Acceleration of electric current-carrying string loop near a Schwarzschild black hole immersed in an asymptotically uniform magnetic field

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We study the acceleration of an electric current-carrying and axially symmetric string loop initially oscillating in the vicinity of a Schwarzschild black hole embedded in an external asymptotically uniform magnetic field. The plane of the string loop is orthogonal to the magnetic field lines and the acceleration of the string loop occurs due to the transmutation effect turning in the deep gravitational field the internal energy of the oscillating strings to the energy of their translational motion along the axis given by the symmetry of the black hole spacetime and the magnetic field. We restrict our attention to the motion of string loop with energy high enough, when it can overcome the gravitational attraction and escape to infinity. We demonstrate that for the current-carrying string loop the transmutation effect is enhanced by the contribution of the interaction between the electric current of the string loop and the external magnetic field and we give conditions that have to be fulfilled for an efficient acceleration. The Schwarzschild black hole combined with the strong external magnetic field can accelerate the current-carrying string loop up to the velocities close to the speed of light $v \sim c$. Therefore, the string loop transmutation effect can potentially well serve as an explanation for acceleration of highly relativistic jets observed in microquasars and active galactic nuclei.

I. INTRODUCTION

Current-carrying string loop represents a simplified 1D model of magnetized-plasma structures [1]. The plasma may exhibit a stringlike structure arising from either dynamics of the magnetic field lines in the plasma [2–4] or thin isolated flux tubes produced in plasma [5–8]. Tension of such a string loop governs an outer barrier of the string loop motion, while its world sheet current introduces an angular momentum barrier preventing the loop from collapse.

Dynamics of an axially symmetric string loop along the axis of symmetry of Kerr black holes has been investigated in [1,9], and extended also to the case of Kerr naked singularities [9]. The string loop dynamics in the spherically symmetric spacetimes has been studied for the case of Schwarzschild–de Sitter (SdS) black holes [10] and brane-world black holes or naked singularities described by the Reissner-Nordstrom spacetimes [11]. Such a configuration was also studied in [12,13].

Quite recently, it has been demonstrated that small oscillations of string loop around a stable equilibrium radii in the Kerr black hole spacetimes can be well applied in

astrophysical situations related to the high-frequency quasiperiodic oscillations observed in microquasars, i.e., binary systems containing a black hole [14].

On the other hand, it has been proposed in [1] that the current-carrying string loops could be relevant in an inverse astrophysical situation, as a model of formation and collimation of relativistic jets in the field of compact objects. The acceleration of jets is possible due to the transmutation effect where the chaotic character of the string loop motion around a central black hole enables transmission of the internal energy in the oscillatory mode to the kinetic energy of the linear mode [1,10]. It has been demonstrated in [11,15] that ultrarelativistic escaping velocities of string loop can be really obtained even in spherically symmetric black hole spacetimes. Efficiency of the transmutation effect is slightly enhanced by the rotation of the central Kerr black hole as demonstrated in [9]. Enhancement of the efficiency of the transmutation process can be substantial in the innermost parts of the Kerr naked singularity spacetimes [9], similarly to the acceleration process occurring in the particle collisions [16].

The string loop accelerated by the gravitational field of nonrotating black holes thus can potentially serve as a new model of ultrarelativistic jets observed in microquasars and active galactic nuclei. This is an important result since the standard models of jet formation are based on the Blandford-Znajek effect [17] that requires rapidly rotating Kerr black holes [18]. The rotation of the central black hole

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is an important aspect also in the alternate model of ultrarelativistic jet formation based on the geodesic collimation along the rotation axis of the Kerr spacetime metric [19,20].

It is generally assumed that the strong interaction of the magnetic field with the gravitational field of a compact object plays a crucial role in the processes of the formation and ejection of relativistic jets in the accreting systems orbiting the compact objects. The magnetohydrodynamics (MHD) of plasma in such processes enables the energy transfer from accreting matter to jets in the standard Blandford-Znajek model [17]. The recent observational data and theoretical study of compact systems show that the magnetic field has to exist in the black hole environment [21,22] (see, e.g., discussion in [23]).

Understanding the dynamics of charged particles in the combined magnetic and gravitational fields is necessary for modeling the MHD processes. The single-particle dynamics is relevant also for collective processes modeled in the framework of kinetic theory [6–8]. Charged particle motion in electromagnetic fields surrounding the black hole or in the field of magnetized neutron stars has been studied in a large variety of works for both equatorial and general motion (see e.g., [24,25]). The special class of the off-equatorial circular motion of charged particles in combined gravitational and electromagnetic fields of compact objects has been studied in papers [26–28]. Detailed analysis of the astrophysics of rotating black holes with electromagnetic fields related to the Penrose process has been discussed in [29,30]. The Blandford-Znajek mechanism applied for a black hole with a toroidal electric current was investigated in [31]. The oscillatory motion of charged particles around equatorial and off-equatorial circular orbits could be relevant in the formation of magnetized string loops [6,32].

For the model of ultrarelativistic jet formation based on the transmutation effect in the string loop dynamics, it is important to clear up the role of the external influences, namely those based on the cosmic repulsion and the large-scale magnetic fields. It has been demonstrated in [15] that the escaping string loops will be efficiently accelerated by the cosmic repulsion behind the so-called static radius [33,34] that plays an important role also in the accretion toroidal structures [35] or motion of gravitationally bound galaxies [36]. Here we shall study the role of an asymptotically uniform external magnetic field in the transmutational acceleration of electrically charged current-carrying string loop in the field of nonrotating black holes. The acceleration of the string loop is assumed along the direction of the vector of the strength of the magnetic field. Because of the axial symmetry of the investigated system of the string loop and the central black hole immersed in the magnetic field, we are able to describe the string loop motion by an effective potential similar to those of the charged particle motion. Using the results of our preceding paper [37], we will show that the presence of even a weak magnetic field can sufficiently increase the possibility and efficiency of conversion of the internal oscillatory energy of the string loop into the kinetic energy of its linear transitional motion.

We focus our attention to the simple case of the asymptotically uniform magnetic field in order to illustrate the large scale role of the magnetic field on the string loop motion. For the transmutation effect itself, the local strength of the magnetic field is relevant, but the subsequent motion is influenced by the large scale structure of the magnetic field. Estimates related to the magnitude of the magnetic field in astrophysically plausible situations related to the magnetic field around neutron stars, stellar black holes and supermassive black holes in the galactic nuclei are presented in Appendix A. According to these estimates, the magnetic field can be always considered as a test field, having a negligible influence of the spacetime structure.

The paper is organized as follows. In Sec. II, the general relativistic description of the string loop model is presented, the fundamental quantities that characterize the electric current-carrying string loop through the action function and the Lagrangian formalism are given. In Sec. III, the motion of the string loop in the combined electromagnetic and gravitational fields of a Schwarzschild black hole immersed in an asymptotically uniform magnetic field is studied. Due to the symmetries of the string loop and the combined electromagnetic and gravitational background, the string loop dynamics can be governed by a properly defined effective potential, which is compared to the effective potential governing motion of charged particles. At the end of Sec. III, the physical interpretation of the string loop model through the superconductivity phenomena of plasmas in accretion disk is discussed. In Sec. IV, the transmutation effect and the acceleration of the string loop are studied. Dependence of the ejection speed (relativistic Lorentz γ factor) on the intensity of the external uniform magnetic field is given. It is shown that the maximal acceleration of the string loop, up to the ultrarelativistic velocities ($v \simeq c$, $\gamma \gg 1$) is possible for the special orientation of an electric current and the case of strong magnetic field. The concluding remarks and discussions are presented in Sec. V. In Appendix A, the estimation of the realistic magnetic field intensity is presented and discussed. In Appendix B, the dimensional analysis of the characteristic parameters of the string loop model is presented, along with estimates of the physical quantities that characterize the string loop.

II. RELATIVISTIC ELECTRIC CURRENT-CARRYING STRING LOOP

Generally, the string loops are assumed to be thin circular objects that carry a current and preserve their axial symmetry relative to the chosen axis, or the axis of the black hole spacetime. The string loops can oscillate, changing their radius in the loop plane, while propagating in the perpendicular direction as shown in Fig. 1. Let us

consider first a string loop moving in a spherically symmetric spacetime with the line element

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2. \quad (1)$$

We will specify the components of the metric tensor $g_{\alpha\beta}$ for the Schwarzschild black hole case in Sec. III. The spherically symmetric Schwarzschild black hole is assumed to be immersed in an asymptotically uniform magnetic field; both the spacetime and the external magnetic field define an axis of symmetry that is considered to be the axis of the string loop.

In order to give the relativistic description for the motion of string loop, thereby enabling the derivation of the equations of motion one may chose the action, which will reflect the properties of both the string loop and external fields. However, before giving the definition of the string action, one has to introduce the string world sheet which is characterized by a scalar function φ and by coordinates $X^\alpha(\sigma^a)$, where $\alpha = 0, 1, 2, 3$ and $a = \tau, \sigma$ [1]. The world sheet is thus a two-dimensional subspace which characterizes the properties of the string loop in a given combined gravitational and electromagnetic background. Therefore, the world sheet of a string loop is an analogue of the worldline of a test particle, i.e., it gives the loci of the events of existence of the string loop in the given background. Thus, we can introduce the world sheet induced metric in the following way [1]:
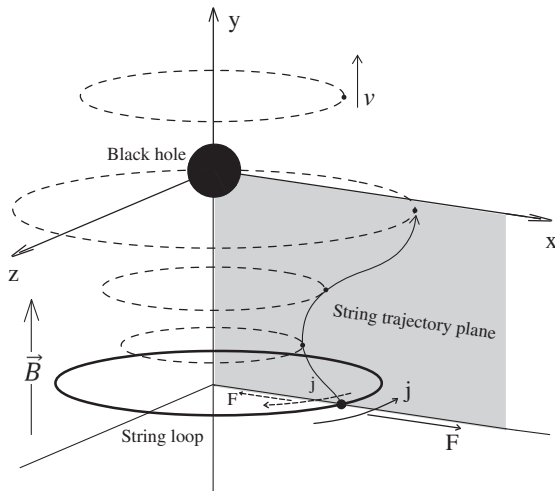

FIG. 1. The schematic picture of the oscillations and the acceleration of the string loop near a black hole embedded in an external uniform magnetic field. Due to the axial symmetry of the system, the string trajectory can be presented by a curve lying on the plane, chosen to be the x-y plane. The direction of the Lorenz force acting on the string loop depends on the orientation of the string loop current with respect to the external magnetic field. The initial position of the string loop is represented by the solid line, while its positions during the motion are represented by the dashed lines.

$$h_{ab} = g_{\alpha\beta}X^\alpha_{|a}X^\beta_{|b}, \quad (2)$$

where $X_{|a} = \partial X/\partial a$. The current of the string loop localized on the world sheet is described by a scalar field φ which depends on the world sheet coordinates τ and σ, but is independent of the choice of the spacetime coordinates. Dynamics of the string loop moving in the combined gravitational and electromagnetic field is described by the action S that should contain along with the term characterizing the freely moving string loop (see, e.g., [1]), also the term characterizing the interaction of the string loop and the external electromagnetic field. This implies the action and the Lagrangian density in the form $\mathcal{L}$ [38]:

$$S = \int \mathcal{L}\sqrt{-h}d\sigma d\tau, \quad (3)$$

$$\mathcal{L} = -\left[\mu/c + \frac{k}{2}h^{ab}(\varphi_{|a} + A_a)(\varphi_{|b} + A_b)\right], \quad (4)$$

where we use the projection

$$A_a = A_\gamma X^\gamma_{|a}, \quad (5)$$

and k is the constant number constrained by the world constants. For the electrically neutral current, the constant k is chosen to be equal to unity, $k = 1$. The first part of (4) represents the classical Nambu-Goto string action with the tension μ, the second part describes the scalar field φ, rescaled according to [1,9], along with the potential A_α of the electromagnetic field, and their interaction.

According to pioneering paper of Goto [39], the constant μ/c^2 can be interpreted as the uniform mass density which prevents expansion of the string loop beyond some radius, while the world sheet current introduces an angular momentum barrier preventing the loop from collapse. This implies that the parameter $\mu > 0$ characterizes the tension of the string loop, or its self-force, which preserves its radius, or squeezes the string loop. We take here the dimensions of the world sheet coordinates as the length dimension. Hereafter, we use the geometric units with $c = G = 1$. In these units, the constant k is equal to unity as well. The complete dimensional analysis and full conversion of the quantities describing the string loop motion from the geometrized units to the Gaussian or CGS units is given in Appendix B of the present paper, along with estimates of the string loop parameters in relation to realistic astrophysical conditions.

In the spherically symmetric spacetimes, the coordinate dependence of the world sheet coordinates (1) can be written in the form [12]

$$X^\alpha(\tau, \sigma) = \{t(\tau), r(\tau), \theta(\tau), \sigma\}, \quad (6)$$

in such a way that new coordinates satisfy the relations

$$\dot{X}^\alpha = (t_{|\tau}, r_{|\tau}, \theta_{|\tau}, 0), \qquad X'^\alpha = (0, 0, 0, 1), \quad (7)$$

where we use the dot to denote derivative with respect to the string loop evolution time τ, and the prime to denote derivative with respect to the space coordinate σ of the world sheet. From the relation (7), we can clearly see that the string loop does not rotate in the spherically symmetric spacetime (1) combined with the external uniform magnetic field.

