8) when the background independent prescription is employed, i.e., $\Phi = -B$ [17]. Therefore we will consider the ten-dimensional $\mathcal{N}=1$ NC $U(1)$ super Yang-Mills theory (2.8) as a nontrivial leading approximation of the supersymmetric completion of the NC DBI action (3.11). The supersymmetric completion with the κ symmetry will be discussed in Sec. V.

IV. AdS/CFT CORRESPONDENCE FROM NC $U(1)$ GAUGE FIELDS

In their well-known paper [17], Seiberg and Witten showed that there exists an equivalent commutative description of the low-energy effective theory for the open string ending on a NC D-brane. From the point of view of an open string sigma model, an explicit form of the effective action depends on the regularization scheme of two-dimensional field theory. The difference due to different regularizations is always in a choice of contact terms, leading to the redefinition of coupling constants that are spacetime fields. So low-energy field theories defined with different regularizations should be related to each other by the field redefinitions in spacetime. Now we will explain how the NC DBI action (3.11) arises from a low-energy effective action in a curved background that will be identified with the HEA speculated by Schwarz [1]. First we identify a commutative description that is SW equivalent to the NC DBI action (3.11). From a conventional approach, the answer is obvious. It is given by the D9-brane action (3.3) (with $p=9$) with the field strength (3.10). But, for our purpose, it is more proper to consider the NC DBI action (3.11) as a particular commutative limit of the full NC D9-brane described by the star product

$$(\hat{f}\star\hat{g})(Y) = e^{\frac{i}{2}\Theta^{MN}\frac{\partial}{\partial Y^M}\otimes\frac{\partial}{\partial Z^N}}\hat{f}(Y)\hat{g}(Z)|_{Y=Z} \tag{4.1}$$

for $\hat{f}(Y),\hat{g}(Y)\in C^\infty(\mathbb{R}^{10})$. We implicitly assumed the Wick rotation, $\mathbb{R}^{9,1}\to\mathbb{R}^{10}$, although it is simply formal because we eventually come back to the space $\mathbb{R}^{3,1}\times\mathbb{R}^6_\theta$. For this purpose, it is convenient to take the split $\Theta^{MN} = (\zeta^{\mu\nu},\theta^{ab})$ where an $SO(10)$ rotation was used to put $\zeta^{\mu a}=0$. We intend to understand the star product (1.8) as a particular case of Eq. (4.1) with $\zeta^{\mu\nu}=0$. Later we will explain why the star product (4.1) is more relevant for our context, especially, from the viewpoint of emergent spacetime. Hence we need to identify a commutative DBI action that is SW equivalent to the NC DBI action (3.11), instead, using the star product (4.1). It is given by the D9-brane action (3.3) with the $U(1)$ field strength

$$\mathcal{F} = \frac{1}{2}\mathcal{F}_{MN}(X)dX^M\wedge dX^N$$
$$= \frac{1}{2}(B_{MN}+F_{MN}(X))dX^M\wedge dX^N = B+F, \tag{4.2}$$

where $B=\Theta^{-1}$ and $\mathrm{rank}(B)=10$. We will assume that $\mathcal{F}$ is also nondegenerate, i.e., $\det(1+F\Theta)\neq 0$.

To derive the HEA, it is enough only to employ the logic expounded in Appendix A in Ref. [12]. Note that $\mathcal{F}$ in Eq. (4.2) is the gauge invariant quantity under the Λ symmetry (3.8). In other words, the dynamical $U(1)$ gauge fields should appear only as the combination (4.2). In particular, we can use the Λ symmetry (3.8) so that the B field in Eq. (4.2) is constant. Then $dB=0$ trivially and B is nondegenerate because of $\mathrm{rank}(B)=10$. Therefore $(\mathbb{R}^{10},B)$ is a symplectic manifold. Moreover, $(\mathbb{R}^{10},\mathcal{F})$ is also a symplectic manifold since $d\mathcal{F}=0$ and $\mathcal{F}$ is nondegenerate by our assumption. Then we can realize an important identity

$$\mathcal{F} = (1+\mathcal{L}_X)B \tag{4.3}$$

as we explained below Eq. (3.8). It implies that there exists a local coordinate transformation $\phi\in\mathrm{Diff}(M)$ such that $\phi^*(\mathcal{F})=B$, i.e., $\phi^* = (1+\mathcal{L}_X)^{-1}\approx e^{-\mathcal{L}_X}$. This statement is the well-known theorem in symplectic geometry known as the Darboux theorem [41,42]. Its global statement is known as the Moser lemma [43]. The Darboux theorem states that it is always possible to find a local coordinate transformation $\phi \in \mathrm{Diff}(M)$, which eliminates dynamical $U(1)$ gauge fields in $\mathcal{F}$. That is, in terms of local coordinates, there exists $\phi\colon Y \mapsto X = X(Y)$ so that

$$(B_{MN} + F_{MN}(X)) \frac{\partial X^M}{\partial Y^P} \frac{\partial X^N}{\partial Y^Q} = B_{PQ}. \tag{4.4}$$

If we represent the local coordinate transformation by

$$X^M(Y) = Y^M + \Theta^{MN} \hat{A}_N(Y), \tag{4.5}$$

Eq. (4.4) can be written as

$$\mathfrak{P}^{MN}(X) \equiv (\mathcal{F}^{-1})^{MN}(X) = \{X^M(Y), X^N(Y)\}_\Theta, \tag{4.6}$$

where we introduced the Poisson bracket defined by

$$\{f(Y), g(Y)\}_\Theta = \Theta^{MN} \frac{\partial f(Y)}{\partial Y^M} \frac{\partial g(Y)}{\partial Y^N} \tag{4.7}$$

for $f, g \in C^\infty(\mathbb{R}^{10})$. We will call $\hat{A}_M(Y)$ in Eq. (4.5) symplectic gauge fields and $X^M(Y)$ covariant (dynamical) coordinates. The field strength of symplectic gauge fields is defined by

$$\hat{F}_{MN} = \partial_M \hat{A}_N - \partial_N \hat{A}_M + \{\hat{A}_M, \hat{A}_N\}_\Theta. \tag{4.8}$$

Then Eq. (4.6) gives us the relation

$$\mathfrak{P}^{MN} = [\Theta(B - \hat{F})\Theta]^{MN}. \tag{4.9}$$

By solving this equation, we yield the semiclassical version of the SW map [18–20],

$$\hat{F}_{MN}(Y) = \left(\frac{1}{1 + F\Theta} F\right)_{MN}(X), \tag{4.10}$$

$$d^{10}Y = d^{10}X \sqrt{\det(1 + F\Theta)}, \tag{4.11}$$

where the second equation is derived from Eq. (4.4) by taking the determinant on both sides.

The coordinate transformation (4.4) leads to the identity

$$g_{MN} + \kappa \mathcal{F}_{MN} = (\mathcal{G}_{PQ} + \kappa B_{PQ}) \frac{\partial Y^P}{\partial X^M} \frac{\partial Y^Q}{\partial X^N}, \tag{4.12}$$

where the dynamical (emergent) metric is defined by

$$\mathcal{G}_{MN} = g_{PQ} \frac{\partial X^P}{\partial Y^M} \frac{\partial X^Q}{\partial Y^N}. \tag{4.13}$$

The identity (4.12) in turn leads to a remarkable identity between DBI actions,

$$\begin{aligned} &\frac{1}{g_s} \int d^{10}X \sqrt{\det(g + \kappa \mathcal{F})} \\ &= \frac{1}{g_s} \int d^{10}Y \sqrt{\det(\mathcal{G} + \kappa B)} \qquad (4.14) \\ &= \frac{1}{G_s} \int d^{10}Y \sqrt{\det(G + \kappa(\hat{F} + \Phi))}. \qquad (4.15) \end{aligned}$$

It is straightforward to derive the second identity (4.15) by using Eqs. (3.13) and (3.14) and the SW maps (4.10) and (4.11). For the derivation of Eq. (4.15), see Eq. (5.10) in Ref. [20] and Sec. 3.4 of Ref. [38]. It may be instructive to check Eq. (4.15) by expanding the right-hand side (RHS) of Eq. (4.14) around the background B field, i.e.,

$$\begin{aligned} \sqrt{\det(\mathcal{G} + \kappa B)} &= \sqrt{\det(\kappa B)} \sqrt{\det\left(1 + \frac{M}{\kappa}\right)} \\ &= \sqrt{\det(\kappa B)} \left(1 - \frac{1}{4\kappa^2} \mathrm{Tr} M^2 - \frac{1}{8\kappa^4} \mathrm{Tr} M^4 + \frac{1}{32\kappa^4} (\mathrm{Tr} M^2)^2 + \cdots \right), \end{aligned} \tag{4.16}$$

where

$$M_N{}^Q = \mathcal{G}_{NP} \Theta^{PQ} \tag{4.17}$$

and

$$\mathrm{Tr} M^2 = \mathrm{Tr}(g\mathfrak{P})^2, \qquad \mathrm{Tr} M^4 = \mathrm{Tr}(g\mathfrak{P})^4. \tag{4.18}$$

But it is not difficult to show that $\mathrm{Tr} M^{2n} = \mathrm{Tr}(g\mathfrak{P})^{2n}, \mathrm{Tr} M^{2n+1} = \mathrm{Tr}(g\mathfrak{P})^{2n+1} = 0$ for $n \in \mathbb{N}$ and thus

$$\det\left(1 + \frac{M}{\kappa}\right) = \det\left(1 + \frac{1}{\kappa} g\mathfrak{P}\right) \tag{4.19}$$

using the expansion of the determinant [see Eq. (4.30) in Ref. [32]]. Then, using the result (4.9), the expansion in Eq. (4.16) can be arranged into the form

$$\begin{aligned} \sqrt{\det(\mathcal{G} + \kappa B)} &= \sqrt{\frac{\det(\kappa B)}{\det G}} \sqrt{\det(G + \kappa(\hat{F} - B))} \\ &= \frac{g_s}{G_s} \sqrt{\det(G + \kappa(\hat{F} - B))}, \end{aligned} \tag{4.20}$$

where

$$G_{MN} = -\kappa^2 (Bg^{-1}B)_{MN}, \quad G_s = g_s \sqrt{\det(\kappa B g^{-1})} \tag{4.21}$$

are the open string metric and coupling constant, respectively, in the background independent prescription, i.e., $\Phi = -B$ [17]. To demonstrate how $2n$-dimensional Kähler metrics arise from $U(1)$ gauge fields, in Appendix A, we will solve the identities (4.14) and (4.15). In particular, it is shown that Calabi-Yau n-folds for $n = 2$ and 3 are emergent from symplectic $U(1)$ instantons in four and six dimensions, respectively.

NC $U(1)$ gauge fields are obtained by quantizing symplectic gauge fields. The quantization in our case is simply defined by the canonical quantization of the Poisson algebra $\mathfrak{P} = (C^\infty(\mathbb{R}^{10}), \{-,-\}_\Theta)$. The quantization map $\mathcal{Q}\colon C^\infty(\mathbb{R}^{10}) \to \mathcal{A}_\theta$ by $f \mapsto \mathcal{Q}(f) \equiv \hat{f}$ is a $\mathbb{C}$-linear algebra homomorphism defined by

$$f \cdot g \mapsto \widehat{f \star g} = \hat{f} \cdot \hat{g} \tag{4.22}$$

and

$$f \star g \equiv \mathcal{Q}^{-1}(\mathcal{Q}(f) \cdot \mathcal{Q}(g)) \tag{4.23}$$

for $f, g \in C^\infty(\mathbb{R}^{10})$ and $\hat{f}, \hat{g} \in \mathcal{A}_\theta$. The above star product is given by Eq. (4.1) [22]. The DBI action (3.11) for the NC D9-brane relevant to the NC $U(1)$ gauge theory (2.8) is then obtained by simply considering a particular NC parameter $\Theta^{MN} = (\zeta^{\mu\nu}, \theta^{ab})$ with $\zeta^{\mu\nu} = 0$. We understand the limit $\zeta^{\mu\nu} \to 0$ as $|\zeta|^2 \equiv G_{\mu\rho} G_{\nu\sigma} \zeta^{\mu\nu} \zeta^{\rho\sigma} = \kappa^2 |\kappa B_{\mu\lambda} g^{\lambda\rho}|^2 \ll \kappa^2$ where the open string metric in Eq. (4.21) was used. This means that $g_{\mu\nu} + \kappa B_{\mu\nu} = (\delta_\mu^\rho + \kappa B_{\mu\lambda} g^{\lambda\rho}) g_{\rho\nu} \approx g_{\mu\nu}$; in other words, the metric part in the DBI background $g_{\mu\nu} + \kappa B_{\mu\nu}$ is dominant so that the B-field part can be ignored.

