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Higher spin versus renormalization group equations

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We present a variation of earlier attempts to relate renormalization group equations to higher spin equations. We work with a scalar field theory in 3 dimensions. In this case we show that the classical renormalization group equation is a variant of the Vasiliev higher spin equations with Kleinians on AdS_4 for a certain subset of couplings. In the large N limit this equivalence extends to the quantum theory away from the conformal fixed points.

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I. INTRODUCTION

In view of possible generalizations of the AdS/CFT correspondence away from conformal fixed points and, perhaps more importantly, deriving it from field theory, an interesting idea is to think of it as a geometric realization of the renormalization group (RG) flow (see, e.g., [1–6] for a selection of various attempts in this direction). One strategy, [7,8] (and more recently, [9]) is to relate higher spin (HS) equations to the RG-equation for a nonlocal mass term in a free field theory.

In this paper we consider a simpler, more restricted model, consisting of a scalar field theory in $2+1$ dimensions. After a short review of Polchinski's form [10] of the Wilsonian RG-equation we then consider the RG-flow for local, quadratic, higher derivative couplings. We first map the linearized RG-flow of these couplings, as functions of the cutoff scale to the *nonpropagating, auxiliary or, topological* sector of the free, nonminimal Vasiliev-type theory, with outer Kleinians [11] on a four-dimensional AdS-background.[1] This mapping turns out to be a straightforward consequence of the simple relation between the 3- and 4-dimensional HS-algebras in the twistor representation [13].

Next we consider the representation of HS-gauge transformations on the RG equations. While we are not able to give an interpretation of generic HS-gauge transformations we will analyze some simple examples in detail. In particular, we will see that while four dimensional HS-gauge transformations do not leave the field theory action invariant in general, the RG equation transforms covariantly with respect to such transformations.

Finally, we describe how nonlinear terms in the RG flow affect the HS-equation of motion. At the classical level these interactions produce an inhomogeneity in the HS equation which nevertheless preserves HS gauge invariance. The reason for this to work is that while the nonlinear flow induces a plethora of other couplings in addition to the higher spin couplings, described so far, these extra couplings do not effect the RG flow of the latter. At quantum level this is no longer the case. Nevertheless, we show that in the usual large N limit of an N-component scalar field theory the classical HS system is merely affected by anomalous dimensions which enter as a HS singlet in the RG equation.

We comment on possible applications of our findings to the HS-AdS/CFT duality at large N [14–17].

II. POLCHINSKI EQUATION

We start with a single scalar field in $d=2+1$ dimensions. Following the conventions of [18] the Polchinski equation, expressed in terms of dimensionless coordinates, momenta and fields takes the form

$$\left(\frac{\partial}{\partial t}+D\varphi\cdot\frac{\delta}{\delta\varphi}\right)\mathcal{S}_t[\varphi]=\frac{1}{2}\frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi]\cdot G\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi]-\frac{1}{2}\frac{\delta}{\delta\varphi}G\cdot\frac{\delta}{\delta\varphi}\mathcal{S}_t[\varphi]. \quad (2.1)$$

Here, $\mathcal{S}_t[\varphi]$ is basically the Wilsonian effective action, minus the kinetic term, at cutoff scale $t=-\ln(\Lambda/\Lambda_0)$, expressed in terms of the couplings and dimensionless fields $\Lambda^\delta\varphi(x\Lambda)\coloneqq\phi(x)$. Furthermore, δ is the canonical dimension of φ. $\mathcal{S}_t[\varphi]$ is essentially local, meaning that the nonlocality has an all-orders Taylor expansion for small p^2 (see, e.g., [19]). The UV-regularized propagator, $\mathcal{K}(p)/p^2$, enters through $G(p^2)=2\mathcal{K}'(p^2)$ with $\mathcal{K}(0)=1$, $\lim_{p^2\to\infty}\mathcal{K}(p^2)=0$. For simplicity we will also assume that $\mathcal{K}(p^2)$ is analytic in the neighborhood of the real positive semiaxis with essentially exponential falloff at infinity. Finally, D is the dilatation operator which acts on φ as

$$D\varphi(p)=-(p\partial_p+d-\delta)\varphi(p). \quad (2.2)$$

Expanding $\mathcal{S}_t$ in the fields and couplings we can extract the renormalization group equations. To illustrate this we set the right-hand side of (2.1) to zero which is justified in the

