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Casimir energy in Kerr space-time

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We investigate the vacuum energy of a scalar massless field confined in a Casimir cavity moving in a circular equatorial orbit in the exact Kerr space-time geometry. We find that both the orbital motion of the cavity and the underlying space-time geometry conspire in lowering the absolute value of the (renormalized) Casimir energy $\langle\epsilon_{\rm vac}\rangle_{\rm ren}$, as measured by a comoving observer, with respect to whom the cavity is at rest. This, in turn, causes a weakening in the attractive force between the Casimir plates. In particular, we show that the vacuum energy density $\langle\epsilon_{\rm vac}\rangle_{\rm ren} \to 0$ when the orbital path of the Casimir cavity comes close to the corotating or counter-rotating circular null orbits (possibly geodesic) allowed by the Kerr geometry. Such an effect could be of some astrophysical interest on relevant orbits, such as the Kerr innermost stable circular orbits, being potentially related to particle confinement (as in some interquark models). The present work generalizes previous results obtained by several authors in the weak field approximation.

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I. INTRODUCTION

The vacuum state of a quantum field exhibits a nontrivial structure in presence of boundaries, which can be real material media or represent some peculiar topology of the space. As a result, we are faced with two main phenomena, namely vacuum polarization and particle creation. The former has a beautiful manifestation in the celebrated (static) Casimir effect, in which an attractive force arises between two parallel uncharged conducting plates. Such a force is due to a negative quantum vacuum energy associated with the field between the plates.

Since the pioneering Casimir's work [1,2], a lot of effort has been devoted to a deeper investigation of vacuum fluctuations in various physical configurations. It has been found that the Casimir force does depend on the topology as well as on the peculiar geometry of the boundaries [3,4].

A number of authors have investigated the influence of external gravitational fields on Casimir vacuum energy [5–11]. It has been proved that Casimir energy gravitates, according to the equivalence principle [12–14]. Inertial effects due to rotation have been considered in [15–18]. Also, quantum vacuum energy has been discussed in cosmological models allowing for field confinement, as in the static Einstein universe [19,20], or in other closed models [21].

In a recent paper, Bezerra *et al.* [22] have evaluated the renormalized vacuum energy density for a massless scalar field in a Casimir cavity placed nearby a massive, spherical, rotating source, having mass M and angular momentum $\vec{J}$. Working in the weak field approximation, they found a $O(m/r)^2$ small correction to the vacuum energy due to the source rotation. Such result agrees with that found in [11], where it was proved that a (weak) gravitomagnetic field does not influence the Casimir vacuum energy at the first order in (m/r). However, a closer look at [22] shows that calculations are performed in a comoving frame, which is assumed to rotate just with the angular velocity Ω required to reduce the space-time metric to its diagonal form. This means that $\Omega = \omega_d$, where $\omega_d = 2J/r^3$ is (in the weak limit) the dragging angular velocity of space-time around the rotating source. In other words, the adopted frame is that of a ZAMO (zero-angular-momentum observer), according to whom the Casimir cavity is at rest and the space-time is locally nonrotating. Now, it seems unlikely that Ω is the same one of the gravitational source, (say, Ω_s), as assumed in [22]. Actually, in the weak limit $\omega_d = 2J/r^3 \simeq \frac{r_{\rm sch}}{r}\Omega_s \to \omega_d \ll \Omega_s$, since, in the adopted weak limit approximation, one must have $r_{\rm sch} \ll r$ ($r_{\rm sch} = 2M$). We conclude that the results presented in [22] seem relevant only if the physical measurements are performed by a ZAMO, comoving with the cavity.

The main purpose of the present paper is to generalize the results of [10,11,22], relaxing the assumptions of (a) weak field approximation and (b) a ZAMO comoving with the Casimir apparatus.

It is well known that the original (flat space-time, nonrotating) Casimir effect for a quantum field confined in a cavity arises from the space-time symmetry breaking induced by the boundaries (the reflecting plates) that change the trivial R^3 space topology into the $R^2 \times [0, L]$ topology (L is the plate separation). Namely, it is the breaking of translational invariance that causes the shift in the vacuum energy of the field inside the Casimir cavity. Here, we are interested in other possible symmetry-breaking mechanisms that could cause further changes in the field vacuum energy.

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An expected breaking is that of the azimuthal reflection symmetry $\phi \to -\phi$, induced by the rotation of a Casimir cavity orbiting with angular velocity Ω with the plates orthogonal to the orbital motion direction.

Considering such orbital motion as an equatorial orbit in the space-time geometry of a rotating gravitational source (Kerr space-time), physical intuition suggests that a comoving observer should detect no shift in the vacuum energy as far as the orbital motion is that of a ZAMO, i.e., if $\Omega = \omega_d$. Actually, according to a ZAMO, the $\phi \to -\phi$ symmetry is restored, since there is no local rotation at all. On the other hand, a comoving observer should measure a growing effect on the field vacuum energy as the difference $|\Omega - \omega_d|$ increases.

We will see that the Casimir vacuum energy density indeed suffers a shift due both to the cavity orbital motion and the gravitational dragging induced by the background space-time. Such a shift causes a lowering of the absolute value of the vacuum energy. In particular, we will show that the (renormalized) vacuum energy density $\langle \epsilon_{\rm vac} \rangle_{\rm ren} \to 0$ when the orbital path of the Casimir cavity comes close to the corotating or counter-rotating circular null orbits (possibly geodesic) allowed by the Kerr geometry.

Because of the intrinsic smallness of the Casimir energy, its coupling with a gravitational background is usually expected to be too tiny to be experimentally detected, mainly in the weak field limit, as we will see in the final discussion. Nevertheless, in presence of strong gravitational fields, as near black holes, the presented effect could be likely to have some astrophysical relevance, being potentially related to particle confinement [5].