Why do we need to take the limit $\zeta^{\mu\nu} \to 0$ instead of simply putting $\zeta^{\mu\nu} = 0$? Actually the answer is involved with the most beautiful aspect of emergent gravity. In the emergent gravity picture, any spacetime structure is not assumed *a priori* but defined by the theory itself. In a sonorous phrase, the theory of emergent gravity must be background independent. Hence it is necessary to define a configuration in the algebra $\mathcal{A}_\theta$, for instance, like Eq. (1.7), to generate any kind of spacetime structure, even for flat spacetime. Emergent gravity then says that the flat spacetime is emergent from the Moyal-Heisenberg algebra (1.7). In other words, even the flat spacetime must have a dynamical origin [12,16,28], which is absent in general relativity. This picture may also be convinced by gazing up at the identity (4.14). Note that the dynamical variables on the RHS of Eq. (4.14) are (emergent) metric fields, $\mathcal{G}_{MN}(Y)$, whereas those on the left-hand side (LHS) are $U(1)$ gauge fields, $F_{MN}(X)$, in a specific background (g, B). Therefore the gravitational fields $\mathcal{G}_{MN}(Y)$ are completely determined by dynamical $U(1)$ gauge fields, and so the former is emergent from the latter. When $U(1)$ gauge fields are turned off, the emergent metric reduces to the flat metric, i.e., $\mathcal{G}_{MN} = g_{MN}$. But the background B field still persists, and it can be regarded as a vacuum gauge field $A_M^{(0)} = -\frac{1}{2} B_{MN} X^N$. Then it is natural to think that the flat metric g_{MN} is emergent from the vacuum gauge fields $A_M^{(0)}$. This remarkable picture can be rigorously confirmed from a background independent formulation, e.g., matrix models [12,16,28]. In consequence, any spacetime structure did not exist *a priori*, but the existence of spacetime requires a coherent condensate of vacuum gauge fields. Nature allows "no free lunch." As a result, the usual commutative spacetime has to be understood as a *commutative* limit of NC spacetime as we advocated above. Indeed, we do not know how to reproduce the NC DBI action (3.11) via the identity (4.14) starting with the $U(1)$ field strength (3.10).[3]

Note that the coordinate transformation (4.4) to a Darboux frame is defined only locally and symplectic or NC gauge fields have been introduced to compensate local deformations of an underlying symplectic structure by $U(1)$ gauge fields, i.e., the Darboux coordinates in $\phi\colon Y \mapsto X = X(Y) \in \mathrm{Diff}(\mathbb{R}^{10})$ obey the relation $\phi^*(B + F) = B$. The identity (4.20) also manifests this local nature of NC gauge fields because they manifest themselves only in a locally inertial frame (in free fall) with the local metric (4.13) [12]. If the gravitational metric in Eq. (4.20) were represented by a global form, e.g.,

$$\mathcal{G}_{MN} = g_{AB} E_M^A E_N^B, \qquad A, B = 0, 1, \ldots, 9, \tag{4.24}$$

where $E^A = E_M^A dx^M$ are elements of a global coframe on an emergent ten-dimensional manifold $\mathcal{M}$, it would be difficult to find an imprint of symplectic or NC gauge fields in the expression (4.24).

Recall that the basic program of differential geometry is that all the world can be reconstructed from the infinitely small. For example, manifolds are obtained by gluing open subsets of Euclidean space. So the differential forms and vector fields on a manifold are defined locally and then glued together to yield a global object. The gluing is possible because these objects are independent of the choice of local coordinates. In reality this kind of globalization of a (spacetime) geometry by gluing local data might be enforced because global comparison devices are not available owing to the restriction of the finite propagation speed. Indeed, the global metric (4.24) can be constructed in a similar way. First note that the D9-brane described by the LHS of Eq. (4.14) supports a line bundle $L \to \mathbb{R}^{10}$ over

[3]Note that the Darboux theorem (4.4) can be applied only to a symplectic form, i.e., a nondegenerate and closed two-form. But the dynamical two-form F does not belong to this category because it usually vanishes at an asymptotic infinity.

a symplectic manifold $(\mathbb{R}^{10}, B)$. Introduce an open covering $\{U_i:\ i \in I\}$ of $\mathbb{R}^{10}$, i.e., $\mathbb{R}^{10} = \cup_{i=I} U_i$, and let $A^{(i)}$ be a connection of the line bundle $L \to U_i$ on an open neighborhood U_i. Consider all compatible coordinate systems $\{(U_i, \varphi_i):\ i \in I\}$ as a family of local Darboux charts where $\varphi_i:\ U_i \to \mathbb{R}^{10}$ are Darboux coordinates on U_i. Then we have the collection of local data $\bigoplus_{i\in I}(A^{(i)}, Y_{(i)})$ on the D9-brane where $Y_{(i)} = \varphi_i(U_i)$ are Darboux coordinates on U_i obeying Eq. (4.4), i.e., $\varphi_i^*(B + F^{(i)}) = B$ where $F^{(i)} = dA^{(i)}$. On an intersection $U_i \cap U_j$, local data $(A^{(i)}, Y_{(i)})$ and $(A^{(j)}, Y_{(j)})$ on Darboux charts U_i and U_j, respectively, are glued together by [44,45]

$$A^{(j)} = A^{(i)} + d\lambda^{(ji)}, \tag{4.25}$$

$$Y_{(j)} = \varphi_{(ji)}(Y_{(i)}), \tag{4.26}$$

where $\varphi_{(ji)}$ is a symplectomorphism on $U_i \cap U_j$ generated by a Hamiltonian vector field $X_{\lambda^{(ji)}}$ obeying $\iota_{X_{\lambda^{(ji)}}} B + d\lambda^{(ji)} = 0$. Note that the symplectomorphism is a canonical transformation preserving the Poisson structure $\Theta = B^{-1}$ and can be identified with a NC $U(1)$ gauge transformation upon quantization [21,22]. Since the local metric (4.13) is the incarnation of symplectic gauge fields in a Darboux frame, the gluing of local Darboux charts can be translated into that of emergent metrics in locally inertial frames from the viewpoint of the RHS of Eq. (4.14). This kind of gluing should be well defined because every manifold can be constructed by gluing open subsets of Euclidean space together and both sides of Eq. (4.14) are coordinate independent, and so local Darboux charts can be consistently glued altogether. See Ref. [46] to illuminate how a nontrivial topology of an emergent manifold can be implemented by gluing local data $\cup_{i=I}(A^{(i)}, Y_{(i)})$.

It is in order to ponder on the results obtained. We showed in Sec. II that the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory on the Coulomb branch (2.2) is equivalent to the ten-dimensional $\mathcal{N} = 1$ supersymmetric NC $U(1)$ gauge theory. And we considered the resulting ten-dimensional NC $U(1)$ gauge theory as a low-energy effective theory of supersymmetric NC D9-brane. Finally we got the important identity (4.20) that the dynamics of NC $U(1)$ gauge fields after ignoring fermion fields is completely encoded into a ten-dimensional emergent geometry described by the metric (4.24). According to the AdS/CFT correspondence, it is natural to expect that the metric (4.24) must describe a ten-dimensional emergent geometry dual to the four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. An immediate question to arise is how to realize the $\mathrm{AdS}_5 \times \mathbb{S}^5$ vacuum geometry in our context.

Since there is no reason to further reside in Euclidean space, let us go back to the Lorentzian spacetime with the NC parameter $\Theta^{MN} = (\zeta^{\mu\nu} = 0, \theta^{ab} \neq 0)$ by Wick rotation. To pose the above question, let us consider a more general vacuum geometry that is conformally flat. That is, we are interested in a background geometry with the metric given by

$$ds^2 = \lambda^2(\eta_{\mu\nu} dx^\mu dx^\nu + dy^a dy^a). \tag{4.27}$$

There are two interesting cases that are conformally flat [12]:

$$\lambda^2 = 1 \Rightarrow \mathcal{M} = \mathbb{R}^{9,1}, \tag{4.28}$$

$$\lambda^2 = \frac{R^2}{\rho^2} \Rightarrow \mathcal{M} = \mathrm{AdS}_5 \times \mathbb{S}^5, \tag{4.29}$$

where $\rho^2 = \sum_{a=1}^6 y^a y^a$ and $R = (4\pi g_s (\alpha')^2 N)^{1/4}$ is the radius of AdS_5 and $\mathbb{S}^5$ spaces. We already speculated before that the flat Minkowski spacetime (4.28) arises from a uniform condensate of vacuum gauge fields $A_M^{(0)} = -\frac{1}{2} B_{MN} X^N$. This can be confirmed by looking at the vacuum configuration (2.2). Note that, from the four-dimensional gauge theory point of view, the vacuum configuration (2.2) simply represents a particular configuration of large N matrices and it is connoted as an extra six-dimensional "emergent" space only in a ten-dimensional description. Its tangible existence must be addressed from the RHS of Eq. (4.14). (See Sec. 1 in Ref. [12] for the rationale underlying this reasoning.) Then it is easy to prove that the emergent metric (4.13) for the vacuum configuration (2.2) is precisely the flat Minkowski spacetime (4.28). Note that a Darboux chart (U, φ) in this case can be extended to the entire spacetime, and so it is not necessary to consider the globalization prescribed before.

Now a perplexing problem is to understand what is the gauge field configuration to realize the vacuum geometry (4.29). To figure out the problem, it is necessary to find a stable configuration of NC or large N gauge fields and so certainly a supersymmetric or Bogomol'nyi-Prasad-Sommerfield state. And this configuration must be consistent with the isometry of the vacuum geometry (4.27), in particular, preserving $SO(6)_R$ Lorentz symmetry as if a hydrogen atom preserves $SO(3)$ symmetry. It was conjectured in [12] that the $\mathrm{AdS}_5 \times \mathbb{S}^5$ geometry arises from the stack of NC Hermitian $U(1)$ instantons at the origin in the internal space $\mathbb{R}^6$ like a nucleus containing a lot of nucleons. The NC Hermitian $U(1)$ instanton obeys the Hermitian Yang-Mills equations [47] given by

$$\hat{F}_{ab} = -\frac{1}{4} \varepsilon_{abcdef} \hat{F}_{cd} I_{ef}, \tag{4.30}$$

where $I = \mathbf{I}_3 \otimes i\sigma^2$ is a 6×6 matrix of the complex structure of $\mathbb{R}^6$ and the field strength is defined by Eq. (2.7). Note that the six-dimensional NC $U(1)$ gauge fields $\hat{A}_a$ in Eq. (4.30) are originally adjoint scalar fields $\Phi_a = p_a + \hat{A}_a$ in four-dimensional $\mathcal{N} = 4$ super Yang-Mills theory. See Eq. (2.4). If true, the vacuum geometry (4.29) will be emergent from the stack of infinitely many NC $U(1)$ instantons obeying Eq. (4.30) according to the identity (4.20).[4] Since we are interested in the approximation of slowly varying fields, $\sqrt{\theta}|\frac{\hat{D}\hat{F}}{\hat{F}}| \ll 1$, ignoring the derivatives of field strengths, the $U(1)$ field strength in Eq. (4.30) can be replaced by Eq. (4.8) in this limit, and so we can use the SW maps (4.10) and (4.11). But, if we include NC corrections containing higher-order derivatives of field strengths, the LHS of Eq. (4.20) will receive derivative corrections introducing a higher-order gravity in the emergent geometry [21].