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[1]See [12] for a review on HS theories.

linearized approximation for a quadratic action and neglecting the cosmological constant. For instance, for $\mathcal{S}_t = m^2 \int \varphi(p)\varphi(-p)$ Eq. (2.1) then gives

$$(\partial_t m^2 - 2m^2(d-\delta)) \int \varphi(p)\varphi(-p) - m^2 \int (p\partial_p\varphi(p)\varphi(-p) + \varphi(p)p\partial_p\varphi(-p)) = 0. \quad (2.3)$$

Using $\delta = \frac{d-2}{2}$ for a scalar with canonical kinetic term and performing a partial integration on the last term we recover $\partial_t m^2 - 2m^2 = 0$. The marginal deformation in $2+1$ dimensions is $\mathcal{S}_t = \lambda \int \varphi(p_1)\cdots\varphi(p_6)\delta^3(p_1 + \cdots + p_6)$ with $D\varphi \cdot \frac{\delta}{\delta\varphi}\mathcal{S}_t = 2(d-3)\mathcal{S}_t$. We should note that in this case the second (quantum) term on the right-hand side of (2.1) acts as a source term for lower order couplings. Similarly, for $\mathcal{S}_t = g^{ab} \int p_a\varphi(p)p_b\varphi(-p)$ we get

$$(\partial_t g^{ab} - 2g^{ab}(d-\delta)) \int p_a\varphi(p)p_b\varphi(-p) - g^{ab} \int (p_a p\partial_p\varphi(p)p_b\varphi(-p) + p_a\varphi(p)p_b p\partial_p\varphi(-p)) = 0, \quad (2.4)$$

which, upon partial integration amounts to $\partial_t g^{ab} = 0$. More generally, we consider higher derivative terms of the form

$$\mathcal{S}_t = g^{a_1\cdots a_n} \int p_{a_1}\cdots p_{a_n}\varphi(p)\varphi(-p), \quad g^{a_1\cdots a_n} = g^{(a_1\cdots a_n)}. \quad (2.5)$$

This is a special case of the more general HS coupling analyzed in [20]. In particular, we can interpret $g^{a_1\cdots a_n}$ as sources for the integrated higher spin currents. Unlike [20], we assume the couplings to be independent of the coordinates of the $2+1$ dimensional field theory because we want to apply standard Wilsonian methods for the RG-flow. Equation (2.1) then yields at linear order

$$(\partial_t g^{a_1\cdots a_n} + (n-2)g^{a_1\cdots a_n}) \frac{\partial}{\partial g^{a_1\cdots a_n}} \mathcal{S}_t[g_t, \varphi] = 0. \quad (2.6)$$

As long as $\mathcal{S}_t$ is quadratic in φ this is exact in the linearized approximation up to the cosmological constant, which receives corrections proportional to $g^{a_1\cdots a_n}$ through the second term on the right-hand side of (2.1).

In what follows we shall concentrate on the RG flow of the *traceless* subset of couplings of the form (2.5), first in linearized and classical approximation. In this case we will show that the RG equation (2.1) is actually a free HS equation on AdS_4. Following [13] we note that the action of D on momenta has a Weyl star-product realization in terms of quadratic products of $(y^-_\alpha, y^{+\alpha})$ satisfying $[y^-_\alpha, y^{+\beta}]_* = \delta^\beta_\alpha$. In particular,

$$D = \frac{1}{2} y^{+\alpha} y^-_\alpha \quad \text{and} \quad P_{\alpha\beta} = i y^-_\alpha y^-_\beta, \quad (2.7)$$

where $P_{\alpha\beta}$ is the translation generator, whereas

$$L^\alpha{}_\beta = y^{+\alpha} y^-_\beta - \frac{1}{2}\delta^\alpha_\beta y^{+\gamma} y^-_\gamma, \quad \text{and} \quad K_{\alpha\beta} = -i y^+_\alpha y^+_\beta \quad (2.8)$$