The plan of the paper is the following. In Sec. II we briefly recall the main properties of the Kerr space-time with special concern for the equatorial orbital motion. We define a local frame, to be employed by a physical observer comoving with the cavity, according to whom the Casimir vacuum energy measurements are performed. In Sec. III we solve (in the local frame of the observer) the Klein-Gordon equation for a massless scalar field confined in a Casimir cavity following an equatorial orbit (not necessarily geodesic). At variance with [22], we assume that the cavity plates are placed orthogonally to the direction of the orbital motion. In such a configuration the breaking of the azimuthal reflection symmetry, $\phi \to -\phi$, induces a distortion of the discretized field modes inside the cavity. After a mode normalization and a subsequent regularization procedure, we get the renormalized vacuum energy density for the field in the Casimir cavity. We point out the fundamental role played by the proper cavity length in obtaining a physically meaningful result (see [10]). In Sec. IV we discuss several relevant cases, considering in particular the geodesic motion of the cavity and the weak field, slow motion limit. Finally, Sec. V is devoted to some concluding remarks. Throughout the text, unless otherwise specified, use has been made of natural units: $\hbar = c = G = 1$. Greek indices take values from 0 to 3, latin ones take values from 1 to 3. The metric signature is -2 with determinant g.

II. ORBITAL MOTION IN KERR SPACETIME: AN OVERVIEW

A. The space-time metric

The Kerr solutions are the only known family of exact, stationary, axisymmetric (with further reflection symmetry with respect to the equatorial plane), asymptotically flat solutions, which are believed to represent the exterior space-time metric outside a rotating massive spherical object. The metric can be given in the Boyer-Lindquist coordinates (t, r, ϕ, θ), namely

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{4Mar}{\Sigma} \sin^2\theta dt d\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{A}{\Sigma} \sin^2\theta d\phi^2, \quad (1)$$

where

$$\Sigma = r^2 + a^2 \cos^2\theta, \quad (2)$$

$$\Delta = r^2 + a^2 - 2Mr, \quad (3)$$

$$A = (r^2 + a^2)^2 - a^2 \Delta \sin^2\theta = (r^2 + a^2)\Sigma + 2Mra^2 \sin^2\theta; \quad (4)$$

M and $a = J/M$ are the mass and the specific angular momentum of the rotating body. The roots $r_\pm$ of $\Delta = 0$ give the Kerr horizons, while those of $g_{tt} = 0$ define the infinite redshift surfaces $r_{0\pm}$,

$$r_\pm = M \pm \sqrt{M^2 - a^2}, \quad (5)$$

$$r_{0\pm} = M \pm \sqrt{M^2 - a^2 \cos^2\theta}. \quad (6)$$

To avoid naked singularities we usually assume $a < M$. In such a case there exists a region $r_+ < r < r_{0+}$, termed ergoregion; $r > r_+$ defines the physically accessible region, outside the horizon r_+.

B. Circular equatorial orbits

The 4-velocity of an observer $\mathbf{w}$ (or a test particle) orbiting in the equatorial plane ($\theta = \pi/2$) in the Boyer-Lindquist coordinates (1) is

$$w^\mu = C(\Omega)(\delta^\mu_t + \Omega \delta^\mu_\phi), \quad (7)$$

where Ω is the orbital angular velocity of $\mathbf{w}$ as seen from spatial infinity and $C(\Omega)$ is obtained from the normalization condition $w^\mu w_\mu = 1$. One has

$$C(\Omega) = [\Omega^2 g_{\phi\phi} + 2\Omega g_{t\phi} + g_{tt}]^{-1/2}. \tag{8}$$

Allowed orbiting observers require $\Omega^2 g_{\phi\phi} + 2\Omega g_{t\phi} + g_{tt} > 0$, so

$$\Omega_-(r) < \Omega < \Omega_+(r), \tag{9}$$

for any allowed orbital radius r, where

$$\Omega_\pm(r) = \omega_d \pm \frac{\Sigma\sqrt{\Delta}}{A}, \qquad r > r_+, \tag{10}$$

and

$$\omega_d = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{A} \tag{11}$$

is the angular velocity with which the space-time is "dragged around" the central rotating body. An observer (7) having $\Omega = \omega_d$ is termed a ZAMO. With respect to a ZAMO, the space-time is locally static. Using (10) we can recast (8) in a more useful form,

$$C(\Omega) = \left[\frac{\Delta\Sigma}{A}\left(1 - \frac{A^2}{\Delta\Sigma^2}(\Omega - \omega_d)^2\right)\right]^{-1/2}. \tag{12}$$

We point out that $\Omega_\pm(r)$ represent, for any R, the angular velocities of massless test particles moving on circular equatorial null orbits. Actually, (10) can also be straightforwardly obtained from (1), requiring $ds^2 = 0$. In Fig. 1 the curves $\nu_\pm \equiv M\Omega_\pm$ (null orbits) and $\nu_d \equiv M\omega_d$ (the ZAMO orbits) have been plotted with respect to the adimensional radial coordinate $\rho = r/M$ for a given value of the adimensional parameter $\alpha = a/M$. The shaded area represents points of the (ρ, ν)-plane corresponding to admissible orbits of given radius and angular velocity. We see that static observers ($\Omega = 0$) are allowed only outside the ergoregion ($\rho > \rho_{0+} = 2$), where both corotating ($\Omega > 0$) and counter-rotating ($\Omega < 0$) orbits exist. Inside the ergoregion only corotating orbits are allowed. Notice that the ZAMO orbits can penetrate the ergoregion, grazing the horizon. We stress that—generally—the shaded area in Fig. 1 does not represent geodesic orbits. Usually, an external force will be required to keep the observer (and the comoving Casimir cavity) in such a given orbit. In Fig. 1 are also plotted the curves representing the loci of corotating and counter-rotating geodesic orbits, namely $\Omega_{g\pm} = \pm\sqrt{M}/(\rho^{3/2} \pm \sqrt{M}a)$ in the Kerr geometry which, in terms of the adimensional parameters, read $\nu_{g\pm} = \pm 1/(\rho^{3/2} \pm \alpha)$. We see that, for a given parameter α, there exist two innermost geodesic orbits, corresponding to the points $G_\pm$ where $\nu_\pm$ and $\Omega_\pm$ meet. Actually, these are the null geodesic, corresponding to the (corotating or counter-rotating) unstable orbits for massless particles moving in the Kerr geometry.

