

**Dirac spectra of two-dimensional QCD-like theories**Mario Kieburg,<sup>1,2,\*</sup> Jacobus J. M. Verbaarschot,<sup>1,†</sup> and Savvas Zafeiropoulos<sup>1,3,‡</sup><sup>1</sup>*Department of Physics and Astronomy, Stony Brook University, Stony Brook, New York 11794-3800, USA*<sup>2</sup>*Fakultät für Physik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany*<sup>3</sup>*Laboratoire de Physique Corpusculaire, Université Blaise Pascal, CNRS/IN2P3, 63177 Aubière Cedex, France*

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We analyze Dirac spectra of two-dimensional QCD-like theories both in the continuum and on the lattice and classify them according to random matrix theories sharing the same global symmetries. The classification is different from QCD in four dimensions because the antiunitary symmetries do not commute with  $\gamma_5$ . Therefore, in a chiral basis, the number of degrees of freedom per matrix element are not given by the Dyson index. Our predictions are confirmed by Dirac spectra from quenched lattice simulations for QCD with two or three colors with quarks in the fundamental representation as well as in the adjoint representation. The universality class of the spectra depends on the parity of the number of lattice points in each direction. Our results show an agreement with random matrix theory that is qualitatively similar to the agreement found for QCD in four dimensions. We discuss the implications for the Mermin-Wagner-Coleman theorem and put our results in the context of two-dimensional disordered systems.

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**I. INTRODUCTION**

It has been well established that chiral symmetry is spontaneously broken in strongly interacting systems of quarks and gluons for a wide range of parameters such as the temperature, the chemical potential, the number of colors, the number of flavors and the representation of the gauge group. In the broken phase the corresponding low energy effective theory is given by a weakly interacting system of pseudo-Goldstone bosons with a Lagrangian that is determined by the pattern of chiral symmetry breaking. In lattice QCD the spontaneous breaking of chiral symmetry is studied by evaluating the Euclidean partition function which is the average of the determinant of the Euclidean Dirac operator weighted by the Euclidean Yang-Mills action. Its low energy limit is given by the partition function of the Euclidean chiral Lagrangian. This theory simplifies drastically [1,2] in the limit where the pion Compton wave-length is much larger than the size of the box. Then the partition function factorizes into a part comprising the modes with zero momentum and a part describing the modes with nonzero momentum. It turns out that the zero momentum part is equivalent to a random matrix theory with the global symmetries of QCD [3].

A particular useful way to study chiral symmetry breaking is to analyze the properties of the eigenvalues of the Dirac operator. Because of the Banks-Casher formula [4] the chiral condensate  $\Sigma = |\langle \bar{\psi}\psi \rangle|$ , the order parameter for

the spontaneous breaking of chiral symmetry, is given by the average spectral density (denoted by  $\rho(\lambda)$ ) near zero of the Dirac operator per unit of the space-time volume  $V$ ,

$$\begin{aligned} \Sigma &\equiv |\langle \bar{\psi}\psi \rangle| = \lim_{a \rightarrow 0} \lim_{m \rightarrow 0} \lim_{V \rightarrow \infty} \frac{1}{V} \int_{-\infty}^{\infty} \frac{2m\rho(\lambda)d\lambda}{\lambda^2 + m^2} \\ &= \lim_{a \rightarrow 0} \lim_{\lambda \rightarrow 0} \lim_{V \rightarrow \infty} \frac{\pi}{V} \rho(\lambda). \end{aligned} \quad (1)$$

Here,  $a$  is the lattice spacing which provides the ultraviolet cut-off. The order of the limits is critical and a different order gives a different result. A better understanding of these limits can be obtained from the behavior of the eigenvalue density of the Dirac operator on the scale of the smallest eigenvalues which according to the Banks-Casher formula is given by

$$\Delta\lambda = \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V}. \quad (2)$$

The so-called microscopic spectral density is defined by [5]

$$\rho_s(x) = \lim_{V \rightarrow \infty} \frac{1}{\Sigma V} \rho\left(\frac{x}{\Sigma V}\right). \quad (3)$$

If the Compton wavelength associated with the Dirac eigenvalues,  $\lambda$ , is much larger than the size of the box,  $L$ , then the partition function that generates the Dirac spectrum factorizes into a zero momentum part and a nonzero momentum part. The zero momentum part is completely determined by the global symmetries of the QCD-(like) partition function and is equivalent to a random matrix

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theory with the same global symmetries. The Compton wave length associated with the Dirac eigenvalues is the Compton wave-length of the corresponding pseudo-Goldstone modes and is given by

$$\frac{2\pi}{m_\pi} = \frac{2\pi F_\pi}{\sqrt{2\lambda\Sigma}}, \quad (4)$$

where  $F_\pi$  is the pion decay constant. The condition  $2\pi/m_\pi \gg L$  implies

$$\lambda \ll \frac{4\pi^2 F_\pi^2}{2L^2 \Sigma} = \lambda_L, \quad (5)$$

meaning that  $\lambda_L$  is the characteristic eigenvalue scale corresponding to the size of the box. In  $d$  dimensions the Euclidean volume is  $V = L^d$  so that the average number of eigenvalues in the universal domain scales as

$$E_{\text{Th}} \propto \frac{\lambda_L}{\Delta\lambda} = 2\pi F_\pi^2 L^{d-2}. \quad (6)$$

This scale is also known as the Thouless energy  $E_{\text{Th}}$ . In two dimensions, the number of eigenvalues in the universal domain remains of  $O(1)$  in the thermodynamic limit.

Arguments have been made that in one-dimensional systems all states become localized for an arbitrary small amount of disorder. The two-dimensional case is marginal. For site disorder all states are exponentially localized whereas for link disorder the situation is less clear [6]. For systems that are both rotational invariant and time reversal invariant (denoted by the Dyson index  $\beta_D = 1$ ) all states seem to be localized. In the case of rotational invariant systems with broken time reversal invariance (denoted by the Dyson index  $\beta_D = 2$ ) states in the center of the band seem to be delocalized and the localization length may be very large in a region around the band center. For nonrotational invariant spin 1/2 systems (denoted by the Dyson index  $\beta_D = 4$ ) states are delocalized for a substantial range of disorder and energies [7–9].

The connection between localization and Goldstone bosons was most clearly formulated by McKane and Stone [10]. They argued that a nonzero density of states around the origin for a disordered system may either indicate the presence of Goldstone bosons or may be due to a nonzero density of the localized states. For dynamical quarks the second alternative is not possible. The reason is that the eigenvalues of localized states are uncorrelated so that the partition function

$$Z(m) = \left\langle \prod_k (i\lambda_k + m) \right\rangle \quad (7)$$

factorizes into single eigenvalue partition functions resulting in a vanishing chiral condensate [11].

If we take the results from the condensed matter literature at face value, also in two-dimensional systems, there may be a finite region of extended states around zero with correlations that are described by chiral random matrix theory or alternatively a partition function with spontaneously broken chiral symmetry. In more than two dimensions we expect that these correlations will remain in the presence of a fermion determinant and the corresponding partition function will be the zero momentum part of a chiral Lagrangian. In two dimensions, the presence of a fermion determinant may push these states beyond the Thouless energy so that *all* states become localized. This would reconcile the numerical results for  $\beta_D = 4$  with the Mermin-Wagner-Coleman theorem which states that a continuous symmetry cannot be broken spontaneously in two or less dimensions in systems with sufficiently short-range interactions. In terms of the supersymmetric formulation of the quenched limit, the Mermin-Wagner-Coleman theorem could be evaded because the symmetry group is noncompact. This has been shown for hyperbolic spin models [12,13]. This opens the possibility to have extended states and universal spectral correlations also in two dimensions.

Another interpretation is possible if the localization length  $\xi$  is large so that we can consider the limiting case

$$1 \ll L \ll \xi. \quad (8)$$

Then the states behave as extended states with eigenvalues that are described by random matrix theory up to the Thouless energy. In this case the problems with the Mermin-Wagner-Coleman theorem can be avoided and a transition to a localized phase only takes place when  $L \sim \xi$ . In such scenario the scalar correlation function may drop off at a similar rate, so that chiral symmetry appears to be broken in the regime (8). This correlation function was studied for the  $N_c$ -color Thirring model [14] with a drop-off of  $1/x^{1/N_c}$ .

A two-dimensional model for which Dirac spectra have been studied in great detail, both analytically and numerically, is the Schwinger model. The eigenvalue correlations of the one-flavor Schwinger model are given by random matrix theory as was shown numerically [15,16] and analytically by calculating the Leutwyler-Smilga sum rules [17]. The two flavor Schwinger model was analyzed in great detail in [18], and after rescaling the eigenvalues by the average level spacing excellent agreement with chiral random matrix theory is observed. No agreement with chiral random matrix theory is found for the quenched Schwinger model [16], while the spectral density seems to diverge for  $\lambda \rightarrow 0$ . The repulsion between the eigenvalues seems to be greatly suppressed indicating that the states are localized.

Let us consider a theory where the mass dependent chiral condensate scales with the quark mass as

$$\Sigma(m) \sim m^\alpha. \quad (9)$$

In the Schwinger model we have that [19]

$$\alpha = \frac{N_f - 1}{N_f + 1}, \quad (10)$$

but the argument given in this paragraph is more general. According to the Gell-Mann-Oakes-Renner relation the mass of the “pions” associated with this condensate is given by

$$m_\pi^2 \sim m^{\alpha+1}. \quad (11)$$

Using the relation (1) between the spectral density and the chiral condensate we find that the eigenvalue density behaves as

$$\rho(\lambda) \sim V\Sigma(\lambda) \sim V\lambda^\alpha. \quad (12)$$

The Thouless energy is given by the scale for which the pion Compton wavelength is equal to the size of the box, i.e.  $m_\pi \propto 1/L$ . Employing the relation (11) we find the mass associated to the Thouless energy,

$$m_{\text{th}} \sim L^{-2/(\alpha+1)}. \quad (13)$$

The integrated spectral density is given by

$$N(\lambda) = \int_{-\lambda}^{\lambda} \rho(\lambda') d\lambda' \sim V\lambda^{\alpha+1}, \quad (14)$$

so that the average number of eigenvalues below the Thouless energy is proportional to

$$N_{\text{th}} = N(m_{\text{th}}) \sim L^{d-2}. \quad (15)$$

due to combination of Eqs. (13) and (14). Remarkably, the number of eigenvalues described by random matrix theory does not depend on  $\alpha$ . In two dimensions this number is constant in the thermodynamic limit but the agreement with chiral random matrix theory seems to improve with larger volumes for the Schwinger model [16]. The corollary of this argument is that correlations of low-lying Dirac eigenvalues in conformal QCD-like theories are given by chiral random matrix theory after unfolding the eigenvalues, i.e.  $\lambda'_k = \lambda_k^{\alpha+1}$ .

The eigenvalues scale with the volume as

$$\lambda \sim V^{-1/(\alpha+1)} \quad (16)$$

via the relation (14) when keeping the average number of eigenvalues fixed. This scaling was studied in [18] where a volume scaling of  $V^{-5/8}$  is observed for two almost massless flavors, c.f. Eq. (10). This would correspond to  $N_f = 4$ , cf. Eq. (10). This is actually correct because the lattice Dirac operator couples only even and odd sites

doubling the number of flavors. Apparently, we need exact massless quarks to push the states in the localized domain.

Another important difference between QCD in four dimensions and QCD in lower dimensions is the index of the Dirac operator. In four dimensions the index is equal to the topological charge of the gauge field configurations. In three dimensions the index is not defined. In two dimensions topology is defined for U(1) and can for example be studied for the Schwinger model [17,20]. However, for higher-dimensional gauge groups the index of the Dirac operator is zero [19,21,22] although unstable instantons do exist [23,24].

In this paper we consider the quenched two-dimensional QCD Dirac operator in the strong coupling limit with the gauge fields distributed according to the Haar measure. Both the continuum limit and the lattice QCD Dirac operator will be discussed. For the lattice Dirac operator we employ naive fermions. Our original motivation for this choice was to understand the transition between different symmetry classes when taking the continuum limit which was observed for staggered fermions in three [25] as well as in four [26] dimensions, but this issue is not addressed in this paper.

The strong coupling lattice model is expected to be equivalent to an interacting theory of mesons and/or baryons. For U(1) gauge theories in two dimensions this has been shown explicitly [27] by means of a color-flavor transformation [28–30], where a gradient expansion generates the various terms of a chiral Lagrangian. In this paper, we do not perform the continuum limit, so that the lattice theory is equivalent to an unrenormalized chiral Lagrangian, and the usual arguments, that the states below the Thouless energy are correlated according to random matrix theory, apply. In the continuum limit and two dimensions, the fluctuations of the hadronic fields will dominate the chiral condensate and the theory renormalizes to a trivial phase without Goldstone bosons. In other words, the theory renormalizes to a localized phase.

The symmetry breaking pattern for the continuum limit in any dimension was discussed in [31] and in the context of topological insulators in [32] and goes back to what is known as Bott-periodicity. The dimensional dependence of the symmetry breaking pattern (see Table I) has its origins in the structure of Clifford algebras. The four-dimensional symmetry breaking pattern and its description in terms of random matrix theory has been known for a long time [33–35]. Because of the absence of the  $\gamma_5$  Dirac matrix in three dimensions the symmetry breaking pattern is different [36–38]. In two dimensions, the  $\gamma_5$  matrix is replaced by the third Pauli matrix,  $\sigma_3$ . This matrix also anticommutes with the Dirac operator, but it does not commute with the charge conjugation matrix given by  $\sigma_2$ , which leads to a different symmetry breaking pattern [31].

TABLE I. Symmetry breaking patterns in two ( $d = 2$ ), three ( $d = 3$ ), and four ( $d = 4$ ) dimensions for different gauge theories and their associated Dyson index  $\beta_D$  which is equal to the level repulsion. The corresponding random matrix theory sharing the same symmetry breaking pattern and its classification according to symmetric spaces is indicated in the last column. The repulsion of the levels from the origin,  $\lambda^{\alpha_D}$ , depends on the topological charge  $\nu$  for QCD-like theories in four dimensions. The case of the two-dimensional  $SU(N_c)$  theory with the fermions in the adjoint representation is particular since the index of the Dirac operator is either 0 or 1 depending on the parity of the dimensions of the Dirac matrix. This results in a repulsion that is either  $\lambda$  or  $\lambda^5$ . The corresponding random matrix theory consists of antisymmetric off-diagonal blocks so that depending on the dimensionality we have either no or one pair of generic zero modes, respectively. For a discussion of the classification of random matrix theories in terms of symmetric spaces we refer to Refs. [39,40]. In this table we do not include the breaking of the axial symmetry.

d	Gauge theory	$\beta_D$	$\alpha_D$	Symmetry Breaking Pattern	RMT
2	$N_c = 2$ , fund.	1	1	$USp(2N_f) \times USp(2N_f) \rightarrow USp(2N_f)$	(CI)
2	$N_c \geq 3$ , fund.	2	1	$U(N_f) \times U(N_f) \rightarrow U(N_f)$	chGUE (AIII)
2	$N_c \geq 2$ , adj.	4	1,5	$O(2N_f) \times O(2N_f) \rightarrow O(2N_f)$	(DIII)
3	$N_c = 2$ , fund.	1	0	$USp(4N_f) \rightarrow USp(2N_f) \times USp(2N_f)$	GOE (AI)
3	$N_c \geq 3$ , fund.	2	0	$U(2N_f) \rightarrow U(N_f) \times U(N_f)$	GUE (A)
3	$N_c \geq 2$ , adj.	4	0	$O(2N_f) \rightarrow O(N_f) \times O(N_f)$	GSE (AII)
4	$N_c = 2$ , fund.	1	$\nu$	$U(2N_f) \rightarrow USp(2N_f)$	chGOE (BDI)
4	$N_c \geq 3$ , fund.	2	$2\nu + 1$	$U(N_f) \times U(N_f) \rightarrow U(N_f)$	chGUE (AIII)
4	$N_c \geq 2$ , adj.	4	$4\nu + 3$	$U(2N_f) \rightarrow O(2N_f)$	chGSE (CII)

The symmetries of the Dirac operator also depend on the parity of the lattice. If the lattice size is even in both directions, even lattice sites are only coupled to odd lattice sites resulting in a ‘‘lattice chiral symmetry’’. Having a lattice that is odd in one direction and even in the other one also puts global constraints on the Dirac operator resulting in different symmetry properties. In this paper we classify lattice theories in terms of random matrix theory. In total we can distinguish 9 classes. First of all they differ in their antiunitary symmetries, namely with no antiunitary symmetry, with an antiunitary symmetry that squares to 1, and with an antiunitary symmetry that squares to  $-1$ . Moreover for each of these three classes we can have an even-even, an even-odd or an odd-odd lattice. For all nine classes we give the spectral properties in the microscopic domain and compare them with lattice simulations of the corresponding lattice theory in the strong coupling limit.