represent Lorentz transformations and special conformal transformations respectively. We then replace $g^{a_1\cdots a_n}(t)p_{a_1}\cdots p_{a_n}$ by

$$g^{(n)}(t|y^\pm) \equiv e^{-2t} g^{a_1\cdots a_n}(t)(\gamma_{a_1})^{\alpha_1\beta_1}\cdots(\gamma_{a_n})^{\alpha_n\beta_n} y^-_{\alpha_1} y^-_{\beta_1}\cdots y^-_{\alpha_n} y^-_{\beta_n}, \quad (2.9)$$

where the factor e^{-2t} is introduced to absorb the spin-independent factor -2 in (2.6). We should mention that expressing p_μ in terms of the twistor variables $(y^-_\alpha, y^{+\alpha})$ is one-to-one only for lightlike momenta. However, this does not mean that the momenta in (2.5) are restricted to be lightlike since we merely use the twistors to represent the action of the dilatation operator on $g^{a_1\cdots a_n}$. For $g^{(n)}(t)$ the left-hand side of (2.1) is then equivalent to

$$D_t g^{(n)} \equiv \partial_t g^{(n)} + [D, g^{(n)}]_*. \quad (2.10)$$

In particular, the linearized RG equation for this class of couplings around the free field fixed point becomes $D_t g^{(n)} = 0$.

III. LINEARIZED RG FLOW AS HS EQUATION ON ADS

In this section we will identify $D_t g^{(n)} = 0$ with the linearized HS equation of motion on AdS_4. To do so we first express $P_{\alpha\beta}$ in terms of 4-dimensional spinor variables through [13]

$$P_{\alpha\beta} = i y^-_\alpha y^-_\beta = \frac{1}{4}(i\bar{y}_\alpha + y_\alpha)(\bar{y}_\beta - i y_\beta) \to \frac{i}{4}\bar{y}_{\dot\alpha}\bar{y}_{\dot\beta} - \frac{i}{4} y_\alpha y_\beta + \frac{1}{4}\bar{y}_{\dot\alpha} y_\beta + \frac{1}{4} y_\alpha \bar{y}_{\dot\beta}. \quad (3.1)$$

Next we express D in terms of 4-dimensional spinor variables,

$$D = \frac{1}{2}\epsilon_{\alpha\beta} y^{+\alpha} y^{-\beta} = \frac{1}{8}\epsilon_{\alpha\beta}(y^\alpha - i\bar{y}^\alpha)(\bar{y}^\beta - i y^\beta) = \frac{1}{4}\epsilon_{\alpha\beta} y^\alpha \bar{y}^\beta \to -\frac{i}{4}(\sigma_2)_{\alpha\dot\beta} y^\alpha \bar{y}^{\dot\beta}. \quad (3.2)$$

We now want to compare (2.10) with the HS equation [11]

$$dC + W * C - C * W = 0, \quad (3.3)$$

where $W = W_\mu dx^\mu$ is a HS gauge potential and C is the field which we will shortly identify with the coupling constants describing perturbations of the RG fixed point. The integrability condition for (3.3) reads $dW + W * W = 0$. The form (3.2) of the dilatation operator D picks the AdS_4 solution to this latter equation. Indeed, the AdS_4 solution is given by

$$W_0(x|Y) = e_0(x|Y) + \omega_0(x|Y), \tag{3.4}$$

where (in Poincaré coordinates)

$$e_0(x|Y) = \frac{1}{4i}\frac{dx^\mu}{z}(\sigma_\mu)_{\alpha\dot\beta} y^\alpha \bar y^{\dot\beta},$$
$$\omega_0(x|Y) = -\frac{1}{4i}\frac{dx^i}{z}((\sigma^{iz})_{\alpha\beta} y^\alpha y^\beta + (\bar\sigma^{iz})_{\dot\alpha\dot\beta}\bar y^{\dot\alpha}\bar y^{\dot\beta}).$$