The first order derivatives of the scalar field φ with respect to the world sheet coordinates determine the current on the string world sheet, $\varphi_{|a} = j_a$. The axial symmetry of the string loop model and the conformal flatness of the two-dimensional world sheet metric h^{ab} allows one to write the scalar field φ in the form [1,38]

$$\varphi = j_\tau \tau + j_\sigma \sigma. \quad (8)$$

In the presence of the external electromagnetic field, the equations of evolution of the scalar field will be influenced by the four-vector potential A_μ; we then define the total current, labeled by the tilde $\tilde{j}_a$, in the following form:

$$\tilde{j}_\tau = j_\tau + A_\alpha X^\alpha_{|\tau}, \qquad \tilde{j}_\sigma = j_\sigma + A_\alpha X^\alpha_{|\sigma}. \quad (9)$$

The variation of the action (4) with respect to the scalar field φ can now be written in the form

$$(\sqrt{-h} h^{ab} \tilde{j}_a)_{|b} = 0. \quad (10)$$

Equations of motion (10) for the scalar field φ, and the string loop axisymmetry imply existence of conserved quantities $\tilde{j}_\tau$ and j_σ. The conserved quantities $\tilde{j}_\tau$ and j_σ correspond to the parameters Ω and n introduced in [38], up to the constant k. Quantities j_τ and $\tilde{j}_\sigma$ generally could not be conserved during string loop motion.

Varying the action (4) with respect to the four-potential A_α [38], we obtain the electromagnetic current density

$$I^\mu = \frac{\delta \mathcal{L}}{\delta A_\mu} = k \tilde{j}_\tau \dot{X}^\mu - k \tilde{j}_\sigma X'^\mu, \quad (11)$$

and we can identify the string loop electric charge density q, and the electric current density j due to the relations

$$q = k \tilde{j}_\tau, \qquad j = k \tilde{j}_\sigma. \quad (12)$$

Up to the constant k, we can consider the parameters $\tilde{j}_\tau$ and $\tilde{j}_\sigma$ to be related to the electric charge and the current densities. We can deduce from (10) that the electric charge density q is conserved during the string loop evolution, but the current density j is changing as it is influenced by the external electromagnetic field.

It is important to specify the world sheet stress-energy tensor which can be found varying the action (4) with respect to the induced metric h_{ab} [1]:

$$\Sigma^{\tau\tau} = \frac{k}{2} \frac{\tilde{j}_\tau^2 + \tilde{j}_\sigma^2}{g_{\phi\phi}} + \mu, \qquad \Sigma^{\sigma\tau} = -\frac{k \tilde{j}_\tau \tilde{j}_\sigma}{g_{\phi\phi}},$$

$$\Sigma^{\sigma\sigma} = \frac{k}{2} \frac{\tilde{j}_\tau^2 + \tilde{j}_\sigma^2}{g_{\phi\phi}} - \mu. \quad (13)$$

The string loop canonical momentum density is defined by the relation

$$\Pi_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = \Sigma^{\tau a} g_{\mu\alpha} X^\alpha_{|a} + k \tilde{j}_\tau A_\mu. \quad (14)$$

From the Lagrangian density (4) we can obtain the Hamiltonian of the string loop dynamics in the combined gravitational and electromagnetic field in the form [38,40]

$$H = \frac{1}{2} g^{\alpha\beta} (\Pi_\alpha - q A_\alpha)(\Pi_\beta - q A_\beta)$$
$$+ \frac{1}{2} g_{\phi\phi} [(\Sigma^{\tau\tau})^2 - (\Sigma^{\tau\sigma})^2]. \quad (15)$$

The string loop dynamics is determined by the Hamilton equations

$$P^\mu \equiv \frac{dX^\mu}{d\zeta} = \frac{\partial H}{\partial \Pi_\mu}, \qquad \frac{d\Pi_\mu}{d\zeta} = -\frac{\partial H}{\partial X^\mu}. \quad (16)$$

The canonical momentum Π^μ (14) is related to the mechanical momentum P^μ by the relation

$$\Pi^\mu = P^\mu + q A^\mu, \quad (17)$$

where we use the string loop charge density definition (12). In the Hamiltonian (15), we use an affine parameter ζ, related to the world sheet coordinate τ by the transformation $d\tau = \Sigma^{\tau\tau} d\zeta$.

III. DYNAMICS OF STRING LOOP IN UNIFORM MAGNETIC FIELD AROUND SCHWARZSCHILD BLACK HOLE

In this section we apply the previous solutions for the case of the Schwarzschild black hole spacetime immersed in an axially symmetric magnetic field that is uniform at the spatial infinity. The plane of the string loop is perpendicular to the lines of strength of the magnetic field. The string loop moves along the axis which is chosen to be the y axis as shown in Fig. 1. The oscillations of the string loop are restricted to the x-z plane (considered in the x axis due to the axisymmetry of the string loop), while its trajectory, because of the symmetry, is considered in the x-y plane. The Schwarzschild black hole spacetime characterized by

the mass parameter M takes in the standard spherical coordinates the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
(18)

where the metric "lapse" function $f(r)$ is defined by

$$f(r) = 1 - \frac{2M}{r}.$$
(19)

Due to the symmetries discussed above for the description of the string loop motion, it is convenient for the proper description of the string loop motion to work with the Cartesian coordinates defined as [1,10]

$$x = r\sin\theta, \qquad y = r\cos\theta.$$
(20)

In our study we assume the static, axisymmetric and asymptotically uniform magnetic field. Since the Schwarzschild spacetime is flat at spatial infinity, the timelike $\xi_{(t)}$ and spacelike $\xi_{(\phi)}$ Killing vectors satisfy the equations $\Box\xi^\alpha = 0$, which exactly correspond to the Maxwell equations

$$\Box A^\alpha = 0,$$
(21)

for the four-vector potential of the electromagnetic field. The solution of the Maxwell equations can be then written in the Lorentz gauge in the form [41]

$$A^\alpha = C_1\xi_{(t)}^\alpha + C_2\xi_{(\phi)}^\alpha.$$
(22)

The first integration constant has to be $C_1 = 0$, because of the asymptotic properties of the Schwarzschild spacetime (18), while the second integration constant takes the form $C_2 = B/2$, where B is the strength of the homogeneous magnetic field at the spatial infinity. The commuting Killing vector $\xi_{(\phi)} = \partial/\partial\phi$ generates rotations around the symmetry axis. Consequently, the only nonzero covariant component of the potential of the electromagnetic field takes the form [41]

$$A_\phi = \frac{B}{2}r^2\sin^2\theta = \frac{B}{2}x^2.$$
(23)

The symmetries of the considered background gravitational and magnetic fields, corresponding to the t and ϕ components of the Killing vector, imply the existence of two constants of the string loop motion [1,37],

$$E = -\xi_{(t)}^\mu\Pi_\mu = -\Pi_t,$$
(24)

$$L = \xi_{(\phi)}^\mu\Pi_\mu = -\tilde{j}_\tau\tilde{j}_\sigma + qA_\phi = -\tilde{j}_\tau j_\sigma.$$
(25)

where we have already set $k = 1$. The angular momentum L is given by two other constants of motions $\tilde{j}_\tau$ and j_σ.

In the spherically symmetric spacetime (18), the Hamiltonian (16) governing the string loop dynamics can be expressed in the form [37]

$$H = \frac{1}{2}f(r)P_r^2 + \frac{1}{2r^2}P_\theta^2 - \frac{E^2}{2f(r)} + \frac{V_{\text{eff}}}{2f(r)},$$
(26)

with an effective potential for the string loop motion in the combined gravitational and magnetic fields

$$V_{\text{eff}} = f(r)\left\{\frac{B^2x^3}{8} + \left(\frac{\Omega JB}{\sqrt{2}} + \mu\right)x + \frac{J^2}{x}\right\}^2.$$
(27)

In accord with [1], we have introduced new parameters that are conserved for the string loop dynamics in the Schwarzschild spacetime combined with the uniform magnetic field

$$J^2 \equiv \frac{j_\sigma^2 + j_\tau^2}{2}, \qquad \omega \equiv -\frac{j_\sigma}{j_\tau}, \qquad \Omega \equiv \frac{-\omega}{\sqrt{1+\omega^2}},$$
(28)

where the parameter J is always positive, $J > 0$, the dimensionless parameter ω runs in the interval $-\infty < \omega < \infty$, and the dimensionless parameter Ω varies in the range $-1 < \Omega < 1$.

For the uniform magnetic field ($A_t = 0$), we can introduce relations between the string loop charge q and the current j densities (12), and the string loop parameters J, Ω, in the form

$$q = j_\tau, \qquad j = j_\sigma + A_\phi = j_\sigma + \frac{B}{2}x^2, \qquad j_\sigma = \sqrt{2}J\Omega.$$
(29)

It is worth recalling that j_τ and j_σ are conserved quantities in the uniform magnetic fields.

Note that in absence of the external magnetic field, $B = 0$, there exists a symmetry which allows for the interchange $\omega \leftrightarrow 1/\omega$; then the interval $-1 < \omega < 1$ covers all possible cases of the string loop motion [1,10]. In the case of nonvanishing magnetic field, such a symmetry does not exist, and ω has to range generally in the interval $(-\infty, \infty)$. The sign of the parameter Ω depends on the choice of the direction of the electric current with respect to the direction of the uniform magnetic field. The case $\Omega = 0$ corresponds to the zero current. The case with $\Omega > 0$, in principle is unstable, since even a small deviation of the string loop from the symmetry axis leads to the appearance of a torque proportional to the product of the current and the magnetic field and potentially the string loop can be overturned to the stable configuration with $\Omega < 0$ [37].

The condition $H = 0$ determines the regions allowed for the string loop motion, and it implies the relation for the effective potential (or energy boundary function) governing

the motion through the string loop and the background (spacetime and magnetic field) parameters:

$$E^2 = V_{\text{eff}}(x, y; J, B, \Omega). \qquad (30)$$

Detailed analysis of the boundaries of the string loop motion has been done in [37], we only recall main results here. The string loop motion boundaries and related types of the motion can be distinguished into four classes according to the possibility of the string loop to escape to infinity or collapse to the central compact object. The first class of the boundaries correspond to the absence of inner and outer boundaries, i.e., the string loop can be captured by the black hole or escapes to infinity. The second class corresponds to the situation with an outer boundary—the string loop must be captured by the black hole. The third class corresponds to the situation when both inner and outer boundaries exist—the string loop is trapped in some region forming a potential "lake" around the black hole. The fourth class corresponds to an inner boundary— the string loop cannot fall into the black hole but it must escape to infinity, see Fig. 2 for the details. For the purposes of the present paper, the first and fourth classes of boundaries, when the string loop can escape to infinity, are relevant.

A. Analogy with test particle motion

The influence of electromagnetic field on the string loop dynamics can be cleared up, if we compare the effective potential of the string loop dynamics with those of the charged test particle motion on circular orbits around a Schwarzschild black hole immersed in the same magnetic field B. The Hamiltonian of the motion of a charged test particle with mass m and charge q is given by the relation [42]
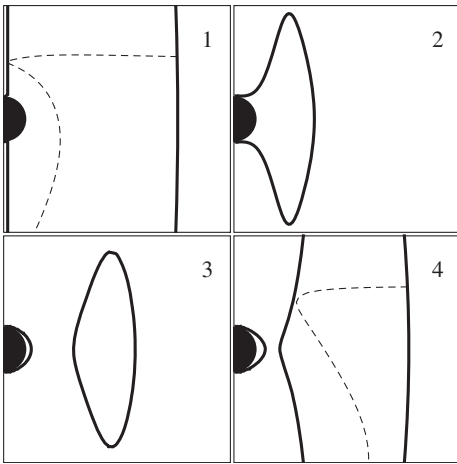

FIG. 2. Four possible types of the boundaries of the string loop motion in the Schwarzschild black hole spacetime and examples of string loop trajectories escaping to the infinity along the y axis.

$$H_{\text{p}} = \frac{1}{2} g^{\alpha\beta}(\Pi_\alpha - qA_\alpha)(\Pi_\beta - qA_\beta) + \frac{1}{2}m^2, \qquad (31)$$

where the mechanical and canonical momenta are again related by

$$P^\mu = \Pi^\mu - qA^\mu. \qquad (32)$$

We suppose the particle moving on a circular orbit at radius x, with constant angular velocity $\omega = P^\phi/m$.

In the homogeneous magnetic field B, the Hamiltonian of the test particle motion (31) can be cast into the (26) form, but with modified effective potential containing a constant axial angular momentum L:

$$V_{\text{eff}}^{(\text{p})} = f(r)\left\{\frac{L}{x} - \frac{qB}{2}x\right\}^2 + m^2. \qquad (33)$$

The effective potentials V_{eff} for both string loop (27) and test particle (33) motion consist of the part given by the geometry, $f(r)$, and the part in braces depending only on the x coordinate. We can compare terms in braces with the same x dependence for both string and particle cases revealing their interpretation.