In conclusion, the AdS/CFT correspondence is a particular example of emergent gravity from NC $U(1)$ gauge fields. And the duality between large N gauge fields and a higher-dimensional gravity is simply a consequence of the novel equivalence principle stating that the electromagnetic force can always be eliminated by a local coordinate transformation as far as spacetime admits a symplectic structure; in other words, a microscopic spacetime becomes NC [12,16].

V. HEA FROM NC $U(1)$ GAUGE FIELDS

Now we are ready to derive the HEA of four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch. According to the conjecture [1], the HEA should be a $U(1)$ gauge theory in the $AdS_5 \times \mathbb{S}^5$ geometry with N units of flux threading $\mathbb{S}^5$. However the original conjecture did not allude to any clue why the HEA on the Coulomb branch must be described by the $U(1)$ gauge theory although the probe-brane approximation requires a large N limit. For the discussion of this problem, see, in particular, Sec. 5 in Ref. [1]. As we emphasized in footnote 2, our approach based on the NC field theory representation of AdS/CFT correspondence will clarify why $N=1$ is the relevant choice for the HEA.

We argued before that the $AdS_5 \times \mathbb{S}^5$ geometry is emergent from the stack of infinitely many NC Hermitian $U(1)$ instantons near the origin in $\mathbb{R}^6$. Thus suppose that the vacuum configuration for the background geometry (4.29) is given by

$$\langle \Phi_a \rangle_{\rm vac} = p_a + \hat{A}_a, \qquad \langle A_\mu \rangle_{\rm vac} = 0, \qquad \langle \lambda^i \rangle_{\rm vac} = 0, \tag{5.1}$$

where $\hat{A}_a$ is a solution of Eq. (4.30) describing N NC Hermitian $U(1)$ instantons in six dimensions. We introduce fluctuations around the vacuum (5.1) and represent them as

$$\hat{D}_\mu = \partial_\mu - i\hat{a}_\mu(x,y), \tag{5.2}$$

$$\hat{D}_a = -i(p_a + \hat{A}_a(y) + \hat{a}_a(x,y)) \equiv \hat{\nabla}_a(y) - i\hat{a}_a(x,y), \tag{5.3}$$

whose field strengths are given by

$$\hat{\mathcal{F}}_{\mu\nu} = \partial_\mu \hat{a}_\nu - \partial_\nu \hat{a}_\mu - i[\hat{a}_\mu, \hat{a}_\nu]_\star \equiv \hat{f}_{\mu\nu}, \tag{5.4}$$

$$\hat{\mathcal{F}}_{\mu a} = \hat{D}_\mu \hat{a}_a - \hat{\nabla}_a \hat{a}_\mu \equiv \hat{f}_{\mu a}, \tag{5.5}$$

$$\begin{aligned}\hat{\mathcal{F}}_{ab} &= -B_{ab} + \hat{F}_{ab} + \hat{\nabla}_a \hat{a}_b - \hat{\nabla}_b \hat{a}_a - i[\hat{a}_a, \hat{a}_b]_\star \\ &\equiv -B_{ab} + \hat{F}_{ab} + \hat{f}_{ab},\end{aligned} \tag{5.6}$$

where $\hat{F}_{ab}(y) - B_{ab} = i[\hat{\nabla}_a, \hat{\nabla}_b]_\star(y)$. We will include fermions later. Note that we assumed that the instanton connection $\hat{\nabla}_a(y)$ depends only on NC coordinates in extra dimensions. Hence the solution has a translational invariance along $\mathbb{R}^{3,1}$, which means that the solution describes extended objects along $\mathbb{R}^{3,1}$. They were conjecturally identified with N D3-branes in [12]. Since the SW relation between commutative and NC gauge theories is true for general gauge fields, we can apply to the gauge fields in Eqs. (5.2) and (5.3) the SW maps

$$\hat{\mathcal{F}}_{MN}(Y) = \left(\frac{1}{1+\mathfrak{F}\Theta}\mathfrak{F}\right)_{MN}(X), \tag{5.7}$$

$$d^{10}Y = d^{10}X\sqrt{\det(1+\mathfrak{F}\Theta)}, \tag{5.8}$$

where $\mathfrak{F} \equiv B + F + f$ is the total $U(1)$ field strength including the background instanton part F_{ab} and the fluctuation part $f_{MN} = \partial_M a_N - \partial_N a_M$. The result will be given by the following equivalence:

$$\begin{aligned}&\frac{1}{g_s}\int d^{10}X\sqrt{-\det(g+\kappa\mathfrak{F})} \\ &\quad = \frac{1}{G_s}\int d^{10}Y\sqrt{-\det(G+\kappa(\hat{\mathcal{F}}+\Phi))}.\end{aligned} \tag{5.9}$$

But we can also apply the Darboux transformation (4.4) to the field strength $\mathfrak{F}$ such that the Darboux coordinates Z^M eliminate only the instanton gauge fields F_{ab}. Then we will get the following identity:

$$g_{MN} + \kappa\mathfrak{F}_{MN} = (\mathcal{G}_{PQ} + \kappa(B+\tilde{f})_{PQ})\frac{\partial Z^P}{\partial X^M}\frac{\partial Z^Q}{\partial X^N}, \tag{5.10}$$

where

[4] Given the metric (4.27) of $AdS_5 \times \mathbb{S}^5$ geometry on the LHS of Eq. (4.20), we may simply assume that we have solved Eq. (4.20) to find some configuration of $U(1)$ gauge fields that gives rise to the $AdS_5 \times \mathbb{S}^5$ geometry. In Appendix A, we will solve Eq. (4.20) to illustrate how $2n$-dimensional Calabi-Yau manifolds arise from $2n$-dimensional symplectic $U(1)$ gauge fields. But it should be remarked that the underlying argument can proceed with impunity if our conjecture is true or not.

$$\mathcal{G}_{MN} = g_{PQ}\frac{\partial X^P}{\partial Z^M}\frac{\partial X^Q}{\partial Z^N},$$
$$\tilde{f}_{MN} = f_{PQ}\frac{\partial X^P}{\partial Z^M}\frac{\partial X^Q}{\partial Z^N} = \frac{\partial \tilde{a}_N}{\partial Z^M} - \frac{\partial \tilde{a}_M}{\partial Z^N} \qquad (5.11)$$

with $\tilde{a}_M = \frac{\partial X^P}{\partial Z^M} a_P$. This leads to an enticing result

$$\frac{1}{g_s}\int d^{10}X\sqrt{-\det(g+\kappa\mathfrak{F})}$$
$$= \frac{1}{g_s}\int d^{10}Z\sqrt{-\det(\mathcal{G}+\kappa(B+\tilde{f}))} \qquad (5.12)$$
$$= \frac{1}{G_s}\int d^{10}Y\sqrt{-\det(G+\kappa(\hat{\mathcal{F}}+\Phi))}. \qquad (5.13)$$

We can check the consistency of the above identities by showing that Eq. (5.13) can be derived from the RHS of Eq. (5.12). Consider a Darboux transformation $\phi_1: Y^M \mapsto Z^M = Y^M + \Theta^{MN}\hat{a}_N(Y)$ satisfying $\phi_1^*(B+\tilde{f}) = B$. Then it leads to the identity

$$\mathcal{G}_{MN} + \kappa(B+\tilde{f})_{MN} = (\mathfrak{G}_{PQ} + \kappa B_{PQ})\frac{\partial Y^P}{\partial Z^M}\frac{\partial Y^Q}{\partial Z^N}, \qquad (5.14)$$

where

$$\mathfrak{G}_{MN} = \mathcal{G}_{PQ}\frac{\partial Z^P}{\partial Y^M}\frac{\partial Z^Q}{\partial Y^N} = g_{PQ}\frac{\partial X^P}{\partial Y^M}\frac{\partial X^Q}{\partial Y^N}. \qquad (5.15)$$

The previous Darboux transformation (5.10) satisfies $\phi_2^*(B+F) = B$ where $\phi_2: Z^M \mapsto X^M = Z^M + \Theta^{MN}\hat{A}_N(Z)$, which, in Eq. (5.15), has been combined with ϕ_1, i.e.,

$$\phi_2 \circ \phi_1: Y^M \mapsto X^M = Y^M + \Theta^{MN}(\hat{A}_N + \hat{a}_N)(Y). \qquad (5.16)$$

Note that we can put $\hat{A}_\mu = 0$ by our assumption. Using the identity (5.14), we can derive the following equivalence between DBI actions:

$$\frac{1}{g_s}\int d^{10}Z\sqrt{-\det(\mathcal{G}+\kappa(B+\tilde{f}))}$$
$$= \frac{1}{g_s}\int d^{10}Y\sqrt{-\det(\mathfrak{G}+\kappa B)}. \qquad (5.17)$$

By applying the same method as Eq. (4.20) and using the coordinates (5.16), it is straightforward to derive Eq. (5.13) from the RHS of Eq. (5.17).

The conformally flat metric (4.27) takes the form

$$ds^2 = R^2\left(\frac{dx\cdot dx + d\rho^2}{\rho^2} + d\Omega_5^2\right), \qquad (5.18)$$

where $dx\cdot dx = \eta_{\mu\nu}dx^\mu dx^\nu$. This form of the metric can be transformed into the metric form used in [1] by a simple inversion $\rho = 1/v$,

$$ds^2 = R^2\left(v^2 dx\cdot dx + v^{-2}dv^2 + d\Omega_5^2\right)$$
$$= R^2(v^2 dx\cdot dx + v^{-2}dv\cdot dv), \qquad (5.19)$$

where $dv\cdot dv = dv^a dv^a$. Note that the four-dimensional supersymmetric gauge theory is defined on the boundary of AdS_5 space where $v\to\infty$ in the metric (5.19) and so the five-sphere $\mathbb{S}^5$ shrinks to a point near the conformal boundary of the AdS space. Then the $SO(6)$ isometry of $\mathbb{S}^5$ is realized as a global symmetry in the gauge theory and the (angular) momenta dual to five-sphere coordinates are given by generators of the $SO(6)$ R symmetry. Since we are interested in the HEA of the boundary theory where the $\mathbb{S}^5$ shrinks to a point, we can thus consider a low-energy limit by ignoring any y dependence for fluctuations, but leaving the background intact. Then the fluctuating $U(1)$ field strengths on the LHS of Eq. (5.17) reduce to

$$\tilde{f}_{\mu\nu}(x,y) \to \partial_\mu\tilde{a}_\nu(x) - \partial_\nu\tilde{a}_\mu(x) \equiv f_{\mu\nu}(x),$$
$$\tilde{f}_{\mu a}(x,y) \to \partial_\mu\tilde{a}_a(x) \equiv \partial_\mu\varphi_a(x),$$
$$\tilde{f}_{ab}(x,y) \to 0. \qquad (5.20)$$