Our conventions for the 4D sigma matrices are $(\sigma_\mu)_{\alpha\dot\beta} = (1,\vec\sigma)_{\alpha\dot\beta}$, $(\bar\sigma_\mu)^{\dot\alpha\beta} = (1,-\vec\sigma)^{\dot\alpha\beta}$. The 3D gamma matrices are then obtained by deleting the matrix with space-time index $\mu = 2$, i.e., $(\gamma_a)_{\alpha\beta} = (1,\sigma_1,\sigma_3)_{\alpha\beta}$. Identifying $t = \ln z$ we then see that expression (3.2) for D is identical with $(W_0)_t(x|Y)$. Thus the tree-level, linearized RG-flow [left-hand side of (2.1)] is identified with the HS equation [11]

$$\partial_t C(x|Y) + [(W_0)_t(x|Y), C(x|Y)]_* = 0, \tag{3.5}$$

where $C(x|Y)$ is given by $g^{(n)}(t|y^\pm)$ as in (2.9) but expressed in terms of 4-dimensional spinor variables using

$$\begin{aligned}4(\gamma^a)^{\alpha\beta} y^-_\alpha y^-_\beta &= (\gamma^a)^{\alpha\beta}(\bar y_\alpha \bar y_\beta - y_\alpha y_\beta - i\bar y_\alpha y_\beta - i y_\alpha \bar y_\beta)\\ &= -i(\gamma^a)^{\alpha\beta}(\bar y_\alpha y_\beta + y_\alpha \bar y_\beta) + \epsilon^{abc}(\gamma_{bc})^{\alpha\beta}(\bar y_\alpha \bar y_\beta - y_\alpha y_\beta)\\ &\to -i(\sigma^a)^{\alpha\dot\beta} y_\alpha \bar y_{\dot\beta} - i(\sigma^a)^{\dot\alpha\beta}\bar y_{\dot\alpha} y_\beta + \epsilon^{abc}((\bar\sigma_{bc})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta} - (\sigma_{bc})^{\alpha\beta} y_\alpha y_\beta)\\ &= -2i(\sigma^a)^{\alpha\dot\beta} y_\alpha \bar y_{\dot\beta} + 2i((\bar\sigma^{za})^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\bar y_{\dot\beta} + (\sigma^{za})^{\alpha\beta} y_\alpha y_\beta),\end{aligned} \tag{3.6}$$

and we defined $\gamma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$. In order to verify the other components of the equation (3.3), i.e.,

$$\partial_a C + (W_0)_a * C - C * (W_0)_a = 0, \tag{3.7}$$

we then observe that (3.6) is proportional to $(W_0)_a$ in (3.4). This then implies that (3.7) is satisfied so long $C(x|Y)$ does not depend on x^a, ie. $C(x|Y) = C(t|Y)$.

To summarize, *Eq.* (3.5) *is the tree-level linearized RG-equation for the couplings* $g^{(n)}$ *around the Gaußian fixed point while* (3.7) *encodes translation invariance*.

Note that in (3.5) and (3.7) the commutator $[(W_0)_a, C]_*$ rather than the twisted commutator $(W_0)_a * C - C * (\tilde W_0)_a$ with $\tilde f(x|y,\bar y) \equiv f(x|-y,\bar y)$ enters in the equation for $C(x|Y)$. This means that $C(x|Y)$ is a nonpropagating (auxiliary) field. This is to be expected since the RG equation is a first order rather than a second order equation. In other words, the auxiliary sector is responsible for couplings (moduli) of the theory. A similar observation was made in 3-dimensional HS theory considered in [21] which describes HS interactions of 3d matter fields of an arbitrary mass. In this case, as explained in that paper, the mass parameter is directly related to the value of some of the auxiliary fields.

To make the relation between HS and RG equations complete we should also consider the Vasiliev equations involving the auxiliary spinor connection, $S(Z,Y,k,\bar k) = s_\alpha dz^\alpha + s_{\dot\alpha} dz^{\dot\alpha}$, that is [11]

$$\begin{aligned}S * S &= -i dz_\alpha dz^\alpha(1 + F(B) * \kappa)\\ &\quad - i dz_{\dot\alpha} dz^{\dot\alpha}(1 + \bar F(B) * \bar\kappa),\\ S * B - B * S &= 0,\\ dS + W * S - S * W &= 0.\end{aligned} \tag{3.8}$$