The first and the second term in braces in the effective potential of the particle motion (33) represent the angular momentum contribution, $\sim x^{-1}$, and the Lorentz force contribution, $\sim x$, respectively.

The first term in braces in the effective potential of the string loop motion (27) represents the pure contribution of the external magnetic field to the "effective" energy— density of the magnetic field energy is proportional to B^2, and the space volume to x^3. The second term in braces in (27), showing the same dependence, $\sim x$, as the Lorentz force, consists from two parts—the pure tension μ, and the part representing interaction between the electric current carried by the string loop and the external magnetic field. This term can be associated to interaction of two magnetic fields, where one of them is the "global," external, magnetic field, $\sim B$, and the other is the "local," self-generated, magnetic field related to the string loop, $\sim J\Omega/x$, in accord with (28). The sign of this term can be either positive or negative, depending on the sign of Ω, i.e., the direction of the electric current on the string loop. We consider Ω to be negative, if the direction of the vector of the self-generated magnetic field coincides with the direction of the vector of the external magnetic field, and positive, if these vectors are directed oppositely. Since the direction of the external magnetic field is given as an initial condition, the direction of the Lorenz force acting on the string is determined by the direction of the current. The last term in braces of the effective potential (27) corresponds to the angular momentum generated by the current of the string loop.

B. Analogy with the superconducting string

The string loop model with the current generated by the scalar field φ demonstrates an interesting analogy with the superconductivity. In the pioneering works [43,44], it has been shown for the cosmic strings that in the case when the electromagnetic gauge invariance is broken, the string can be considered as a superconductor carrying large currents and charges, up to the order of the string mass scale. Under such circumstances, the carriers of the electric charge can be either bosons or fermions, depending on the energetic favor for the charged particle [45]. In a series of papers [46,47], the dynamics of the superconducting strings has been considered in the framework of the Nambu-Goto string theory (see, e.g., [48]), and the so-called "vortex" theory [46]. In the recent paper [49] the oscillations of the superconducting cosmic strings has been proposed as a possible model of the generation of the observed highly beamed radio bursts. Similarly to the current of the string loop, the density of the superconducting electric current j_s is proportional to the gradient of scalar function Φ, which is identified to the phase of the wave function of the superconducting Cooper pairs [50,51]:

$$j_\alpha^s = \frac{2\hbar n_s e}{m_s}\left(\partial_\alpha \Phi - \frac{2e}{\hbar c}A_\alpha\right), \qquad (34)$$

where n_s and m_s represent the concentration and mass of the Cooper pairs, e is the charge of electron and $\hbar$ is the Planck constant.

In addition to zero resistivity, the superconductors are characterized by the existence of the so-called Meissner effect according to which the magnetic field either is expelled from a type-I superconductor or penetrates into a type-II superconductor as an array of vortices. The phase transition between the superconducting and normal states can be caused by increase of either temperature or magnetic field. The highest strength of the magnetic field in a state with given temperature under which a material remains superconducting is called critical superconductivity strength. Further, the superconducting states are possible only if temperature is lower than the critical temperature T_c, i.e., the temperature of the phase transition to the superconducting state. It has to be underlined that the value of the critical temperature T_c strongly depends on the pressure, and, e.g., in the neutron star crust, reaches the values up to $10^9 \sim 10^{10}$ K [52]. The critical magnetic field at any temperature below the critical temperature is given by the relation

$$B_c \approx B_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right], \qquad (35)$$

where $B_c(0)$ is the critical magnetic field at zero temperature.

Suppose superconducting material is in the external magnetic field $B < B_c$ at $T > T_c$ and start lowering the temperature. When $T < T_c$, the medium separates into two phases: superconducting regions without magnetic flux (flux expulsion) and normal regions with concentrated strong magnetic field that suppresses superconductivity. Consequently, magnetic flux can pass through the superconducting material, separated into superconducting regions without magnetic flux and normally conducting regions into which the magnetic flux is concentrated, and as we discussed the nature of the nonsuperconducting regions depends on the type of superconductor as type I or type II.

Large-scale, ordered magnetic fields in central parts of accretion disks around black holes are desirable for explaining e.g., collimated jet production. However, inward advection of vertical flux in a turbulent accretion disk is problematic if there is an effective turbulent diffusivity. A new mechanism to resolve this problem was predicted in [53]. Turbulent flux expulsion leads to concentration of the large-scale vertical flux into small patches of strong magnetic field and because of their large field strengths, the patches experience higher angular-momentum loss rate via magnetic braking and winds. As a result, patches rapidly drift inward, carrying the vertical flux with them. The accumulated vertical flux aggregates in the central flux bundle in the inner part of the accretion disk and accretion flow through the bundle changes its character. It is necessary to underline that the new phenomenon called *turbulent diamagnetism* when the magnetic field is expelled from regions of strong turbulence and is concentrated between turbulent cells was first predicted by Zeldovich [54] and Parker [55].

Hence, the possibility of appearance of superconductors near the horizon of black holes, discussed in [53], has an important analogy with the turbulence in the accretion disk, namely, the mechanism of efficient transport of the large-scale external magnetic flux inward through a turbulent flow can play a role of superconductivity in the accretion disk and according to [53], suppression of turbulence by a strong magnetic field is analogous to lifting of the superconductor by an applied magnetic field.

For an electric current-carrying string loop immersed in a magnetic field there exists a similar effect of the vanishing of the electric current when the magnetic field is reaching a critical value. In the theory of the string loops no thermodynamic features are contained, and the critical phenomena are simply relating the magnetic field to the electric current (and angular momentum) parameters of the string loop. To find the critical value of the magnetic field, let us consider a stationary string loop located at a fixed radius r_0 in the flat spacetime. The energy per length of the string loop in an asymptotically uniform magnetic field B, given by Eq. (30) where the lapse function is reduced to $f(r) = 1$ in the flat spacetime, reads

$$\frac{E}{r_0} = \mu + \frac{1}{2}\frac{q^2}{r_0^2} + \frac{1}{2}\frac{j^2}{r_0^2} = \mu + \frac{J^2}{r_0^2} + \frac{\Omega J B}{\sqrt{2}} + \frac{B^2 r_0^2}{8}. \quad (36)$$

The final expression is the sum of the terms responsible for the string tension, the electric current, the Lorenz force and the energy of the magnetic field, respectively.

If one increases the strength of the background magnetic field B while keeping the string loop at the initial position r_0 with fixed energy E, the electric current of the string loop has to be modified accordingly. Using the relation (29), we can write the electric current density j in a form corresponding to those related to the superconducting current (34), and relate the current to the string loop parameters by

$$j = \varphi_{|\sigma} + A_\phi = \pm\sqrt{-q^2 + 2Er_0 - 2\mu r_0^2}. \quad (37)$$

The dependence of the current $j_\sigma = \varphi_{|\sigma}$ (being constant of the motion) on the strength of the magnetic field B is then determined by the relation

$$j_\sigma = j - \frac{B}{2}r_0^2. \quad (38)$$

According to Eq. (38), the magnitude of the current j_σ has to be evidently decreasing with increasing magnetic field B. Therefore, we have to consider the possibility of disappearance of the electric j_σ current when magnitude of the magnetic field reaches the critical value. The critical magnitude of the magnetic field for a string loop located at a given radius r_0 with energy E and charge density q takes the form

$$B_{cr} = \pm\frac{2}{r_0^2}\sqrt{-q^2 + 2Er_0 - 2\mu r_0^2}. \quad (39)$$

The presence of the critical magnetic field is provided by the existence of the last term of Eq. (36) representing the energy density of the magnetic field. If the last term in Eq. (36) vanishes, the current of the string loop could decrease to arbitrarily small values, but it cannot reach zero value. The presence of the last term of Eq. (36) also plays an important role for the benefit of the superconduction–string loop analogy. The so-called Meissner effect of the exclusion of the magnetic field from the superconductor means that the supercurrent generates a magnetic field having strength that is exactly the same as the strength of the external magnetic field. In other words, the term which is proportional to the square of B in Eq. (36) can be in the case of the superconducting string loop interpreted as the self-magnetic field generated by the string loop.

IV. STRING LOOP ACCELERATION

Explanation of relativistic jets in active galactic nuclei (AGN) and microquasars could be one of the most important astrophysical applications of the string loop model. It is possible because of the acceleration and fast ejection of the string loop from the black hole neighborhood by the transmutation effect, i.e., transmission of the energy of the string loop oscillatory motion in the x direction to the energy of the linear translation motion along the y direction related to the string loop symmetry axis [1,12,15].

In the analysis of the acceleration process, it is convenient to use dimensionless coordinates and string loop parameters. We thus make rescaling of the coordinates, $x \to x/M, y \to y/M$, and the string loop and background parameters,

$$J \to J/\sqrt{\mu}M, \quad E \to E/\sqrt{\mu}M, \quad B \to BM/\sqrt{\mu}. \quad (40)$$

We will return to the Gaussian units in Appendix B.

A. Maximal acceleration

Since the Schwarzschild spacetime is asymptotically flat, we have to study the string loop motion in the flat spacetime in order to understand the transmutation process. This enables clear definition of the string loop acceleration process and establishing of maximal acceleration that is available in a given uniform magnetic field. The energy of the string loop (30) in the flat spacetime with uniform magnetic field B, expressed in the Cartesian coordinates, reads

$$E^2 = \dot{y}^2 + \dot{x}^2 + V_{flat}(x; B, J, \Omega) = E_y^2 + E_x^2, \quad (41)$$

where the dot denotes derivative with respect to the affine parameter ζ and $V_{flat}(x; B, J, \Omega)$ is the effective potential of the string loop motion in the flat spacetime:

$$V_{flat} = \left\{\frac{B^2 x^3}{8} + \frac{\Omega J B x}{\sqrt{2}} + \left(1 + \frac{J^2}{x^2}\right)x\right\}^2. \quad (42)$$

The plots of the effective potential for the flat spacetime $V_{flat}(x)$ are illustrated in Fig. 3 for the different values of magnetic field. The energy related to the motion in the x and the y directions is given by the relations [15]

$$E_y^2 = \dot{y}^2, \quad E_x^2 = \dot{x}^2 + V_{flat}(x; B, J, \Omega) = E_0^2. \quad (43)$$

The energy in the x direction E_0 can be interpreted as an internal energy of the oscillating string, consisting from the potential and kinetic parts; in the limiting case of coinciding minimal and maximal extension of the string loop motion, $x_i = x_o$, the internal energy has zero kinetic component. The string internal energy can in a well-defined way represent the rest energy of the string moving in the y direction in the flat spacetime [1,15].

The final Lorentz factor of the transitional motion along the y axis of an accelerated string loop as observed in the asymptotically flat region is determined by the relation [1,15]

$$\gamma = \frac{E}{E_0}, \tag{44}$$

where E is the total energy of the string loop having the internal energy E_0 and moving in the y direction with the velocity corresponding to the Lorentz factor γ. Clearly, the maximal Lorentz factor of the transitional motion of the string loop is related to the minimal internal energy that can the string loop have, i.e., those with vanishing kinetic energy of the oscillatory motion [15]

$$\gamma_{max} = \frac{E}{E_{0(min)}}. \tag{45}$$

It should be stressed that rotation of the black hole (or naked singularity) is not a relevant ingredient of the acceleration of the string loop motion due to the transmutation effect [15], contrary to the Blandford-Znajek effect [17] usually considered in modeling acceleration of jetlike motion in AGN and microquasars.

Extremely large values of the gamma factor given by Eq. (45) can be obtained by setting the initial energy E very large or by adjusting properly the string loop parameters J, Ω and the magnetic field strength B in order to obtain very low minimal internal energy related to infinity, $E_{0(min)}$. It is crucial to examine properties of the energy function $E_0(x; J, \Omega, B)$, primarily its minimal allowed value, $E_{0(min)}(J, \Omega, B)$, given by the local minimum of the effective potential V_{flat} (42).

Recall that the nonmagnetic case, $B = 0$, implies quite simple properties of the effective potential V_{eff}, as there is only one local minimum with the location and the energy level determined by [15]

$$x_{min} = J, \qquad E_{0(min)} = 2J. \tag{46}$$

The effective potential V_{flat} is always positive in this case.