Since we assumed that the low-energy theory does not depend on the coordinates y^a of extra dimensions, we will try to reduce the ten-dimensional theory to a four-dimensional effective field theory. For this purpose, first let us consider the block matrix

$$\mathcal{G}_{MN} + \kappa(B+\tilde{f})_{MN} = \begin{pmatrix} \lambda^2\eta_{\mu\nu} + \kappa f_{\mu\nu} & \kappa\partial_\mu\varphi_a \\ -\kappa\partial_\mu\varphi_a & \lambda^2\delta_{ab} + \kappa B_{ab}\end{pmatrix}, \qquad (5.21)$$

where we put $B_{\mu\nu} = 0$ according to the reasoning explained in Sec. IV. Even we may take the approximation $\lambda^2\delta_{ab} + \kappa B_{ab} \approx \lambda^2\delta_{ab}$ because $\lambda^2 = R^2v^2 \to \infty$ and the low-energy limit applied to Eq. (5.20) is basically equivalent to $\theta^{ab}\to 0$, and so the metric part is dominant similar to the reasoning below Eq. (4.23). Considering the fact that NC corrections in NC gauge theory correspond to $1/N$ expansions in large N gauge theory [21], the approximation considered can be interpreted as the planar limit in AdS/CFT correspondence. Using the determinant formula for a block matrix

$$\det\begin{pmatrix} A & B \\ C & D\end{pmatrix} = \det D \det(A - BD^{-1}C), \qquad (5.22)$$

we get the following relation:

$$\sqrt{-\det(\mathcal{G}+\kappa(B+\tilde{f}))} = \sqrt{\det(\lambda^2+\kappa B)}\sqrt{-\det\left(\lambda^2\eta_{\mu\nu}+\kappa f_{\mu\nu}+\kappa^2\partial_\mu\varphi_a\left(\frac{1}{\lambda^2+\kappa B}\right)^{ab}\partial_\nu\varphi_b\right)}$$
$$\approx \lambda^6\sqrt{-\det\left(\lambda^2\eta_{\mu\nu}+\kappa^2\lambda^{-2}\partial_\mu\varphi\cdot\partial_\nu\varphi+\kappa f_{\mu\nu}\right)}. \quad (5.23)$$

Suppose that a D3-brane is embedded in ten-dimensional target spacetime $\mathcal{M}$ with local coordinates $X^M=(x^\mu,\phi^a)$ whose metric is given by $\mathcal{G}_{MN}(X)$. To be specific, we consider $\mathcal{M}=\mathrm{AdS}_5\times\mathbb{S}^5$ and choose a static gauge for the embedding functions, i.e., $X^M(\sigma)=(x^\mu(\sigma),\phi^a(\sigma))=(\delta^\mu_\alpha\sigma^\alpha, v^a+\frac{\kappa}{R^2}\varphi^a(x))$ where $v^a\equiv\langle\phi^a\rangle_{\mathrm{vac}}$ are vevs of world-volume scalar fields. The fact that the world-volume scalar fields ϕ^a are originated from NC $U(1)$ gauge fields in Eq. (5.3) implies that the vevs $v^a=\langle\phi^a\rangle_{\mathrm{vac}}$ can be identified with the Coulomb branch parameters p_a in Eq. (2.2). Then we see that the symmetric part in Eq. (5.23) is precisely the induced world-volume metric (3.1), i.e.,

$$h_{\mu\nu}=\mathcal{G}_{MN}\partial_\mu X^M\partial_\nu X^N = R^2(v^2\eta_{\mu\nu}+v^{-2}\partial_\mu\phi\cdot\partial_\nu\phi), \quad (5.24)$$

where $\lambda^2=R^2 v\cdot v=R^2/\rho^2$. Therefore, in the approximation considered above, we get the identity

$$\sqrt{-\det{}_{10}(\mathcal{G}+\kappa(B+\tilde{f}))}=\lambda^6\sqrt{-\det{}_4(h+\kappa f)}, \quad (5.25)$$

where the subscript in the determinant indicates the size of matrix. Using the identity (5.25), we can reduce the ten-dimensional DBI action in $\mathrm{AdS}_5\times\mathbb{S}^5$ geometry to a four-dimensional DBI action given by

$$-T_{D9}\int d^{10}Z\sqrt{-\det{}_{10}(\mathcal{G}+\kappa(B+\tilde{f}))} = \left(\frac{g_sN}{4\pi}\right)^{\frac{3}{2}}L(\epsilon,R)\left[-T_{D3}\int_W d^4x\sqrt{-\det{}_4(h+\kappa f)}\right], \quad (5.26)$$

where $(\frac{g_sN}{4\pi})^{\frac{3}{2}}=\frac{T_{D9}R^6}{T_{D3}}\int_{\mathbb{S}^5}\mathrm{vol}(\mathbb{S}^5)$ and

$$L(\epsilon,R)\equiv\int_\epsilon^R\frac{dv}{v}=\ln\frac{R}{\epsilon} \quad (5.27)$$

is a regularized integral along the AdS radius. We identify the DBI action in the bracket in Eq. (5.26) with the world-volume action of a probe D3-brane in $\mathrm{AdS}_5\times\mathbb{S}^5$ geometry. Schwarz speculated in [1] that the probe D3-brane action can be interpreted as the HEA of four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch. We want to emphasize we directly derived the HEA from the four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch although we have not incorporated fermions yet. One caveat is that our HEA is slightly different from Eq. (12) in Ref. [1] where our v^2 was replaced by ϕ^2. But one needs to recall that v^2 is coming from the background geometry and the probe brane approximation involves neglecting the backreaction of the brane on the geometry and other background fields (which requires that N is large). In this description, the $\mathrm{AdS}_5\times S^5$ geometry is regarded as a background, and so it remains to be fixed against the fluctuations of world-volume fields. Thus the ϕ^2 in the denominator in Eq. (12) of Ref. [1] can be replaced by v^2 in the probe-brane approximation.

A demanding task is to understand how to derive the coupling (3.6) of background RR gauge fields from the four-dimensional $\mathcal{N}=4$ superconformal field theory. Actually this issue is closely related to our previous conjecture for a possible realization of D3-branes in terms of NC Hermitian $U(1)$ instantons. Hence we will only draw a plausible picture based on this conjecture. If the conjecture is true, N D3-branes correspond to a stack of N NC Hermitian $U(1)$ instantons at the origin of $\mathbb{R}^6$. Then, this instanton configuration generates a topological invariant given by (up to normalization)

$$I\sim\int_{\mathbb{R}^6}\hat{F}\wedge\hat{F}\wedge\Omega=\int_{\mathbb{S}^5}\left(\hat{A}\wedge\hat{F}-\frac{1}{3}\hat{A}\wedge\hat{A}\wedge\hat{A}\right)\wedge\Omega, \quad (5.28)$$

where Ω is a Kähler form on $\mathbb{R}^6$. The topological invariant I refers to the instanton number N, and so we identify $I=2\pi N$. Since the "instanton flux" is threading $\mathbb{S}^5=\partial\mathbb{R}^6$ and the instanton flux emanating from the origin is regarded as a background field, we make a simple identification for the five-form in Eq. (5.28),

$$\mu_3F_5 := \frac{1}{g_{YM}^2}\left(\hat{A}\wedge\hat{F}-\frac{1}{3}\hat{A}\wedge\hat{A}\wedge\hat{A}\right)\wedge\Omega = \mu_3k_3\mathrm{vol}(\mathbb{S}^5), \quad (5.29)$$

where μ_3 is the basic unit of D3-brane charge and k_3 is a coefficient depending on the normalization convention. In the AdS/CFT correspondence, F_5 is the self-dual RR five-form of N D3-branes given by

$$F_5=k_3(\mathrm{vol}(\mathrm{AdS}_5)+\mathrm{vol}(\mathbb{S}^5))=dC_4. \quad (5.30)$$

Although we do not pin down the origin of the self-duality, the self-duality is necessary for the conjecture to be true because it implies that the topological charge of NC $U(1)$ instantons can be interpreted as the RR charge of D3-branes, i.e.,

$$\mu_3 \int_{\mathbb{S}^5} F_5 = \mu_3 \int_{\mathrm{AdS}_5} dC_4 = \mu_3 \int_W C_4, \tag{5.31}$$

where $W = \partial(\mathrm{AdS}_5)$. Besides the background instanton gauge fields, there exist world-volume $U(1)$ gauge fields, and they can induce a well-known topological instanton coupling given by

$$\frac{\chi}{8\pi} \int_W f \wedge f. \tag{5.32}$$

Combining these two couplings leads to a moderate (if any) suggestion for the Wess-Zumino coupling in Eq. (3.6) given by [1]

$$S_2 = \mu_3 \int_W C_4 + \frac{\chi}{8\pi} \int_W f \wedge f. \tag{5.33}$$

Now we will include the Majorana-Weyl fermion $\hat{\Psi}(Y)$ in the HEA. This means that we are considering a supersymmetric D9-brane that respects the local κ symmetry [29–34]. Thus we use the κ symmetry to eliminate half of the (ψ_1, ψ_2) coordinates where $\psi_{1,2}$ are two Majorana-Weyl spinors of the same chirality. We adopt the gauge choice, $\psi_1 = 0$, used in Refs. [29,30] and rename $\psi_2 \coloneqq \psi$. It was shown in [29,30] that in this gauge the supersymmetric extension of ten-dimensional DBI action has a surprisingly simple form. The supersymmetric case also respects the identity (5.12) with the following replacement:

$$\mathfrak{F}_{MN} \to \mathfrak{F}_{MN} + i\bar{\psi}\Gamma_M \partial_N \psi - \frac{\kappa}{4}\bar{\psi}\Gamma^P \partial_M \psi \bar{\psi}\Gamma_P \partial_N \psi \equiv \mathfrak{F}_{MN} + \Upsilon_{MN}, \tag{5.34}$$

$$\tilde{f}_{MN} \to \tilde{f}_{MN} + i\bar{\psi}\tilde{\Gamma}_M \tilde{\partial}_N \psi - \frac{\kappa}{4}\bar{\psi}\tilde{\Gamma}^P \tilde{\partial}_M \psi \bar{\psi}\tilde{\Gamma}_P \tilde{\partial}_N \psi \equiv \tilde{f}_{MN} + \xi_{MN}, \tag{5.35}$$

where $\tilde{\Gamma}_M = \Gamma_P \frac{\partial X^P}{\partial Z^M}$ and $\tilde{\partial}_M = \frac{\partial}{\partial Z^M}$. Again we can apply the Darboux transformation $\phi_1\colon\ Y^M \mapsto Z^M = \Theta^{MN}(B_{NP}Y^P + \hat{a}_N(Y))$ satisfying $\phi_1^*(B + \tilde{f}) = B$. Then it leads to the following identity:

$$\mathcal{G}_{MN} + \kappa(B + \tilde{f} + \xi)_{MN} = (\mathfrak{G}_{PQ} + \kappa(B + \tilde{\xi})_{PQ}) \frac{\partial Y^P}{\partial Z^M} \frac{\partial Y^Q}{\partial Z^N}, \tag{5.36}$$

where

$$\tilde{\xi}_{MN} = \xi_{PQ} \frac{\partial Z^P}{\partial Y^M} \frac{\partial Z^Q}{\partial Y^N} = \Upsilon_{PQ} \frac{\partial X^P}{\partial Y^M} \frac{\partial X^Q}{\partial Y^N}. \tag{5.37}$$

The above identity (5.36) leads to the following equivalence between DBI actions:

$$\frac{1}{g_s} \int d^{10}Z \sqrt{-\det(\mathcal{G} + \kappa(B + \tilde{f} + \xi))} = \frac{1}{g_s} \int d^{10}Y \sqrt{-\det(\mathfrak{G} + \kappa(B + \tilde{\xi}))}. \tag{5.38}$$

Let us expand the RHS of Eq. (5.38) around the background B field as the bosonic case (4.16),

$$\sqrt{-\det(\mathfrak{G} + \kappa(B + \tilde{\xi}))} = \sqrt{-\det(\kappa B)} \sqrt{\det\left(1 + \frac{M}{\kappa}\right)}, \tag{5.39}$$

where

$$M_N{}^Q = (\mathfrak{G} + \kappa\tilde{\xi})_{NP}\Theta^{PQ} = (g + \kappa\Upsilon)_{RS} \frac{\partial X^R}{\partial Y^N} \frac{\partial X^S}{\partial Y^P} \Theta^{PQ}. \tag{5.40}$$