Here, $Z = (z^\alpha, z_{\dot\alpha})$ is pair of auxiliary twistor variable z^α with $[z^\alpha, z^\beta]_* = -2i\epsilon^{\alpha\beta}$ and $dz^\alpha dz^{\dot\alpha} = -dz^{\dot\alpha} dz^\alpha$ are anti-commuting differentials. The field $B(x|Y,Z)$ is such that $B(x|Y,Z)|_{Z=0} = C(x|Y)$ and otherwise determined by the Vasiliev equations. $F(B)$ is so far an arbitrary function and $\kappa = kK$ where k is a Kleinian with the property

$$kf(z^\alpha, dz^\alpha, y^\alpha, z^{\dot\alpha}, dz^{\dot\alpha}, y^{\dot\alpha}) = f(-z^\alpha, -dz^\alpha, -y^\alpha, z^{\dot\alpha}, dz^{\dot\alpha}, y^{\dot\alpha})k \tag{3.9}$$

and $K = e^{iz_\alpha y^\alpha}$ is an inner Kleinian for the $*$-product with the properties

$$\begin{aligned}f(y,z) * K &= f(-z,-y)K, \quad \text{and}\\ K * f(y,z) &= Kf(z,y).\end{aligned} \tag{3.10}$$

To 0th order in B, the first equation in (3.8) is solved by $S_0 = z_\alpha dz^\alpha + z_{\dot\alpha} dz^{\dot\alpha}$. The second equation then implies that B is independent of Z at leading order (which is the case). To 0th order in B the last equation is then again identically satisfied. On the other hand, for nonvanishing $F(B)$ this equation implies a correction to W. In the last section we will see that in pour case interactions enter in a different way leaving the connection W invariant.

IV. GAUGE TRANSFORMATIONS

A complete understanding of how to represent arbitrary HS gauge transformations in RG equations is still lacking. We can nevertheless develop some intuition by considering examples. The HS gauge transformations are of the form [13]

$$\delta B(x|Y,Z) = -\epsilon(x|Y,Z) * B(x|Y,Z) + B(x|Y,Z) * \epsilon(x|Y,Z),$$
$$\delta W(x|Y,Z) = d\epsilon(x|Y,Z) + [W,\epsilon]_*(x|Y,Z). \quad (4.1)$$

Let us first assume that $\epsilon = \epsilon(t|y^\alpha, \bar{y}^{\dot{\alpha}})$. In that case the first line in (4.1) reduces to

$$\delta g^{(n)} = [g^{(n)}, \epsilon]_*. \quad (4.2)$$

In order to obtain a 3-dimensional interpretation of $\delta g^{(n)}$ we should express y^α and $\bar{y}^{\dot{\alpha}}$ in terms of $y^{\pm\alpha}$. For instance, if we take $\epsilon = b^a(t)K_a$ where K_a is given by (2.8) expressed in 4D spinor variables, this then amounts to

$$g^{(n)} \to g^{(n)} + 2(g\cdot b)^{(n-1)}D + 2i(g^{[a}b^{b]})^{(n-1)}L_{ab}, \quad (4.3)$$

which can then be realized in field theory as lower dimensional, momentum dependent couplings in $\mathcal{S}_t$. In (4.3) symmetric (Weyl) ordering is understood. This transformation does not leave $\mathcal{S}_t$ invariant, generically,[2] although some terms may vanish. For instance, for n odd $\mathcal{S}_t$ does not depend on $g^{(n)}$ as we already noted. Similarly, the Lorentz term on the right-hand side of (4.3) vanishes in $\mathcal{S}_t$ for n odd while the dilatation term does contribute a coupling with n even. The connection, on the other hand, transforms as

$$D \to D + \partial_t b^a K_a - b^a K_a. \quad (4.4)$$

Clearly, $W_0 + \delta W$ is no longer of the AdS-form, not even modulo coordinate transformations which are parametrized by $\epsilon(x)$. Nevertheless, (2.10) transforms covariantly by construction. Explicitly,

$$[D,\delta g^{(n)}] + [\delta D, g^{(n)}]$$
$$= 2n(\delta g)^{(n-1)} - 2(\dot{b}\cdot g)^{(n-1)}D - 2i(\dot{b}^{[a}g^{b]})^{(n-1)}L_{ab} \quad (4.5)$$

and

$$\partial_t \delta g^{(n)} = 2(\dot{b}\cdot g)^{(n-1)}D + 2i(\dot{b}^{[a}g^{b]})^{(n-1)}L_{ab}$$
$$+ 2(b\cdot\dot{g})^{(n-1)}D + 2i(b^{[a}\dot{g}^{b]})^{(n-1)}L_{ab}. \quad (4.6)$$