Now we have to discuss in detail properties of the effective potential V_{flat} for the string loop dynamics in the homogeneous magnetic field in flat spacetime. It is enough to consider $B > 0$ as the situation with opposite sign of B can be described by inversing the sign of Ω. Naturally, also $x > 0$. The effective potential diverges for $x \to 0(\sim J^2/x)$ and $x \to \infty(\sim B^2 x^3)$. It is convenient to define a new parameter b reflecting the interaction of the magnetic field and the string loop current, modified space coordinate $\tilde{x}$, and modified energy $\tilde{E}$, by the relations

$$b = BJ, \qquad \tilde{x} = \frac{x}{J}, \qquad \tilde{E} = \frac{E}{2J}. \tag{47}$$

The zero points of the effective potential are then governed by

$$\tilde{x}^2_{(z)\pm}(b, \Omega) = \frac{-F \pm \sqrt{F^2 - 8b}}{b^2}, \tag{48}$$

$$F = 4 + 2\sqrt{2}b\Omega. \tag{49}$$

It is immediately clear that both solutions $\tilde{x}^2_{(z)\pm}(b, \Omega)$ can be negative only, being thus physically unrealistic. Therefore, the effective potential is always positive, $V_{flat} > 0$. It has only one local minimum for all values of the string loop parameters, and the magnetic field intensity $B > 0$. The extremum, given by $\partial V_{flat}/\partial x = 0$ is located at

$$\tilde{x}^2_{min} = \frac{x^2_{min}}{J^2} = \frac{2\sqrt{2}}{3b^2}\{\sqrt{D} - (\sqrt{2} + b\Omega)\}, \tag{50}$$

$$D = 3b^2 + (\sqrt{2} + b\Omega)^2, \tag{51}$$

that exist for all considered values of the parameters J, B, Ω or b, Ω. The minimal energy of the string loop in the flat spacetime with the homogeneous magnetic field B is then given by

$$\tilde{E}_{0(min)} \equiv \frac{E_{0(min)}}{2J} = \frac{\sqrt{42}}{3\sqrt{3}} \frac{\sqrt{D}(\sqrt{2} + b\Omega) + 6b^2 - D}{b\sqrt{\sqrt{D} - (\sqrt{2} + b\Omega)}}. \tag{52}$$

Positions of the local minima of the effective potential $\tilde{x}^2_{min}$, and the minimal energy $\tilde{E}_{0(min)}$, respectively, are illustrated as a function of b for characteristic values of the string loop parameter Ω in Fig. 4(a), and in Fig. 4(b), respectively. The related local extrema of the $\tilde{x}_{min}(b), \tilde{E}_{0(min)}(b)$ functions are given by the relation

$$2\sqrt{D}(b\Omega + \sqrt{2} - \sqrt{D}) + b^2(\Omega^2 + 3) - b\sqrt{D}\Omega + \sqrt{2}b\Omega = 0, \tag{53}$$

for $\tilde{x}_{min}(b)$, and the relation

$$b^4(-2\Omega^4 - 3\Omega^2 + 9) + 2b^3\Omega(\sqrt{D}\Omega^2 - \sqrt{2}(\Omega^2 - 6))$$
$$+ b^2(-9\sqrt{2}\sqrt{D} + 12\Omega^2 + 30) + 4b(5\sqrt{2} - 3\sqrt{D})\Omega$$
$$- 8\sqrt{2}\sqrt{D} + 16 = 0 \tag{54}$$

for $\tilde{E}_{0(min)}(b)$.

In the limit of $B \to 0$, we find the minimum location at $\tilde{x}_{min} = 1$, and the minimal energy $\tilde{E}_{0(min)} = 1$, i.e., the values obtained for the empty flat spacetime, given by Eq. (46). However, these limit values are reached also for a special value of the magnetic field strength B, or parameter b, in dependence on the other string loop parameter Ω. Therefore, it is relevant to discuss the conditions

$$\tilde{x}_{min}(b, \Omega) = 1, \qquad \tilde{E}_{0(min)}(b, \Omega) = 1 \tag{55}$$

in dependence on the interaction of the string loop and the magnetic field expressed by the parameter $b = BJ$. This enables one to distinguish qualitatively different string loop
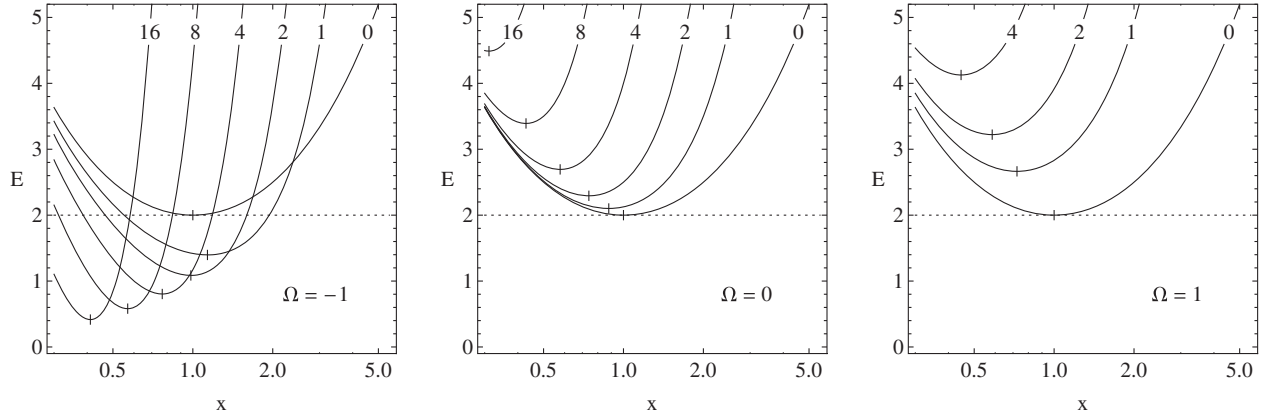
FIG. 3. String loop effective potential $V_{\text{flat}}(x)$ for the flat spacetime with and without a uniform magnetic field. The effective potential is presented for three representative values of the string loop parameter $\Omega \in \{-1, 0, 1\}$, keeping constant the current parameter $J = 1$, and for various values of the strength of the magnetic field B (denoted by numbers in the plot). Minimum of all the effective potentials is denoted by the small vertical line, the dotted line corresponds to the string loop energy in the nonmagnetic case $B = 0$.

configurations from the point of view of the acceleration process. Namely, the condition $\tilde{E}_{0(\text{min})}(b, \Omega) = 1$ separates the string loop configurations enabling efficiency of the acceleration process in the magnetic field to be higher in comparison with those related to the nonmagnetic case, from those where the efficiency is lower. Equation (55) can be expressed in the form

$$-3b^2 - 2\sqrt{2}b\Omega + 2\sqrt{2}\sqrt{D} - 4 = 0, \qquad (56)$$

$$2b^3(\Omega^2 - 1)^2 + 2\sqrt{2}b^2\Omega(3\Omega^2 + 5)$$
$$+ 12b\Omega^2 + b + 4\sqrt{2}\Omega = 0. \qquad (57)$$

The numerically determined solutions of the equations governing the local extrema and the "flat limit" values of the position and energy functions, given in terms of the functions $b(\Omega)$, are represented in Fig. 4(c). We can see that the solutions exist only in the range of the string loop parameter $\Omega \in \langle -1, 0 \rangle$. We denote the solutions of equations (56) and (57) by $b_{1(x)}(\Omega)$ and $b_{1(E)}(\Omega)$, while $b_{E(x)}(\Omega)$ and $b_{E(E)}(\Omega)$ are referred to solutions of (53) and (54).

For string loop with fixed parameter Ω the maximal distance of the $\tilde{x}_{\text{min}}(b, \Omega)$ function from the $B = 0$ position, $\tilde{x}_{\text{min}} = 1$, is given by points in Fig. 4(a) and denoted by $\tilde{x}_{\text{min(max)}}$. It is also useful to give the minimal energy $\tilde{E}_{0(\text{min})}(b, \Omega)$ of the effective potential at the extremal cases assuming the parameter Ω fixed and given by the $b_{E(E)}(\Omega)$ dependence. The results are illustrated by a sequence of points in Fig. 4(b)—the minimal values of the energy $\tilde{E}_{0(\text{min})}(b, \Omega)$ are denoted by circles, while the $\tilde{E}_{0(\text{min})}(b, \Omega)$ values obtained at the maximal distance from the $\tilde{x}_{\text{min(max)}}$ point are denoted by squares.
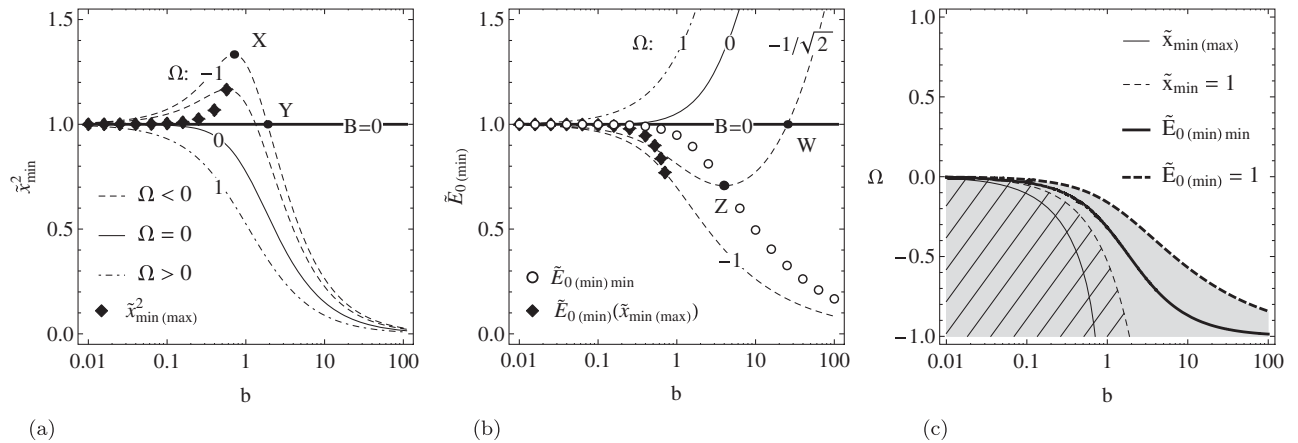

FIG. 4. Properties of the effective potential $V_{\text{flat}}(x)$ for the string loop dynamics in the flat spacetime with a uniform magnetic field. Position of the $V_{\text{flat}}(x)$ minima $\tilde{x}_{(0)\,\text{min}}$ (a), and the minimal energy $\tilde{E}_{(0)\,\text{min}}$ (b) are given as a function of b for some characteristic values of the string loop parameter Ω. In (c), properties of the functions $\tilde{x}_{(0)\,\text{min}}(b, \Omega), \tilde{E}_{(0)\,\text{min}(b,\Omega)}$ are demonstrated.

The loci $\tilde{x}_{\min}$ of the effective potential V_{flat} local minima depend on the string loop parameter Ω, and the parameter b combining the role of the magnetic field intensity B and the string loop parameter J. Keeping the string loop parameters Ω and J constant, we can obtain a unique position of the minimum $\tilde{x}_{\min}$ for each magnitude of magnetic field $B > 0$ in the case of $\Omega \in \langle 0, 1 \rangle$. Such minima are located at $x_{(\min)} < 1$, i.e., closer to the coordinate origin as compared to the nonmagnetic ($B = 0$) case, see Fig. 4(a).

On the other hand, for $\Omega \in \langle -1, 0 \rangle$ we can obtain a "binary" behavior of the effective potential, if the other parameters are properly tuned. For $b < b_{1(x)}(\Omega)$, we can obtain the same location of the effective potential minimum, $x_{(\min)}$, for two different values of parameter b. These two different string loop configurations at a given radius will differ in the string loop energy E, and have to be located at $\tilde{x}_{\min} > 1$, i.e., at larger distance from the origin than the minimum of the effective potential in the $B = 0$ case. Such binary string loop configurations can exist only for subcritical values of the magnetic parameter

$$b < b_{\mathrm{bin}} = 4\sqrt{2}/3 \qquad (58)$$

determined by the condition $b_{1(x)}(-1) = 1$—see the point Y in Fig. 4(a). The maximal difference of the location of the local minimum of the effective potential from the position $\tilde{x}_{\min} = 1$ corresponding to the nonmagnetic case, $B = 0$, is given by the point X in Fig. 4(a), i.e., by the local maximum of the $\tilde{x}_{\min}(b)$ function determined by Eq. (53), taken for $\Omega = -1$. For this maximum we obtain

$$b_{\mathrm{E(x)}}(-1) = 1/\sqrt{2}, \qquad \tilde{x}_{\mathrm{bin(max)}}^2 = \tilde{x}_{\min}^2(1/\sqrt{2}) = 4/3. \qquad (59)$$

The extremal efficiency of the string loop transmutation process is governed by the minimum of the effective potential at the flat spacetime containing a homogeneous magnetic field, $\tilde{E}_{(0)\min}(b, \Omega)$, see Eq. (45). If the minimal energy $E_{0(\min)}$ is zero (for example in $J = 0, B = 0$ case) the maximal possible acceleration of the string loop diverges, $\gamma_{\max} \to \infty$.

For positive values of the string loop parameter Ω, range $\Omega \in (0, 1\rangle$, the minimal energy function $\tilde{E}_{0(\min)}(b)$ is monotonically increasing with increasing parameter b and diverges for $b \to \infty$, see Fig. 4(b). Since there is $\tilde{E}_{0(\min)} > 1$ for all values of $b > 0$, $\Omega \in (0, 1\rangle$, such configurations are not advantageous for the string loop acceleration as compared to the nonmagnetic case $B = 0$.