In Sec. II we discuss the microscopic Dirac spectrum and chiral symmetry breaking pattern for the continuum limit of two-dimensional QCD. The two-dimensional lattice gauge theory for three different values of the Dyson index is analyzed in Sec. III, and concluding remarks are made in Sec. IV. In the appendices we derive several random matrix results that have been used in the main text. Part of this work was presented at the Lattice 2013 conference [41].

## II. CONTINUUM DIRAC OPERATOR

The Euclidean Dirac operator of QCD-like theories is given by

$$\mathcal{D} = \gamma^\mu (\partial_\mu + iA_\mu^a \lambda_a), \quad (17)$$

where  $A_\mu^a$  are the gauge fields,  $\gamma_\mu$  are the Euclidean  $\gamma$ -matrices and  $\lambda_a$  are the generators of the gauge group. For

an even number of dimensions the Dirac operator in a chiral basis reduces to a  $2 \times 2$  block structure

$$\mathcal{D}^{(2/4)} = \begin{bmatrix} 0 & \mathcal{W}^{(2/4)} \\ -\mathcal{W}^{(2/4)\dagger} & 0 \end{bmatrix}, \quad (18)$$

where in two dimensions the operator  $\mathcal{W}^{(2)}$  is given by

$$\mathcal{W}^{(2)} = \partial_1 + i\partial_2 + (iA_1^a - A_2^a)\lambda_a, \quad (19)$$

and in four dimensions the operator  $\mathcal{W}^{(4)}$  can be written as

$$\mathcal{W}^{(4)} = i\sigma_\mu (\partial_\mu + iA_\mu^a \lambda_a) \quad (20)$$

employing the standard chiral representation of Euclidean  $\gamma$ -matrices with  $\sigma_\mu = (\sigma_k, -i1_4)$  and  $\sigma_k$  the Pauli matrices.

In three dimensions, the Dirac operator is given by

$$\mathcal{D} = \sum_{k=1}^3 \sigma^k (\partial_k + iA_k^a \lambda_a). \quad (21)$$

There is no involution that anticommutes with the Dirac operator so that there is no chiral block structure. This is the crucial difference with even-dimensional theories and was already studied in Ref. [42].

In addition to chiral symmetry the Dirac operator has other symmetries depending on the representation of the gauge group which is discussed in the ensuing subsections. In subsection II A we recall the discussion of the global symmetries of the QCD Dirac operator in three and four dimensions and extend it to the two-dimensional theory as well. This symmetry classification is summarized in Table I. In subsection II B we discuss the corresponding random matrix theories. Thereby we summarize the



classification of the random matrix theories for three- and four-dimensional continuum QCD and supplement this with the random matrix theories for two-dimensional QCD. In subsection II C we recall the symmetry breaking patterns.

### A. Antiunitary symmetries of the QCD Dirac operator

The antiunitary symmetries of the Dirac operator depend on the representation of the generators  $\lambda_a$  of the gauge group  $SU(N_c)$ . We consider three different gauge theories, namely with the gauge group  $SU(N_c = 2)$  and fermions in the fundamental representation denoted by the Dyson index  $\beta_D = 1$ , with the gauge group  $SU(N_c > 2)$  and fermions in the fundamental representation which is  $\beta_D = 2$ , and with the gauge group  $SU(N_c \geq 2)$  and fermions in the adjoint representation labelled by  $\beta_D = 4$ .

#### I. $\beta_D = 1$

Let us consider the first case which is QCD with two colors ( $N_c = 2$ ) and fermions in the fundamental representation. Then the  $\lambda_a$  are given by the three Pauli matrices  $\tau_a$  acting in color space. Hence each covariant derivative

$$\mathcal{D}_\mu = \partial_\mu + iA_\mu^a \tau_a \quad (22)$$

is pseudoreal (quaternion) and anti-Hermitian, i.e.

$$\mathcal{D}_\mu^\dagger = -\mathcal{D}_\mu \quad \text{and} \quad [\mathcal{D}_\mu, \tau_2 K]_- = \mathcal{D}_\mu \tau_2 K - \tau_2 K \mathcal{D}_\mu = 0 \quad (23)$$

with  $K$  the complex conjugation operator. The corresponding Dirac operator has the antiunitary symmetry

$$[i\mathcal{D}^{(d)}, \tau_2 C K]_- = 0, \quad (24)$$

where  $C$  is the charge conjugation matrix. In four dimensions the charge conjugation matrix reads  $C \equiv \gamma_2 \gamma_4$  and in two and three dimensions it is given by  $C = \sigma_2$ .

A crucial point is that the antiunitary operator satisfies

$$(C\tau_2 K)^2 = 1. \quad (25)$$

Therefore one can always find a gauge field independent basis for which the Dirac operator is real [35,39]. This is the reason why this case is denoted by the Dyson index  $\beta_D = 1$  (one degree of freedom per matrix element). Collecting everything, the continuum Euclidean QCD Dirac operator for QCD with two fundamental fermions fulfills three global symmetries in four and two dimensions namely anti-Hermiticity, chiral symmetry, and a reality condition, i.e.

$$\mathcal{D}^{(4)\dagger} = -\mathcal{D}^{(4)}, \quad [\mathcal{D}^{(4)}, \gamma_5]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(4)}, \tau_2 \gamma_2 \gamma_4 K]_- = 0 \quad (26)$$

for four dimensions, see Ref. [35], and

$$\mathcal{D}^{(2)\dagger} = -\mathcal{D}^{(2)}, \quad [\mathcal{D}^{(2)}, \sigma_3]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(2)}, \tau_2 \sigma_2 K]_- = 0 \quad (27)$$

for two dimensions. For three dimensions there is no chiral symmetry but the rest remains the same as in the even-dimensional case

$$\mathcal{D}^{(3)\dagger} = -\mathcal{D}^{(3)} \quad \text{and} \quad [i\mathcal{D}^{(3)}, \tau_2 \sigma_2 K]_- = 0, \quad (28)$$

see Ref. [42]. Next we discuss the implications of these symmetries.

In four dimensions, Eq. (26) implies that we can construct a gauge field independent basis for which the Dirac operator decomposes into a chiral block structure or a basis for which the Dirac operator becomes real. This can be done at the same time if the projection onto a chiral basis commutes with the antiunitary symmetry. This is the case in four dimensions where

$$\left[ \frac{1 \pm \gamma_5}{2}, \tau_2 \gamma_2 \gamma_4 K \right]_- = 0. \quad (29)$$

The corresponding random matrix ensemble is the chiral Gaussian orthogonal ensemble (chGOE), see Refs. [35].

Equation (29) does not carry over to the two-dimensional theory. In this case the projectors onto a chiral basis are given by  $(1 \pm \sigma_3)/2$ , playing the role of  $(1 \pm \gamma_5)/2$ , but the commutator with the antiunitary operator does not vanish,

$$\left[ \frac{1 \pm \sigma_3}{2}, \tau_2 \sigma_2 K \right]_- \neq 0. \quad (30)$$

Therefore, one cannot find a basis for which the two-dimensional Dirac operator decomposes into real chiral blocks.

Choosing a chiral basis for  $\mathcal{D}^{(2)}$  the antiunitary symmetry yields a different condition

$$\left[ \left( \begin{array}{cc} 0 & i\tau_2 K \\ -i\tau_2 K & 0 \end{array} \right), \left( \begin{array}{cc} 0 & i\mathcal{W}^{(2)} \\ -i\mathcal{W}^{(2)\dagger} & 0 \end{array} \right) \right]_- = 0, \quad (31)$$

which is equivalent to

$$\mathcal{W}^{(2)} = -\tau_2 \mathcal{W}^{(2)T} \tau_2. \quad (32)$$

Thus the operator is anti-self-dual and complex since we have no additional symmetries. After a unitary transformation one obtains an equivalent Dirac operator with an off-diagonal block  $\tau_2 \mathcal{W}^{(2)}$  which is complex symmetric. The corresponding random matrix is known as the first Bogolyubov-de Gennes ensemble denoted by the Cartan symbol CI, see Ref. [40], and has been applied to the normal-superconducting transitions in mesoscopic physics [43].

In three dimensions we can construct a gauge field independent basis for which the matrix elements of the operator  $i\mathcal{D}^{(3)}$  become real symmetric. The corresponding random matrix ensemble is the Gaussian orthogonal ensemble (GOE), see Ref. [42].

### 2. $\beta_D = 2$

In the case of three or more colors ( $N_c \geq 3$ ) with the fermions in the fundamental representation the symmetry under complex conjugation (23) is lost. Only anti-Hermiticity and, for even dimensions, chiral symmetry survive. The global symmetries of the Dirac operator are

$$\mathcal{D}^{(2)\dagger} = -\mathcal{D}^{(2)} \quad \text{and} \quad [\mathcal{D}^{(2)}, \sigma_3]_+ = 0 \quad (33)$$

in two dimensions and

$$\mathcal{D}^{(4)\dagger} = -\mathcal{D}^{(4)} \quad \text{and} \quad [\mathcal{D}^{(4)}, \gamma_5]_+ = 0 \quad (34)$$

in four dimensions. Since there are no antiunitary symmetries the operator  $\mathcal{W}^{(2/4)}$  is generically complex both in two and four dimensions. This is the reason why we denote this case by the Dyson index  $\beta_D = 2$ . Therefore the random matrix ensemble corresponding to the Dirac operator  $\mathcal{D}^{(2)}$  as well as  $\mathcal{D}^{(4)}$  is given by an ensemble of chiral, complex, anti-Hermitian random matrices which can be chosen with Gaussian weights. This ensemble is known as the chiral Gaussian Unitary Ensemble (chGUE), see Ref. [35].

In three dimensions we only have the anti-Hermiticity condition,

$$\mathcal{D}^{(3)\dagger} = -\mathcal{D}^{(3)}. \quad (35)$$

Hence the operator  $i\mathcal{D}^{(3)}$  is Hermitian and its analogue in random matrix theory is the Gaussian Unitary Ensemble (GUE). The three-dimensional case was discussed in Ref. [42] and its predictions for the microscopic Dirac spectrum have been confirmed by various lattice simulations [25].

### 3. $\beta_D = 4$

The third case is for fermions in the adjoint representation with two or more colors ( $N_c \geq 2$ ). In this case the generators of the gauge group are antisymmetric and purely imaginary. This results in two symmetry relations for the covariant derivatives

$$\mathcal{D}_\mu^\dagger = -\mathcal{D}_\mu \quad \text{and} \quad [K, i\mathcal{D}_\mu]_- = 0. \quad (36)$$

The corresponding Dirac operator fulfills the antiunitary symmetry

$$[i\mathcal{D}^{(d)}, CK]_- = 0, \quad (37)$$

where the antiunitary operator satisfies

$$(CK)^2 = -1 \quad (38)$$

for all dimensions. This allows us to construct a gauge field independent basis for which the matrix elements of the Dirac operator can be grouped into real quaternions. This case is denoted by the Dyson index  $\beta_D = 4$ .

Collecting all global symmetries of the Dirac operator we have

$$\mathcal{D}^{(4)\dagger} = -\mathcal{D}^{(4)}, \quad [\mathcal{D}^{(4)}, \gamma_5]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(4)}, \gamma_2 \gamma_4 K]_- = 0 \quad (39)$$

for four dimensions,

$$\mathcal{D}^{(2)\dagger} = -\mathcal{D}^{(2)}, \quad [\mathcal{D}^{(2)}, \sigma_3]_+ = 0, \quad \text{and} \quad [i\mathcal{D}^{(2)}, \sigma_2 K]_- = 0 \quad (40)$$

for two dimensions, and

$$\mathcal{D}^{(3)\dagger} = -\mathcal{D}^{(3)} \quad \text{and} \quad [i\mathcal{D}^{(3)}, \sigma_2 K]_- = 0 \quad (41)$$

for three dimensions. The last case is the simplest. There is no chiral symmetry, but we can construct a basis for which the matrix elements of the Hermitian operator  $i\mathcal{D}^{(3)}$  can be grouped into real quaternions. The associated random matrix ensemble is the Gaussian Symplectic Ensemble (GSE) pointed out for the first time in Ref. [42].

In two and four dimensions we have again to consider the commutator of the projection operators onto the eigenspaces of  $\gamma_5$  and the antiunitary operator. As is the case for  $\beta_D = 1$ , the commutator vanishes in four dimensions,

$$\left[ \frac{1 \pm \gamma_5}{2}, \gamma_2 \gamma_4 K \right]_- = 0. \quad (42)$$

Therefore we can construct a basis for which  $\mathcal{D}^{(4)}$  decomposes into chiral blocks with quaternion real elements. Such Dirac operators are in the universality class of the chiral Gaussian Symplectic Ensemble (chGSE), see Ref. [35].

In two dimensions the commutator of the antiunitary symmetry and the chiral projector does not vanish, i.e.

$$\left[ \frac{1 \pm \sigma_3}{2}, \sigma_2 K \right]_- \neq 0. \quad (43)$$

Therefore, there is no gauge field independent basis for which  $\mathcal{D}^{(2)}$  decomposes into quaternion real chiral blocks. Nevertheless, we can find a basis for which one of these properties holds. In a chiral basis the antiunitary symmetry (37) reads

$$\left[ \left( \begin{array}{cc} 0 & iK \\ -iK & 0 \end{array} \right), \left( \begin{array}{cc} 0 & i\mathcal{W}^{(2)} \\ -i\mathcal{W}^{(2)\dagger} & 0 \end{array} \right) \right]_- = 0 \quad (44)$$

and results into

$$\mathcal{W}^{(2)T} = -\mathcal{W}^{(2)}. \quad (45)$$

Thus the operator  $\mathcal{W}^{(2)}$ , see Eq. (19), is complex anti-symmetric. In random matrix theory this symmetry class is known as the second Bogolyubov-de Gennes ensemble denoted by the Cartan symbol DIII [40]. This ensemble also plays an important role in mesoscopic physics [43].