Since $\dot{g}^{(n)} = -ng^{(n)}$, (2.10) is satisfied without using further properties of $\mathcal{S}_t$. If $b(t)$ satisfies the RG equation then the connection D in (4.4) is invariant so that (2.10) in its original form is valid for $g^{(n)}$ as well as $\delta g^{(n)}$. It is interesting to note that the RG equation displays an $o(3,2)$ covariance even if the action parametrized by $\{g^{(n)}\}$ is not conformal. Of course, we have only considered the linearized flow so far.

As an example of a z^α-dependent gauge transformation we take $\epsilon = K$ as in (3.10). Then $\delta B = -K * B + B * K$ or, equivalently

$$\delta g^{(n)} = -K * g^{(n)} + g^{(n)} * K$$
$$= -K(z,y)g^{(n)}(z^\alpha, \bar{y}^{\dot{\alpha}}) + g^{(n)}(-z^\alpha, \bar{y}^{\dot{\alpha}})K(z,y). \quad (4.7)$$

Remember that $g^{(n)}$ was independent of z^α in our construction but this is not a gauge-invariant statement. Since there is no interpretation for z^α in 3-dimensions I suspect that the correct interpretation in 3 dimensions is to set $z^\alpha = 0$. Thus $\delta g^{(n)} = 0$.

V. ADDING THE INHOMOGENEITY

Let us finally return to the right-hand side of (2.1). In [8] it was suggested to include this terms in a redefinition of W_0. However, we shall see below that this is realized differently in the HS description. For a free field theory the quantum term on the right-hand side can only contribute to the cosmological constant which we already discussed and which is not part of the higher spin spectrum. Let us now turn to the first (tree-level) term on the right-hand side of (2.1) which corresponds to the graph in Fig. 1a. This graph contributes a source term to (2.6) of the form[3]

$$\int \varphi(-p)g^{a_1\cdots a_m}g^{b_1\cdots b_n}\mathcal{K}'(p^2)p_{a_1}\cdots p_{a_m}p_{b_1}\cdots p_{b_n}\varphi(p). \quad (5.1)$$

Of course, the product of two traceless couplings as in (5.1) need not be traceless so that traceful couplings will be

[2]It is conceivable that an invariant action can nevertheless be found along the lines [22] where a consistent coupling of a scalar to higher spin external fields is constructed.

[3]In x-space this sources a nonlocal coupling through

$$\partial_t g^{a_q\cdots a_{n+m}}(x)(\partial_{a_1}\cdots\partial_{a_{m+n}}\varphi(y))\varphi(x+y)$$
$$= g^{a_1\cdots a_n}g^{a_{n+1}\cdots a_{n+m}}\mathcal{K}'(x)(\partial_{a_1}\cdots\partial_{a_{m+n}}\varphi(y))\varphi(x+y).$$

Taking the symmetrized, traceless part of both sides this amounts to

$$D_t g^{(n+m)}(t,x)$$
$$= \mathcal{K}'(x)e^{-2t}g^{a_1,\cdots,a_n}(t)g^{a_{n+1},\cdots,a_{n+m}}(t)\gamma^{\alpha_1\beta_1}_{a_1}\cdots\gamma^{\alpha_{n+m}\beta_{n+m}}_{a_{n+m}}y^-_{\alpha_1}\cdots y^-_{\beta_{n+m}}.$$