Note that $\mathrm{Tr}M \neq 0$ unlike the bosonic case. Using the formula $\det(1 + A) = \exp \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k} \mathrm{Tr}A^k$, it is not difficult to show that

$$\det\left(1 + \frac{M}{\kappa}\right) = \det\left(1 + \frac{1}{\kappa}(g + \kappa\Upsilon)\mathfrak{P}\right), \tag{5.41}$$

where

$$(\Upsilon\mathfrak{P})_M{}^N = -i\left(\delta_M^P + \frac{i\kappa}{4}\bar{\psi}\Gamma^P \partial_M \psi\right)\bar{\psi}\Gamma_P \{X^N, \psi\}_\Theta. \tag{5.42}$$

In terms of the matrix notation, the matrix on the RHS of Eq. (5.41) can be read as

$$1+\frac{1}{\kappa}(g+\kappa\Upsilon)\mathfrak{P} = B(1+\kappa G^{-1}(\hat{\mathcal{F}}-B)+\Theta\Upsilon\mathfrak{P}B)\Theta = BG^{-1}(G+\kappa(\hat{\mathcal{F}}-B)+G\Theta\Upsilon\mathfrak{P}B)\Theta, \tag{5.43}$$

where the NC field strengths $\hat{\mathcal{F}}_{MN}$ including an instanton background are given by Eqs. (5.4)–(5.6). Using the result (5.42), one can calculate the fermionic term $G\Theta\Upsilon\mathfrak{P}B = -\kappa^2 Bg^{-1}\Upsilon\mathfrak{P}B$, which takes the form

$$\begin{aligned} &-i\kappa^2(Bg^{-1})_M{}^P\left(\delta_P^Q+\frac{i\kappa}{4}\bar{\psi}\Gamma^Q\partial_P\psi\right)\bar{\psi}\Gamma_Q D_N\psi \\ &\equiv -\kappa^2(Bg^{-1})_M{}^P\hat{\Upsilon}_{PN} \\ &\approx -i\kappa\bar{\psi}\boldsymbol{\Gamma}_M D_N\psi + \mathcal{O}(\kappa^2), \end{aligned} \tag{5.44}$$

where $\boldsymbol{\Gamma}_M \equiv \kappa B_{MN}g^{NP}\Gamma_P$ obey the Dirac algebra $\{\boldsymbol{\Gamma}_M,\boldsymbol{\Gamma}_N\} = 2G_{MN}$ and

$$D_N\psi = \partial\psi/\partial Y^N + \{\hat{A}_N+\hat{a}_N,\psi\}_\Theta. \tag{5.45}$$

In the end, we get the supersymmetric version of Eqs. (5.12) and (5.13),

$$\begin{aligned} &\frac{1}{g_s}\int d^{10}X\sqrt{-\det(g+\kappa(\mathfrak{F}+\Upsilon))} \\ &=\frac{1}{g_s}\int d^{10}Z\sqrt{-\det(\mathcal{G}+\kappa(B+\tilde{f}+\xi))} \end{aligned} \tag{5.46}$$

$$=\frac{1}{G_s}\int d^{10}Y\sqrt{-\det(G+\kappa(\hat{\mathcal{F}}+\Phi)-\kappa^2Bg^{-1}\hat{\Upsilon})}. \tag{5.47}$$

Let us redefine the fermion field, $\Psi \equiv (\kappa T_9)^{\frac{1}{2}}\psi$, and use the approximation (5.44) to take the expansion like Eq. (3.15). With this normalization, we correctly reproduce the action (2.8) at leading orders. As before, we consider the limit $\Theta^{MN} \to (\zeta^{\mu\nu}=0, \theta^{ab}\neq 0)$. Then it is easy to see that, at nontrivial leading orders, Eq. (5.47) reproduces the ten-dimensional $\mathcal{N}=1$ supersymmetric NC $U(1)$ gauge theory (2.8) in the instanton background (5.1). As we demonstrated in Sec. II, the action (2.8) is equivalent to the four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch. And we argued in this section that fluctuations in $AdS_5\times\mathbb{S}^5$ background geometry are described by the ten-dimensional $\mathcal{N}=1$ supersymmetric NC $U(1)$ gauge theory in the background of NC Hermitian $U(1)$ instantons obeying Eq. (4.30). According to our construction, we thus declare that the RHS of Eq. (5.46) has to describe the fluctuations in $AdS_5\times\mathbb{S}^5$ geometry. Therefore we expect that the supersymmetric HEA for the $\mathcal{N}=4$ superconformal field theory on the Coulomb branch would be derived from a dimensional reduction of the RHS of Eq. (5.46) similar to Eq. (5.26).

Before proceeding further, let us first address some subtle issues regarding the equivalence in Eqs. (5.46) and (5.47). The first one is that an interpretation for the factor $(\delta_P^Q+\frac{i\kappa}{4}\bar{\psi}\Gamma^Q\partial_P\psi)$ in $\hat{\Upsilon}_{PN}$ is not clear from the point of view of NC $U(1)$ gauge theory. Note that $\partial_P\psi = \partial\psi/\partial X^P$ and the Darboux transformations did not touch the factor. Hence this factor behaves like a background part induced from the backreaction of fermions at higher orders. Therefore a plausible picture from the viewpoint of NC $U(1)$ gauge fields is to interpret this factor as vielbeins $\mathfrak{E}_M^A = (\delta_M^A - \frac{i\kappa}{4}\bar{\psi}\Gamma^A\partial_M\psi)$ with an effective metric $\mathfrak{G}_{MN} = \mathfrak{E}_M^A\mathfrak{E}_N^B g_{AB}$ and write

$$\kappa^2(Bg^{-1})_M^P\hat{\Upsilon}_{PN} = i\kappa\bar{\psi}\mathfrak{T}_M D_N\psi, \tag{5.48}$$

where

$$\mathfrak{T}_M \equiv \kappa B_{MN}g^{NP}\mathfrak{E}_P^A\Gamma_A. \tag{5.49}$$

Then the gamma matrices $\mathfrak{T}_M$ satisfy the Dirac algebra

$$\{\mathfrak{T}_M,\mathfrak{T}_N\} = -2\kappa^2(Bg^{-1}\mathfrak{G}g^{-1}B)_{MN} \equiv 2\mathbb{G}_{MN}. \tag{5.50}$$

Of course, if we ignore the backreaction from the fermions, we recover the previous Dirac term (5.44) in flat spacetime. Another issue is how to glue local Darboux charts now involved with fermions as well as bosons. We argued before that the global metric (4.24) can be constructed via the globalization in terms of the gluing of local Darboux charts described by Eqs. (4.25) and (4.26). Or the local frames in the metric (5.11) are replaced by global vielbeins [12],

$$\frac{\partial X^A}{\partial Z^M} \to E_M^A. \tag{5.51}$$

Then the gamma matrices in Eq. (5.35) will also be replaced by $\Gamma_M \equiv E_M^A\Gamma_A$ and $\Gamma^M \equiv E_A^M\Gamma^A$.[5] Now it is also necessary to glue the fermions defined on local Darboux patches by local Lorentz transformations

$$\psi^{(j)} = S_{(ji)}\psi^{(i)} \tag{5.52}$$

acting on fermions on an intersection $U_i\cap U_j$. As usual, we introduce a spin connection $\omega_M = \frac{1}{2}\omega_{MAB}\Gamma^{AB}$ to covariantize the local gluing (5.52). This means that the fermionic terms in Eq. (5.52) are now given by

$$\xi_{MN} \to i\bar{\psi}E_M^A\Gamma_A\nabla_N\psi - \frac{\kappa}{4}\bar{\psi}\Gamma^A\nabla_M\psi\bar{\psi}\Gamma_A\nabla_N\psi, \tag{5.53}$$

where the covariant derivative is defined by

[5] They should not be confused with the gamma matrices in Eq. (5.34) that are defined on the flat spacetime $\mathbb{R}^{9,1}$ while those in Eq. (5.35) are now defined on a curved spacetime.

$$\nabla_M\psi = (\partial_M + \omega_M)\psi. \tag{5.54}$$

The spin connections ω_M are determined by the metric (5.18).

Therefore the block matrix (5.21) for the supersymmetric case is replaced by

$$\mathcal{G}_{MN} + \kappa(B + \tilde{f} + \xi)_{MN} \approx \begin{pmatrix} \lambda^2\eta_{\mu\nu} + \kappa(f_{\mu\nu} + \xi_{\mu\nu}) & \kappa(\partial_\mu\varphi_a + \xi_{\mu a}) \\ -\kappa(\partial_\mu\varphi_a - \xi_{a\mu}) & \lambda^2\delta_{ab} + \kappa(B_{ab} + \xi_{ab}) \end{pmatrix}. \tag{5.55}$$

Since we are interested in the HEA of the four-dimensional supersymmetric gauge theory defined on the boundary of AdS_5 space, the dimensional reduction similar to Eq. (5.20) was adopted too for fermionic excitations, i.e.,

$$\xi_{\mu\nu} = i\bar{\psi}\Gamma_\mu\nabla_\nu\psi, \qquad \xi_{ab} = i\bar{\psi}\Gamma_a\omega_b\psi,$$
$$\xi_{\mu a} = i\bar{\psi}\Gamma_\mu\omega_a\psi, \qquad \xi_{a\mu} = i\bar{\psi}\Gamma_a\nabla_\mu\psi, \tag{5.56}$$

where $\Gamma_M = E_M^A\Gamma_A$ and we ignored the quartic term in Eq. (5.53). To get a four-dimensional picture after the dimensional reduction (5.26), it is convenient to decompose the 16 components of the Majorana-Weyl spinor ψ into the four Majorana-Weyl gauginos $\lambda^i (i = 1, \ldots, 4)$ as follows:

$$\psi = \begin{pmatrix} P_+\lambda^i \\ P_-\tilde{\lambda}_i \end{pmatrix} \quad \text{with} \quad P_\pm = \frac{1}{2}(I_4 \pm \gamma_5) \quad \text{and} \quad \tilde{\lambda}_i = -C\bar{\lambda}^{iT}, \qquad \Gamma^A = (\gamma^{\hat{\mu}} \otimes I_8, \gamma_5 \otimes \gamma^{\hat{a}}), \qquad \Gamma_{11} = \gamma_5 \otimes I_8, \tag{5.57}$$

where C is the four-dimensional charge conjugation operator and the hat is used to indicate tangent space indices. We take the four- and six-dimensional Dirac matrices in the chiral representation

$$\gamma^{\hat{\mu}} = \begin{pmatrix} 0 & i\sigma^{\hat{\mu}} \\ -i\bar{\sigma}^{\hat{\mu}} & 0 \end{pmatrix}, \qquad \sigma^{\hat{\mu}} = (I_2, \vec{\sigma}) = (\sigma^{\hat{\mu}})_{\alpha\dot{\beta}}, \qquad \bar{\sigma}^{\hat{\mu}} = (-I_2, \vec{\sigma}) = (\bar{\sigma}^{\hat{\mu}})^{\dot{\alpha}\beta}, \tag{5.58}$$

$$\gamma^{\hat{a}} = \begin{pmatrix} 0 & \Sigma^{\hat{a}} \\ \bar{\Sigma}^{\hat{a}} & 0 \end{pmatrix}, \qquad \Sigma^{\hat{a}} = (\vec{\eta}, i\vec{\bar{\eta}}) = \Sigma^{\hat{a},ij}, \qquad \bar{\Sigma}^{\hat{a}} = (\Sigma^{\hat{a}})^\dagger = (-\vec{\eta}, i\vec{\bar{\eta}}) = \bar{\Sigma}^{\hat{a}}_{ij}, \tag{5.59}$$

where $\vec{\sigma}$ are Pauli matrices and the 4×4 matrices $(\vec{\eta}, \vec{\bar{\eta}})$ are self-dual and anti–self-dual 't Hooft symbols. Then the fermion bilinear terms in Eq. (5.56) read as

$$\xi_{\mu\nu} = iv^{-1}(\bar{\lambda}_i\bar{\sigma}_{\hat{\mu}}\nabla_\nu\lambda^i - \lambda^i\sigma_{\hat{\mu}}\nabla_\nu\bar{\lambda}_i),$$
$$\xi_{ab} = \partial_c v^{-1}(\bar{\lambda}\Sigma_{\hat{a}}\bar{\Sigma}_{\hat{b}\hat{c}}\bar{\lambda} - \lambda\bar{\Sigma}_{\hat{a}}\Sigma_{\hat{b}\hat{c}}\lambda),$$
$$\xi_{\mu a} = 2i\partial_b v^{-1}(\bar{\lambda}\bar{\sigma}_{\hat{\mu}}\Sigma_{\hat{a}\hat{b}}\lambda),$$
$$\xi_{a\mu} = v^{-1}(\bar{\lambda}\Sigma_{\hat{a}}\nabla_\mu\bar{\lambda} - \lambda\bar{\Sigma}_{\hat{a}}\nabla_\mu\lambda), \tag{5.60}$$

where

$$\bar{\Sigma}^{\hat{a}\hat{b}} \equiv \frac{1}{2}(\bar{\Sigma}^{\hat{a}}\Sigma^{\hat{b}} - \bar{\Sigma}^{\hat{b}}\Sigma^{\hat{a}}), \qquad \Sigma^{\hat{a}\hat{b}} \equiv \frac{1}{2}(\Sigma^{\hat{a}}\bar{\Sigma}^{\hat{b}} - \Sigma^{\hat{b}}\bar{\Sigma}^{\hat{a}}), \tag{5.61}$$

and the spin connection for the background geometry (4.27) is given by

$$\omega_\mu = -\Gamma^{\hat{\mu}\hat{a}}\partial_a \ln v, \qquad \omega_a = -\Gamma^{\hat{a}\hat{b}}\partial_b \ln v. \tag{5.62}$$