### B. Random matrix theory for continuum QCD

As was outlined in Refs. [5,35] a random matrix theory for the Dirac operator is obtained by replacing its matrix elements by random numbers while maintaining the global unitary and antiunitary symmetries of the QCD(-like) theory. Within a wide class, the distribution of the eigenvalues on the scale of the average level spacing does not depend on the probability distribution of the matrix elements. This allows us to choose the probability distribution to be Gaussian. The random matrix partition function is thus given by

$$Z_{N_f}^\nu = \int d[D] \exp \left[ -\frac{n\beta_D}{2} \text{tr} D^\dagger D \right] \prod_{k=1}^{N_f} \det(D + m_k \mathbb{1}). \quad (46)$$

In even dimensions, in particular for  $d = 2, 4$ , the Dirac operator has the chiral block structure

$$D = \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}, \quad (47)$$

while in three dimensions the Dirac operator is still anti-Hermitian but the block structure is absent. The mass matrix for the  $N_f$  quarks is given by  $M = \text{diag}(m_1, \dots, m_{N_f})$ . The measure  $d[D]$  is the product of all real independent differentials of the matrix elements of  $D$ .

In three dimensions, the random matrix ensemble is  $n \times n$ -dimensional for  $\beta_D = 1, 2$  and  $2n \times 2n$ -dimensional for  $\beta_D = 4$ . The random matrix  $iD$  is either real symmetric ( $\beta_D = 1$ ), Hermitian ( $\beta_D = 2$ ), or Hermitian self-dual ( $\beta_D = 4$ ). From the corresponding joint probability density of the eigenvalues [44],

$$p_{d=3}(\Lambda) \prod_{1 \leq j \leq n} d\lambda_j \propto |\Delta_n(\Lambda)|^{\beta_D} \prod_{1 \leq j \leq n} \exp \left[ -\frac{n\beta_D}{2} \lambda_j^2 \right] d\lambda_j, \quad (48)$$

one can already read off many important spectral properties of the QCD-Dirac operator  $\mathcal{D}^{(3)}$  in the microscopic limit, cf. Table I. Recall the Vandermonde determinant

$$\Delta_n(\Lambda) = \prod_{1 \leq a < b \leq n} (\lambda_a - \lambda_b) = (-1)^{n(n-1)/2} \det[\lambda_a^{b-1}]_{1 \leq a, b \leq n}. \quad (49)$$

Thus, in three dimensions the eigenvalues are not degenerate apart from the Kramers degeneracy of QCD with adjoint fermions. Moreover, the eigenvalues of  $\mathcal{D}^{(3)}$  repel each other like  $|\lambda_a - \lambda_b|^{\beta_D}$  and have no repulsion from the origin [42].

In four dimensions, the operator  $\mathcal{W}^{(4)}$  is replaced by an  $n \times (n + \nu)$  real ( $\beta_D = 1$ ) or complex ( $\beta_D = 2$ ) random matrix  $W$  or a  $2n \times 2(n + \nu)$  quaternion matrix for  $\beta_D = 4$ . Then the Dirac operator has exactly  $\nu$  and  $2\nu$  zero modes for  $\beta_D = 1, 2$  and  $\beta_D = 4$ , respectively. Therefore,  $\nu$  is identified as the index of the Dirac operator. Due to the axial symmetry the nonzero eigenvalues always come in pairs  $\pm i\lambda$ . Moreover, because of the quaternion structure, the eigenvalues of  $\mathcal{D}^{(4)}$  as well as of the corresponding random matrix Dirac operator are degenerate for QCD with adjoint fermions. The joint probability density of the eigenvalues of the random matrix  $D$  reads [35]

$$p_\chi(\Lambda) \prod_{1 \leq j \leq 2n} d\lambda_j \propto |\Delta_{2n}(\Lambda^2)|^{\beta_D} \prod_{1 \leq j \leq 2n} \exp \left[ -\frac{n\beta_D}{2} \lambda_j^2 \right] \lambda_j^{\alpha_D} d\lambda_j, \quad (50)$$

cf. Table I. Again we can read off the behavior of the eigenvalues of  $D$  which, in the microscopic limit, are shared with the behavior of the low-lying eigenvalues of the QCD Dirac operator. The eigenvalues again repel each other as  $|\lambda_a - \lambda_b|^{\beta_D}$ . The difference with the three-dimensional case is the level repulsion from the origin  $\lambda_a^{\alpha_D} = \lambda_a^{\beta_D(\nu+1)-1}$  which results from the generic zero modes and the chiral structure of the Dirac operator. The global symmetries of the four-dimensional QCD Dirac operator and their impact on the microscopic spectrum were discussed in Refs. [5,35].

In two dimensions, rather than choosing a basis for which the Dirac operator becomes real or quaternion real for  $\beta_D = 1$  and  $\beta_D = 4$ , respectively, we insist on a chiral basis that preserves the chiral block structure of the Dirac operator. This results in a random matrix theory for which the matrix  $\tau_2 W$  is complex symmetric for  $\beta_D = 1$ ,  $\tau_2 W = (\tau_2 W)^T \in \mathbb{C}^{2n \times 2n}$ , and complex antisymmetric for  $\beta_D = 4$ ,  $W = -W^T \in \mathbb{C}^{n \times n}$ , cf. Eqs. (32) and (45), respectively. For QCD with three or more colors and the fermions in the fundamental representation ( $\beta_D = 2$ ), the two-dimensional Dirac operator has the same symmetries as the four-dimensional theory resulting in the same random matrix theory.

Another important difference between two and four dimensions is the topology of the gauge field configurations. For QCD with fundamental fermions the homotopy class is  $\Pi_1(\text{SU}(2)) = 0$ . Hence, no stable instanton solutions exist [22,45] (unstable instanton solutions are still possible [23,24,46]). Also the index of the Dirac operator is necessarily zero. Suppose that the two-dimensional Dirac operator has an exact zero mode

$$\mathcal{D}^{(2)}\phi = 0 \quad (51)$$

with definite chirality

$$\sigma_3\phi = \pm\phi. \quad (52)$$

Then, because of the antiunitary symmetry, we also have that

$$\mathcal{D}^{(2)}\sigma_2\tau_2K\phi = 0, \quad (53)$$

which generates another zero mode unless  $\sigma_2\tau_2K\phi$  and  $\phi$  are linearly dependent. This exactly happens in the four-dimensional theory. However in two dimensions  $\phi$  and  $\sigma_2\tau_2K\phi$  have opposite chiralities

$$\sigma_3\sigma_2\tau_2K\phi = -\sigma_2\tau_2K\sigma_3\phi = \mp\sigma_2\tau_2K\phi \quad (54)$$

implying that they have to be linearly independent states. We conclude that the index of the Dirac operator is zero for two-dimensional QCD in the fundamental representation and with two colors.

Although the index is trivial we still have a linear repulsion of the spectrum from the origin resulting from the chiral structure of  $\mathcal{D}$ . The joint probability density of the corresponding random matrix ensemble was first derived in the context of mesoscopic physics [43] and is given by Eq. (50). For completeness we give a derivation of this result in appendix A 1. Since we have a linear repulsion from the origin we have  $\alpha_D = 1$ . The level repulsion is also linear, i.e.  $\sim|\lambda_a - \lambda_b|$ , and the eigenvalues show no generic degeneracy.

For quarks in the adjoint representation the gauge group is given by  $SU(N_c)/\mathbb{Z}_{N_c}$  with the homotopy group  $\Pi_1(SU(N_c)/\mathbb{Z}_{N_c}) = \mathbb{Z}_{N_c}$  [45]. If  $\phi$  is a zero mode with positive chirality, then  $\sigma_2K\phi$  is a zero mode with negative chirality. Therefore, the index of the Dirac operator is zero. Using a bosonization approach it can be shown that the chiral condensate is nonzero for all  $N_c$  [47], which is consistent with having at most one pair of zero modes. Indeed, in a chiral basis, the nonzero off-diagonal block of the Dirac matrix is a square antisymmetric matrix, and generically has one zero mode if the matrix is odd-dimensional and no zero modes if the matrix is even dimensional. In Ref. [45], in the sector of topological charge  $k = 0, \dots, N_c - 1$ , a total of  $2k(N_c - k)$  zero modes are found, half of them right-handed and the other half left-handed. However, these zero modes are only obtained after complexifying the  $SU(N_c)$  algebra and are irrelevant in the present context. The corresponding random matrix theory for this universality class also has an antisymmetric off-diagonal block with no zero modes or one zero mode.

The joint probability density of the eigenvalues is given by the form (50) where the level repulsion is  $|\lambda_a - \lambda_b|^4$  since all eigenvalues are Kramers degenerate (because the

antiunitary symmetry operator satisfies  $(\sigma_2K)^2 = -1$ ). We rederive this joint probability density in appendix A 2 and relate it to the QCD Dirac operator in the microscopic limit. The repulsion of the eigenvalues from the origin is either linear ( $\alpha_D = 1$ ) for an even-dimensional  $W$  or quintic ( $\alpha_D = 5$ ) for an odd-dimensional  $W$ . We emphasize that the pair of zero modes for odd-dimensional matrices is not related to topology.

### C. Symmetry breaking pattern

In Table I, we also summarize the symmetry breaking patterns for continuum QCD in two, three, and four dimensions (see [31] for a discussion of general dimensions). We recall the results for the cases considered in our work and show that they also apply to the random matrix ensembles introduced in the previous subsection. We restrict ourselves to the two-dimensional case with the Dyson index  $\beta_D = 1, 4$ . The other symmetry breaking patterns and their relation to random matrix theory were extensively discussed in Refs. [35,42].

For  $\beta_D = 1$ , the off-diagonal block is symmetric after a unitary transformation,  $(\tau_2\mathcal{W}^{(2)})^T = \tau_2\mathcal{W}^{(2)}$ . Then we have

$$\bar{\psi}_R\tau_2\mathcal{W}^{(2)}\psi_R = \frac{1}{2}\begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & \tau_2\mathcal{W}^{(2)} \\ -\tau_2\mathcal{W}^{(2)} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}, \quad (55)$$

where  $\psi_R = (\mathbb{1} + \sigma_3)\psi/2$  is the right handed component of a quark field  $\psi$ . We obtain a similar expression for the other off-diagonal block,  $\mathcal{W}^{(2)\dagger}$ , of the Dirac operator  $\mathcal{D}^{(2)}$  with  $\psi_R \rightarrow \psi_L = (\mathbb{1} - \sigma_3)\psi/2$ , i.e.

$$\bar{\psi}_L(\tau_2\mathcal{W}^{(2)})^\dagger\psi_L = \frac{1}{2}\begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}^T \begin{pmatrix} 0 & (\tau_2\mathcal{W}^{(2)})^\dagger \\ -(\tau_2\mathcal{W}^{(2)})^\dagger & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}, \quad (56)$$

Therefore, the chiral symmetry is  $USp(2N_f) \times USp(2N_f)$  and acts on the doublets via the transformation  $(\bar{\psi}_R, \psi_R^T) \rightarrow (\bar{\psi}_R, \psi_R^T)U_R$  and  $(\bar{\psi}_L, \psi_L^T) \rightarrow (\bar{\psi}_L, \psi_L^T)U_L$  with  $U_{R/L} \in USp(2N_f)$ . In terms of these doublets the chiral condensate can be written as

$$\bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R = \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}. \quad (57)$$

A nonzero expectation value of the chiral condensate requires that the unitary symplectic matrices fulfill the constraint

$$U_R \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} U_L^T = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \quad (58)$$

so that the chiral symmetry is broken to  $USp(2N_f)$ .



TABLE II. Random matrix theories for the two-dimensional naive lattice QCD Dirac operator with gauge group listed in the first column. The Dyson index  $\beta_D$  refers to the antiunitary symmetry of the Dirac operator in the continuum. Because of additional symmetries the power of the Vandermonde determinant,  $\beta_D^{(\text{eff})}$ , is generally different from the continuum theory and thus, the level repulsion as well. Moreover the repulsion of the levels from the origin, namely  $\lambda^{\alpha_D^{(\text{eff})}}$ , the generic degeneracy of the eigenvalues (third to last column, “Deg”), and the number of generic zero modes (second to last column, “ZM”) generally change as well. The third column refers to whether  $L_1$  or  $L_2$  are even (e) or odd (o). All discretizations are classified according to the ten-fold classification of random matrix theories (fourth column) which share the same pattern of chiral symmetry breaking with the lattice QCD Dirac operator (we do not consider axial symmetry breaking). Notice that the symmetry breaking pattern and, therefore, the global symmetries of the lattices where  $L_1$  and  $L_2$  are both odd is the same as the two-dimensional QCD Dirac operator in continuum, cf. Table I.

Gauge theory	$\beta_D$	Lat.	Sym. Class	$\beta_D^{(\text{eff})}$	$\alpha_D^{(\text{eff})}$	Deg	ZM	Symmetry breaking pattern
$N_c = 2$ , fund.	1	ee	CII	4	3	4	0	$U(4N_f) \rightarrow O(4N_f)$
$N_c = 2$ , fund.	1	eo	C	2	2	2	0	$USp(4N_f) \rightarrow U(2N_f)$
$N_c = 2$ , fund.	1	oo	CI	1	1	1	0	$USp(2N_f) \times USp(2N_f) \rightarrow USp(2N_f)$
$N_c > 2$ , fund.	2	ee	AIII	2	1	2	0	$U(2N_f) \times U(2N_f) \rightarrow U(2N_f)$
$N_c > 2$ , fund.	2	eo	A	2	0	1	0	$U(2N_f) \rightarrow U(N_f) \times U(N_f)$
$N_c > 2$ , fund.	2	oo	AIII	2	1	1	0	$U(N_f) \times U(N_f) \rightarrow U(N_f)$
$N_c \geq 2$ , adj.	4	ee	BDI	1	0	2	0	$U(4N_f) \rightarrow USp(4N_f)$
$N_c \geq 2$ , adj.	4	eo	D	2	0	2	0	$O(4N_f) \rightarrow U(2N_f)$
$N_c \in 2\mathbb{N} + 1$ , adj.	4	oo	DIII (even-dim)	4	1	2	0	$O(2N_f) \times O(2N_f) \rightarrow O(2N_f)$
$N_c \in 2\mathbb{N}$ , adj.	4	oo	DIII (odd-dim)	4	5	2	2	$O(2N_f) \times O(2N_f) \rightarrow O(2N_f)$

This result can be derived by an explicit calculation for the corresponding random matrix model, see appendix A 1 b, and was also found in Ref. [31] for general QCD-like theories and in Ref. [40] for random matrix theories.