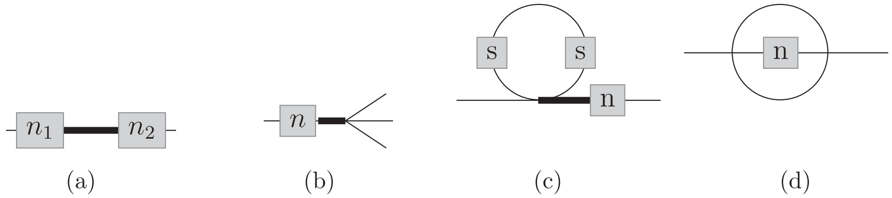

FIG. 1. Classical and quantum contributions the right-hand side of (2.1). A fat line represents an insertion of $\mathcal{K}'(0)$ and the squares represent traceless, quadratic higher derivative couplings, $g^{(n)}$.

generated along the RG flow. In order to clarify the algebraic structure it is then useful to define a product "$\cdot$" on the space of couplings through

$$g^{a_1\cdots a_m}\cdot g^{b_1\cdots b_n}=g^{(a_1\cdots a_m}g^{b_1\cdots b_n)}, \tag{5.2}$$

where $(\cdots)$ stands for symmetrization of the indices. This product is clearly associative. If we furthermore divide out the ideal generated by elements containing traceful couplings (corresponding to deformations of $\mathcal{S}_t$ that involve powers of the d'Alembertian, or, p^2) then the set of traceless symmetric higher derivative couplings with the above product form an Abelian algebra.[4]

In order to describe the contribution from (5.1) to the flow of the traceless couplings we expand $\mathcal{K}'$ as $\mathcal{K}'(p^2)=\mathcal{K}'(0)+O(p^2)$. Then, recalling that representation of the momenta p_a in term of spinor variable as in (2.7) takes care of symmetrization and projection onto vanishing trace automatically the nonlinear correction to (3.5) can be written as

$$\partial_z C(t|Y)+[(W_0)_z(t|Y),C(t|Y)]_*=\mathcal{K}'(0)C(t|Y)C(t|Y). \tag{5.3}$$

As it stands this equation does not look covariant under higher spin gauge transformations (4.1) not even when restricted to gauge parameters $\epsilon(x|Y,Z)$ that are independent of Z. Note, however, that we may as well replace $C(t|Y)C(t|Y)$ by the star products $C(t|Y)*C(t|Y)$ since on functions that depend only on y^-_α both products agree. Thus (5.3) is, in fact, covariant under higher spin gauge transformations. Finally, the traceful couplings sourced by (5.1) and sitting in the ideal defined above, do not "backreact" on the traceless couplings at tree level.

Let us now analyze what happens in the presence of interactions. Here we focus on the quartic term $\lambda\int\varphi(p_1)\cdots\varphi(p_4)\delta^3(p_1+\cdots+p_4)$ in $\mathcal{S}_t$ which is interesting in connection with the $O(N)$ model. At tree level, the RG flow (2.1) produces, among higher order couplings, also traceless couplings of the form (Fig. 1b)

$$2\lambda\mathcal{K}'(0)g^{a_1\cdots a_n}\int q_{a_1}\cdots q_{a_n}\varphi(q)\varphi(p_2)\cdots\varphi(p_4)\times\delta^3(q+\cdots+p_4) \tag{5.4}$$

which are not captured by the higher spin system (5.3) but, again do not effect the running of the traceless, quadratic couplings at tree level and thus the system (5.3) is indeed closed at tree level.

At quantum level [2nd term on the right-hand side of (2.1)] the above decoupling no longer takes place. Indeed, at one-loop the iterated flow equations will involve contributions to the flow of $g^{a_1\cdots a_n}$ that include the graph displayed in Fig. 1c, that is

$$g^{a_1\cdots a_n}\lambda\mathcal{K}'(0)(g^{(s)},g^{(s)})\int(p^2)^s(\mathcal{K}'(p^2))^3, \tag{5.5}$$

where $(g^{(s)},g^{(s)})=g^{a_1\cdots a_s}g_{a_1\cdots a_s}$ is an element in the ideal. This, and other contributions of this type to the flow can be incorporated by noticing that they affect the dimension $\gamma(g^{(s)},\lambda)$ of $g^{a_1\cdots a_n}$ as a HS singlet. They are thus consistent with our HS equation (5.3) provided we modify the definition (2.9) as