Since we are considering the HEA of the four-dimensional supersymmetric gauge theory defined on the boundary of the AdS_5 space where $v \to \infty$ and the $\mathbb{S}^5$ shrinks to a point, we can ignore ξ_{ab} and $\xi_{\mu a}$ in Eq. (5.60) as well as the spin connections $\omega_M \to 0$.

After applying the formula (5.22) to the matrix (5.55), it is straightforward to yield the supersymmetric completion of the bosonic HEA obtained in Eq. (5.26), and it is given by

$$\sqrt{-\det(\mathcal{G} + \kappa(B + \tilde{f} + \xi))} = \sqrt{\det(\lambda^2 + \kappa B)}\sqrt{-\det\left(\lambda^2\eta_{\mu\nu} + \kappa(f_{\mu\nu} + \xi_{\mu\nu}) + \kappa^2\partial_\mu\varphi_a\left(\frac{1}{\lambda^2 + \kappa B}\right)^{ab}(\partial_\nu\varphi_b - \xi_{b\nu})\right)}$$
$$\approx \lambda^6\sqrt{-\det\left(h_{\mu\nu} + \kappa\left(f_{\mu\nu} + \xi_{\mu\nu} - v^{-2}\partial_\mu\phi^a\xi_{a\nu}\right)\right)}. \tag{5.63}$$

One may drop the last term since it is of $\mathcal{O}(v^{-3})$. As the bosonic case (5.26), the ten-dimensional supersymmetric DBI action (5.46) in $\mathrm{AdS}_5 \times \mathbb{S}^5$ geometry is thus reduced to a four-dimensional supersymmetric DBI action given by

$$\begin{aligned} &-T_{D9}\int d^{10}Z\sqrt{-\det{}_{10}(\mathcal{G}+\kappa(B+\tilde{f}+\xi))} \\ &\quad=\left(\frac{g_s N}{4\pi}\right)^{\frac{3}{2}} L(\epsilon,R)\left[-T_{D3}\int_W d^4x\sqrt{-\det{}_4(h_{\mu\nu}+\kappa(f_{\mu\nu}+\xi_{\mu\nu}-v^{-2}\partial_\mu\phi^a\xi_{a\nu}))}\right]. \end{aligned} \tag{5.64}$$

If the quartic term in Eq. (5.53) is included, it contributes an extra term given by $\frac{\kappa^2 v^2}{4}(\xi_{\lambda\mu}\xi^\lambda{}_\nu + \xi_{a\mu}\xi^a{}_\nu)$ inside the determinant. Since the metric (5.19) becomes flat when $v=1$, the result in this case should be equal to the action of a supersymmetric D3-brane. One can see that the action (5.64) is actually the case. See Eq. (88) in Ref. [30]. According to the identity (5.46), the left-hand side of Eq. (5.64) is equal to the world-volume action of a Bogomol'nyi-Prasad-Sommerfield D9-brane of type IIB string theory after fixing the κ symmetry, which is invariant under the supersymmetry transformations given by Eqs. (90) and (91) in Ref. [30]. Since Eq. (5.46) is a mathematical identity, the action on the left-hand side of Eq. (5.64) will also be supersymmetric. Its supersymmetry transformations basically take the form replacing the ordinary derivatives in Eqs. (90) and (91) in Ref. [30] by covariant derivatives on the $\mathrm{AdS}_5 \times \mathbb{S}^5$ space. But an explicit check of supersymmetry is somewhat lengthy though straightforward. Its detailed exposition from the perspective of HEA deserves to pursue a separate work, which will be reported elsewhere. Note that, after the gauge fixing, $\psi_1 = 0$, for the κ symmetry, the Wess-Zumino term for the supersymmetric case is the same as the bosonic one (5.33) [30]. The final result can be interpreted as the world-volume action of a supersymmetric probe D3-brane in the $\mathrm{AdS}_5 \times \mathbb{S}^5$ background geometry. According to the conjecture in Ref. [1], it can be reinterpreted as the HEA of four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch. We emphasize that we directly derived the HEA from the four-dimensional $\mathcal{N}=4$ superconformal field theory on the Coulomb branch defined by the NC space (2.3).

VI. DISCUSSION

We want to emphasize that NC spacetime should be regarded as a more fundamental concept from which classical spacetime should be derived as quantum mechanics is a more fundamental theory and the classical phenomena are emergent from quantum physics. Then the NC spacetime requires us to take a radical departure from the 20th century physics. First of all, it introduces a new kind of duality, known as the gauge/gravity duality, as formalized by the identity (4.20). But we have to recall that quantum mechanics has already illustrated such a kind of novel duality where the NC phase space obeying the commutation relation $[x^i, p_j] = i\hbar\delta^i_j$ is responsible for the so-called wave-particle duality. Remarkably there exists a novel form of the equivalence principle stating that the electromagnetic force can always be eliminated by a local coordinate transformation as far as spacetime admits a symplectic structure. The novel equivalence principle is nothing but the famous mathematical theorem known as the Darboux theorem or the Moser lemma in symplectic geometry [41,42]. It proves the equivalence principle for the gravitational force in the context of emergent gravity. Therefore we may conclude [12,16] that the NC nature of spacetime is the origin of the gauge/gravity duality and the first principle for the duality is the equivalence principle for the electromagnetic force.

The AdS/CFT correspondence [3,26,27] is a well-tested gauge/gravity duality and a typical example of emergent gravity and emergent space. But we do not understand yet why the duality should work. We argued that the AdS/CFT correspondence is a particular example of emergent gravity from NC $U(1)$ gauge fields and the duality between large N gauge fields and a higher-dimensional gravity is simply a consequence of the novel equivalence principle for the electromagnetic force. We note [12,16] that the emergent gravity from NC $U(1)$ gauge fields is an inevitable conclusion as far as spacetime admits a symplectic structure; in other words, a microscopic spacetime becomes NC. Moreover, the emergent gravity is much more general than the AdS/CFT correspondence because it holds for general background spacetimes as exemplified by the identity (5.17). Therefore we believe that the emergent gravity from NC gauge fields provides a lucid avenue to understand the gauge/gravity duality or large N duality.

For example, it is interesting to notice that the transformation (4.20) between NC $U(1)$ gauge fields and an emergent gravitational metric holds even locally. Thus one may imagine an (infinitesimal) open patch U where the field strength F_U of fluctuating $U(1)$ gauge fields has a maximal rank such that (U, F_U) is a symplectic Darboux chart. Then one can apply the Darboux theorem on the local patch to transform the local $U(1)$ gauge fields into a corresponding local spacetime geometry supported on U. But this local geometry is unfledged yet to be materialized into a classical spacetime geometry. Hence this kind of immature geometry describes a bubbling geometry or spacetime foams that intrinsically correspond to a quantum geometry. Even we may consider fluctuating $U(1)$ gauge fields on a local patch U whose field strengths F_U do not support the maximal rank. The dimension of emergent bubbling geometry will be determined by the rank of F_U on U. This implies that the dimension of quantum geometries is not fixed but fluctuates. This picture is in a sense well-known folklore in quantum gravity.

Then one may raise a question why NC spacetime reproduces all the results in string theory. The connection between string theory and symplectic geometry becomes most manifest by Gromov's J-holomorphic curves. See Sec. 7 in Ref. [12] for this discussion. The J-holomorphic curve for a given symplectic structure is nothing but the minimal world sheet in string theory embedded in a target spacetime. Moreover, α' corrections in string theory correspond to derivative corrections in NC gauge theory. In this sense the string theory can be regarded as a stringy realization of symplectic geometry or more generally Poisson geometry. But the NC spacetime provides a more elegant framework for the background independent formulation of quantum gravity in terms of matrix models [16,28], which is still elusive in string theory.

We showed that the world-volume effective action of a supersymmetric probe D3-brane in $\mathrm{AdS}_5 \times \mathbb{S}^5$ geometry can be directly derived from the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on the Coulomb branch defined by the NC space (1.7). Since our result, for example, described by the identity (5.17) should be true for general $U(1)$ gauge fields in an arbitrary background geometry, the remaining problem is to identify a corresponding dual (super)gravity whose solution coincides with the emergent metric $\mathfrak{G}_{MN}$. One may use the method in Refs. [48,49] to attack this problem. See also [50]. It was shown there that the world-volume effective action of a probe D3-brane is a solution to the Hamilton-Jacobi equation of type IIB supergravity defined by the Arnowitt-Deser-Misner formalism adopting the radial coordinate as time for type IIB supergravity reduced on $\mathbb{S}^5$. In particular, the radial time corresponds to the vev of the Higgs field in the dual Yang-Mills theory as our case. It will be interesting to find the relation between the DBI action obtained in Refs. [48,49] and the HEA derived in this paper. Also there are several works [7–11] to address the relation of the HEA with the low-energy effective actions of $\mathcal{N} = 4$ super Yang-Mills theory on the Coulomb branch. Thus it may be a vital project to understand any relation between our approach based on the Coulomb branch defined by the NC space and other approaches for the HEA cited above.

Recently there have been some developments [51,52] that describe D-branes in the framework of generalized geometry. A D-brane including fluctuations in a static gauge is identified with a leaf of foliations generated by the Dirac structure of a generalized tangent bundle, and the scalar fields and vector fields on the D-brane are unified as a generalized connection [51]. It was also argued in [52] that the equivalence between commutative and NC DBI actions is naturally encoded in the generalized geometry of D-branes. In particular, when considering a D-brane as a symplectic leaf of the Poisson structure, describing the noncommutativity, the SW map is naturally interpreted in terms of the corresponding Dirac structure. Thus NC gauge theories can be naturally interpreted within the generalized geometry. Since the Darboux transformation relating the deformation of a symplectic structure with diffeomorphism symmetry is one of the pillars for emergent gravity, we think that the emergent gravity from NC gauge fields can be formulated in a natural way within the framework of generalized geometry. It will be interesting to inquire further into this idea.