For two-dimensional QCD with adjoint fermions ( $\beta_D = 4$ )  $\mathcal{W}^{(2)T} = -\mathcal{W}^{(2)}$  is antisymmetric so that the coupling of the gauge fields and the quarks can be rewritten as

$$\bar{\psi}_R \mathcal{W}^{(2)} \psi_R = \frac{1}{2} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{W}^{(2)} \\ \mathcal{W}^{(2)} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}, \quad (59)$$

and

$$\bar{\psi}_L \mathcal{W}^{(2)\dagger} \psi_L = \frac{1}{2} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}^T \begin{pmatrix} 0 & \mathcal{W}^{(2)\dagger} \\ \mathcal{W}^{(2)\dagger} & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}. \quad (60)$$

The corresponding chiral symmetry is  $O(2N_f) \times O(2N_f)$  with the transformation  $(\bar{\psi}_R, \psi_R^T) \rightarrow (\bar{\psi}_R, \psi_R^T) A O_R A^{-1}$  and  $(\bar{\psi}_L, \psi_L^T) \rightarrow (\bar{\psi}_L, \psi_L^T) A O_L A^{-1}$  with  $O_{R/L} \in O(2N_f)$  and

$$A A^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (61)$$

Invariance of the nonzero chiral condensate,

$$\bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R = \begin{pmatrix} \bar{\psi}_R^T \\ \psi_R \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\psi}_L^T \\ \psi_L \end{pmatrix}, \quad (62)$$

requires

$$O_R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} O_L \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (63)$$

such that the symmetry is broken to  $O(2N_f)$ . Also this case agrees with results of Refs. [31,40].

### III. TWO-DIMENSIONAL LATTICE QCD WITH NAIVE FERMIONS AT STRONG COUPLING

In this section we consider the microscopic limit of naive fermions in the strong coupling limit and the corresponding random matrix theories. Thus the links, the gauge group elements on the lattice, are distributed according to the Haar-measure of the gauge group. In Secs. III A we discuss the general effect of the parity of the lattice on the global symmetries of the Dirac operator. This discussion is combined with the specific antiunitary symmetries of the QCD-like theories in Secs. III B, III C, and III D. In particular, we classify each lattice Dirac operator according to a random matrix ensemble, which is summarized in Table II together with some spectral properties. These random matrix theory predictions are compared with two-dimensional lattice simulations confirming that the parity of the lattice has an important effect on the properties of the smallest eigenvalues. This was observed before in the condensed matter literature [6].

#### A. General lattice model

The covariant derivatives that enter in the lattice QCD Dirac operator can be readily constructed via the translation matrices. Before doing so we introduce the lattice. Let  $|j\rangle$  be the  $j$ 'th site in one direction of a lattice written in Dirac's bra-ket notation. Then the dual vector is  $\langle j|$ . The translation matrices of an  $L_1 \times L_2$  lattice in the directions  $\mu = 1, 2$  are given by

$$T_\mu = \begin{cases} \sum_{\substack{1 \leq i \leq L_1 \\ 1 \leq j \leq L_2}} U_{1ij} \otimes |i\rangle\langle i+1| \otimes |j\rangle\langle j|, & \mu = 1, \\ \sum_{\substack{1 \leq i \leq L_1 \\ 1 \leq j \leq L_2}} U_{2ij} \otimes |i\rangle\langle i| \otimes |j\rangle\langle j+1|, & \mu = 2. \end{cases} \quad (64)$$

The matrices  $U_{\mu ij}$  are given in some representation of the special unitary group  $SU(N_c)$  and are weighted with the Haar-measure of  $SU(N_c)$ . Hence, the translation matrices  $T_\mu$  are unitary.

Note that our lattices have a toroidal geometry. We have numerically looked at the effect of periodic and antiperiodic fermionic boundary conditions on the spectrum of the Dirac operator. Indeed, the universality class remains unaffected since the global symmetries are independent of the boundary conditions. Only the Thouless energy marginally changes.

The Dirac operator on a two-dimensional lattice is given by

$$\begin{aligned} D &= \sigma_\mu (T_\mu - T_\mu^\dagger) \\ &= \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & T_x - T_x^\dagger + i(T_y - T_y^\dagger) \\ T_x - T_x^\dagger - i(T_y - T_y^\dagger) & 0 \end{bmatrix}. \end{aligned} \quad (65)$$

Due to the lattice structure, an additional symmetry can exist in each direction if the number of sites in a direction is even. Then the matrix elements of the Dirac operator between even and odd sites are nonvanishing while there is no direct coupling between an even and an even lattice site and between an odd and an odd site. For a two-dimensional lattice we can distinguish three cases. First, the number of lattice sites  $L_1$  and  $L_2$  are both odd. Then, there are no additional symmetries such that the lattice Dirac operator is in the same symmetry class as the continuum theory. The other two cases are, second,  $L_1$  even and  $L_2$  odd or the reverse, and third, both  $L_1$  and  $L_2$  are even. We analyze these two cases in detail for each antiunitary symmetry class separately. Thereby we assume that both  $L_1$  and  $L_2$  are larger than 2 because only then the low-lying eigenvalues of the Dirac operator show a generic behavior.

Let us define the operators

$$\Gamma_5^{(\mu)} = \begin{cases} \sum_{\substack{1 \leq i \leq L_1 \\ 1 \leq j \leq L_2}} (-1)^i \mathbb{1}_{N_c} \otimes |i\rangle\langle i| \otimes |j\rangle\langle j|, & \mu = 1, \\ \sum_{\substack{1 \leq i \leq L_1 \\ 1 \leq j \leq L_2}} (-1)^j \mathbb{1}_{N_c} \otimes |i\rangle\langle i| \otimes |j\rangle\langle j|, & \mu = 2. \end{cases} \quad (66)$$

Then one can show that the operator  $\Gamma_5^{(\mu)}$  fulfills the relation

$$\Gamma_5^{(\mu)} T_\omega \Gamma_5^{(\mu)} = (-1)^{\delta_{\mu\omega}} T_\omega \quad (67)$$

if  $L_\mu$  is even. Hereby we employ the Kronecker symbol  $\delta_{\mu\omega}$  in the exponent of the sign.

Let us consider the simplest case where  $L_1$  and  $L_2$  are odd. Then  $W$  has no additional symmetries resulting from the lattice structure. Therefore, the Dirac operator will have the same unitary and antiunitary symmetries as in the continuum limit discussed in Sec. II, in particular it is anti-Hermitian and chirally symmetric,

$$D = -D^\dagger \quad \text{and} \quad [\sigma_3, D]_+ = 0. \quad (68)$$

Therefore the Dirac operator has the structure

$$D = \begin{pmatrix} 0 & W \\ -W^\dagger & 0 \end{pmatrix}, \quad (69)$$

where  $W$  may fulfill some additional antiunitary symmetries because of the representation of the gauge theory.

In the second case, we have in one direction an even number of lattice sites and in the other direction an odd number of lattice sites. Let us assume that without loss of generality  $L_1 \in 2\mathbb{N}$  and  $L_2 \in 2\mathbb{N} + 1$ . Then the lattice Dirac operator fulfills the global symmetries

$$D = -D^\dagger, \quad [\sigma_3, D]_+ = 0, \quad \text{and} \quad [\Gamma_5^{(1)} \sigma_2, D]_- = 0 \quad (70)$$

plus possible antiunitary symmetries depending on the representation of the gauge group. From the first two symmetries it follows that the Dirac operator has the chiral structure (69). The last symmetry relation of Eq. (70) tells us that the matrix  $W$  is  $\Gamma_5^{(1)}$ -Hermitian, i.e.

$$W^\dagger = \Gamma_5^{(1)} W \Gamma_5^{(1)}. \quad (71)$$

Hence the Dirac operator for this kind of lattices takes the form

$$\begin{aligned} D &= \begin{pmatrix} 0 & \Gamma_5^{(1)} H \\ -H \Gamma_5^{(1)} & 0 \end{pmatrix} \\ &= \text{diag}(\Gamma_5^{(1)}, \mathbb{1}) \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix} \text{diag}(\Gamma_5^{(1)}, \mathbb{1}), \end{aligned} \quad (72)$$

with  $H$  a Hermitian matrix. This matrix  $H$  may be restricted to a subspace of the Hermitian matrices if we take into account the antiunitary symmetries resulting from the representation of the gauge theory. The unitary matrix  $\text{diag}(\Gamma_5^{(1)}, \mathbb{1})$  does not change the eigenvalue spectrum of  $D$  and can be omitted.

One can also derive the structure (72) by employing the projection operators  $(1 \pm \Gamma_5^{(1)})/2$ . They project the lattice

onto sub-lattices associated to the even and odd lattice sites in the direction  $\mu = 1$ . In such a basis, the translation matrix  $T_1$  maps the even lattice sites to the odd ones and vice versa while the translation matrix  $T_2$  maps the two sub-lattices onto themselves.

In the third case the lattice has an even number of lattice sites in both directions. This is exactly the situation of staggered fermions. The corresponding Dirac operator for naive fermions has the symmetries

$$D = -D^\dagger, \quad [\sigma_3, D]_+ = 0, \\ [\Gamma_5^{(1)}\sigma_2, D]_- = 0, \quad \text{and} \quad [\Gamma_5^{(2)}\sigma_1, D]_- = 0. \quad (73)$$

Again the Dirac operator has the chiral structure (69), but the symmetry relation of the matrix  $W$  is given by

$$W^\dagger = \Gamma_5^{(1)}W\Gamma_5^{(1)} \quad \text{and} \quad [\Gamma_5^{(1)}\Gamma_5^{(2)}, W]_+ = 0. \quad (74)$$

The first symmetry restricts  $W$  to a  $\Gamma_5^{(1)}$ -Hermitian matrix whereas the second relation reflects the even-odd symmetry of the Dirac operator. Therefore the lattice Dirac operator has the structure

$$D = \text{diag}(\Gamma_5^{(1)}, 1) \left( \begin{array}{cc|cc} 0 & & 0 & X \\ & & X^\dagger & 0 \\ \hline 0 & -X & & \\ -X^\dagger & 0 & & 0 \end{array} \right) \text{diag}(\Gamma_5^{(1)}, 1), \quad (75)$$

where  $X$  is a complex matrix that may fulfill antiunitary symmetries depending on the representation of the gauge fields. The double degeneracy is immediate and is eliminated for staggered fermions.

Again one can also explicitly construct the form of the lattice Dirac operator (75) by employing the four projection operators  $(1 \pm \Gamma_5^{(1)})/2$  and  $(1 \pm \Gamma_5^{(2)})/2$ . They split the lattice into four sub-lattices which are coupled via the translation matrices  $T_{1/2}$ .

Adding the antiunitary symmetries to the symmetries (68), (70), and (73) will give rise to further constraints on  $W$ . In Table II we summarize these cases for each antiunitary symmetry class. In general, the symmetry class will differ from the symmetry class in continuum. Therefore the corresponding random matrix ensemble and the symmetry breaking pattern will also change. In particular, one has to replace the indices  $\beta_D$  (Dyson index = level repulsion) and  $\alpha_D$  (= repulsion of the levels from the origin) in the joint probability densities of the eigenvalues of the random matrix model, cf. Eqs. (48) and (50), by effective values,

$$\beta_D \rightarrow \beta_D^{(\text{eff})} \quad \text{and} \quad \alpha_D \rightarrow \alpha_D^{(\text{eff})}. \quad (76)$$

This impacts the spectral properties of the Dirac operator in the microscopic limit.

There are additional conditions on the off-diagonal block  $W$  of the lattice Dirac operator  $D$  which are independent of the gauge configurations. For example the traces of  $W$  satisfy the relations

$$\text{tr}W^2 = \text{tr}W^{2l+1} = 0 \quad \text{and} \\ \text{tr}WW^\dagger = \begin{cases} 2N_c L_1 L_2, & \text{fundamental fermions,} \\ 2(N_c^2 - 1)L_1 L_2, & \text{adjoint fermions} \end{cases} \quad (77)$$

with  $l = 0, 1, 2, \dots$  such that  $l \leq \min\{L_1, L_2\}/2 - 1$ . They result from the fact that the translation matrices (64) are unitary and have no diagonal elements. The conditions of the kind (77) are expected to have no influence on the microscopic spectrum in the limit of large matrices. Nevertheless, they may give rise to finite volume corrections which turn out to be particularly large for the simulations of SU(3) gauge theory with fermions in the fundamental representation and choosing  $L_1$  even and  $L_2$  odd, see subsection III D 2. The effect of such conditions can also be studied with random matrix theory and we do this for the simplest condition, namely that  $W$  is traceless, i.e.  $\text{tr}W = 0$ .

## B. SU(2) and fermions in the fundamental representation

When studying the two-color theory in its fundamental representation the translation matrix fulfills exactly the same antiunitary symmetry as the covariant derivative in the continuum theory,

$$[iT_\mu, \tau_2 K]_- = 0, \quad (78)$$

cf. Eq. (23). This symmetry carries over to the symmetry

$$[iD, \tau_2 \sigma_2 K]_- = 0, \quad (79)$$

for the lattice Dirac operator meaning that there is always a gauge field independent basis where the Dirac operator appears real. However, as is the case in the continuum theory, the symmetry (79) may not commute with the symmetries (68), (70), and (73). In the continuum theory we showed that the antiunitary symmetry results in a symmetry of the off-diagonal block  $W$ ,

$$W = -\tau_2 W^T \tau_2. \quad (80)$$

cf. Eq. (32). This carries over to the lattice theory as well and together with the symmetries (68), (70), and (73) yields the symmetry classification given in Table 2. This is worked out in detail in the subsections III B 1, III B 2, and III B 3 for  $(L_1, L_2)$  odd-odd, even-odd, and even-even, respectively.

### 1. The odd-odd case

As already discussed before, this case does not have any additional symmetries and the pattern of chiral symmetry breaking as well as the distribution of the eigenvalues in the microscopic domain has to be the same as in the continuum limit which was discussed in Sec. III A. The symmetries of the Dirac operator are summarized in Eq. (27) which translates in terms of the lattice Dirac operator as in Eqs. (68) and (80). That corresponds to a chiral random matrix theory with symmetric complex off-diagonal blocks. In the Cartan classification of symmetric spaces, this is denoted by the symbol CI. The corresponding microscopic level density is given by  $(x = \lambda V\Sigma)$  [48]

$$\rho(x) = \frac{x}{2} [J_1^2(x) - J_0(x)J_2(x)] + \frac{1}{2} J_0(x)J_1(x). \quad (81)$$

The symmetry breaking pattern is therefore the same as in the continuum, namely  $U(2N_f) \rightarrow O(2N_f)$ .

In Fig. 1(a) we compare the prediction (81) for the low-lying Dirac spectrum with lattice QCD data at strong coupling. The size of the lattices is quite small. Nevertheless the agreement of the analytical prediction for the microscopic level density and the simulations around the origin is good. In particular, the linear repulsion of the eigenvalues from the origin is confirmed. Also the

degree of degeneracy and the number of generic zero modes, which are in this case one and zero, respectively, are verified. The lattice results are obtained from an ensemble of about  $10^5$  independent configurations with the links generated by the Haar measure of the gauge group  $SU(2)$ .

### 2. The even-odd case

For definiteness we choose  $L_1$  even and  $L_2$  odd. Then, the Dirac operator is of the form (72). We combine the intermediate result  $W = \Gamma_5^{(1)} H$  with a Hermitian matrix  $H$  and the antiunitary symmetry (80). Therefore we can find a gauge field independent rotation, namely  $U_5^{(1)} = \exp[\pi i (\Gamma_5^{(1)} - \mathbb{1}_{N_c L_1 L_2}) / 4]$ , where  $\tilde{H} = U_5^{(1)} H U_5^{(1)-1}$  becomes an anti-self-dual Hermitian matrix ( $\tilde{H} = \tilde{H}^\dagger = -\tau_2 \tilde{H}^T \tau_2 = -\tau_2 \tilde{H}^* \tau_2$ ). This is the class C of the tenfold classification [40] and  $\tilde{H}$  is an element in the Lie-algebra of the group  $USp(N_c L_1 L_2)$ . In this basis, the Dirac operator reads

$$D = \text{diag}(U_5^{(1)}, U_5^{(1)-1}) \begin{pmatrix} 0 & \tilde{H} \\ -\tilde{H} & 0 \end{pmatrix} \text{diag}(U_5^{(1)-1}, U_5^{(1)}). \quad (82)$$

Note that  $\Gamma_5^{(1)} = U_5^{(1)2}$ .