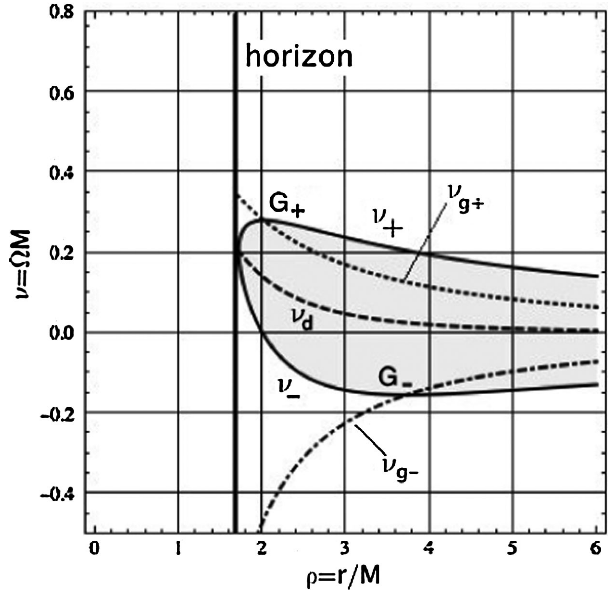

FIG. 1. The curves ν_+ and ν_- (solid lines) represent the allowed null equatorial circular orbits for massless particles in the Kerr geometry. The shaded area, delimited by $\nu_+ \cup \nu_-$, represents the allowed timelike circular orbits (ρ, ν) for massive particles in the Kerr geometry outside the horizon r_+, for $\alpha = a/M = 0.7$. Also plotted are the ZAMO ν_d (dashed line), the corotating ν_{g+} (dotted line), and the counter-rotating ν_{g-} (dot-dashed line) geodesic orbits. The points $G_\pm = \nu_\pm \cap \nu_{g\pm}$ represent the null geodesics (see text).

C. Comoving coordinates

Let $\mathbf{w}$ be an admissible observer (7), orbiting in an equatorial orbit with angular velocity Ω. The coordinate transformation $\phi' = \phi - \Omega t$ changes the metric into $ds^2 = g'_{\mu\nu} dx'^\mu dx'^\nu$, where

$$g'_{tt} = C^{-2}(\Omega), \tag{13}$$

$$g'_{t\phi'} = g_{t\phi} + \Omega g_{\phi\phi} = g_{\phi\phi}(\Omega - \omega_d) \tag{14}$$

$$g'_{rr} = g_{rr}, \qquad g'_{\phi\phi} = g_{\phi\phi}, \qquad g'_{\theta\theta} = g_{\theta\theta}. \tag{15}$$

It is straightforward to check that the observer (7) is now static in the new coordinates in which the metric has been rewritten. Actually, the observer 4-velocity reads now

$$w'^\mu = C(\Omega)\delta^\mu_t. \tag{16}$$

Notice that—as stated above—(16) becomes a ZAMO if $\Omega = \omega_d$, since in that case $\mathbf{w}$ is static in the coordinates in which the Kerr metric takes a diagonal form, as $g'_{t\phi} = 0$.

D. Comoving local frame

Assume that the observer carries a local frame defined by three spatial rectangular coordinates (x, y, z). Take the x axis tangent to the orbit and the y axis along the outward radial direction. Then, locally we have $dx = rd\phi'$, $dy = dr$, $dz = rd\theta$. Let us call x, y, z locally comoving coordinates. In such a comoving local frame the metric reads $ds^2 = \hat{g}_{\mu\nu} dx^\mu dx^\nu$, where

$$\hat{g}_{tt} = C^{-2}(\Omega), \tag{17}$$

$$\hat{g}_{tx} = -\frac{A}{r\Sigma}(\Omega - \omega_d) = -\frac{A}{r^3}(\Omega - \omega_d), \tag{18}$$

$$\hat{g}_{xx} = -\frac{A}{r^2\Sigma} = -\frac{A}{r^4}, \tag{19}$$

$$\hat{g}_{yy} = -\frac{\Sigma}{\Delta} = -\frac{r^2}{\Delta}, \tag{20}$$

$$\hat{g}_{zz} = -\frac{\Sigma}{r^2} = -1 \tag{21}$$

(recall that, in the equatorial plane, $\Sigma = r^2$). Notice that $\hat{g} = \det(\hat{g}_{\mu\nu}) = -1$. The inverse metric $\hat{g}^{\mu\nu}$ reads

$$\left(\frac{\partial}{\partial s}\right)^2 = \frac{A}{r^2\Delta}\left(\frac{\partial}{\partial t}\right)^2 + 2\frac{A}{r\Delta}(\omega_d - \Omega)\frac{\partial}{\partial t}\frac{\partial}{\partial x} - \frac{r^2}{\Delta}C^{-2}(\Omega)\left(\frac{\partial}{\partial x}\right)^2 - \frac{\Delta}{r^2}\left(\frac{\partial}{\partial y}\right)^2 - \left(\frac{\partial}{\partial z}\right)^2. \tag{22}$$

Note that, in the local frame, $\hat{w}^\mu = w'^\mu$.