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APPENDIX: KÄHLER MANIFOLDS FROM $U(1)$ GAUGE FIELDS

In this appendix we will illustrate how to determine four- and six-dimensional Kähler metrics from $U(1)$ gauge fields by solving the identities (4.14) and (4.15) between DBI actions. For this purpose, let us introduce $d = 2n$-dimensional complex coordinates

$$z^i = x^{2i-1} + ix^{2i}, \qquad \bar{z}^i = x^{2i-1} - ix^{2i}, \qquad i = 1, \ldots, n, \tag{A1}$$

and corresponding complex $U(1)$ gauge fields

$$A_i = \frac{1}{2}(A_{2i-1} - iA_{2i}), \qquad \bar{A}_{\bar{i}} = \frac{1}{2}(A_{2i-1} + iA_{2i}). \tag{A2}$$

Then the field strengths of (2,0) and (1,1) parts are, respectively, given by

$$F_{ij} = \frac{1}{4}(F_{2i-1,2j-1} - F_{2i,2j}) - \frac{i}{4}(F_{2i-1,2j} + F_{2i,2j-1}), \tag{A3}$$

$$F_{i\bar{j}} = \frac{1}{4}(F_{2i-1,2j-1} + F_{2i,2j}) + \frac{i}{4}(F_{2i-1,2j} - F_{2i,2j-1}). \tag{A4}$$

If $U(1)$ gauge fields in Eq. (A2) are the connection of a holomorphic vector bundle, i.e., $F_{ij} = F_{\bar{i}\bar{j}} = 0$, Eq. (A3) leads to the following relations:

$$F_{2i-1,2j-1} = F_{2i,2j}, \qquad F_{2i-1,2j} = -F_{2i,2j-1}, \quad i,j = 1,\ldots,n. \tag{A5}$$

The connections of a holomorphic line bundle can be obtained by solving the condition $F_{ij} = F_{\bar{i}\bar{j}} = 0$, and they are given by

$$A_i = -i\frac{\partial\phi(z,\bar{z})}{\partial z^i} \coloneqq -i\partial_i\phi(z,\bar{z}), \qquad \bar{A}_{\bar{i}} = i\frac{\partial\phi(z,\bar{z})}{\partial\bar{z}^i} = i\bar{\partial}_{\bar{i}}\phi(z,\bar{z}), \tag{A6}$$

where $\phi(z,\bar{z})$ is a real smooth function on $\mathbb{C}^n$. Then the (1,1) field strength (A4) is given by

$$F_{i\bar{j}} = 2i\partial_i\bar{\partial}_{\bar{j}}\phi(z,\bar{z}). \tag{A7}$$

Similarly the condition for a Hermitian metric, i.e., $\mathcal{G}_{ij} = \mathcal{G}_{\bar{i}\bar{j}} = 0$, can be solved by

$$\mathcal{G}_{2i-1,2j-1} = \mathcal{G}_{2i,2j}, \qquad \mathcal{G}_{2i-1,2j} = -\mathcal{G}_{2i,2j-1}. \tag{A8}$$

If we further impose the Kähler condition, $d\Omega = 0$, for the Hermitian metric $ds^2 = \mathcal{G}_{i\bar{j}}dz^i d\bar{z}^j$ where $\Omega = i\mathcal{G}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a Kähler form, the metric is solely determined by a Kähler potential $K(z,\bar{z})$ as

$$\mathcal{G}_{i\bar{j}} = \partial_i\bar{\partial}_{\bar{j}}(2K(z,\bar{z}) - K_0), \tag{A9}$$

where $K_0 = \bar{z}^k z^k$ and our choice of Kähler potential is just for a later convenience.

To deduce Kähler metrics from $U(1)$ gauge fields obeying Eqs. (4.14) and (4.15), let us take their local form given by

$$\sqrt{\det(g + \kappa\mathcal{F})} = \sqrt{\det(\mathcal{G} + \kappa B)} \tag{A10}$$

$$= \frac{g_s}{G_s}\sqrt{\det(G + \kappa(\hat{F} - B))}. \tag{A11}$$

For our case at hand, $g_{\mu\nu} = G_{\mu\nu} = \delta_{\mu\nu}, \mu,\nu = 1,\ldots,d = 2n$, and $B_{\mu\nu} = -\frac{2}{\kappa}\mathbf{1}_n \otimes i\sigma^2$ in Eqs. (A10) and (A11). We will choose the same complex structure as (A1) for all DBI densities in Eqs. (A10) and (A11). In terms of complex coordinates, their nonvanishing components are given by $g_{i\bar{j}} = G_{i\bar{j}} = \delta_{i\bar{j}}$ and $B_{i\bar{j}} = -\frac{i}{\kappa}\delta_{i\bar{j}}$ for $i,j = 1,\ldots,n$. Thus they are Kähler metrics and a Kähler form on $\mathbb{C}^n$, i.e., $g_{i\bar{j}} = G_{i\bar{j}} = \partial_i\bar{\partial}_{\bar{j}}K_0$ and $B_{i\bar{j}} = -\frac{i}{\kappa}\partial_i\bar{\partial}_{\bar{j}}K_0$ with $K_0 = \bar{z}^k z^k$, respectively. However, the RHS of Eq. (A10) needs some care since $\mathcal{G}_{\mu\nu}(x)$ is regarded as a nontrivial metric on a Riemannian manifold. For this case, it is convenient to distinguish local coordinate indices $(\mu,\nu,\ldots)$ from tangent space indices $(a,b,\ldots)$ by introducing vielbeins E^a_μ, i.e., $E^a_\mu E^a_\nu = \mathcal{G}_{\mu\nu}$. Let us split both coordinate indices into holomorphic and antiholomorphic ones: $\mu = (\alpha,\bar{\alpha})$, $\nu = (\beta,\bar{\beta})$, $a = (i,\bar{i})$, $b = (j,\bar{j})$. The Hermitian condition (A8) can be solved by taking the vielbeins as

$$E^i_{\bar{\alpha}} = E^{\bar{i}}_\alpha = 0, \qquad E^{\bar{\alpha}}_i = E^\alpha_{\bar{i}} = 0. \tag{A12}$$

Then the nonvanishing components of the B field in Eq. (A10) are given by $B_{i\bar{i}} = E^\alpha_i E^{\bar{\beta}}_{\bar{i}} B_{\alpha\bar{\beta}}$ where $B_{\alpha\bar{\beta}} = -i\delta_{\alpha\bar{\beta}}$.

Our primary concern is to find $U(1)$ gauge fields that give rise to the Kähler metric (A9). This means that the RHS of Eq. (A10) is purely of (1,1) type. Therefore, to satisfy Eq. (A10), the $U(1)$ gauge fields on the LHS must be connections of a holomorphic line bundle obeying $F_{ij} = F_{\bar{i}\bar{j}} = 0$. Moreover, $\mathcal{F} = \mathcal{F}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a nondegenerate, closed (1,1) form, and so a Kähler form, i.e.,[6]

$$\mathcal{F} = i\partial_i\bar{\partial}_{\bar{j}}(2\phi(z,\bar{z}) - K_0)dz^i \wedge d\bar{z}^j, \tag{A13}$$

because B is a symplectic two-form and F in Eq. (A7) satisfies the Bianchi identity, $dF = 0$. By the same reasoning, we have to impose a similar condition $\hat{F}_{ij} = \hat{F}_{\bar{i}\bar{j}} = 0$ for symplectic $U(1)$ gauge fields in Eq. (A11). This condition is equivalent to Eq. (A5) and replaced F by $\hat{F}$. Before proceeding to the particular dimensions we are interested in, let us first discuss general properties of the above determinant equation. Suppose that S and A are $d \times d$ symmetric and antisymmetric matrices, respectively. Then we have the relation

$$P(S,A) \equiv \det(S + A) = \det(S - A) = (-1)^d \det(-S + A). \tag{A14}$$

This means that the polynomial $P(S,A)$ has only even powers in A, or equivalently, only even (odd) powers of S appear in $P(S,A)$ for $d =$ even (odd). When S is a Hermitian metric $\mathfrak{H}$ on an n-dimensional (i.e., $d = 2n$) complex manifold M, there is a remarkable property. As we noticed above, the DBI densities in Eqs. (A10) and (A11) are involved only with (1,1)-type quantities when we restrict ourselves to the Kähler metric (A9). The polynomial $P(\mathfrak{G},A)$ can then be written as the form

$$\det(\mathfrak{G}_{\mu\nu} + A_{\mu\nu}) = |\det(\mathfrak{G}_{\alpha\bar{\beta}} + A_{\alpha\bar{\beta}})|^2, \tag{A15}$$

where $\mathfrak{G}_{\alpha\bar{\beta}} + A_{\alpha\bar{\beta}}$ is an $n \times n$ complex matrix.

The proof goes as follows. Take the LHS of Eq. (A15) as the form, $\det(\mathfrak{G} + A) = \det\mathfrak{G}\det(1 + M)$ where $M^\mu{}_\nu = \mathfrak{G}^{\mu\lambda}A_{\lambda\nu}$. Because of the Hermiticity property of $\mathfrak{G}$ and A, we have the following split:

[6] Note that $F_{i\bar{j}}$ alone in Eq. (A7) cannot be a Kähler form because it becomes degenerate, e.g., at an asymptotic infinity. This is a reason why the symplectic B field is necessary to attain a Kähler form.

$$M^{\mu}{}_{\nu} = \begin{cases} \mathfrak{m}^{\alpha}{}_{\beta} \equiv \mathfrak{G}^{\alpha\bar{\gamma}} A_{\bar{\gamma}\beta}, & \mu = \alpha, \\ \bar{\mathfrak{m}}^{\bar{\alpha}}{}_{\bar{\beta}} \equiv \mathfrak{G}^{\bar{\alpha}\gamma} A_{\gamma\bar{\beta}}, & \mu = \bar{\alpha}, \end{cases} \tag{A16}$$

where $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ are now regarded as $n \times n$ matrices. A critical step is to use the determinant formula, $\det(1+M) = \exp \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k} \mathrm{Tr} M^k$. Then the split (A16) induces the same split for the trace,

$$\mathrm{Tr}_{2n} M^k = \mathrm{Tr}_n \mathfrak{m}^k + \mathrm{Tr}_n \bar{\mathfrak{m}}^k, \tag{A17}$$

where the subscript in the trace denotes the size of the matrix. Therefore we get the result

$$\det(1+M) = \det(1+\mathfrak{m}) \det(1+\bar{\mathfrak{m}}). \tag{A18}$$

Similarly the formula, $\det \mathfrak{G} = \exp \mathrm{Tr} \ln \mathfrak{G}$, leads to the result, $\det \mathfrak{G}_{\mu\nu} = \det \mathfrak{G}_{\alpha\bar{\beta}} \det \mathfrak{G}_{\bar{\alpha}\beta}$. Combining all together, we finally get the formula (A15).