Configurations $\tilde{E}_{0(\min)} < 1$ implying possibility of more efficient string loop acceleration than in the nonmagnetic case $B = 0$, can exist only for negative values of the parameter Ω. For $\Omega \in (-1, 0)$, the minimum energy function $\tilde{E}_{0(\min)}(b)$ decreases with increasing b for small

enough values of the parameter b reaching a minimum for the magnetic parameter $b_{\mathrm{E(E)}}$ [point Z in Fig. 4(b)] and increases with further increasing of b, crossing the $\tilde{E}_{0(\min)} = 1$ line at $b_{1(\mathrm{E})}$ [point W in Fig. 4(b)] and for larger values of parameter b increases towards infinity. In the special case of $\Omega = -1$, the minimum energy function $\tilde{E}_{0(\min)}(b)$ is monotonically decreasing and tends to zero value as b increases to infinity.

For negative values of the string loop parameter, $\Omega \in \langle -1, 0 \rangle$, and for a given string loop parameter J, we can always find a properly large values of magnetic field B to obtain the minimal energy $E_{0(\min)}$ of the effective potential smaller then in the nonmagnetic case where $E_{0(\min)} = 2J$. The ratio of the minimal energy $\tilde{E}_{(0)\min}$ considered in the nonmagnetic and magnetic cases can be put arbitrarily close to zero for large enough values of the parameter b. This implies the possibility of an extremely efficient transmutation effect leading to accelerations of the string loop up velocities corresponding to ultrahigh Lorentz factor of the motion of electric current-carrying string loop with $J > 0$ in the combined Schwarzschild gravitational field and the uniform magnetic field—see Fig. 4(c).

B. Acceleration in combined gravitational and magnetic fields

Clearly, $E_x = E_0$ and E_y are constants of the string loop motion in the flat spacetime and no transmission between these energy modes is possible. However, in vicinity of black holes or naked singularities, the internal kinetic energy of the oscillating string can be transmitted into the kinetic energy of the translational linear motion (or vice versa) due to the chaotic character of the string loop dynamics [1,15].

In order to get a strong acceleration, the string loop has to pass the region of strong gravity near the black hole horizon or in vicinity of the naked singularity, where the string loop transmutation effect $E_x \leftrightarrow E_y$ can occur with maximal efficiency. However, during the acceleration process, all energy of the E_x mode cannot be transmitted into the E_y energy mode—there always remains the inconvertible internal energy of the string, $E_{0(\min)}$, being the minimal energy hidden in the E_x energy mode, corresponding to the minimum of the effective potential.

The opposite case corresponds to amplitude amplification of the oscillations in the x direction and deceleration of the linear motion in the y direction; in this case the translational kinetic energy is partially converted to the internal oscillatory energy of the string. All energy of the transitional (E_y) energy mode can be transmitted to the oscillatory (E_x) energy mode—oscillations of the string loop in the x direction and the internal energy of the string loop will increase maximally in such a situation, while the string loop will stop moving in the y direction. We shall focus our attention to the case of accelerating string loop.

First, we have to discuss the properties of the effective potential of the string loop motion in the combined gravitational and magnetic field. The effective potential is a simple combination of the lapse function $f(r)$ of the spacetime metric, and the effective potential of the string loop motion in the flat spacetime with the uniform magnetic field V_{flat}; therefore,

$$V_{\text{eff}} = f(r)V_{\text{flat}}. \tag{60}$$

The zero point of the effective potential is given by the vanishing of the lapse function $f(r) = 0$, i.e., at the Schwarzschild black hole horizon at $r = 2$. The divergence occurs at infinity $x \to \infty$, as in the flat spacetime with an homogeneous magnetic field. The local extrema of the effective potential cannot be located off the equatorial plane given by the spherically symmetric spacetime and the uniform magnetic field. In the equatorial plane $y = 0$ they are given by the condition [37]

$$\frac{3}{8}B^2x^5 - \frac{5}{8}B^2x^4 + \left(1 + \frac{BJ\Omega}{\sqrt{2}}\right)(x^3 - x^2)$$
$$- J^2x + 3J^2 = 0. \tag{61}$$

The local extrema can thus be determined by the condition related to the angular momentum parameter of the string loop

$$J = J_{\text{E}\pm}(x; B, \Omega) \equiv \frac{B\Omega x^2(x-1) \mp \sqrt{G}}{2\sqrt{2}(x-3)}, \tag{62}$$

where

$$G = B^2(x-1)^2x^2\Omega^2 + B^2(x-3)(3x-5)x^2$$
$$+ 8(x-3)(x-1). \tag{63}$$

There are no zero points of the functions $J_{\text{E}\pm}(x; B, \Omega)$ because the condition

$$\frac{1}{8}B^2x^4(3x-5) + x^2(x-1) > 0 \tag{64}$$

is satisfied at $x > 2$ for all values of the parameter B.

The positive branch of the solution $J_{\text{E}+}$ is real and positive above the radius of the photon circular geodesic, at $x > 3$, for all combinations of parameters B and Ω. For

$$\Omega < -\sqrt{\frac{3}{11}} \doteq -0.52, \quad B > \sqrt{\frac{3 + \sqrt{33}}{18}} \doteq 0.70, \tag{65}$$

the solution $J_{\text{E}+}$ can be real and positive also below the photon circular geodesic, at $x > 2$. Also the solution $J_{\text{E}-}$ can be for (65) real and positive, but only in the region $2 < x < 3$. The behavior of the functions $J_{\text{E}\mp}(x; B, \Omega)$ is demonstrated in Fig. 5.

A given current parameter J represented by a line has possible multiple intersections with the function $J_{\text{E}\mp}(x)$ that determine positions of the local extrema of effective potential $V_{\text{eff}}(x)$ function in equatorial plane. The local extrema of the $J_{\text{E}\mp}^2(x)$ function, given by the condition $\partial_r J_{\text{E}\mp} = 0$, enable us to distinguish maxima and minima of effective potential $V_{\text{eff}}(x)$. There can exist only local minimum $J_{E\mp(\min)}$ for all combination of parameters B and Ω, see Fig. 5.

For the string loop immersed in the combined uniform magnetic field and the spherically symmetric gravitational field described by the Schwarzschild spacetime, we can have two intersection points of the $J = \text{const}$ line with the function $J_{\text{E}\mp}(x, B, \Omega)$ (maxima and minima of V_{eff}) for the parameter $J > J_{\text{E}\mp(\min)}$, one intersection point (inflex point of V_{eff}) for $J = J_{\text{E}\mp(\min)}$, and none intersection point for $J < J_{\text{E}\mp(\min)}$ (no extrema of V_{eff})—the situation is the same as in the Schwarzschild spacetime without magnetic field [10].

At the minima (maxima) of the effective potential, stable (unstable) equilibrium positions of the string loop occur. Note that energy of the string loop at the stable equilibrium
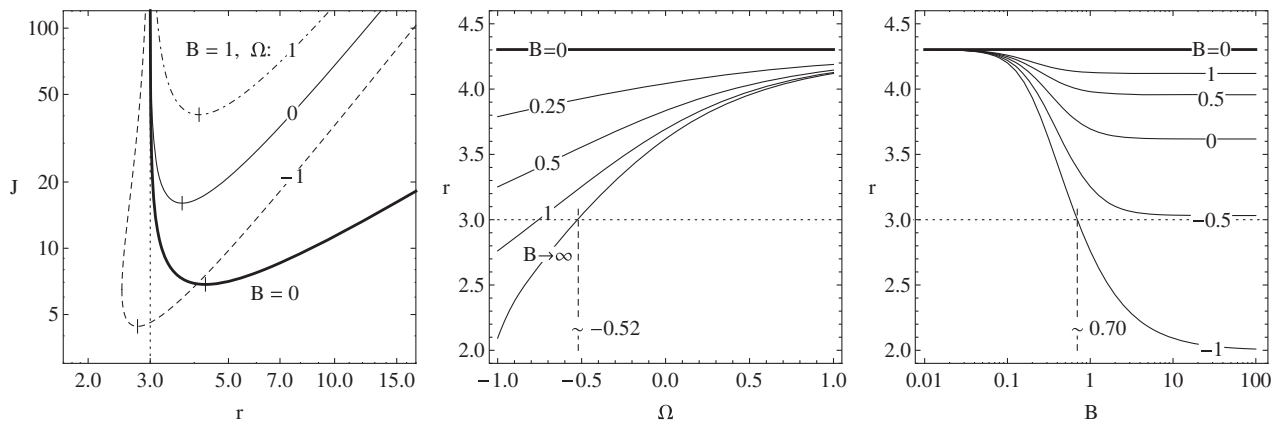

FIG. 5. Plots of the $J_{\text{E}\mp}(r; B, \Omega)$ function (left figure), position of the extrema of $J_{\text{E}\mp}(r; B, \Omega)$ in dependence on the string loop parameter Ω (central figure), and dependence of the extrema on the strength of the magnetic field B (right figure).

positions governs oscillatory motion around the equilibrium state, but it is not relevant for the maximal acceleration of the string loop in the transmutation process—the maximal acceleration is given by the local minimum of the effective potential in the flat spacetime [11].

C. Numerical simulations of the transmutation process

In the previous section, the maximal possible acceleration of the string loop has been determined, in dependence on the parameters J, B, Ω, by finding the minima of the effective potential of the string loop dynamics in the uniform magnetic field in the flat spacetime that reflects the asymptotic properties of the combined Schwarzschild gravitational field and uniform magnetic field. However, in realistic transmutation processes in the vicinity of the black hole horizon, the efficiency is usually lower than the maximally allowed efficiency corresponding to the maximally accelerated string loop when their oscillatory motion is fully suppressed.

Due to the chaotic character of the string loop equations of motion, even tiny change in the initial conditions of the motion can change completely character of the string loop trajectory. In order to demonstrate the effect of the magnetic field B on the string loop acceleration, it is useful to compare the set of trajectories with or without magnetic field B in dependence on the string loop parameter Ω introducing the qualitative differences of the character of the transmutation process, namely in the potential efficiency of the transmutation process reflected by the maximal acceleration determined by $\gamma_{\max}$.

The effect of the uniform magnetic field on the string loop acceleration will be first illustrated by sending the string loop with fixed current parameters J and Ω towards the black hole from rest in the initial position with coordinate x_s adjusted in order to have fixed string loop energy $E = 25$, but with freely varying impact parameter $y_s \in (0, 13)$—giving displacement from the equatorial plane, see Fig. 6. Trajectories with large ejection velocity are assumed to appear for large maximal Lorentz factors, $\gamma_{\max}$—due to Eq. (45) this should occur for small values of the string loop current parameter J, and large values of the string loop energy E; we will use $J \sim 2$ and $E \sim 25$. Of course, we can test different initials parameters, but our choice is quite reasonable and illustrative [15]. The current J should be minimized, but since the minimum of the effective potential V_{eff} is located at $x_{\min} \sim J$, it is difficult for the trajectories of the string loop with $J < 2$ to jump over the black hole horizon (see Fig. 6) and many of the string loop trajectories will end inside the black hole. The initial starting point at x_s corresponds to an initial stretching of the string loop—for larger x_s, we will start with larger energy E that also could provide for larger Lorentz factor γ of the string loop far away from the black hole.

For simplicity, we first study the role of the string loop parameter Ω on the transmutation process in a fixed magnetic field with intensity $B = 0.2$, considering the characteristic values of $\Omega = -1, 0, 1$ and compare the results to the case of vanishing magnetic field $B = 0$. The scattering function $\gamma(y_s)$, plotted in Fig. 7, demonstrates some regular scattering regions [for example $y_s \in (5.2, 11)$ in the $B = 0$ case], where the γ depends on y_s in a quite regular way, combined with chaotic scattering regions (chaotic bands [56]), where γ depends on y_s in a completely chaotic way when there is no regular prediction of final output from neighboring initial positions [for example $y_s \in (4.9, 5.1)$ in the $B = 0$ case]. The reason why there exist such unregular outcomes from some regions of the initial positions is the presence of unstable periodic
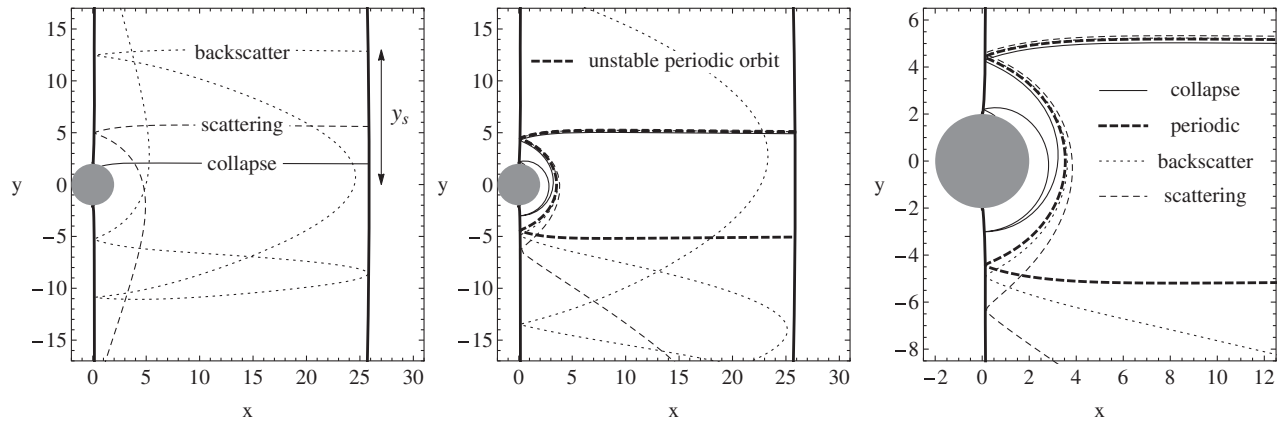