III. THE CASIMIR CAVITY

The orbiting observer **w** carries with him a small Casimir cavity. As discussed in the introduction, we are interested in the breaking of the azimuthal reflection symmetry $\phi \to -\phi$, suffered by the field modes, possibly due to the orbital motion and/or to the underlying space-time geometry. Such breaking can be obtained placing the cavity plates orthogonally to the x axis (see Fig. 2). We will also assume that one of the plates is at the origin of the local observer's frame. Let L be the coordinate (i.e., not physical) separation between the plates. Assuming—reasonably—that the typical size of the cavity is much smaller than the orbital radius, we may assume that the metric $\hat{g}_{\mu\nu}$ is almost constant inside the cavity.

Actually, calling $x_0^\lambda \equiv (t, x_0^i)$ the orbital coordinates of the Casimir cavity, we may expand the metric around x_0^λ, writing

$$\hat{g}_{\mu\nu}(t, x^i) = \hat{g}_{\mu\nu}(t, x_0^i) + \left.\frac{\partial \hat{g}_{\mu\nu}}{\partial x^i}\right|_{x^i = x_0^i} \cdot \delta x^i + \cdots, \tag{23}$$

where $x^i = x_0^i + \delta x^i$.

For admissible orbits ($r > r_+ = M + \sqrt{M^2 - a^2}$), the relevant gradients of the metric tensor are $\sim M/r^2$, so that

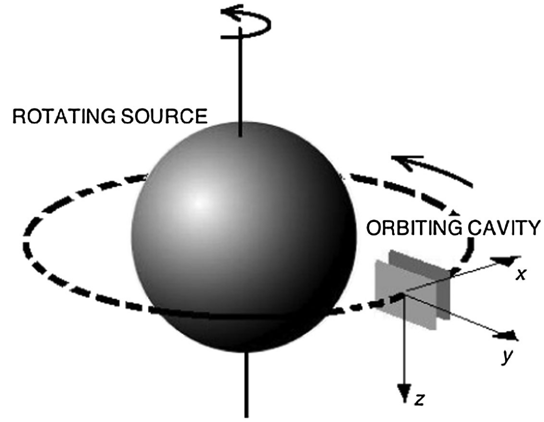

FIG. 2. A sketch of the Casimir cavity orbiting around a massive, rotating source. At variance with [22], the cavity plates are placed orthogonally to the orbital direction. The typical size of the cavity is assumed small with respect to the orbital radius, so that $\hat{g}_{\mu\nu} \simeq$ const inside the cavity.

the metric variations $\delta\hat{g}_{\mu\nu} \simeq \frac{\partial \hat{g}_{\mu\nu}}{\partial x^i} L$ inside the cavity are of order $ML/r^2 \ll 1$, provided that $L \ll r$. The latter condition is obviously fulfilled for any realistic Casimir cavity (where $L \sim 1$ μm), and even more in the subatomic realm, where the Casimir effect could have some role [5]. This justifies the assumption of an almost constant $g_{\mu\nu}$.

A. Scalar field modes

Consider a massless scalar field ψ confined in the Casimir cavity. It obeys the Klein-Gordon equation [23]

$$\frac{1}{\sqrt{-\hat{g}}}\partial_\mu[\sqrt{-\hat{g}}\hat{g}^{\mu\nu}\partial_\nu]\psi + \xi R\psi = 0. \tag{24}$$

In a vacuum $R = 0$. Also, taking into account the approximation $\hat{g}_{\mu\nu} \simeq$ const, we simply have $\hat{g}^{\mu\nu}\partial_\mu\partial_\nu\psi = 0$. Being the field confined in the cavity, we adopt as usual the Dirichlet boundary conditions at the cavity plates, hence requiring

$$\psi(x = 0, y, z, t) = \psi(x = L, y, z, t) = 0. \tag{25}$$

Let us seek for solutions of the form

$$\psi \sim e^{-i\omega_n t} e^{i\vec{k}_\perp \cdot \vec{x}_\perp} \chi(x), \tag{26}$$

where $\vec{k}_\perp = (k_y, k_z)$ and $\vec{x}_\perp = (y, z)$. Substituting (26) in the Klein-Gordon equation and using the inverse metric (22) we get the following solution for $\chi(x)$:

$$\chi(x) \sim e^{-\frac{i\omega_n A(\omega_d - \Omega)}{r^3 C^{-2}(\Omega)}x} \sin\left(\frac{n\pi}{L}x\right), \quad n \in N, \tag{27}$$

where the eigenfrequencies ω_n satisfy

$$\left[\omega_n^2 - C^{-2}(\Omega)\left(\frac{\Delta}{r^2}k_y^2 + k_z^2\right)\right] = \left(\frac{n\pi}{L}\right)^2 \frac{r^2}{\Delta} C^{-4}(\Omega), \quad (28)$$

yielding

$$\omega_n = \frac{r}{\sqrt{\Delta}} C^{-2}(\Omega)\left[\left(\frac{n\pi}{L}\right)^2 + \frac{\Delta}{r^2} C^2(\Omega)\left(\frac{\Delta}{r^2}k_y^2 + k_z^2\right)\right]^{1/2} \quad (29)$$

(the formal asymmetry between the k_y- and the k_z- sectors is due to the Boyer-Lindquist coordinates we started with, which are nonisotropic).

Let us define, for further convenience, the following parameters:

$$b_n = \omega_n C^2(\Omega) = \frac{r}{\sqrt{\Delta}}\left[\left(\frac{n\pi}{L}\right)^2 + \frac{\Delta}{r^2} C^2(\Omega)\left(\frac{\Delta}{r^2}k_y^2 + k_z^2\right)\right]^{1/2}, \quad (30)$$

$$\beta_n = \frac{b_n A}{r^3}(\omega_d - \Omega). \quad (31)$$

Then

$$\chi(x) \sim e^{-i\beta_n x} \sin\left(\frac{n\pi}{L}x\right), \quad (32)$$

and the full mode solutions for the field in the Casimir cavity read

$$u_n(t, x, \vec{x}_\perp) = N_n e^{-i\omega_n t} e^{i\vec{k}_\perp \cdot \vec{x}_\perp} e^{-i\beta_n x} \sin\left(\frac{n\pi}{L}x\right), \quad (33)$$

where N_n is a normalization constant, to be determined below.