What does this imply for the spectrum of the Dirac operator? The antiunitary symmetry leads to a pair of

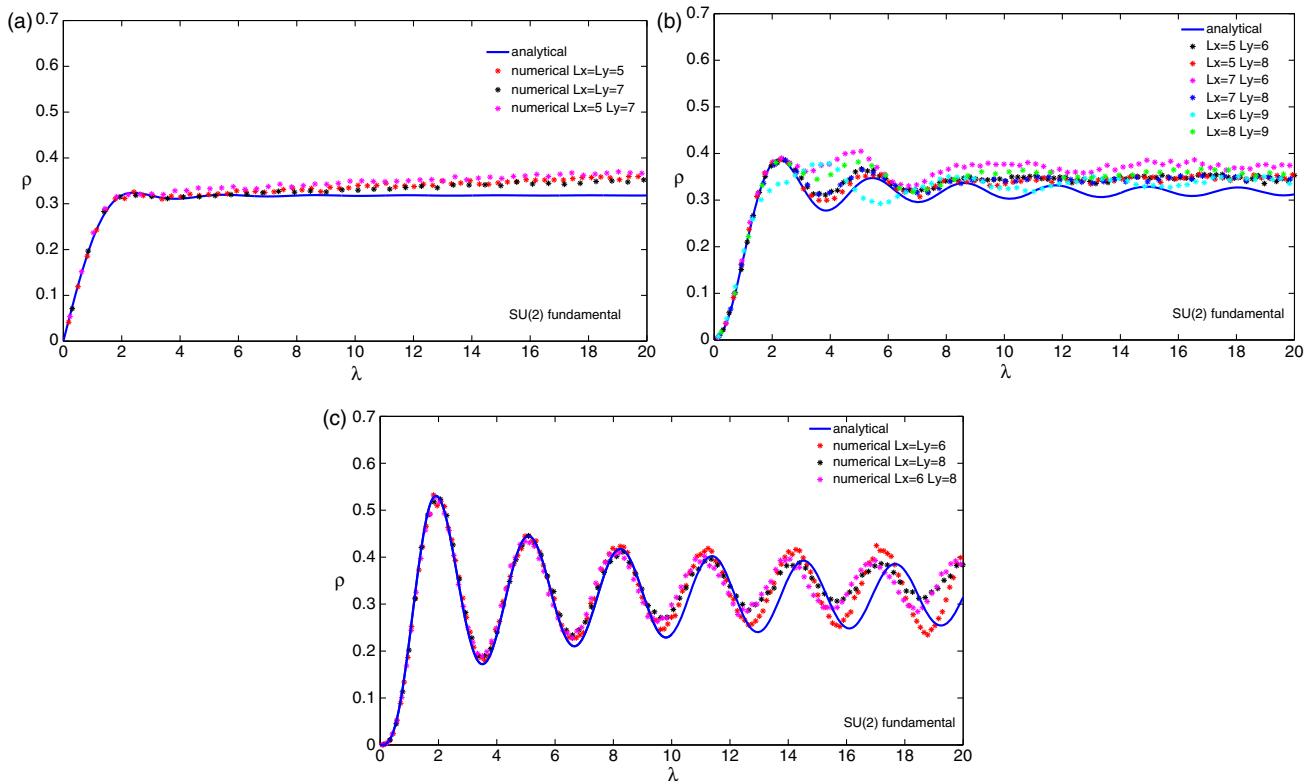


FIG. 1 (color online). Comparison of the microscopic level densities of lattice QCD data in the strong coupling limit at various lattice sizes (stars) and the analytical results given by the corresponding random matrix theories (solid curves). The plotted lattice gauge theories are: a)  $SU(2)$  fundamental and  $L_1, L_2$  odd, b)  $SU(2)$  fundamental and  $L_1 + L_2$  odd, and c)  $SU(2)$  fundamental and  $L_1, L_2$  even.



eigenvalues  $\pm\lambda$  of the Hermitian matrix  $\tilde{H}$ . Indeed, if  $\lambda$  is an eigenvalue of  $\tilde{H}$  with the eigenvector  $|\phi\rangle$ ,

$$\tilde{H}|\phi\rangle = \lambda|\phi\rangle, \quad (83)$$

then the state  $\tau_2|\phi^*\rangle$  is an eigenvector with eigenvalue  $-\lambda$ ,

$$\tilde{H}\tau_2|\phi^*\rangle = -\lambda\tau_2|\phi^*\rangle. \quad (84)$$

Therefore the Dirac operator (82) has the eigenvalues  $\pm i\lambda$  which are doubly degenerate. This leads to a doubling of the number of flavors and the spectrum of  $D$  is twice the spectrum of  $i\tilde{H}$ . In addition, because of

$$\bar{\psi}\tilde{H}\psi = \frac{1}{2}(\bar{\psi}\tilde{H}\psi - (\tau_2\psi)^T\tilde{H}\tau_2\bar{\psi}^T), \quad (85)$$

the flavor symmetry is enhanced to  $\text{USp}(4N_f)$ , cf. Eq. (55). Because

$$\begin{aligned} \det(D + m\mathbb{1}) &= \det(\tilde{H}^2 + m^2\mathbb{1}) \\ &= \det(\tilde{H} + im\mathbb{1})\det(\tilde{H} - im\mathbb{1}), \end{aligned} \quad (86)$$

a nonzero eigenvalue density of  $\tilde{H}$  leads to a nonzero eigenvalue density of the Dirac operator,  $D$ . The symmetry  $\text{USp}(4N_f)$  is thus spontaneously broken by the formation of a condensate with  $m$  as source term. However this condensate is still invariant under a  $\text{U}(2N_f)$  subgroup of  $\text{USp}(4N_f)$

$$\begin{aligned} &[\text{diag}(U, U^*)]^T \begin{pmatrix} 0 & m\mathbb{1}_{2N_f} \\ -m\mathbb{1}_{2N_f} & 0 \end{pmatrix} \text{diag}(U, U^*) \\ &= \begin{pmatrix} 0 & m\mathbb{1}_{2N_f} \\ -m\mathbb{1}_{2N_f} & 0 \end{pmatrix}. \end{aligned} \quad (87)$$

Thus the symmetry breaking pattern is  $\text{USp}(4N_f) \rightarrow \text{U}(2N_f)$  in agreement with the symmetry breaking pattern of the corresponding random matrix ensemble [40].

The joint probability distribution of the symmetry class C coincides with the distribution of the nonzero eigenvalues of the chGUE for  $\nu = 1/2$ . The microscopic level density is thus given by [43,48,49]

$$\rho(x) = \frac{1}{\pi} - \frac{\sin(2x)}{2\pi x}. \quad (88)$$

In Fig. 1(b) we compare this result to lattice simulations. We find only good agreement to about one eigenvalue spacing. The reason for the strong disagreement above the average position of the first eigenvalue is not clear. Nevertheless, the quadratic repulsion of the eigenvalues from the origin, the double degeneracy of the eigenvalues, and the fact that there are no generic zero modes are confirmed by the lattice simulations.

### 3. The even-even case

Finally, we consider the case with both  $L_1$  and  $L_2$  even. Then the Dirac operator has the structure given in Eq. (75). After combining the chiral structure of  $W$  with the antiunitary symmetry (80) the Dirac operator takes the form

$$\begin{aligned} D &= \text{diag}(U_5^{(1)}, U_5^{(1)-1}) \begin{pmatrix} 0 & \tilde{W} \\ \tilde{W}^\dagger & 0 \end{pmatrix} \\ &\times \text{diag}(U_5^{(1)-1}, U_5^{(1)}) \end{aligned} \quad (89)$$

with  $\tilde{W}^* = \tau_2\tilde{W}\tau_2$  a quaternion matrix without any further symmetries. The unitary transformation  $\text{diag}(U_5^{(1)-1}, U_5^{(1)})$  is exactly the same as in the previous subsection and keeps the spectrum invariant such that the global symmetries of the lattice Dirac operator  $D$  essentially coincide with the continuum Dirac operator in four dimensions with the fermions in the adjoint representation. Therefore, the random matrix ensemble corresponding to this type of lattice theory is the chGSE with the chiral symmetry breaking pattern  $\text{U}(4N_f) \rightarrow \text{O}(4N_f)$ . The degeneracy of the eigenvalues is four because of Kramers degeneracy and the doubling of flavors. In Table II we summarize the main properties of this ensemble.

The microscopic level density of the lattice QCD Dirac operator in this class is given by the  $\nu = 0$  result of the chGSE [48,50] (note that  $\tilde{W}$  is a square matrix),

$$\rho(x) = x[J_0^2(2x) + J_1^2(2x)] - \frac{1}{2}J_0(2x) \int_0^{2x} J_0(\tilde{x})d\tilde{x}. \quad (90)$$

There are no generic zero modes and the levels show a cubic repulsion from the origin.

In Fig. 1(c) we compare the result (90) to lattice simulations of the two-dimensional Dirac operator for QCD with two colors. There is an excellent agreement for the first few eigenvalues confirming our predictions. In this case we also checked that the finite size corrections of the chiral condensate behave as  $\Sigma = c_1 + c_2/V$  with  $c_1 = 0.375$  and  $c_2 = 2.5$ .

### C. $\text{SU}(N_c)$ and fermions in the adjoint representation

For the fermions in the adjoint representation of the gauge group  $\text{SU}(N_c \geq 2)$  the translation matrices are real and, hence, satisfy the antiunitary symmetry

$$[K, T_\mu]_- = T_\mu. \quad (91)$$

On a  $L_1 \times L_2$  lattice, the translation matrices are represented by a subset of matrices in the orthogonal group  $\text{O}((N_c^2 - 1)L_1L_2)$ . The symmetry (91) carries over to the two-dimensional lattice Dirac operator

$$[iD, \sigma_2 K]_- = 0, \quad (92)$$

and its off-diagonal block matrix

$$W = -W^T. \quad (93)$$

We combine this symmetry with the symmetries (68), (70), and (73) along the same lines as shown in subsection III C. Thereby we discuss the odd-odd, even-odd, and even-even lattices in subsections III C 1, III C 2, and III C 3, respectively.

### 1. The odd-odd case

In the case where both the number of lattice sites  $L_1$  and  $L_2$  are odd, the Dirac operator has the same symmetries as in the continuum limit resulting in the same pattern of chiral symmetry breaking ( $O(2N_f) \times O(2N_f) \rightarrow O(2N_f)$ ) and the same microscopic spectral properties (see Table II). Depending on the number of colors the off-diagonal block  $W$  of the lattice Dirac operator is either even or odd dimensional and the corresponding symmetry class is given by the second Bogolyubov–de Gennes ensemble DIII, see Ref. [40,43], which can be also either even or odd, respectively. The microscopic level density was obtained in Ref. [48] and is given by

$$\rho(x) = \frac{x}{2} [2J_1^2(2x) + J_0^2(2x) - J_0(2x)J_2(2x)] + \frac{1}{2}J_1(2x) \quad (94)$$

for  $N_c$  odd and

$$\rho(x) = 2\delta(x) + \frac{x}{2} [2J_1^2(2x) + J_0^2(2x) - J_0(2x)J_2(2x)] - \frac{1}{2}J_1(2x) \quad (95)$$

for  $N_c$  even. Notice that the lattice Dirac operator has one additional pair of generic zero-modes if the number of colors is even otherwise there are no generic zero modes. Therefore the repulsion of the eigenvalues from the origin is stronger. However, the level repulsion is always quartic (see Table II). Moreover, the full spectrum is Kramers degenerate. This is a characteristic for ensembles associated to the Dyson index  $\beta_D = 4$ .

In Figs. 2(a) and 2(b) we compare the low lying lattice Dirac spectra and the analytical results of (94) and (95) for two and three colors, respectively. The agreement is good for the first few eigenvalues and becomes better when increasing the number of colors.

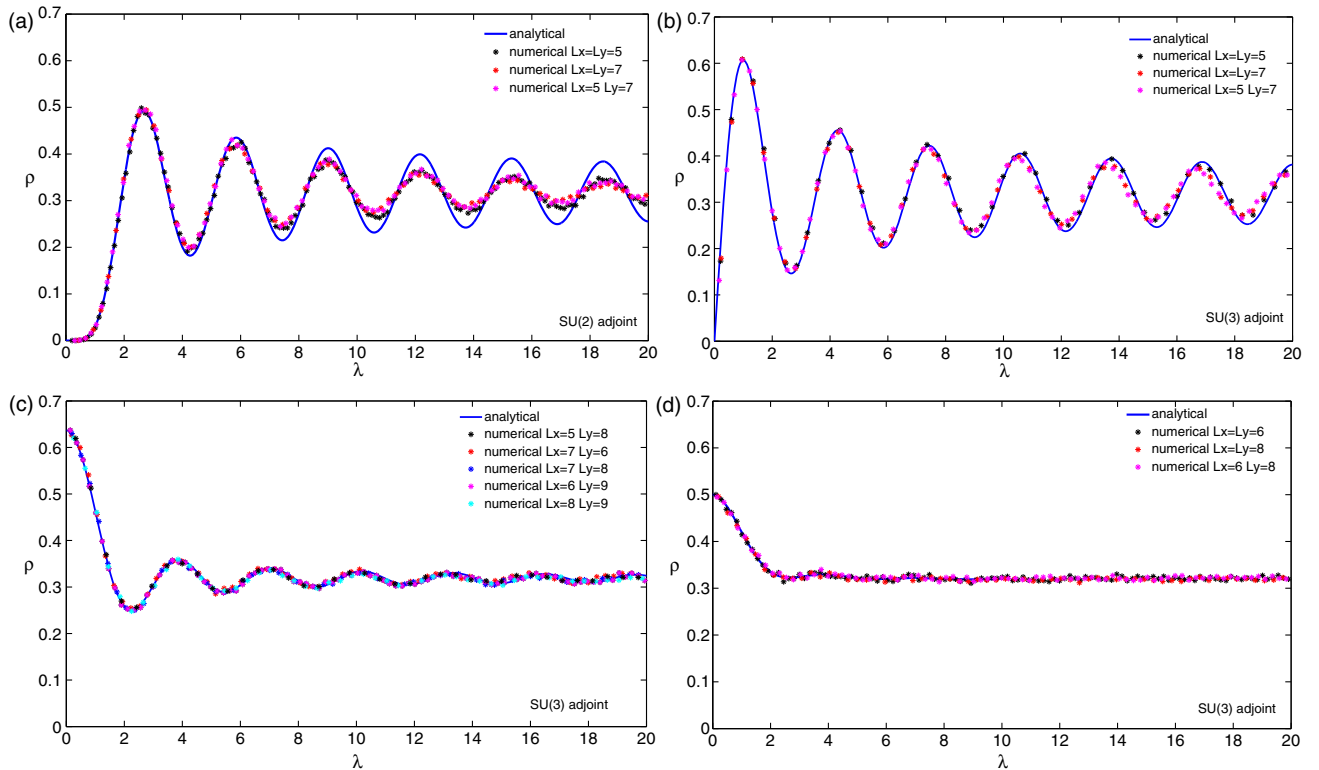


FIG. 2 (color online). Comparison of the microscopic level densities of the lattice QCD Dirac operator in the strong coupling limit at various lattice sizes (stars) and the analytical results derived from the corresponding random matrix theories (solid curves). Results are shown for the lattice theories: (a) SU(2) adjoint and  $L_1, L_2$  odd, (b) SU(3) adjoint and  $L_1, L_2$  odd, (c) SU(3) adjoint and  $L_1$  odd and  $L_2$  even, and (d) SU(3) adjoint and  $L_1, L_2$  even.