FIG. 6.   Different types of the string loop motion in the combined gravitational Schwarzschild field and uniform magnetic field B. The string loop trajectories are represented for appropriately chosen characteristic cases (scattering, backscattering, collapse). The thick line represents boundary of the motion given by the energy boundary function $E = E_b(x, y)$, gray is the dynamical region below the black hole horizon. Sensitivity of the string loop motion to the initial conditions is observed in neighborhood of the unstable periodic orbit; an example is shown in the second and third figures (being enlargements of the first one) by the thick dashed line constructed for $y_s \doteq 5.059$. We continuously vary the impact parameter y_s and determine by numerical calculations the resulting gamma factor $\gamma(y_s)$; for the characteristic values of B and Ω the results are given in Fig. 7 and Tables I–III.
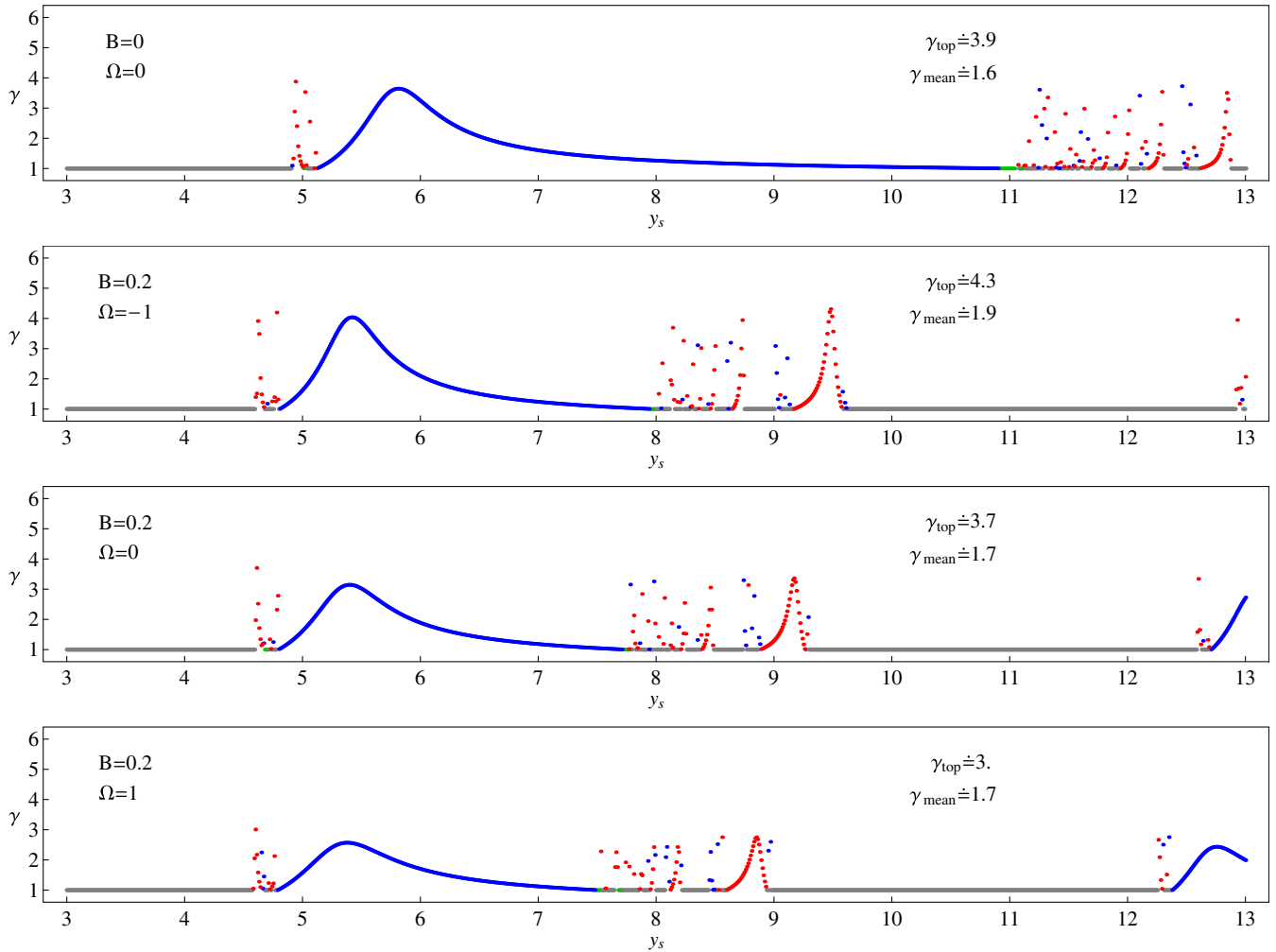
FIG. 7 (color online). The asymptotic Lorentz factor γ obtained due to the transmutation of the string loop energy in the Schwarzschild backgrounds with or without uniform magnetic field B is given for three characteristic values of string loop parameter Ω. The Lorentz factor γ (vertical axis) is calculated for string loop with energy $E = 25$ and current $J = 2$, starting from the rest with varying initial position in the coordinate $y_0 \in (0, 13)$ (horizontal axis), while the coordinate x_0 of the initial position is calculated from Eq. (30). Equation (45) for the maximal acceleration implies the limiting gamma factor $\gamma_{\max} = 6.25$ for the nonmagnetic case $B = 0$. We give the topical Lorentz factor, γ_{top}, that is numerically found in the sample, and also the efficiency of the transmutation effect expressed by the mean value of the Lorentz factor, γ_{mean}, in the sample. The blue (red) color depicts results of the scattering (rescattering) of the string loop, the gray color depicts its collapse to the black hole.

orbits. An illustrative example on the unstable periodic orbits and behavior of the string loop trajectories in its vicinity is presented in Fig. 6 for our set of the initial conditions. The results of the study assuming fixed energy E are reflected in Fig. 7 by the values of the Lorentz factors γ_{top} and γ_{mean}, giving the maximal and mean values of the Lorentz factor obtained from the considered sample of the initial conditions of the string loop motion in the given magnetic field. The results confirm expectation based on the properties of the function $\gamma_{\max}(B, J, \Omega)$ given by Eq. (45) that the transmutation process is most efficient for negative values of the string loop parameter Ω. Notice that for string loop in the magnetic field, the factor γ_{top} is higher than in the nonmagnetic case $B = 0$ only for $\Omega = -1$, but slightly lower even for $\Omega = 0$, and substantially lower for $\Omega = 1$. On the

other hand, the value of γ_{mean} is higher in the magnetic field for all three cases of Ω.

Our study seems to be very similar to the problem of chaotic scattering [56]—the initial vertical displacement, y_s, plays the role of the impact parameter, while we use the resulting string loop Lorentz factor γ instead of the scattering angle. One can assume that, due to the chaotic nature of the string loop motion, the numerical simulations will be able to find the trajectories with Lorentz factor having almost the maximal value, i.e., $\gamma_{\text{top}} \sim \gamma_{\max}$. However, our numerical simulations show that such an assumption is not generally true, as we obtained at least $\gamma_{\text{top}} \sim 0.5\gamma_{\max}$, or slightly larger values—see Fig. 7 or Tables I–III. This is a general effect of the string loop transmutation process in the field of black holes, related to

TABLE I.　The characteristic values of the string loop asymptotic Lorentz factor, γ_{top} and γ_{mean}, numerically obtained for the set of trajectories in the acceleration scenario 1 are compared to the maximal Lorentz factor γ_{max}. In scenario 1 we keep energy $E = 25$ and current $J = 2$, while coordinate x_s is varied according to increase of the strength of the magnetic field B.

		x_0	γ_{max}	γ_{top}	γ_{mean}
$B = 0$		25.8	6.3	3.9	1.6
$B = 0.1$	$\Omega = -1$	19.5	6.7	4.5	1.7
	$\Omega = 0$	18.4	6.2	3.8	1.7
	$\Omega = 1$	17.3	5.8	3.3	1.6
$B = 0.2$	$\Omega = -1$	14.7	7.2	4.3	1.9
	$\Omega = 0$	13.7	6.2	3.7	1.7
	$\Omega = 1$	12.7	5.5	3.0	1.7

TABLE III.　The characteristic values of the string loop asymptotic Lorentz factor, γ_{top} and γ_{mean}, numerically obtained for the set of trajectories in the acceleration scenario 3 are compared to the maximal Lorentz factor γ_{max}. In scenario 3 we keep coordinate $x_s = 25$ and current $J = 2$, while energy E is varied according to increase of the strength of the magnetic field B.

		E	γ_{max}	γ_{top}	γ_{mean}
$B = 0$		24.2	6.0	4.0	1.5
$B = 0.1$	$\Omega = -1$	39.6	10.6	6.6	1.7
	$\Omega = 0$	43.0	10.7	6.9	1.7
	$\Omega = 1$	46.4	10.8	6.6	1.7
$B = 0.2$	$\Omega = -1$	92.6	26.8	15.5	2.2
	$\Omega = 0$	99.4	24.6	14.8	2.3
	$\Omega = 1$	106.2	23.3	12.1	2.1

the existence of the event horizon capturing the string loop entering the region of the most efficient transmutation—in the naked singularity spacetimes, where the efficient transmutation occurs in regions containing no event horizon and no capturing of the string loop occurs, we observe $\gamma_{top} \sim \gamma_{max}$ frequently [9,11].

Now we have to discuss the influence of the magnitude of the magnetic field on the transmutation process and compare the results to those related to the case of $B = 0$. The physical reason for distinction of the transmutations process efficiency in the Schwarzschild black hole without or with magnetic field B comes from the behavior of the minimum of the effective potential in the flat spacetime $E_{0(min)}$, giving the behavior of the accelerated string loop far away from the black hole. For $B = 0$ we have the simple relation $E_{0(min)} = 2J$, while for $B \neq 0$ the value of $E_{0(min)}$ is modified by the magnetic field intensity and the string loop parameter Ω. However, the realistic transmutation process is also strongly influenced by the presence of the black hole horizon, since the region of minimum of the effective potential enabling high efficiency of the transmutation process is close to the black hole horizon [11,15]. Large values of the Lorentz factor γ can be achieved by enlarging the (initial, and conserved) energy E, or by lowering the

minimal string loop energy at infinity, $E_{0(min)}$, by lowering J—see Eq. (45). We can achieve low values of the energy $E_{0(min)}$ (see Fig. 3), and hence large acceleration, by increasing magnitude of the magnetic field B, or by using very low values of the current magnitude J of the string loop with parameter $\Omega < 0$. However, such string loops minima are very close to the black hole horizon suppressing thus the probability of observable acceleration process.

In testing the role of the magnetic field, we have to reflect the problem of the initial conditions that have to be adjusted in order to enable the comparison with the case of $B = 0$. The initial conditions for the string loop motion are given by the initial position and the initial speed $x_s, y_s, \dot{x}_s, \dot{y}_s$, the internal string loop parameters E, J, Ω, and the external parameter—intensity of the magnetic field B. We cannot choose arbitrarily all the parameters determining the initial conditions, for the string loop starting from the rest the parameters are related by Eq. (30).

If we want to demonstrate the influence of the magnetic field on the string loop acceleration process by varying the external parameter B, we have to modify some of the internal string loop parameters E, J, Ω, or the initial position and the initial speed. We will discuss three scenarios for the string loop starting its motion from rest state, distinguished according to what parameter is varied due to increase of the external parameter B, assuming in all the scenarios the parameter Ω fixed:

(1) Initial position x_s is varied, E and J are fixed.
(2) Current parameter J is varied, x_s and E are fixed.
(3) String energy E is varied, x_s and J are fixed.

The behavior of the effective potential in these three scenarios is represented in Fig. 8.