B. Mode normalization

We now proceed to normalize the modes u_n by means of the usual Klein-Gordon scalar product, defined as [23]

$$\langle u_n, u_m\rangle = i\int_S [(\partial_\mu u_n)u_m^* - u_n(\partial u_m^*)]\sqrt{\hat{g}_S} n^\mu dS, \quad (34)$$

where S is a spacelike Cauchy surface, $\hat{g}_s = -\hat{g}/\hat{g}_{tt}$ is the determinant of the induced metric on S, and $\mathbf{w}$ is a timelike future-directed unit vector orthogonal to S. Putting $dS = dxdydz$, we easily find

$$n^\mu = r\sqrt{\frac{\Delta}{A}}\left[\frac{A}{r^2\Delta}, \frac{A}{r\Delta}(\omega_d - \Omega), 0, 0\right]. \quad (35)$$

From the orthonormality requirement

$$\langle u_n, u_m\rangle = \delta^{(2)}(\vec{k}_{\perp,n} - \vec{k}_{\perp,m})\delta_{mn}, \quad (36)$$

we get

$$N_n^2 = \frac{1}{(2\pi)^2 L\omega_n} \cdot \frac{1}{r}\sqrt{\frac{A}{\Delta}} C^{-3}(\Omega). \quad (37)$$

C. Casimir vacuum energy

The mean vacuum energy density $\langle\epsilon_{\rm vac}\rangle$ for the scalar field inside the cavity (actually, the Casimir energy density) according to the comoving observer $\mathbf{w}$ [see (16)] reads

$$\langle\epsilon_{\rm vac}\rangle = \frac{1}{V_p}\int_V dxdydz\sqrt{g_S}\hat{w}^\mu \hat{w}^\nu \langle 0|T_{\mu\nu}|0\rangle, \quad (38)$$

where $V_p = \int_V dxdydz\sqrt{g_S}$ is the proper volume of the cavity. Since, in the present approximations, $\hat{g}_{\mu\nu} \simeq$ const, the only dependence on the spatial coordinates is contained in $\langle 0|T_{\mu\nu}|0\rangle$, through the field modes (33). Hence, we have

$$\langle\epsilon_{\rm vac}\rangle = \frac{C^2(\Omega)}{L}\int_0^L dx \sum_n \int d^2k_\perp T_{tt}[u_n, u_n^*]. \quad (39)$$

The bilinear form

$$T_{tt}[u_n, u_n^*] = \partial_t u_n \partial_t u_n^* - \frac{1}{2}\hat{g}_{tt}[\hat{g}^{\mu\nu}\partial_\mu u_n \partial_\nu u_n^*] \quad (40)$$

reduces, after some algebra, to

$$T_{tt}[u_n, u_n^*] = \frac{1}{2}N_n^2\left[\mathcal{F}_n \sin^2\left(\frac{n\pi}{L}x\right) + \mathcal{G}_n \cos^2\left(\frac{n\pi}{L}x\right)\right], \quad (41)$$

where

$$\mathcal{F}_n = \omega_n^2 + \frac{\omega_n}{b_n}\left[\frac{\Delta}{r^2}k_y^2 + k_z^2\right], \quad (42)$$

$$\mathcal{G}_n = \frac{\omega_n^2 r^2}{b_n^2 \Delta}\left(\frac{n\pi}{L}\right)^2. \quad (43)$$

Substituting (41) in (39) and performing the dx integration we get

$$\langle\epsilon_{\rm vac}\rangle = \frac{C^2(\Omega)}{4}\sum_n \int d^2k_\perp N_n^2[\mathcal{F}_n + \mathcal{G}_n] = \frac{1}{(2\pi)^2 2L} \cdot \frac{1}{r}\sqrt{\frac{A}{\Delta}} C^{-1}(\Omega)\sum_n \int d_\perp^2 \omega_n, \quad (44)$$

where use has been made of (37) and of (30).

D. Vacuum energy density renormalization

The above expression for the vacuum energy density $\langle\epsilon_{\rm vac}\rangle$ is divergent, due both to the infinite sum and to the $d^2k_\perp$ integration. Such divergence is readily removed by means of dimensional regularization. To be definite, let us rewrite [see (29)]

$$\sum_n \int d^2k_\perp \omega_n = \sum_n \int \frac{d^2k_\perp}{C(\Omega)} \left[\frac{r^2}{\Delta C^2(\Omega)}\left(\frac{n\pi}{L}\right)^2 + \left(\frac{\Delta}{r^2}k_y^2 + k_z^2\right)\right]^{1/2}. \quad (45)$$

After a change $\tilde{k}_y = \frac{\sqrt{\Delta}}{r}k_y$, we have

$$\sum_n \int d^2k_\perp \omega_n = C^{-1}(\Omega)\frac{r}{\sqrt{\Delta}}\sum_n \int d^2\tilde{k}_\perp \sqrt{s + \tilde{k}_\perp^2}, \quad (46)$$

where $\tilde{k}_\perp^2 = \tilde{k}_y^2 + k_z^2$, $d^2\tilde{k}_\perp = d\tilde{k}_y dk_z$ and

$$s = \frac{r^2}{\Delta C^2(\Omega)}\left(\frac{n\pi}{L}\right)^2. \quad (47)$$

We easily perform the $d^2\tilde{k}_\perp$ integration employing the Schwinger proper-time method [24,25]. Also, the infinite sum can be evaluated using a standard Riemann ζ-function regularization procedure. We get [recall that $\zeta(-3) = 1/120$]

$$\sum_n \int d^2k_\perp \omega_n = -\frac{2\pi}{3}C^{-1}(\Omega)\frac{r}{\Delta}s^{-3/2} = -\frac{\pi^4 r^4}{180\Delta^2 C^4(\Omega)L^3}. \quad (48)$$