$$g^{(n)}(t)\equiv\Gamma(t)e^{-2t}g^{a_1\cdots a_n}(t)(\gamma_{a_1})^{\alpha_1\beta_1}\cdots\times(\gamma_{a_n})^{\alpha_n\beta_n}y^-_{\alpha_1}y^-_{\beta_1}\cdots y^-_{\alpha_n}y^-_{\beta_n}, \tag{5.6}$$

where $\Gamma(t)$ is the iterative solution to $\partial_t\Gamma=\gamma(t)\Gamma$. The corresponding HS-RG equation can again be brought to the form (5.3) by a reparametrization $s=f(z)$ with $f'(z)=\Gamma^{-1}(z)$. The HS covariance of (5.3) then follows from the fact that $\gamma(g^{(s)},\lambda)$ is invariant under HS symmetry transformations. Higher loop corrections to γ in λ arise among others from the "bubble diagrams" (see, e.g., [23]).

Note that all diagrams discussed so far and which contribute to the running of $g^{(n)}$ are proportional to $\mathcal{K}'(0)$. This implies, in particular that, if we choose a cutoff function such that $\mathcal{K}'(0)=0$, then these contributions vanish. The scalar coupling is an exception since there is a contribution to the running of the mass $g^{(0)}$ due to standard mass renormalization in φ^4-theory. Of course, there are many more diagrams that contribute to the running of $g^{(n)}$ even for $\mathcal{K}'(0)=0$. One such diagram is depicted in Fig. 1d. While this latter contribution can be absorbed in

[4] If we allow the couplings to depend on the coordinates of the $2+1$ dimensional field theory this algebra is no longer Abelian. In that case, upon suitable ordering of the derivatives as in [22] the couplings generate the higher spin algebra.

the anomalous dimension as in (5.5) we do not expect this to be the case for all higher diagrams of this type.

A substantial simplification occurs, however, if we consider the large N limit of this vector model with Lagrangian (see, e.g., [23])

$$\mathcal{L} = \frac{1}{2}\partial_\alpha\varphi^i\partial^\alpha\varphi^i + \frac{1}{2}m^2\varphi^i\varphi^i + \frac{\lambda}{4!N}(\varphi^i\varphi^i)^2. \tag{5.7}$$

Contributions of the form Fig. 1d are then suppressed in the large N limit. Thus we conclude that

In the large N limit the RG equations for the traceless higher derivative couplings $g^{(n)}$ are precisely given by (5.3) except the mass term $g^{(0)}$. Furthermore, for $\mathcal{K}'(0) = 0$ equation (5.3) reduces to the free equation (3.5).

It is reassuring to note that the higher spin couplings $g^{(n)}$, $n \geq 2$ are (marginally) irrelevant in the IR so that the IR fixed point is independent of the value of $\mathcal{K}'(0)$.

VI. DISCUSSION

In this paper we found that the classical RG flow for traceless higher derivative couplings that are quadratic in the fields is an interacting HS equation of motion for the auxiliary, nonpropagating sector on an AdS background. In the large N limit of an N-component interacting scalar field theory this equivalence extends to the full quantum theory even away from the conformal fixed points. The fact that the auxiliary sector is related to couplings in the RG equation seems physically sensible since the RG flow is a first order flow. Although our derivation was done for a scalar field in $2+1$ dimensions other types of fields can be treated on same footing. Generalizations to other space-time dimensions should also be possible but the 3-dimensional case is particularly intuitive due to the simple relationship between 3- and 4-dimensional twistor variables. An interesting question is whether the relation found here can be extended to finite values of N.

Another interesting question is whether the HS nature of the RG equation is useful in explaining the origin of the HS/O(N)-model duality which involves the physical, propagating sector of the Vasiliev theory. Perhaps progress can be made in combining the map constructed her here with previous ideas developed in [20] for instance. One possible hint comes from the observation that in a covariant formulation of the interaction HS theory, the auxiliary HS modes which were related to the couplings in this paper source physical HS modes [11]. From our point of view it is therefore natural to interpret the latter modes as vevs. We hope to come back to this issue in the future.

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