For the first scenario (initial position x_s varied with B), the results of the numerical calculations of the Lorentz γ factor are summarized in Table I for two characteristic values of B. As we can see from Table I, the string loop is forced to start closer to the black hole horizon for $B > 0$, if we wish to keep its current J and energy E, the initial position x_s decreases with increasing B and decreases with
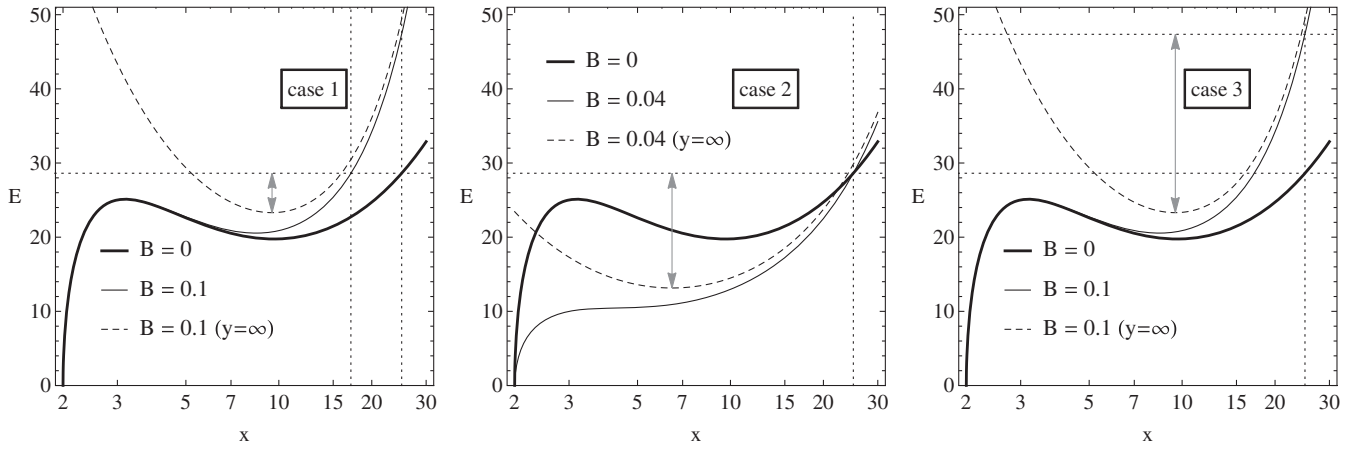
TABLE II.　The characteristic values of the string loop asymptotic Lorentz factor, γ_{top} and γ_{mean}, numerically obtained for the set of trajectories in the acceleration scenario 2 are compared to the maximal Lorentz factor γ_{max}. In scenario 2 we keep coordinate $x_s = 25$ and energy $E = 25$, while current J is varied according to increase of the strength of the magnetic field B.

		J	γ_{max}	γ_{top}	γ_{mean}
$B = 0$		5.0	2.5	2.5	1.4
$B = 0.01$	$\Omega = -1$	7.2	1.8	1.8	1.3
	$\Omega = 0$	4.5	2.8	2.7	1.4
	$\Omega = 1$	2.8	4.5	3.2	1.5
$B = 0.02$	$\Omega = -1$	9.4	1.4	1.4	1.1
	$\Omega = 0$	2.3	5.5	4.0	1.6
	$\Omega = 1$	0.6	23.4	4.0	1.6

FIG. 8.   The string loop effective potential $V_{\rm eff}(x)$ plotted for various combinations of the parameters J, B, Ω are illustrated for the flat or Schwarzschild spacetime in a way giving the energy that can be used for the string loop acceleration during the transmutation process in dependence on the magnetic field strength B. In the Schwarzschild spacetime we use the thick curve for $B = 0$, and the thin curve for $B > 0$; the case of the magnetic field in the flat spacetime is dashed. We will demonstrate the influence of the magnetic field on the string loop acceleration in the three scenarios of modifying the initial conditions because of the increase of B: (1) we keep the position x_i but change energy E; (2) we keep the energy E, but change position x_i; (3) we keep the position x_i and the energy E, but change the current J. In all figures the string loop parameter $\Omega = 0$.

increasing Ω. There is a slight increase of the maximal allowed acceleration, $\gamma_{\rm max}$, with increasing parameter B for $\Omega < 0$, while it decreases for $\Omega > 0$, in accord with the discussion of the properties of the function $E_{0(\rm min)}(B, J, \Omega)$ in the previous section. For the string loop with $\Omega = -1$, the topical value of the Lorentz factor, $\gamma_{\rm top}$, exceeds the

value corresponding to the case $B = 0$, while it is lower for $\Omega = 0, 1$. The value of the $\gamma_{\rm top}$ decreases with increasing B in all the cases of $\Omega = -1, 0, 1$.

For the second scenario (string loop current parameter J varied with B), the data of the numerical simulations giving the Lorentz γ factors are given in Table II for two
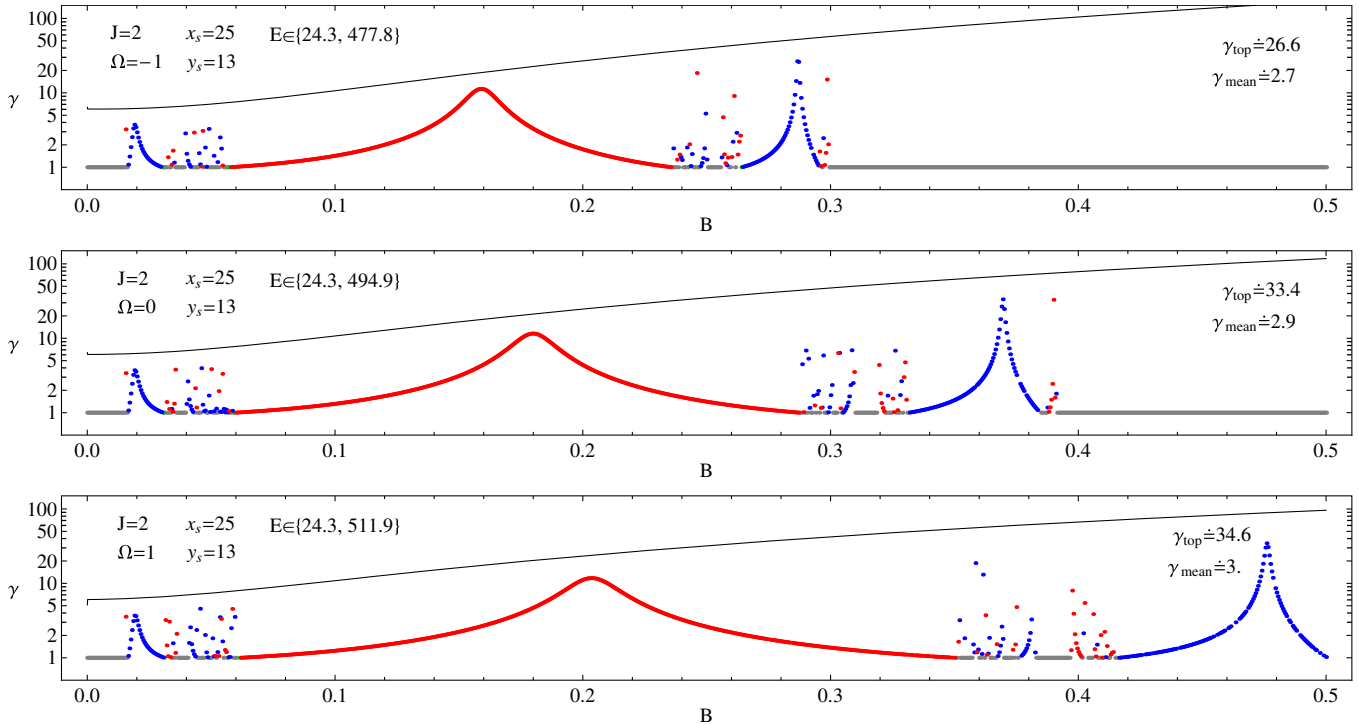

FIG. 9 (color online).   The Lorentz factor γ (vertical axis), calculated for the string loop with current $J = 2$, starting from rest state at a fixed initial position with coordinates x_0, y_0, while the strength of the magnetic field B is varied. The maximal acceleration given by Eq. (45) is plotted as the black curve. The string loop energy E is calculated from Eq. (30) and increases with increasing B.

characteristic values of B that are by one order smaller than in the previous case. We have to use smaller values of the magnetic field B, since a critical value B_{crit} of the magnetic field intensity exists for which the current $J = 0$, see Sec. III B. In this scenario, the parameter J decreases significantly with increasing Ω and increasing B. We have found an increase of the maximal allowed acceleration γ_{max} in comparison to the case $B = 0$ due to a decrease of the current J for $\Omega = 0, 1$, while for $\Omega = -1$ the current J increased and γ_{max} decreased. The value of γ_{top} decreases with increasing B for string loop with $\Omega = -1$ being lower than in the $B = 0$ case, while it increases with increasing B for $\Omega = 0, 1$ due to the strong decrease of J.
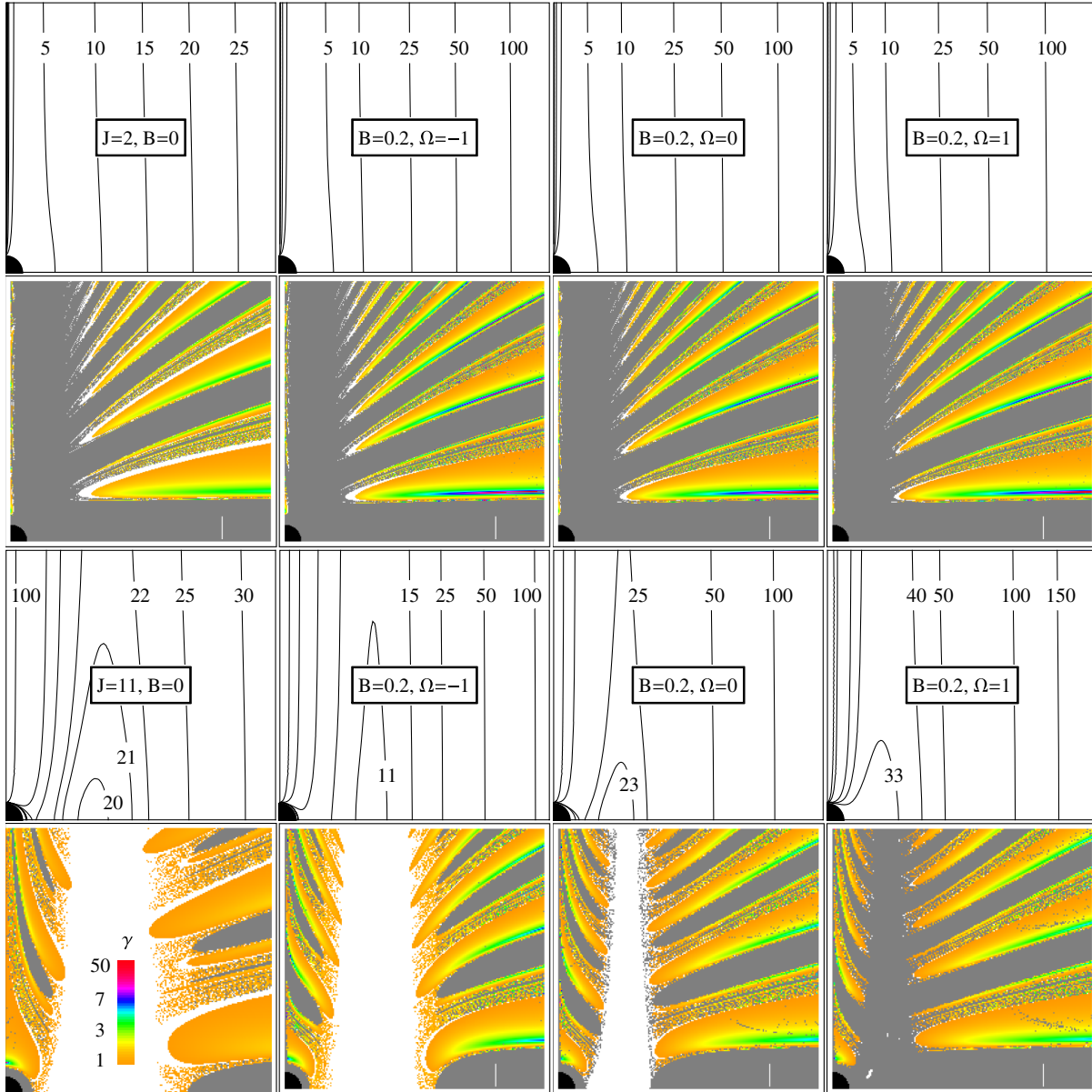

FIG. 10 (color online). String loop acceleration in the Schwarzschild spacetime with and without the external homogeneous magnetic field with $B = 0.2$, plotted for various starting points of the string loop and correspondingly modified energy. The string loop is starting from rest at the points from the region $x_s \in (0.1, 30.1)$, $y_s \in (0.1, 30.1)$ with the angular momentum parameter fixed to values of $J = 2$ (first two rows) or $J = 11$ (second two rows), but with varying energy E determined by the starting point (the energy levels are demonstrated in the first, $J = 2$, and the third, $J = 11$, row). In the distribution of the Lorentz factor γ, in the second row ($J = 2$) and the fourth row ($J = 11$), we colored every point according to the asymptotic Lorentz factor of the translational string loop motion in the y direction; the code for the colors is presented in the fourth row. Black color denotes regions below the horizon, grey regions correspond to the string loop collapsed to the black hole. Regions of white color correspond to string loop that do not reach "infinity" located at $r = 1000$ for given time $\zeta = 200$ and remain oscillating around the black hole.

For the third scenario (string loop energy E varied with B), the results of the numerical simulations for the Lorentz γ factor are summarized in Table III for the same two characteristic values of B as in the first scenario. In comparison to the case of $B = 0$, we need an increase of the energy E and we observe a large increase of the maximal allowed acceleration $\gamma_{\max}$ that increases with increasing Ω for the smaller value of $B = 0.1$, but it decreases with increasing Ω for $B = 0.2$. The Lorentz factor topical value γ_{top} also demonstrates a substantial increase in comparison to the case $B = 0$, especially for the larger magnetic field $B = 0.2$. Large γ factors are observed for the $\Omega = -1$ case, while for $\Omega = 0, 1$ the increase is smaller; in both cases it is caused by the large increase of the string energy E.