## 2. The even-odd case

Next we consider the mixed situation where the lattice has an even  $L_1$  and an odd  $L_2$ . The combination of the symmetries (70) and (93) can be again simplified via the same unitary transformation  $\text{diag}(U_5^{(1)-1}, U_5^{(1)})$  as introduced in subsection III B 2. Then the lattice Dirac operator can be written as

$$D = \text{diag}(U_5^{(1)}, U_5^{(1)-1}) \begin{pmatrix} 0 & \tilde{H} \\ -\tilde{H} & 0 \end{pmatrix} \text{diag}(U_5^{(1)-1}, U_5^{(1)}), \quad (96)$$

where  $\tilde{H}$  is purely imaginary and antisymmetric. Thus the symmetry class is equivalent to a random matrix ensemble with the matrices in the Lie-algebra of the orthogonal group  $O(L_1 L_2 (N_c^2 - 1))$  which is denoted by the Cartan symbol D [40]. Although for this ensemble one also has to distinguish between even and odd matrix size because of an additional pair of generic zero modes, the lattice Dirac operator always yields an even sized matrix  $\tilde{H}$ . The reason is that  $\tilde{H}$  is  $L_1 L_2 (N_c^2 - 1) \times L_1 L_2 (N_c^2 - 1)$  dimensional, where  $L_1$  is even. Therefore we expect a quadratic level repulsion, no repulsion of the levels of  $D$  from the origin and no generic zero modes, cf. Table II. The number of flavors is doubled because of the particular block structure (96).

The quark bilinear can be written as

$$\bar{\psi} \tilde{H} \psi = \frac{1}{2} (\bar{\psi} \tilde{H} \psi + \psi^T \tilde{H} \bar{\psi}^T), \quad (97)$$

so that the symmetry group is  $O(4N_f)$ . As was shown in the case  $\beta_D = 1$ , see subsection III B 2, a nonzero eigenvalue density of  $\tilde{H}$  results in a nonzero eigenvalue density of the Dirac operator yielding a chiral condensate with source term  $m$ . This condensate breaks the  $O(4N_f)$  symmetry group to the subgroup satisfying

$$O^T \begin{pmatrix} 0 & \mathbb{1}_{2N_f} \\ -\mathbb{1}_{2N_f} & 0 \end{pmatrix} O = \begin{pmatrix} 0 & \mathbb{1}_{2N_f} \\ -\mathbb{1}_{2N_f} & 0 \end{pmatrix} \quad (98)$$

This equation enforces the matrix  $O$  to a block structure

$$O = \begin{pmatrix} O_1 & O_2 \\ -O_2 & O_1 \end{pmatrix}. \quad (99)$$

The orthogonality of  $O$  requires that

$$(O_1 + iO_2)^\dagger (O_1 + iO_2) = \mathbb{1} \quad (100)$$

so that  $O$  is equivalent to a unitary transformation. Moreover each unitary matrix  $U \in U(2N_f)$  can be decomposed into the real matrices  $O_1 = \frac{1}{2}(U + U^*)$  and  $O_2 = -i(U - U^*)$ . Hence the remaining group invariance

is equal to  $U(2N_f)$  yielding the symmetry breaking pattern  $O(4N_f) \rightarrow U(2N_f)$ .

The microscopic level density can be calculated from the corresponding random matrix ensemble in class D and is given by [43,48]

$$\rho(x) = \frac{1}{\pi} + \frac{\sin(2x)}{2\pi x}. \quad (101)$$

In Fig. 2(c) we compare this analytical result to strong coupling lattice simulations for naive quarks in the adjoint representation of SU(3). The lattice data show excellent agreement for the low-lying Dirac spectrum. Moreover the simulations confirm the double degeneracy of the Dirac operator (eigenvalues have also the degeneracy two) and the fact that there are no generic zero modes.

## 3. The Even-Even Case

Let  $L_1$  and  $L_2$  be even. This is the case related to the staggered Dirac operator. With help of the symmetries (73) and (93) the lattice Dirac operator can, by choosing a particular gauge field independent basis, be brought to the form

$$D = \text{diag}(U_5^{(1)}, U_5^{(1)-1}) \begin{pmatrix} 0 & \tilde{W} \\ \tilde{W}^\dagger & 0 \end{pmatrix} \times \text{diag}(U_5^{(1)-1}, U_5^{(1)}), \quad (102)$$

where  $\tilde{W}$  is a real  $L_1 L_2 (N_c^2 - 1)/2 \times L_1 L_2 (N_c^2 - 1)/2$  matrix without any additional restrictions. The additional chiral structure is related to the parity of the lattice sites.

The unitary transformation  $\text{diag}(U_5^{(1)-1}, U_5^{(1)})$  does not change the spectrum. Therefore the naive lattice Dirac operator (102) is in the class of the chGOE with index  $\nu = 0$  (because  $\tilde{W}$  is a square matrix). The Dirac spectrum is doubly degenerate which is taken care of when constructing the staggered Dirac operator. The symmetry breaking pattern is  $U(4N_f) \rightarrow \text{USp}(4N_f)$  [35] and the microscopic spectral density is given by the  $\nu = 0$  result of the chGOE [51]

$$\rho(x) = \frac{x}{2} [J_0^2(x) - J_1^2(x)] + \frac{1}{2} J_0(x) \left[ 1 - \int_0^x J_0(\tilde{x}) d\tilde{x} \right], \quad (103)$$

Therefore the level repulsion is linear, the levels have no repulsion from the origin and there are no generic zero modes. The analytical result (103) is compared with lattice data in Fig. 2(d) showing a perfect agreement.

## D. QCD with more than two colors and fermions in the fundamental representation

In this case there are no antiunitary symmetries. The structure and the symmetry class of the Dirac operator are only related to the parity of the lattice. Hence, we have to take the structure of the naive lattice Dirac operator as shown in Eqs. (68), (70), and (73).

The odd-odd and even-even lattices are in the same universality class and are both discussed in subsection III D 1. The case of one even number of lattice sites and one odd number is considered in subsection III D 2.

### 1. The odd-odd and even-even case

If the parity of both directions is odd, there are no additional symmetries and we are in the universality class of the chGUE with the symmetry breaking pattern  $U(N_f) \times U(N_f) \rightarrow U(N_f)$ . The Dirac operator has the form (69). The eigenvalues of  $D$  show no degeneracies and the microscopic spectral density is given by the  $\nu = 0$  result of the chGUE [42]

$$\rho(x) = \frac{x}{2} [J_0^2(x) + J_1^2(x)]. \quad (104)$$

Note that the two-dimensional Dirac operator has no zero modes. Therefore the level repulsion is quadratic and the repulsion from the origin is linear.

If both numbers of lattice sites,  $L_1$  and  $L_2$ , are even, the off-diagonal block  $W$  becomes itself chiral and the Dirac operator takes the form (75). Since we have no additional symmetries the symmetry class is again the one of the chGUE. The only difference with the odd-odd case is a doubling of the number of flavors with the chiral symmetry breaking pattern  $U(2N_f) \times U(2N_f) \rightarrow U(2N_f)$ . Apart from an additional degeneracy from the doubling of the flavors, the spectral properties remain the same. In particular, the

microscopic spectral density has index  $\nu = 0$  and is given by Eq. (104).

In Fig. 3(a) we show lattice data for the spectral density of the Dirac operator in the case that both  $L_1$  and  $L_2$  are either odd or even. There is an excellent agreement with the analytical random matrix result (104). Also the degree of degeneracy and the fact that there are no zero modes is confirmed by the lattice simulations. There has been a numerical study with overlap fermions [52] both in two and four dimensions which has reached similar conclusions for this case.

### 2. The even-odd case

The situation changes if  $L_1 + L_2$  is odd. Then the Dirac operator  $D$  follows the structure (72) where the  $L_1 L_2 N_c \times L_1 L_2 N_c$  matrix  $H$  is Hermitian. The corresponding symmetry class is represented by the GUE and denoted by the Cartan symbol A [40]. Due to the structure (72) and the Hermiticity of  $H$ , the flavor symmetry is doubled to  $U(2N_f)$ . However the eigenvalues of  $D$  are not doubly degenerate but come in complex conjugate pairs  $\pm i\lambda$  because  $H$  appears in the off-diagonal blocks.

The lattice Dirac operator is in the same universality class as the three-dimensional continuum theory. Hence the symmetry breaking pattern for this case is already known from QCD in three dimensions [36]. A nonzero spectral density of  $H$  results in a nonzero spectral density of the Dirac operator (see the discussion in subsection III B 2) resulting in a chiral condensate with source term  $m$ . The chiral condensate is invariant under a transformation with the unitary matrix  $U \in U(2N_f)$  if it fulfills

$$U^\dagger \begin{pmatrix} 0 & \mathbb{1}_{N_f} \\ -\mathbb{1}_{N_f} & 0 \end{pmatrix} U = \begin{pmatrix} 0 & \mathbb{1}_{N_f} \\ -\mathbb{1}_{N_f} & 0 \end{pmatrix}. \quad (105)$$

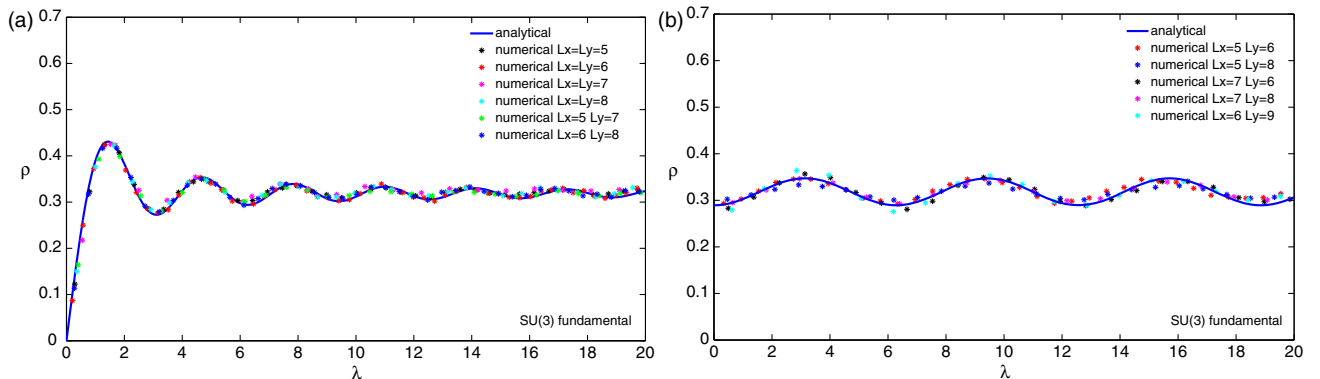


FIG. 3 (color online). Comparison of the microscopic level density of lattice QCD data in the strong coupling limit at various lattice sizes (stars) and the analytical results predicted by the corresponding random matrix theories (solid curves). The presented lattice gauge theories are (a) SU(3) fundamental with  $L_1 + L_2 = \text{even}$  and (b) SU(3) fundamental with  $L_1 + L_2 = \text{odd}$ . Note that in (b) we have a strong oscillation on top of the universal result which is a constant equal to  $1/\pi$ . Therefore we plotted the GUE result with its first correction in a  $1/n$  expansion in its matrix size  $n$ . Astoundingly also this nonuniversal term seems to fit the lattice data quite well.



This breaks chiral symmetry according to the pattern  $U(2N_f) \rightarrow U(N_f) \times U(N_f)$ .

The microscopic level density including the  $O(1/n)$  corrections of a  $2n \times 2n$  GUE is given by

$$\rho(x) = \frac{1}{\pi} \left[ 1 + \frac{\cos 2x}{8n} \right]. \quad (106)$$

In order to obtain a better fit of the analytical result to the lattice data, we have included the correction term multiplied by a fitting parameter. In Fig. 3(b) we compare the microscopic level density of GUE and lattice results. The lattice data exhibit much larger oscillations than the ones given by the  $O(1/n)$  correction in Eq. (106). One possible mechanism that may contribute to this enhancement is the condition that the off-diagonal block  $H$  of  $D$  is traceless,  $\text{tr} H = 0$ , since the translation matrices (64) have no diagonal elements. In appendix B we evaluate the spectral density for the random matrix ensemble that interpolates between the GUE and the traceless GUE. The result is given by

$$\rho_t(x) = \frac{1}{\pi} \left( 1 + \frac{1}{8n} \exp \left[ \frac{2t}{t+1} \right] \cos [2x] \right), \quad (107)$$

which shows oscillations that are enhanced by a factor of  $e^2 \approx 7.4$  for a traceless random matrix ( $t \rightarrow \infty$ ) in comparison to the original GUE ( $t = 0$ ). Because the lattice Dirac operator is sparse the effective value of  $n$  is expected much less than the size of the matrix. Nevertheless we would also expect that  $n$  still increases with the lattice size. However when using  $n$  in Eq. (107) as a fitting parameter we find that  $n \approx 7$  for almost all simulations. It is not clear why the amplitude of the oscillations does not depend on the lattice size which should be analyzed in more detail. Also other conditions such as the fixed Euclidean norm of  $H$ , i.e.  $\text{tr} H^2 = 4N_c L_1 L_2$ , may contribute to the amplitude of the oscillations.

#### IV. CONCLUSIONS

We have analyzed quenched two-dimensional lattice QCD Dirac spectra at strong coupling. The main differences with QCD in four dimensions are the absence of Goldstone bosons, the absence of topology corresponding to the Atiyah-Singer index theorem, and the non-commutativity of the antiunitary symmetries and the axial symmetry. As is the case in four dimensions, the symmetries of the Dirac operator depend on the parity of the number of lattice points in each direction. However in two dimensions we find a much richer classification of symmetry breaking patterns. As is the case in four dimensions, the corresponding random matrix class is determined by the antiunitary and the involutive symmetries. This is consistent with the maximum spontaneous breaking of chiral symmetry.

The simulations were performed with periodic boundary conditions in both directions even though we have also checked the effect of antiperiodicity in one direction. Our results remain unaffected in terms of the identifications of the universality class. Only a marginal increase of the Thouless energy was observed by this modification. We have checked that the finite size corrections of the chiral condensate scale as  $1/V$ .

Notwithstanding the Mermin-Wagner-Coleman theorem, we find that the agreement with random matrix theory is qualitatively the same in two and four dimensions. The agreement is particularly good if the Goldstone manifold contains a  $U(1)$  or  $O(1) \approx \mathbb{Z}_2$  group (i.e for the classes  $D$ ,  $DII$ ,  $BDI$ ,  $CII$  and  $AIII$ ). This raises the possibility that the long range correlations that give rise to random matrix statistics are related to the topological properties of the Goldstone manifold [53].