Inserting this result in (44) we obtain the renormalized vacuum energy density

$$\langle\epsilon_{\rm vac}\rangle_{\rm ren} = -\frac{\pi^2 r^3}{1440\Delta^2 L^4 C^5(\Omega)}\sqrt{\frac{A}{\Delta}}. \quad (49)$$

E. Proper cavity length

The renormalized vacuum energy density (49) is still expressed in terms of local coordinates. In particular, the quantity L appearing in (49) is a mere coordinate length, hence unphysical. In order to get a physical content we need to express L in terms of the proper cavity length, L_p, as measured by the comoving observer **w**. Now, recalling that $\hat{g}_{\mu\nu} \simeq$ const inside the cavity, we have [26]

$$L_p = \int_0^L dx\left[-\hat{g}_{xx} + \frac{\hat{g}_{tx}^2}{\hat{g}_{tt}}\right]^{1/2} \simeq L\cdot\left[-\hat{g}_{xx} + \frac{\hat{g}_{tx}^2}{\hat{g}_{tt}}\right]^{1/2}. \quad (50)$$

From (21) and (12) we get

$$L_p = C(\Omega)\frac{\sqrt{\Delta}}{r}L. \quad (51)$$

Solving (51) for L and substituting in (49) we finally obtain the desired result,

$$\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w = -\frac{\pi^2}{1440L_p^4}\sqrt{\frac{A}{r^2\Delta}}C^{-1}(\Omega) = -\frac{\pi^2}{1440L_p^4}\left[1 - \frac{A^2}{r^4\Delta}(\Omega - \omega_d)^2\right]^{1/2}. \quad (52)$$

We recognize in the first factor of (52) the well-known result for the vacuum Casimir energy density in flat space-time, namely $\epsilon_{\rm vac}^{(0)} = -\frac{\pi^2}{1440L_p^4}$, so

$$\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w = \epsilon_{vac}^{(0)}\left[1 - \frac{A^2}{r^4\Delta}(\Omega - \omega_d)^2\right]^{1/2}. \quad (53)$$

We see that the space-time geometry and the orbital motion of the cavity both modify the (negative) Casimir energy density, generally lowering its absolute value, hence increasing the energy content of the whole system (recall that $\Delta \geq 0$ in the region $r \geq r_+$).

IV. DISCUSSION

In what follows we will analyze some particular cases considering, for a given value of the radial coordinate, the ratio between the Casimir energy density and its corresponding flat space-time, nonrotating limit, namely

$$R(r;\Omega,M,a) = \frac{\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w}{\epsilon_{\rm vac}^{(0)}} = \left[1 - \frac{A^2}{r^4\Delta}(\Omega - \omega_d)^2\right]^{1/2}. \quad (54)$$

As a first remark, let us note that $R(r;\Omega,M,a)$ depends basically on the quantity $(\Omega - \omega_d)^2$ [27]. In the static case ($\Omega = 0$) a weak field analysis performed up to the first order in the (small) quantities (M/r) and (a/r) gives no shift effect upon the Casimir energy. This agrees with the results of, e.g., [10,11,22]. On the other hand, if $\Omega \neq 0$, we are faced with a sort of coupling between the rotation and the gravitational space-time dragging (ω_d). This can give rise to a non-null result also at the first order in (M/r) and (a/r), as we will see below in Sec. E.

A. Flat space-time

This case refers to a cavity moving on a circular trajectory in a Minkowskian background. We obtain

$$R(r;\Omega,0,0)=\sqrt{1-r^2\Omega^2}, \tag{55}$$

i.e., $\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w=\epsilon^{(0)}_{\rm vac}\sqrt{1-r^2\Omega^2}\to 0$ as the cavity velocity approaches the limit value $\Omega r=1$. It is interesting to compare the above result with the energy $\mathcal{E}$ of a particle having mass m, moving on a circular path of radius r with angular velocity Ω. For a comoving observer $\mathcal{E}=m\sqrt{\hat{g}_{tt}}=m\sqrt{1-r^2\Omega^2}$ (see, e.g., [28]).

B. Schwarzschild geometry

1. Static Casimir cavity, $\Omega=0$

Consider now a nonrotating cavity in the Schwarzschild field. We simply obtain $R(r;0,M,0)=1$, i.e., $\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w$ has exactly the same value as in the flat space-time case. Actually, in the present case the azimuthal symmetry $\phi\to-\phi$ is preserved. It is worth noting that this result basically agrees with that obtained in the weak field approximation [10,11,22].

2. Orbiting Casimir cavity, $\Omega\neq 0$

Now the symmetry $\phi\to-\phi$ is broken by the orbital motion of the cavity; we have

$$R(r;\Omega,M,0)=\left(1-\frac{2M}{r}\right)^{-1/2}\left(1-\frac{2M}{r}-r^2\Omega^2\right)^{1/2}. \tag{56}$$

As $\Omega\to\pm\frac{1}{r}\sqrt{1-2M/r}$ [see (10) with $a=0$ for the Schwarzschild case], $R\to 0$. So $\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w\to 0$, too. This could happen, for example, when the orbital motion of the cavity comes close to a null geodesic orbit at $r=3M$ (see Fig. 3). As another interesting example, for a Casimir cavity orbiting on an ISCO (innermost stable circular orbit), at $r=6M$, we have $R=\sqrt{3}/2$.