There is another interesting representation of the determinant (A14) that was used to formulate the kappa symmetry of supersymmetric D-branes [29–34]. The polynomial $P(\mathfrak{G}, A)$ can be written as the form

$$\det(\mathfrak{G} + A) = \rho_{\mathfrak{G}}(A)^{\dagger} \rho_{\mathfrak{G}}(A), \tag{A19}$$

where

$$\rho_{\mathfrak{G}}(A) = \sum_{l=0}^{[\frac{d}{2}]} \frac{1}{2^l l! (d-2l)!} A_{\mu_1\mu_2} \cdots A_{\mu_{2l-1}\mu_{2l}} \gamma_{\mu_{2l+1}\cdots\mu_d} \varepsilon^{\mu_1\cdots\mu_d}. \tag{A20}$$

Here γ matrices on M are defined as usual as $\gamma_\mu = E^a_\mu \gamma_a$ and the γ matrices γ_a obey the Dirac algebra $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ on flat space. For the proof of Eq. (A19), see, in particular, Appendix A in Ref. [30] and Appendix B in Ref. [31]. See also [53] [Eq. (2.18)]. It is convenient to introduce the skew-exponential function [34] (the usual exponential function with completely skew-symmetrized indices of gamma matrices at every order in the expansion)

$$\mathrm{se}^{-\mathbb{A}} = \sum_{l=0}^{[\frac{d}{2}]} \frac{(-1)^l}{2^l l!} \gamma^{\mu_1\cdots\mu_{2l}} A_{\mu_1\mu_2} \cdots A_{\mu_{2l-1}\mu_{2l}} \tag{A21}$$

to rewrite $\rho_{\mathfrak{G}}(A)$ as

$$\rho_{\mathfrak{G}}(A) = \mathrm{se}^{-\mathbb{A}} \Gamma_{\mathfrak{G}}, \tag{A22}$$

where $\mathbb{A} \equiv \frac{1}{2} \gamma^{\mu\nu} A_{\mu\nu}$ and

$$\Gamma_{\mathfrak{G}} = \varepsilon^{\mu_1\cdots\mu_d} \gamma_{\mu_1\cdots\mu_d} = (-i)^{\frac{d(d-1)}{2}} \sqrt{\det \mathfrak{G}} \gamma_{d+1}. \tag{A23}$$

Using the formula (A22), we get the skew exponentials for each DBI density,

$$\rho_g(\mathcal{F}) = (-i)^{\frac{d(d-1)}{2}} \mathrm{se}^{-\gamma^{i\bar{j}} \mathcal{F}_{i\bar{j}}} \gamma_{d+1}, \tag{A24}$$

$$\rho_{\mathcal{G}}(B) = (-i)^{\frac{d(d-1)}{2}} \sqrt{\det \mathcal{G}} \mathrm{se}^{-\gamma^{\alpha\bar{\alpha}} B_{\alpha\bar{\alpha}}} \gamma_{d+1}, \tag{A25}$$

$$\rho_G(\hat{\mathcal{F}}) = (-i)^{\frac{d(d-1)}{2}} \mathrm{se}^{-\gamma^{i\bar{j}} \hat{\mathcal{F}}_{i\bar{j}}} \gamma_{d+1}, \tag{A26}$$

where $\hat{\mathcal{F}} \equiv \hat{F} - B$ and $\gamma_{d+1}^2 = 1$. We set $\kappa = 1$ for convenience.

Note that, using the results (A9) and (A13), we get the expression

$$g_{i\bar{j}} + \mathcal{F}_{i\bar{j}} = i\partial_i \bar{\partial}_{\bar{j}} (2\phi - K_0 - iK_0), \tag{A27}$$

$$\mathcal{G}_{i\bar{j}} + B_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} (2K - K_0 - iK_0), \tag{A28}$$

where we did not discriminate curved and flat space indices because it is no longer necessary. Now, using the relation (A15), we can phrase the equivalence (A10) in terms of Kähler potentials (up to holomorphic gauge transformations),

$$\phi(z, \bar{z}) = K(z, \bar{z}). \tag{A29}$$

The real function $\phi(z, \bar{z})$ and so the Kähler potential $K(z, \bar{z})$ will be determined by solving the equations of motion of either commutative or NC $U(1)$ gauge fields. We remark that the relation (A29) is completely consistent with that in Ref. [54] [see Eqs. (30) and (31)] for the equivalence between hyper-Kähler manifolds and symplectic $U(1)$ instantons. (See also [55].) Therefore the relation (A29) generalizes the one in [54,55] to general $2n$-dimensional Kähler manifolds. Recall that the Ricci tensor and the Ricci form for a $2n$-dimensional Kähler manifold are given by

$$R_{i\bar{j}} = -\frac{\partial^2 \ln \det \mathfrak{G}_{k\bar{l}}}{\partial z^i \partial \bar{z}^j}, \qquad \rho = -i\partial\bar{\partial} \ln \det \mathfrak{G}_{i\bar{j}}, \tag{A30}$$

respectively. In particular, the Ricci tensor (A30) vanishes if $\det \mathfrak{G}_{i\bar{j}}$ is constant, and so the Kähler manifold reduces to a $2n$-dimensional Calabi-Yau manifold. Hence we can translate the statement for Kähler manifolds into that for $U(1)$ gauge theory and vice versa using the relation (A29). For example, one may wonder what is the gauge theory object that gives rise to the $2n$-dimensional Calabi-Yau manifold. It was verified in [54,55] for the four-dimensional case that it is the commutative limit of NC $U(1)$ instantons [56]. Later it was conjectured in [12] that Calabi-Yau threefolds arise from a semiclassical limit of NC Hermitian $U(1)$ instantons in six dimensions.

Now we will show that the conjecture in [12] is true. First we will illustrate our method with the four-dimensional case since this case was well established in [54,55]. Then we will generalize our approach to the six-dimensional case. Consider four-dimensional symplectic $U(1)$ instantons as the commutative limit of NC $U(1)$ instantons [56] obeying the self-duality equations

$$\hat{F}_{\mu\nu} = \pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}\hat{F}_{\rho\sigma} \tag{A31}$$

or in a compact notation

$$P_{\mp}\hat{\mathbb{F}} = 0, \tag{A32}$$

where $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$ and $\hat{\mathbb{F}} = \frac{1}{2}\gamma^{\mu\nu}\hat{F}_{\mu\nu}$. In terms of complex coordinates (A1), the self-duality equations (A31) can be stated as[7]

$$\hat{F}_{ij} = \hat{F}_{\bar{i}\bar{j}} = 0, \tag{A33}$$

$$\hat{F}_{i\bar{i}} = 0. \tag{A34}$$

To see what kind of condition the instanton equations (A33) and (A34) impose on the Kähler metric $\mathcal{G}_{i\bar{j}}$, let us apply the SW map (4.10) to them. An important part is to note that $\theta^{i\bar{j}} = -i\delta^{i\bar{j}}$ or $\theta^{2i-1,2j} = \frac{1}{2}\delta^{ij}$ due to the relation $B_{\mu\lambda}\theta^{\lambda\nu} = \delta_{\mu}^{\nu}$. Then it is easy to see that Eq. (A33) can be solved by $F_{ij} = F_{\bar{i}\bar{j}} = 0$ for which $N_{\mu}{}^{\nu} \equiv \delta_{\mu}^{\nu} + F_{\mu\lambda}\theta^{\lambda\nu}$ is split into holomorphic and antiholomorphic parts such as Eq. (A16). In particular, $N_i{}^j = \delta_i^j + F_{i\bar{k}}\theta^{\bar{k}j} = \delta_{i\bar{j}} + iF_{i\bar{j}} = -\partial_i\bar{\partial}_{\bar{j}}(2\phi - K_0) = -\mathcal{G}_{i\bar{j}}$ where Eqs. (A7) and (A29) were used. Then we can easily solve Eq. (A34),

$$\begin{aligned}\hat{F}_{i\bar{i}} &= (N^{-1})_i{}^k F_{k\bar{i}} = -i(N^{-1})_i{}^k(N_k{}^i - \delta_k^i) \\ &= -i(2 - \mathrm{Tr}N^{-1}) = 0.\end{aligned} \tag{A35}$$

Using the relation $\mathrm{Tr}N^{-1} = \mathrm{Tr}N/\det N$, we get $\mathrm{Tr}N = 2\det N$. Motivated by this relation, we define a new matrix $\mathfrak{G}$ as $N = \frac{1}{2}(1 + \mathfrak{G})$ so that $\det\mathfrak{G}_{i\bar{j}} = 1$. In consequence, the Kähler metric $\mathfrak{G}_{i\bar{j}}$ is Ricci flat because of the formula (A30). In other words, the four-manifold described by the metric $\mathfrak{G}_{i\bar{j}}$ is a hyper-Kähler manifold or a Calabi-Yau twofold. In the end we have checked the equivalence between symplectic $U(1)$ instantons and Calabi-Yau twofolds in [54,55].[8]

Now we consider the six-dimensional case. The analysis is almost the same as the four-dimensional case. We consider six-dimensional symplectic $U(1)$ instantons satisfying the Hermitian Yang-Mills equations [47]

$$\hat{F}_{\mu\nu} = -\frac{1}{4}\varepsilon_{\mu\nu}{}^{\rho\sigma\alpha\beta}\hat{F}_{\rho\sigma}I_{\alpha\beta}, \tag{A36}$$

where $I = \mathbf{1}_3 \otimes i\sigma^2$ is a complex structure of $\mathbb{R}^6$. They can be written with the complex coordinates (A1), and the result takes the same form as Eqs. (A33) and (A34). The same argument shows that Eq. (A33) can be solved by $F_{ij} = F_{\bar{i}\bar{j}} = 0$ and Eq. (A34) leads to the result $\hat{F}_{i\bar{i}} = (N^{-1})_i{}^k F_{k\bar{i}} = -i(3 - \mathrm{Tr}N^{-1}) = 0$, i.e., $\mathrm{Tr}N^{-1} = 3$. The trace of 3×3 complex matrix N^{-1} is given by

$$\begin{aligned}\det N\mathrm{Tr}N^{-1} &= N_{1\bar{1}}N_{2\bar{2}} + N_{2\bar{2}}N_{3\bar{3}} + N_{3\bar{3}}N_{1\bar{1}} \\ &\quad - (N_{1\bar{2}}N_{2\bar{1}} + N_{2\bar{3}}N_{3\bar{2}} + N_{3\bar{1}}N_{1\bar{3}}).\end{aligned} \tag{A37}$$

By a similar reasoning to the four-dimensional case, we introduce a new metric $\mathfrak{G}$ defined by $N = \frac{1}{3}(1 + \mathfrak{G})$. A straightforward calculation shows that $\mathrm{Tr}N^{-1} = 3$ can be written as the form

$$\det\mathfrak{G} = 2 + \mathrm{Tr}\mathfrak{G}. \tag{A38}$$

Note that $\varphi \equiv i\mathfrak{G}_{i\bar{j}}dz^i \wedge d\bar{z}^j$ is a closed two-form of type (1,1), and so we may assume, up to an addition of an exact two-form, that φ is harmonic. And the trace $\mathrm{Tr}\mathfrak{G}$ is equal to the contraction of φ with the Kähler form $\omega \equiv \frac{1}{2}I_{\mu\nu}dx^{\mu} \wedge dx^{\nu}$, i.e., $\mathrm{Tr}\mathfrak{G} = (\varphi, \omega)$. Since φ is a harmonic (1,1)-form, its trace $\mathrm{Tr}\mathfrak{G}$ is then constant [57] (see 2.33). In consequence, the six-manifold described by the metric $\mathfrak{G}_{i\bar{j}}$ is a Ricci flat and Kähler manifold, i.e., a Calabi-Yau threefold. Therefore we confirm the conjecture in [12] for the equivalence between Hermitian $U(1)$ instantons and Calabi-Yau threefolds.

To check our conjecture for the $\mathrm{AdS}_5 \times \mathbb{S}^5$ geometry, it is necessary to sum up the stack of Hermitian $U(1)$ instantons obeying (A36). This may be a challenging problem, and we do not know yet how to sum up the lump of infinitely many Hermitian $U(1)$ instantons near the origin of $\mathbb{R}^6$. We leave this problem and an explicit construction of emergent Kähler metrics for future works.

[7]The complex structure in Eq. (A33) is correlated with the self-dual structure in Eq. (A31). In this appendix we will fix the complex structure with the coordinates (A1). Instead, we will flip the orientation for the definition of the self-duality equations (A31), e.g., $\varepsilon^{12\cdots(2n)(2n-1)} = 1$ for the self-dual case and $\varepsilon^{12\cdots(2n-1)2n} = 1$ for the anti–self-dual case.

[8]Note that we are solving the determinant equations (A10) and (A11), and so $\mathcal{G}_{i\bar{j}} = -N_{i\bar{j}}$ leads to the relation $\mathcal{G}_{\mu\nu} = \frac{1}{2}(\delta_{\mu\nu} + \mathfrak{G}_{\mu\nu})$ according to the formula (A15), which was used in [54,55] to identify a gravitational metric $\mathfrak{G}_{\mu\nu}$ from the emergent metric $\mathcal{G}_{\mu\nu}$ determined by $U(1)$ gauge fields.

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