In this paper all numerical results are at nonzero lattice spacing. We did not attempt to perform an extrapolation to the continuum limit. Based on a bosonized form of the QCD partition function in terms of hadronic fields, one would expect a domain of low-lying eigenvalues that is dominated by the fluctuations of the zero momentum modes so that they are correlated according to random matrix theory. In the continuum limit the two-dimensional theory is expected to renormalize to a theory without spontaneous symmetry breaking. What is disturbing is that we do observe a qualitative similar behavior between QCD in two and four dimensions.

Since quenched spectra are obtained by a supersymmetric extension of the partition function, our results seem to favor the suggestion by Niedermaier and Seiler that noncompact symmetries can be broken spontaneously in two dimensions. One of the signatures of this type of spontaneous symmetry breaking is an order parameter that wanders off to infinity. Indeed, in [16] it was found that the chiral condensate of the quenched Schwinger model seems to diverge in the thermodynamic limit. On the other hand, the Dirac spectrum of the  $N_f = 1$  Schwinger model behaves as predicted by random matrix theory. It is clear that the chiral condensate is determined by the anomaly and does not involve any noncompact symmetries. Because of the absence of massless excitations the partition function of the one flavor Schwinger model must be smooth as a function of the quark mass. This implies that the condensate due to the nonzero Dirac eigenvalues must be the same as the condensate from the one-instanton configurations in the massless limit. This suggests that the eigenfunctions of the low-lying nonzero mode states must be delocalized and that the eigenvalue fluctuations are described by random matrix theory, so that the supersymmetric partition function that generates the Dirac spectrum looks like it has spontaneous symmetry breaking.

An alternative scenario arises because of the finiteness of the Thouless energy in units of the average level spacing.

The fermion determinant due to massless quarks may push all eigenvalues beyond the Thouless energy into the localized domain resulting in a partition function with no spontaneous breaking of chiral symmetry. To find out if this is the case we would have to study two-dimensional lattice QCD with dynamical quarks. This scenario is not favored by simulations of the Schwinger model. Both the one- and two-flavor Schwinger model show excellent agreement with random matrix statistics and the agreement improves with increasing volumes which also excludes the possibility that the localization length is larger than the size of the box.

Our study raises many questions. The most fundamental issue is the reconciliation of the agreement with random matrix theory and the implied spontaneous breaking of chiral symmetry with the Mermin-Wagner-Coleman theorem. In particular, can the noncompact symmetry of the supersymmetric generating function for the Dirac spectrum of two-dimensional QCD-like theories be spontaneously broken? To address this we have to analyze the approach to the thermodynamic limit and the continuum limit. Such studies could also settle whether or not the localization length of the low-lying states exceeds the size of the box used in the present work. This is supported by Dirac spectra of the quenched Schwinger model which deviate more from random matrix theory with increasing volume [16], but there is no hint of this in our results. Another intriguing question is the possibility that all states become localized beyond a critical number of flavors. A final issue concerns the number of generic zero modes of the QCD Dirac operator for fermions in the adjoint representation. Based on chiral perturbation theory and random matrix theory we predict that the Dirac operator may have no or only two generic zero modes of opposite chirality. In future work we hope to address the nature of these zero modes and the possible relation with the complexified zero modes found in Ref. [45].

## ACKNOWLEDGMENTS

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## APPENDIX A: RANDOM MATRIX THEORIES OF TWO-DIMENSIONAL CONTINUUM QCD

In this appendix we evaluate the joint probability density of the eigenvalues and the pattern of chiral symmetry breaking of random matrix theory corresponding to the continuum limit of two-dimensional QCD. The case of two colors with fundamental fermions is worked out in the subsection A 1 and the case with two or more colors with

fermions in the adjoint representation is discussed in the subsection A 2. The case with three or more colors with fermions in the fundamental representation follows the same pattern in two and four dimensions and is not discussed here. Although the results of this appendix are known, discussing them in the present framework will add to the readability of this paper.

### 1. Random matrix theory for two-dimensional QCD with two colors in the fundamental representation

For two colors with the quarks in the fundamental representation we can find a gauge field independent basis for which the Dirac operator becomes real. In two dimensions this transformation does not commute with the transformation to a block structure reflecting its chiral symmetry, see Sec. II A. We choose to preserve the chiral structure of the Dirac operator. Then the consequence of the antiunitary symmetry is that the off-diagonal block of the Dirac operator is complex anti-self-dual which is unitarily equivalent to a random matrix theory with an off-diagonal block that is complex symmetric. The corresponding chiral random matrix theory is given by

$$D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix}, \quad W = -\tau_2 W^T \tau_2 \in \mathbb{C}^{2n \times 2n}, \quad (\text{A1})$$

or equivalently by

$$D' = \begin{bmatrix} 0 & W\tau_2 \\ -W^\dagger\tau_2 & 0 \end{bmatrix}, \quad (W\tau_2)^T = W\tau_2 \in \mathbb{C}^{2n \times 2n}. \quad (\text{A2})$$

The probability distribution is taken to be Gaussian,

$$P(W)d[W] \propto \exp[-n \text{tr} WW^\dagger] \prod_{1 \leq i \leq j \leq 2n} d\text{Re}W_{ij} d\text{Im}W_{ij}. \quad (\text{A3})$$

In the subsection A 1 a we calculate the joint eigenvalue probability density of this theory (see Ref. [43]). In the subsection A 1 b we rederive its partition function which was already summarized for all chiral ensembles in Refs. [40,48].

#### a. Joint probability density

The joint probability density of the eigenvalues of the random matrix  $D$  denoted by  $p(\Lambda)$  is defined by

$$\int_{\mathbb{C}^{2n \times 2n}} f(D) P(W) d[W] = \int_{\mathbb{R}_+^{2n}} f(\pm i\Lambda) p(\Lambda) \prod_{1 \leq j \leq 2n} d\lambda_j \quad (\text{A4})$$

for any function  $f$  invariant under

$$f(D) = f(VDV^\dagger) \quad (\text{A5})$$

for all  $V = \text{diag}(\tilde{V}, \tau_2 \tilde{V}^* \tau_2)$  or equivalently  $W \rightarrow \tilde{V}W\tau_2\tilde{V}^T\tau_2$  with  $\tilde{V} \in \text{U}(2n)$ .

The characteristic polynomial of  $D$  can be rewritten as

$$\det(D - i\lambda\mathbb{1}_{4n}) = \det(WW^\dagger - \lambda^2\mathbb{1}_{2n}) = \det(W^\dagger W - \lambda^2\mathbb{1}_{2n}). \quad (\text{A6})$$

Let  $U \in \text{U}(2n)/\text{U}^{2n}(1)$  be the matrix diagonalizing  $WW^\dagger$ , i.e.  $WW^\dagger = U\Lambda^2U^\dagger$  with the positive definite, diagonal matrix  $\Lambda^2 \in \mathbb{R}_+^{2n}$ . Then we can relate the eigenvectors of  $WW^\dagger$  to those of  $W^\dagger W$ . Let

$$WW^\dagger U = (W\tau_2)(W\tau_2)^\dagger U = U\Lambda^2, \quad (\text{A7})$$

then complex conjugation results in

$$(W\tau_2)^*(W\tau_2)^T U^* = U^*\Lambda^2, \quad (\text{A8})$$

and because of the symmetry of  $W\tau_2$ , we also have

$$(W\tau_2)^\dagger W\tau_2 U^* = U^*\Lambda^2. \quad (\text{A9})$$

Hence the eigenvalue decomposition of  $W^\dagger W$  reads

$$(W\tau_2)^\dagger (W\tau_2) = U^*\Lambda^2U^T. \quad (\text{A10})$$

The combination of this decomposition with  $WW^\dagger = U\Lambda^2U^\dagger$  yields a singular value decomposition of  $W$ ,

$$W\tau_2 = UZU^T, \quad (\text{A11})$$

with the complex, diagonal matrix  $Z \in \mathbb{C}^{2n}$  such that  $|Z| = \Lambda$  and  $U \in \text{U}(2n)/\text{U}^{2n}(1)$ . The number of degrees of freedom is  $2n(2n+1)$  on both sides of Eq. (A11). Hence, the right hand side of Eq. (A11) can be used as a parametrization of  $W$ . The phases of  $Z$  can be absorbed in  $U$  so that  $W$  can be parametrized as

$$W\tau_2 = U\Lambda U^T \quad (\text{A12})$$

with the positive definite, diagonal matrix  $\Lambda \in \mathbb{R}_+^{2n}$  and  $U \in \text{U}(2n)$ .

In the next step we calculate the invariant length element which directly yields the Haar measure of  $W$  in the coordinates (A12),

---


$$\begin{aligned} \text{tr}dWdW^\dagger &= \text{tr}d(W\tau_2)d(W\tau_2)^\dagger \\ &= \text{tr}d\Lambda^2 + \text{tr}(U^\dagger dU\Lambda + \Lambda(U^\dagger dU)^T)(U^\dagger dU\Lambda + \Lambda(U^\dagger dU)^T)^\dagger \\ &= \sum_{1 \leq i \leq 2n} (d\lambda_i^2 + 4\lambda_i^2 (U^\dagger dU)_{ii}^2) \\ &\quad + \sum_{1 \leq i < j \leq 2n} [(U^\dagger dU)_{ij}, (U^\dagger dU)_{ij}^*] \begin{bmatrix} \lambda_i \lambda_j & -\frac{\lambda_i^2 + \lambda_j^2}{2} \\ -\frac{\lambda_i^2 + \lambda_j^2}{2} & \lambda_i \lambda_j \end{bmatrix} \begin{bmatrix} (U^\dagger dU)_{ij} \\ (U^\dagger dU)_{ij}^* \end{bmatrix}. \end{aligned} \quad (\text{A13})$$

Note that the Pauli matrix  $\tau_2$  drops out. Moreover we have used the anti-Hermiticity of  $U^\dagger dU$ . From the invariant length (A13) we find the joint probability density

$$p(\Lambda) \prod_{1 \leq j \leq 2n} d\lambda_j \propto |\Delta_{2n}(\Lambda^2)| \prod_{1 \leq j \leq 2n} \exp[-n\lambda_j^2] \lambda_j d\lambda_j, \quad (\text{A14})$$

cf. Refs. [43,48]. This coincides with the joint probability density of the nonzero eigenvalues of the chiral GOE with  $\nu = 1$ , which has one zero mode while the present model has no zero modes at all. Its microscopic spectral density has a linear slope at the origin and the level repulsion is also linear at small distances, cf. Fig 1(a).

### b. Partition function

The partition function with  $N_f$  flavors is defined by

$$Z(N_f) = \int d[W] \prod_{k=1}^{N_f} \det(D + m_k \mathbb{1}_{4n}) P(W). \quad (\text{A15})$$

Due to the decomposition (A12) we multiply  $D$  by the unitary matrix  $\text{diag}(\mathbb{1}_{2n}, \tau_2)$  from the left and from the right which keeps the spectrum invariant. To evaluate the average (A15) we first rewrite the determinants as Gaussians over Grassmann variables,

$$\begin{aligned} Z(M) &\propto \int d[W, V] \exp[-n\text{tr}W\tau_2(W\tau_2)^\dagger] \\ &\quad \times \exp[\text{tr}V_R^\dagger W\tau_2 V_L - \text{tr}V_L^\dagger (W\tau_2)^\dagger V_R] \\ &\quad \times \exp[\text{tr}M(V_R^\dagger V_R + V_L^\dagger V_L)], \end{aligned} \quad (\text{A16})$$

with the mass matrix  $M = \text{diag}(m_1, \dots, m_{N_f})$ . The matrices  $V_R$  and  $V_L$  are both  $2n \times N_f$  rectangular matrices comprising independent Grassmann variables as matrix elements. Because  $W\tau_2$  is symmetric we have to symmetrize the matrices  $V_L V_R^\dagger$  and  $V_R V_L^\dagger$ . After integrating over  $W$  we obtain

$$\begin{aligned}
Z(M) &\propto \int d[V] \exp \left[ -\frac{1}{4n} \text{tr} (V_L V_R^\dagger - V_R^* V_L^T) (V_R V_L^\dagger - V_L^* V_R^T) \right] \\
&\quad \times \exp [\text{tr} M (V_R^\dagger V_R + V_L^\dagger V_L)] \\
&\propto \int d[V] \exp \left[ \frac{1}{4n} \text{tr} (\tilde{\tau}_2 \otimes \mathbb{1}_{N_f}) \sigma (\tilde{\tau}_2 \otimes \mathbb{1}_{N_f}) \sigma^T \right] \\
&\quad \times \exp [\text{tr} (\mathbb{1}_2 \otimes M) \sigma], \tag{A17}
\end{aligned}$$

where  $\tau_2$  completely drops out. The second Pauli matrix  $\tilde{\tau}_2$  acts on flavor space and should not be confused with  $\tau_2$  which acts on color space for QCD and its analogue in random matrix theory. The dyadic super matrix

$$\sigma = \begin{bmatrix} V_R^\dagger \\ -V_L^T \end{bmatrix} \begin{bmatrix} V_R & V_L^* \end{bmatrix}. \tag{A18}$$

is nilpotent and can be replaced by a unitary matrix  $U \in \text{U}(2N_f)$  via the superbosonization formula [54–56]. By rescaling  $U \rightarrow 2nU$  and introducing the rescaled mass matrix  $\hat{M} = 2nM$ , we arrive at

$$\begin{aligned}
Z(\hat{M}) &\propto \int_{\text{U}(2N_f)} \exp [n \text{tr} (\tilde{\tau}_2 \otimes \mathbb{1}_{N_f}) U (\tilde{\tau}_2 \otimes \mathbb{1}_{N_f}) U^T] \\
&\quad \times \exp [\text{tr} (\mathbb{1}_2 \otimes \hat{M}) U] \det^{-2n} U d\mu(U), \tag{A19}
\end{aligned}$$

where  $d\mu$  is the normalized Haar-measure.

In the microscopic limit ( $n \rightarrow \infty$  and  $\hat{M}$  fixed) we can apply the saddlepoint approximation. The saddlepoint equation is given by

$$U^{-1} = (\tilde{\tau}_2 \otimes \mathbb{1}_{N_f}) U^T (\tilde{\tau}_2 \otimes \mathbb{1}_{N_f}). \tag{A20}$$

Since  $U \in \text{U}(2N_f)$  Eq. (A20) implies  $U \in \text{USp}(2N_f)$ . The final result is given by

$$\begin{aligned}
Z(\hat{M}) &= \int_{\text{USp}(2N_f)} \exp [\text{tr} (\mathbb{1}_2 \otimes \hat{M}) U] d\mu(U) \\
&= \int_{\text{USp}(2N_f)} \exp \left[ \frac{1}{2} \text{tr} (\mathbb{1}_2 \otimes \hat{M}) (U + U^{-1}) \right] d\mu(U). \tag{A21}
\end{aligned}$$

Although the joint probability density of the eigenvalues coincides with the chGOE, the chiral symmetry breaking pattern ( $\text{USp}(2N_f) \times \text{USp}(2N_f) \rightarrow \text{USp}(2N_f)$ ) turns out to be different and agrees with Refs. [31,40]. Especially there are no zero modes such that the partition function does not vanish at  $M = 0$  which would be the case for the chGOE with index  $\nu = 1$ , see Ref. [35].