C. Kerr geometry

1. Static Casimir cavity, $\Omega=0$

In the Kerr geometry such a static case can be realized only outside the ergoregion ($r>r_{0+}$). In this case, the symmetry $\phi\to-\phi$ is broken by the gravitational dragging and we get $R(r;\Omega,M,a)\leq 1$, i.e., in the Kerr field a static Casimir cavity will experience an increase in the (negative) vacuum energy density. The complete expression for R is not so illuminating. However, its limit for $r\to\infty$ is

$$R(r;\Omega,M,a)=1-\frac{2J^2}{r^4},\qquad r\to\infty. \tag{57}$$

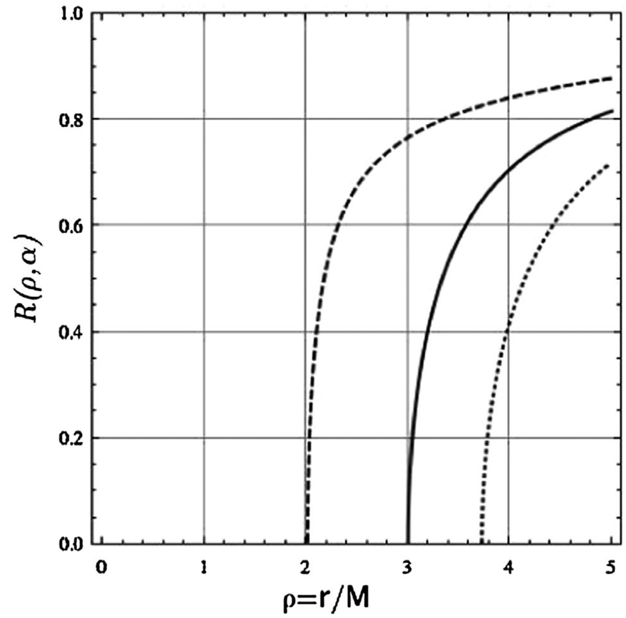

FIG. 3. Plots of the functions $R_+(\rho,\alpha)$ (dashed line), $R_-(\rho,\alpha)$ (dotted line) for the circular corotating and counter-rotating geodesic orbits in the Kerr geometry. Also plotted, for comparison, is $R(\rho,0)$ for the geodesic orbits in the Schwarzschild geometry (continuous line). In both the considered geometries, the zeroes of $R(\rho,\alpha)$ are in correspondence of the geodesic (unstable) null orbit for a massless particle, i.e., $r=3M$ (Schwarzschild space-time) and $r_+\simeq 2M$, $r_-\simeq 3.7M$ (Kerr space-time with $\alpha=0.7$).

So, as expected, the angular momentum of the source causes a shift in the vacuum energy, reducing its absolute value. This gravitomagnetic effect (quadratic in $a=J/M$) is dominating in the asymptotic region of space, when compared to the bare gravitoelectric effect of the source.

2. Orbiting Casimir cavity, $\Omega\neq 0$ and the ZAMO

As in the case of the Schwarzschild geometry, $\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w\to 0$ when the angular velocity of the orbiting Casimir cavity approaches the values $\Omega_\pm(r)$ of the corotating or counter-rotating circular null orbits [see (10)]. On the other hand, when $\Omega=\omega_d$, namely when the cavity is comoving with a ZAMO, such a latter observer will experience no shift in the Casimir energy. This result is an interesting example of the relevant role played by zero-angular-momentum frames in the Kerr geometry. Actually, according to a ZAMO, the $\phi\to-\phi$ symmetry is restored, since there is no local rotation at all.

D. Geodesic orbital motion

We found a vanishing Casimir energy density ($R\to 0$) as the cavity orbital motion approaches the boundary $\nu_+\cup\nu_-$ of the shaded region in Fig. 1. Although—as stated above—such a region does not generally represent the locus of geodesic orbital motion, we see that there are indeed two points ($G_\pm$; see Fig. 1) in which the curves corresponding to the geodesic circular orbits meet the curves $\nu_\pm$. As pointed out in Sec. II A, such points represent null geodesic orbits.

Putting $\Omega_{g\pm}$ in (54) and rewriting the resulting expression in terms of the adimensional quantities ρ and α, we get the ratio between the vacuum energy density and its flat-nonrotating value in the case of a Casimir cavity following a general geodesic orbit, namely

$$R_\pm(\rho;\alpha) = \left[1 - \frac{(\pm 2\alpha\sqrt{\rho} - \rho^2 - \alpha^2)^2}{(\rho^2 + \alpha^2 - \rho)(\rho^{3/2} \pm \alpha)^2}\right]^{1/2}, \quad (58)$$

where the ($\pm$) sign refers to the corotating or counter-rotating motion. In Fig. 3 we have plotted $R_\pm$ for the two limiting cases: a Schwarzschild black hole (where $\alpha = 0$ and the two functions obviously coincide: $R_+ = R_- \equiv R$); and a Kerr black hole with $\alpha = 0.7$. In Fig. 4 we see that (as pointed out in Sec. IV B) in the Schwarzschild geometry the vacuum energy density vanishes when the geodesic orbit approaches the prehorizon limit at $\rho = 3$ ($r = 3M$), which is the (unstable) null geodesic orbit followed by massless particles, as the photons. We also find a similar behavior in the Kerr background (dotted and dashed lines). Solving $R_\pm(\rho,\alpha) = 0$ with respect to ρ, we finally get the radii of the innermost null geodesic orbits (having $\Omega = \Omega_\pm$) as a function of the Kerr parameter α, namely $\rho_\pm(\alpha)$. A plot of such functions is given in Fig. 4.