In the framework of the third scenario we give also data of the numerically calculated Lorentz γ factors in dependence on the magnetic field intensity B for the string loop parameters $\Omega = -1, 0, 1$ and fixed $J = 2$, fixed initial position of the string loop starting at rest, with energy E varied correspondingly. The data are plotted in Fig. 9; we observe increasing of the Lorentz γ factor with increasing B. We also observe combination of the regions of regular behavior of the function $\gamma(B)$, with regions of quite chaotic character, similar to those occurring in Fig. 7. Notice that γ_{top} takes highest value for the value of $\Omega = 1$ due to highest increase of the energy E parameter. On the other hand, it takes the lowest value for $\Omega = -1$; moreover, in this case the efficient acceleration can occur only in the field with $B < 0.3$—clearly, for larger values of B the efficient transmutation processes occur very close to the horizon and the string loop is efficiently captured by the black hole.

The transmutation between the oscillatory motion and the transitional accelerated motion can be properly represented by their distribution in the space of initial states $x_s - y_s$. The numerical simulations are giving the resulting Lorentz γ factor for two characteristic values of the magnetic field $B = 0$ and $B = 0.2$, two characteristic values of the string loop parameter $J = 2$ and $J = 11$, and the three characteristic values of the string loop parameter $\Omega = -1, 0, 1$. The results are given in Fig. 10, along with the $E = \text{const}$ levels for the considered cases of the internal and external parameters of the string loop in the combined gravitational and magnetic fields. We observe that strong acceleration and large final Lorentz γ factors can be obtained.

In astrophysically realistic situations, our results are valid up to the regions distant from the black hole where the magnetic field can be well approximated as uniform. Clearly, the condition of magnetic field uniformity assumed to be valid in the black hole vicinity and reasonably large distance will be violated in regions very distant from the black hole. Then a crucial question arises—how the string loop dynamics will be influenced by decreasing intensity of the magnetic field in distant regions described by the flat spacetime. We plan to study this problem in a future paper.

Nevertheless, we can expect that no change will occur in the flat spacetime far away from the black hole in the energy of the translational motion E_y. This energy has to be conserved since no energy transmutation is possible in the flat spacetime, and the energy mode E_y is independent on the intensity of the magnetic field, i.e., the string loop translational velocity along the y axis has to be conserved as well. However, we can expect the change in the energy of the string loop oscillations in the x direction since both the total energy, E, and the x-mode energy, E_x, will be modified by decreasing intensity of the magnetic field.

Another interesting question for future study arises, if we assume the uniform magnetic field that is not parallel to the string loop axis. Then a new force arises that turns the string loop axis to be parallel with the magnetic field. The nonparallel magnetic field case also opens up the question of stability of the string loops in $\Omega > 0$ configuration—any perturbation may overturn the string loop to the $\Omega < 0$ configuration that is energetically more favorable.

V. CONCLUSION

We have investigated acceleration of the electric current-carrying string loop due to the transmutation process in the gravitational field of the Schwarzschild black hole combined with an asymptotically uniform magnetic field. We have pointed out a physical interpretation of the string loop model through the superconductivity phenomena of plasma in accretions disks. We also give correspondence of the parameters of the string loop model of jets to real physical quantities and estimate such quantities in realistic astrophysical conditions.

In the pure spherically symmetric gravitational field the string loop dynamics is degenerated, being independent of the string loop motion constant Ω. In the combined gravitational and magnetic field that have a common axial symmetry only, the degeneration is canceled, and the dynamics is strongly dependent on the parameter Ω. The effective potential of the string loop dynamics allows for one stable and one unstable equilibrium points of the string loop. The maximal acceleration of the string loop is, however, determined by the minima of the effective potential of the string loop dynamics in the uniform magnetic field immersed in the flat spacetime.

The numerical analysis given in Tables I–III confirms significant acceleration (large γ factor) in the cases, where large $\gamma_{\max}$ is possible. The string loop acceleration is given by the transmutation process governed by two key ingredients: possibility of the string loop to escape with large ratio of the initial energy E to the minimum energy at infinity $E_{0(\min)}$, and existence of the transmutation region of strong gravitational (magnetic) fields where the chaotic regime of the string loop dynamics occurs and transmission of energy of the oscillatory motion to the energy of the translational motion is possible. It has been proved that presence of the external homogeneous magnetic field B

allows the string loop to escape to infinity with large Lorentz γ factor, the magnetic field can significantly increase the maximal acceleration given by $\gamma_{\max}$.

We have demonstrated that for the positive values of the parameter Ω, the presence of the magnetic field decreases the efficiency of the transmutation effect, while it increases the efficiency for negative values of the parameter Ω. We have shown that for the intensity of the magnetic field high enough, the string loop with negative parameter Ω can be strongly accelerated up to ultrarelativistic velocities of their translational motion. Therefore, the string loops accelerated in the field of magnetized Schwarzschild black holes could serve as an acceptable model of ultrarelativistic jets observed in active galactic nuclei. The black hole fast rotation is thus not necessary in the framework of the string loop acceleration model.

One of the most important consequences of our current paper, considered from the point of view of astrophysics and observable phenomena, is that the magnetic field substantially increases the efficiency of the acceleration mechanism of the string loop. The ultrarelativistic acceleration necessary in modeling the jets observed in microquasars and active galactic nuclei is shown to be possible even for nonrotating black holes in the string loop model. This is a clear opposite to the model of ultrarelativistic jets based on the Blandford-Znajek process that requires fast rotating black holes. Therefore, this difference can potentially give clear signature of relevance of string loop models.

APPENDIX A: ESTIMATION OF MAGNETIC FIELD INTENSITY IN THE VICINITY OF BLACK HOLES

Throughout the present paper we assume that the external electromagnetic field is weak in the sense that it is test one and does not effect the background black hole geometry (18).

The rough estimation indicates that the local spacetime curvature produced by the energy of the magnetic field of intensity B is of the order of GB^2/c^4, whereas the spacetime curvature is of the order of $1/M^2$ near the event horizon of a black hole with the total mass M. The critical value B_M of the magnetic field which starts to contribute to the spacetime curvature at the reasonable level can be found from the simple estimation [23]:

$$\frac{GB_M^2}{c^4} \sim \frac{c^4}{G^2 M^2}. \tag{A1}$$

In other words, the local curvature of the spacetime generated by the magnetic field B is of the same order or larger than the gravitational curvature of the black hole spacetime when [23]

$$B > B_M \approx \frac{c^4}{G^{3/2} M_\odot} \left(\frac{M_\odot}{M} \right) \approx 10^{19} \frac{M_\odot}{M} \text{ G}. \tag{A2}$$

Consequently for the test magnetic field $B \ll B_M$ when its influence on the spacetime curvature is totally negligible. According to the estimations made in [25,57], the maximal strength of the magnetic field in the vicinity of the event horizon of astrophysically realistic stellar mass black holes or supermassive black holes in the active galactic nuclei can be approximated as

$$B \approx 10^8 \text{ G}, \quad \text{for } M \approx 10 M_\odot, \tag{A3}$$

$$B \approx 10^4 \text{ G}, \quad \text{for } M \approx 10^9 M_\odot. \tag{A4}$$

The main condition $B \ll B_M$ is well satisfied for stellar mass and supermassive black holes both and the magnetic field practically cannot affect motion of neutral particles.

The self-magnetic field related to the current-carrying string loop has been estimated in our preceding paper [37], where we have demonstrated that the self-magnetic field of the string loop is much smaller than the external magnetic field, and its influence on the string loop motion can be abandoned.

APPENDIX B: DIMENSIONAL ANALYSIS AND ESTIMATES OF STRING LOOP PARAMETERS

In the geometrized units the gravitational constant G and the speed of light c are taken to be dimensionless units. Their values in the Gaussian units or so-called CGS units are

$$G = 6.67 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}, \qquad c = 3 \times 10^{10} \text{ cm s}^{-1}. \tag{B1}$$

Along with G and c, other world constants are also set to unity in some unit systems, e.g., Coulomb or electrostatic

constant $k_e = 1/(4\pi\varepsilon_0) = 1$. The conversions of the fundamental quantities characterizing the string loop motion from the geometrized units to the Gaussian units are shown in the Table IV. The table allows one to perform transformation from the geometrized units to the CGS units, and vice versa, for any dynamical quantity describing the string loop dynamics.

The string loop model enables one to apply and compare the derived solutions for different physical mechanisms to obtain the estimates of the parameters characterizing the string loop dynamics. In order to make the estimate of the tension μ strength, one can use, e.g., the similarity between the role of the parameter μ and the Lorentz force acting on a charged particle in the action governing the string loop dynamics. By comparison of the forces, one can obtain the tension for the string loop which is generated by charged particles orbiting in the vicinity of the black hole. In other words, for such a comparison the tension of the string loop is considered as an analogue of the Lorentz force acting on the string loop. Using Eqs. (27), (33) and (A3), this estimate gives in the Gaussian units the tension in order

$$\mu_{(L)} = 7.2 \times 10^8 \text{ dyn.} \tag{B2}$$

For the tension taken as (B2), one can find the values of the current and charge densities related to a stable electric current-carrying string loop. According to our previous papers [10,11,37], the stable configuration of the string loop implies the values of the dimensionless quantity cJ^2/μ in the order

$$\frac{cJ^2}{\mu} \sim 10, \tag{B3}$$

i.e., we obtain for the parameter $J^2 \sim 0.24 \text{ g s}^{-1}$. For the electric current-carrying string loop, i.e., when the parameters j_σ and j_τ are expressed in units given by Table IV, one has to set in the definition of the action (4) the parameter

$k = 1/c^3$ and rewrite the definition of the current parameter J including the constant c as

$$J^2 = \frac{k}{2}(j_\sigma^2 + c^2 j_\tau^2). \tag{B4}$$

In a particular case given by the relation $j_\sigma^2 = c^2 j_\tau^2$, i.e., when $\omega = \pm 1$, one gets the following estimated values for the charge density and the current of the string loop:

$$j_{\tau(L)} \approx 8.5 \times 10^4 \text{ statC} \cdot \text{cm}^{-1},$$
$$j_{\sigma(L)} \approx 2.5 \times 10^{15} \text{ statA.} \tag{B5}$$

Note, that the estimates given in (B2) and (B5) are obtained on the assumption that the current carriers by the string loop are the elementary particles like electrons or protons, i.e., the string loop is generated by individual charged particles. We call it a "Lorentz" case, and it can be considered as the lower limit of our string loop model, giving minimal estimates of the values of the fundamental parameters, i.e., the tension, the current, and the charge density, characterizing the stable, electric current-carrying string loop.

On the other hand, we can find estimates of the fundamental string loop parameter values related to the so-called cosmic strings, giving the upper limit of the application of the string loop model. The cosmic strings are theoretical constructions describing topological defects occurring in the very early universe with ultralarge mass densities due to the spontaneous symmetry breaking of fundamental physical interactions [44]. The recent summary of the study of the cosmic strings [58] demonstrates that the Nambu-Goto type cosmic strings have an upper limit given by the following dimensionless quantity $G\mu/c^4 < 2.6 \times 10^{-7}$, which implies an upper limit for the tension given by

$$\mu_{(CS)} < 3.15 \times 10^{42} \text{ dyn.} \tag{B6}$$

TABLE IV.   Units and dimensions of the physical quantities of the string loop in Gaussian (CGS) and geometrized system of units.

Quantity	Symbol	Gaussian	Geometrized	Conv.
Length	r	1 cm	1 cm	1
σ coordinate	σ	1 cm	1 cm	1
τ coordinate	$[\tau] = [ct]$	1 cm	1 cm	1
Time	t	1 s	2.99×10^{10} cm	c
Mass	m	1 g	7.42×10^{-29} cm	G/c^2
Energy	E	1 erg	8.26×10^{-50} cm	G/c^4
Tension	μ	1 dyn	8.26×10^{-50}	G/c^4
Neutral current	$k\varphi_{|a}^2$	1 g·s^{-1}	2.48×10^{-39}	G/c^3
Electric current	j_σ	1 statA	9.59×10^{-36}	$\sqrt{G}/c^3$
Charge	q	1 statC	2.87×10^{-25} cm	$\sqrt{G}/c^2$
Charge density	j_τ	1 statC·cm^{-1}	2.87×10^{-25}	$\sqrt{G}/c^2$
Magnetic field	B	1 Gs	8.16×10^{-15} cm^{-1}	$\sqrt{G}/c$

Applying again the relation (B3) for the tension, we can find the upper limits for the charge density and the current carried on the cosmic strings as

$$j_{\tau(CS)} < 5.6 \times 10^{21} \text{ statC} \cdot \text{cm}^{-1},$$

$$j_{\sigma(CS)} < 1.7 \times 10^{32} \text{ statA}. \tag{B7}$$

Therefore, we can conclude that the minimal range of applicability of the string loop model for the stable string loop is approximately determined as

$$\mu_{(L)} \leq \mu < \mu_{(CS)}, \tag{B8}$$

$$j_{\tau(L)} \leq j_\tau < j_{\tau(CS)}, \qquad j_{\sigma(L)} < j_\sigma \leq j_{\sigma(CS)}. \tag{B9}$$

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