## 2. Two-dimensional QCD in the adjoint representation

For two-dimensional QCD with quarks in the adjoint representation the antiunitary symmetry of the Dirac operator allows us to choose a gauge field independent basis for which the Dirac operator becomes quaternion real. However, when performing this transformation we will lose the chiral block structure. We choose to preserve this structure. Then the antiunitary symmetry requires that the off-diagonal block of the Dirac operator becomes antisymmetric. The corresponding random matrix theory is given by

$$D = \begin{bmatrix} 0 & W \\ -W^\dagger & 0 \end{bmatrix}, \quad W = -W^T \in \mathbb{C}^{(2n+\nu) \times (2n+\nu)}. \tag{A22}$$

with the probability distribution

$$P(W) d[W] \propto \exp [-n \text{tr} W W^\dagger] \prod_{1 \leq i < j \leq (2n+\nu)} d\text{Re} W_{ij} d\text{Im} W_{ij}. \tag{A23}$$

Because odd-dimensional antisymmetric matrices have one generic zero eigenvalue we have to distinguish the even- and odd-dimensional case (denoted by  $\nu = 0$  and  $\nu = 1$ , respectively).

In subsection A 2 a we evaluate the joint probability density of the eigenvalues and in subsection A 2 b we discuss the corresponding partition function for  $\nu = 0, 1$ . These results were obtained previously in Refs. [31,40,43,48].

### a. Joint Probability Distribution

The joint probability density  $p(\Lambda)$  is defined as in Eq. (A4) while the arbitrary function  $f$  has the invariance

$$\begin{aligned}
f(D) &= f(V D V^\dagger), \quad \forall V = \text{diag}(\tilde{V}, \tilde{V}^*) \quad \text{with} \\
\tilde{V} &\in \text{U}(2n + \nu). \tag{A24}
\end{aligned}$$

Let  $\nu = 0$ , i.e.  $W$  is even dimensional. Analogous to the discussion in subsection A 1 a we can quasi-diagonalize  $W$ , i.e.

$$W = U (\tau_2 \otimes \Lambda) U^T \tag{A25}$$

with a positive definite, diagonal matrix  $\Lambda \in \mathbb{R}_+^n$  and a unitary matrix  $U \in \text{U}(2n)/\text{SU}^n(2)$ . The division with the subgroup  $\text{SU}^n(2)$  is the result of the identity  $\tilde{U} \tau_2 \tilde{U}^T = \tau_2$  for all  $\tilde{U} \in \text{SU}(2)$ .

The matrix  $\tau_2 \otimes \Lambda$  has  $\pm \lambda_j$  as eigenvalues. We can use the result (A14) by replacing  $\text{diag}(\lambda_1, \dots, \lambda_{2n}) \rightarrow \text{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n)$  and taking care of the fact that some degrees of freedom of  $\text{U}(2n)$  are missing. We can



apply the result (A14) because the invariant length element is calculated similar to the ( $\beta_D = 1, d = 2$ )-case. Hence, we find the joint probability density

$$p(\Lambda) \prod_{1 \leq j \leq 2n} d\lambda_j \propto \Delta_n^4(\Lambda^2) \prod_{1 \leq j \leq n} \exp[-n\lambda_j^2] \lambda_j d\lambda_j, \quad (\text{A26})$$

cf. Ref. [43,48]. This density coincides with the chGSE for  $\nu = -1/2$ .

Let us consider the case with an odd dimension,  $W = -W^T \in \mathbb{C}^{(2n+1) \times (2n+1)}$ . Since an odd-dimensional antisymmetric matrix has one generic zero mode we have to modify the decomposition (A25) according to

$$W = U \text{diag}(\tau_2 \otimes \Lambda, 0) U^T, \quad (\text{A27})$$

where  $\Lambda \in \mathbb{R}_+^n$  and  $U \in \text{U}(2n+1)/[\text{SU}^n(2) \times \text{U}(1)]$ . Hence, the joint probability density (A26) becomes [43,48]

$$p(\Lambda) \prod_{1 \leq j \leq 2n} d\lambda_j \propto \Delta_n^4(\Lambda^2) \prod_{1 \leq j \leq n} \exp[-n\lambda_j^2] \lambda_j^5 d\lambda_j \quad (\text{A28})$$

by employing the result (A14) with the replacement  $\text{diag}(\lambda_1, \dots, \lambda_{2n}) \rightarrow \text{diag}(\lambda_1, \dots, \lambda_n, -\lambda_1, \dots, -\lambda_n, 0)$  and taking care of the subgroup  $\text{SU}^n(2) \times \text{U}(1)$  that is divided out. This coincides with the joint probability density of the nonzero eigenvalues of the chGSE with  $\nu = 1/2$ .

### b. Partition Function

The partition function with  $N_f$  fermionic flavors (A15) can be again mapped to flavor space via the rectangular  $(2n + \nu) \times N_f$  matrices  $V_R$  and  $V_L$  comprising Grassmann variables only. The analogue of Eq. (A17) is given by

$$\begin{aligned} Z(M) &\propto \int d[V] \exp \left[ -\frac{1}{4n} \text{tr}(V_L V_R^\dagger + V_R^* V_L^T)(V_R V_L^\dagger + V_L^* V_R^T) \right] \\ &\quad \times \exp \left[ \text{tr} M (V_R^\dagger V_R + V_L^\dagger V_L) \right] \\ &\propto \int \exp \left[ \frac{1}{4n} \text{tr} \sigma \sigma^T + \text{tr}(\tilde{\tau}_1 \otimes M) \sigma \right] d[V] \end{aligned} \quad (\text{A29})$$

with the dyadic supermatrix

$$\sigma = \begin{bmatrix} -V_L^T \\ V_R^\dagger \end{bmatrix} \begin{bmatrix} V_R & V_L^* \end{bmatrix}. \quad (\text{A30})$$

The first Pauli matrix  $\tilde{\tau}_1$  acts on flavor space. The superbosonization formula [54–56] yields

$$\begin{aligned} Z(\hat{M}) &\propto \int_{\text{U}(2N_f)} \exp [n \text{tr} U U^T + \text{tr}(\tilde{\tau}_1 \otimes \hat{M}) U] \\ &\quad \times \det^{-2n-\nu} U d\mu(U). \end{aligned} \quad (\text{A31})$$

In the microscopic limit by taking  $n$  to infinity we have to solve the saddlepoint equation

$$U^{-1} = U^T. \quad (\text{A32})$$

Therefore we end up with an integral over the group  $\text{O}(2N_f)$ , i.e.

$$\begin{aligned} Z(\hat{M}) &= \int_{\text{O}(2N_f)} \exp [\text{tr}(\tilde{\tau}_1 \otimes \hat{M}) U] \det^\nu U d\mu(U), \\ &= \int_{\text{O}(2N_f)} \exp [\text{tr}(\mathbb{1}_2 \otimes \hat{M}) U] \det^\nu U d\mu(U), \\ &= \int_{\text{O}(2N_f)} \exp \left[ \frac{1}{2} \text{tr}(\mathbb{1}_2 \otimes \hat{M})(U + U^{-1}) \right] \det^\nu U d\mu(U) \end{aligned} \quad (\text{A33})$$

with  $\hat{M} = 2nM$ . For  $\nu = 0$  the partition function is of order one for  $\hat{M} \ll 1$  while for  $\nu = 1$ , the sum over two disconnected components of  $\text{O}(2N_f)$ , results in a partition function  $Z(\hat{M}) \propto \hat{M}^2$  for  $\hat{M} \ll 1$ . This property as well as the symmetry breaking pattern  $\text{O}(2N_f) \times \text{O}(2N_f) \rightarrow \text{O}(2N_f)$  underlines the difference of the random matrix ensemble (A22) with the chGSE, see Refs. [35,40]. The sum over  $\nu = 0$  and  $\nu = 1$  gives an integral over  $\text{SO}(2N_f)$  corresponding to the symmetry breaking pattern of the full partition function [31].

As was shown in Ref. [45] gauge fields with nonzero topology exist for two-dimensional QCD with adjoint fermions and both partition functions for  $\nu = 0$  and  $\nu = 1$  are realized. The argument of Ref. [45] predicting additional values of  $\nu$  for  $N_c > 2$  seems to be in conflict with chiral perturbation theory [31] and random matrix theory, but we hope to address this puzzle in future work. In lattice QCD at strong coupling in the case of an odd-odd lattice only  $\nu = 0$  and  $\nu = 1$  are realized for  $N_c$  odd and  $N_c$  even, respectively. Our simulations confirm this prediction, see Fig. 2.

## APPENDIX B: CORRECTIONS TO THE TRACELESS ENSEMBLE

In this appendix we calculate the eigenvalue density including  $1/n$  corrections for an even-dimensional GUE with the additional condition that the trace of the matrices may vanish. This condition is implemented via a Lagrange multiplier. The level density is thus given by the random matrix integral,

$$\rho_t^{(n)}(x) = \frac{\int_{\text{Herm}(2n)} d[H] \exp[-\text{tr}H^2/(4n) - t\text{tr}^2H/(8n^2)] \text{tr}\delta(H - x\mathbb{1}_{2n})}{\int_{\text{Herm}(2n)} d[H] \exp[-\text{tr}H^2/(4n) - t\text{tr}^2H/(8n^2)]}. \quad (\text{B1})$$

The parameter  $t$  interpolates between the traceless condition ( $t \rightarrow \infty$ ) and the ordinary GUE ( $t \rightarrow 0$ ). The square of the trace in  $H$  can be linearized by a Gaussian integral over an auxiliary scalar variable  $\lambda$ , meaning that we can trace back the whole problem to ordinary GUE,

$$\rho_t^{(n)}(x) = \int_{-\infty}^{\infty} \sqrt{\frac{1+t}{2t\pi}} d\lambda \exp\left[-\frac{1+t}{2t}\lambda^2\right] \rho_0^{(n)}(x+i\lambda). \quad (\text{B2})$$

The level density of GUE is given in terms of Hermite polynomials,  $H_j(x) = x^j + \dots$ , in the following formula [44],

$$\begin{aligned} \rho_0^{(n)}(x) &= \frac{(2n)!}{\sqrt{4\pi n}} \exp\left[-\frac{x^2}{4n}\right] \left(\frac{(2n)^{2n-1}}{((2n-1)!)^2} H_{2n-1}^2\left(\frac{x}{\sqrt{2n}}\right) \right. \\ &\quad \left. - \frac{(2n)^{2n-1}}{(2n)!(2n-2)!} H_{2n}\left(\frac{x}{\sqrt{2n}}\right) H_{2n-2}\left(\frac{x}{\sqrt{2n}}\right)\right). \end{aligned} \quad (\text{B3})$$

The large  $n$  asymptotics of  $H_{2n}(x/\sqrt{2n})$  where  $x$  is fixed can be obtained by the relation between Hermite polynomials with an even order and the associated Laguerre polynomials,  $L_n^{(-1/2)}(x) = x^n + \dots$ ,

$$H_{2n}\left(\frac{x}{\sqrt{2n}}\right) = 2^n L_n^{(-1/2)}\left(\frac{x^2}{2n}\right). \quad (\text{B4})$$

Note that we employ for both polynomials the monic normalization. The associated Laguerre polynomials  $L_n^{(\nu)}$  with a positive integer index  $\nu$  have a simple representation as a contour integral,

$$L_n^{(\nu)}\left(\frac{x^2}{2n}\right) = \frac{n!}{(2n)^n} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\nu\varphi} \left(1 - \frac{e^{-i\varphi}}{2n}\right)^{n+\nu} \exp[x^2 e^{i\varphi}], \quad (\text{B5})$$

which can be expanded asymptotically as

$$\begin{aligned} L_n^{(\nu)}\left(\frac{x^2}{2n}\right) &\stackrel{n \gg 1}{\approx} \frac{1}{(2n)^n} \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{i\nu\varphi} \\ &\quad \times \exp\left[x^2 e^{i\varphi} - (n+\nu)\left(\frac{e^{-i\varphi}}{2n} + \frac{e^{-2i\varphi}}{8n^2}\right)\right] \\ &\approx \frac{1}{(2n)^n} \left(\frac{J_{-\nu}(x)}{x^\nu} - \frac{\nu J_{1-\nu}(x)}{2nx^{\nu-1}} - \frac{J_{2-\nu}(x)}{8nx^{\nu-2}}\right). \end{aligned} \quad (\text{B6})$$

The functions  $J_j$  are the Bessel functions of the first kind and can be analytically continued in their index  $j$ . For  $\nu = -1/2$  the expansion for the Hermite polynomials reads

$$\begin{aligned} &\frac{1}{n!} H_{2n}\left(\frac{x}{\sqrt{2n}}\right) \\ &\stackrel{n \gg 1}{\approx} \frac{1}{n^n} \left(\sqrt{x} J_{1/2}(x) + \frac{x^{3/2} J_{3/2}(x)}{4n} - \frac{x^{5/2} J_{5/2}(x)}{8n}\right) \\ &= \frac{1}{n^n} \sqrt{\frac{2}{\pi}} \left(\sin x + \frac{1}{8n}((x^2-1)\sin x + x \cos x)\right). \end{aligned} \quad (\text{B7})$$

From this asymptotic expansion it also follows

$$\begin{aligned} &\frac{1}{(n-1)!} H_{2n-1}\left(\frac{x}{\sqrt{2n}}\right) \\ &= \sqrt{\frac{n}{2}} \frac{\partial}{\partial x} \frac{1}{n!} H_{2n}\left(\frac{x}{\sqrt{2n}}\right) \\ &\stackrel{n \gg 1}{\approx} \frac{1}{n^n} \sqrt{\frac{n}{\pi}} \left(\cos x + \frac{1}{8n}(x \sin x + x^2 \cos x)\right). \end{aligned} \quad (\text{B8})$$

and

$$\begin{aligned} &\frac{1}{(n-2)!} H_{2n-2}\left(\frac{x}{\sqrt{2n}}\right) \\ &= \frac{\sqrt{2n}(n-1)}{2n-1} \frac{\partial}{\partial x} \frac{1}{(n-1)!} H_{2n-1}\left(\frac{x}{\sqrt{2n}}\right) \\ &\stackrel{n \gg 1}{\approx} \frac{n-1}{(2n-1)n^{n-1}} \sqrt{\frac{2}{\pi}} \\ &\quad \times \left(-\sin x + \frac{1}{8n}((1-x^2)\sin x + 3x \cos x)\right), \end{aligned} \quad (\text{B9})$$

with help of the recurrence relation of the Hermite polynomials and the Stirling formula including subleading corrections,

$$n! \stackrel{n \gg 1}{\approx} \sqrt{2\pi n} n^n e^{-n} \left(1 + \frac{1}{12n}\right). \quad (\text{B10})$$

Summarizing all these asymptotic expansions and plugging everything into the level density (B3) we find the first correction to the GUE asymptotics,

$$\begin{aligned}
\rho_0^{(n)}(x) &\stackrel{n \gg 1}{\approx} \frac{1}{\pi} \left(1 - \frac{x^2}{4n}\right) \left(1 + \frac{1}{8n}\right) \left(\cos^2 x + \frac{1}{4n} \cos x (x \sin x + x^2 \cos x)\right. \\
&\quad \left. - \left(-\sin^2 x + \frac{1}{4n} \sin x ((1 - x^2) \sin x + x \cos x)\right)\right) \\
&\approx \frac{1}{\pi} \left(1 + \frac{\cos 2x}{8n}\right).
\end{aligned} \tag{B11}$$

One can now perform the integral (B2) which yields the result (107).

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