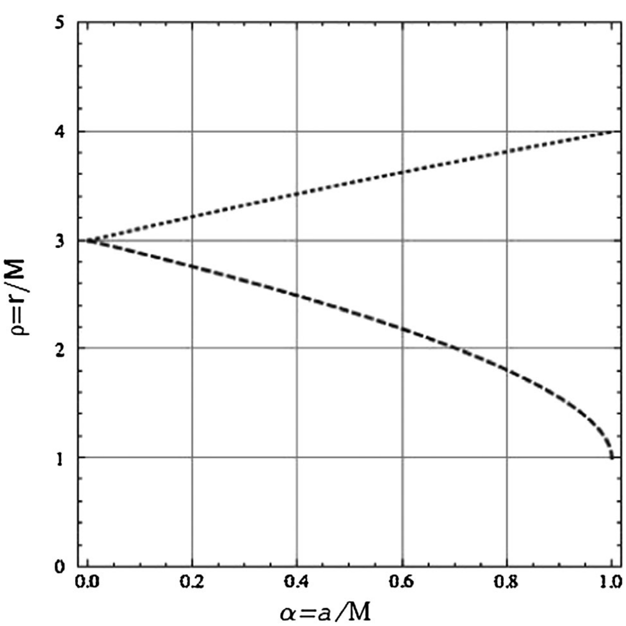

FIG. 4. Plots of the functions $\rho_+(\alpha)$ (dashed line) and $\rho_-(\alpha)$ (dotted line), respectively, for the innermost circular corotating and counter-rotating null geodesic orbits in the Kerr geometry. A cavity moving nearby these orbits will be characterized by a vanishing vacuum energy density.

We point out that the innermost (unstable) null geodesic orbits plotted in Fig. 4 can be actually occupied only by massless particles. Hence, they are not described by physically admissible observers, or by any realistic (massive) Casimir cavity.

Nevertheless, a system cavity + observer moving nearby such orbits may be conceivable; in that case too a comoving observer should detect a vanishing value of the Casimir energy, e.g., measuring an almost zero attractive force between the cavity plates.

E. Weak field limit

Let us consider, as a simple example, the case of a Casimir apparatus placed at rest at the equator of a rapidly spinning neutron star. Up to the first order in the (small) quantities M/r and a/r we rewrite (54) as

$$R(r;\Omega,M,a) \simeq \left(1 - \frac{2M}{r}\right)^{-1/2}\left[1 - \frac{2M}{r} - r^2\Omega^2 + \frac{4J\Omega}{r}\right]^{1/2}, \quad (59)$$

where r, $J = Ma$, and Ω now represent the radius, the angular momentum, and the angular velocity of rotation of the star. For typical values $M \simeq 1.4M_\odot$, $r \simeq 10^4$ m, $\Omega \simeq 190$ rad/s, we obtain (using $J = (2/5)Mr^2\Omega$)

$$R(r;\Omega,M,a) \simeq \left[1 - r^2\Omega^2\left(1 - \frac{8M}{5r}\right)\left(1 - \frac{2M}{r}\right)^{-1}\right]^{1/2} \simeq 1–2.3 \times 10^{-5}, \quad (60)$$

which represents indeed an extremely small first order correction to the flat-nonrotating value of the Casimir energy. In spite of its smallness, the result is theoretically interesting. Actually, looking at (59) we find that at the lowest order of approximation in the weak field limit all the influence of the background gravitational field disappears when the rotation of the cavity is stopped. In other words, the $\phi \to -\phi$ symmetry breaking, due to cavity rotation, appears as a higher order effect, as previously stated.

Yet, we found in Sec. IV C 1 that if $\Omega = 0$ such symmetry breaking can take place as well, when discussing the asymptotic behavior of the Casimir energy in the Kerr background. But in that case the effect appears at the second order in (a/r) [see (57)].

V. CONCLUDING REMARKS

In this paper we have analyzed the influence of the Kerr geometry on the vacuum energy density in a Casimir cavity orbiting in the equatorial plane of a massive rotating gravitational source. Assuming a typical cavity size much smaller than the orbital radius, we succeeded in evaluating the shift in the vacuum energy using the exact form of the Kerr solution. We found that both the orbital motion of the cavity and the underlying space-time geometry conspire in reducing the absolute value of the Casimir energy, as measured by a comoving observer, with respect to whom the cavity is at rest. This, in turn, causes a weakening in the attractive force between the Casimir plates.

We have considered in some detail a few limiting cases, showing that, on general grounds, $\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w \to 0$ when the angular velocity of the orbiting Casimir cavity approaches the values $\Omega_\pm(r)$ of the corotating or counter-rotating circular null orbits. On the other hand, we have shown that a ZAMO comoving with the Casimir cavity does not experience any shift in the vacuum energy. This is an expected result, since the Kerr space-time is locally static with respect to a ZAMO. (Incidentally, let us note that the corresponding non-null result found in [22] is basically due to the assumed nonuniformity of the gravitational field inside the cavity.)

We have pointed out that the energy shift is related to the $\phi \to -\phi$ symmetry breaking suffered by the quantum field, due both to the cavity orbital motion and to the dragging effect induced by the Kerr geometry.

On lack of rotation (i.e., $|\Omega - \omega_d| = 0$, according to a locally nonrotating observer), vacuum fluctuations have null stress-energy *flux*, $T^{tx} = 0$ (i.e., no net linear momentum). Roughly speaking, the linear momenta of the virtual field particles inside the cavity cancel each other, yielding no net coupling with the external gravitomagnetic field of the rotating massive source M. On the other hand, when $|\Omega - \omega_d| \neq 0$, $\phi \to -\phi$ symmetry breaking appears, $T^{tx} \neq 0$ and we have a non-null gravitomagnetic coupling, resulting in a modification of the field vacuum energy.

Although a physical (timelike) Casimir cavity cannot move on a null geodesic orbit, it is however worth noting that $\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w$ can suffer an important modification also on very relevant orbits, as on the ISCOs of almost extremal Kerr black holes $(a/M \simeq 1)$ [29], where the ratio $R(r;\Omega,M,a) = \frac{\langle\epsilon_{\rm vac}\rangle_{\rm ren}|_w}{\epsilon^{(0)}_{\rm vac}}$ is likely to reach very small values as well.

The effect we have considered could be of some astrophysical interest, being potentially related to particle confinement (as in models based on string interquark potentials [5]).

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