

Cosmological applications of $F(T, T_G)$ gravityGeorgios Kofinas^{1,*} and Emmanuel N. Saridakis^{2,3,†}¹*Department of Information and Communication Systems Engineering, Research Group of Geometry, Dynamical Systems and Cosmology, University of the Aegean, Karlovassi 83200, Samos, Greece*²*Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece*³*Instituto de Física, Pontificia Universidad de Católica de Valparaíso, Casilla 4950, Valparaíso, Chile*
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We investigate the cosmological applications of $F(T, T_G)$ gravity, which is a novel modified gravitational theory based on the torsion invariant T and the teleparallel equivalent of the Gauss-Bonnet term T_G . $F(T, T_G)$ gravity differs from both $F(T)$ theories as well as from $F(R, G)$ class of curvature modified gravity, and thus its corresponding cosmology proves to be very interesting. In particular, it provides a unified description of the cosmological history from early-times inflation to late-times self-acceleration, without the inclusion of a cosmological constant. Moreover, the dark energy equation-of-state parameter can be quintessence or phantomlike, or experience the phantom-divide crossing, depending on the parameters of the model.

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I. INTRODUCTION

Since theoretical arguments and observational data suggest that the Universe passed through an early-times inflationary stage and resulted in a late-times accelerated phase, a large amount of research was devoted to explain this behavior. In general, one can follow two ways to achieve it. The first direction is to alter the universe content by introducing additional fields, canonical scalar, phantom scalar, both scalars, vector fields etc, that is introducing the concepts of the inflaton and/or the dark energy, which can be extended in a huge class of models (see [1–3] and references therein). The second way is to modify the gravitational sector instead (see [4] and references therein). Note however that one can in principle transform from one approach to the other, since the important point is the number of degrees of freedom beyond standard model particles and general relativity (GR) [5].

In modified gravitational theories one usually extends the curvature-based Einstein-Hilbert action. However, a different and interesting class of gravitational modification arises when one extends the action of the equivalent torsional formulation of general relativity. In particular, since Einstein’s years it was known that one can construct the so-called “teleparallel equivalent of general relativity” (TEGR) [6–10], that is attributing gravity to torsion instead of curvature, by using instead of the torsion-less Levi-Civita connection the curvature-less Weitzenböck one. In such a formulation the gravitational Lagrangian results from contractions of the torsion tensor and is called the “torsion scalar” T , similarly to the general relativity Lagrangian, i.e., the “curvature scalar” R , which is

constructed by contractions of the curvature tensor. Hence, similarly to the $f(R)$ extensions of general relativity [11,12], one can construct $f(T)$ extensions of TEGR [13,14]. The interesting feature in this extension is that $f(T)$ does not coincide with $f(R)$ gravity, despite the fact that TEGR coincides with general relativity. Since it is a new gravitational modification class, its corresponding cosmological behavior and black hole solutions have been studied in detail [13–19].

However, apart from the simple modifications of curvature gravity, one can construct more complicated actions introducing higher-curvature corrections such as the Gauss-Bonnet combination G [20,21] or arbitrary functions $f(G)$ [22–24], Weyl combinations [25], Lovelock combinations [26,27], etc. Hence, one can follow the same direction starting from the teleparallel formulation of gravity, and construct actions involving higher-torsion corrections. Indeed, in our recent work [28] we first constructed the teleparallel equivalent of the Gauss-Bonnet term T_G (which is a new quartic torsional scalar which reduces to a topological invariant in four dimensions), and then, using also the torsion scalar T , we constructed $F(T, T_G)$ gravity (see [29–31] for different constructions of torsional actions). This is a new class of gravitational modification, since it is different from both $f(T)$ gravity as well as from $f(R, G)$ gravity.

Since $F(T, T_G)$ gravity is a novel modified gravity theory, in the present work we are interested in investigating its cosmological applications. In particular, after extracting the Friedmann equations, we define the effective dark energy sector and the various observables such as the density parameters and the dark energy equation-of-state parameter. Then, considering specific $F(T, T_G)$ ansatzes we investigate the inflation realization and the late-times acceleration. The plan of the work is as follows: In Sec. II

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we review $F(T, T_G)$ gravity. In Sec. III we apply it in a cosmological framework, extracting the corresponding equations and defining the various observables, while in Sec. IV we analyze some specific cases. Finally, Sec. V is devoted to the conclusions.

II. $F(T, T_G)$ GRAVITY

Let us give a brief review of $F(T, T_G)$ gravity [28]. Although in this manuscript we are interested in its cosmological application, we will present the formulation in D -dimensions where it is nontrivial, and then discuss $F(T, T_G)$ gravity in $D = 4$. In the teleparallel formulation of gravity theories, the dynamical variables are the vielbein field $e_a(x^\mu)$ and the connection 1-forms $\omega^a_b(x^\mu)$ which defines the parallel transportation.¹ We can express them in components in terms of coordinates as $e_a = e_a^\mu \partial_\mu$ and $\omega^a_b = \omega^a_{b\mu} dx^\mu = \omega^a_{bc} e^c$, while we define the dual vielbein as $e^a = e^a_\mu dx^\mu$. The vielbein commutation relations read

$$[e_a, e_b] = C^c_{ab} e_c, \quad (1)$$

where the structure coefficients functions C^c_{ab} are written as

$$C^c_{ab} = e_a^\mu e_b^\nu (e^c_{\mu,\nu} - e^c_{\nu,\mu}), \quad (2)$$

with a comma denoting differentiation. Thus, we can define the torsion tensor, expressed in tangent components as

$$T^a_{bc} = \omega^a_{cb} - \omega^a_{bc} - C^a_{bc}, \quad (3)$$

while the curvature tensor is

$$\begin{aligned} R^a_{bcd} &= \omega^a_{bd,c} - \omega^a_{bc,d} + \omega^e_{bd} \omega^a_{ec} \\ &\quad - \omega^e_{bc} \omega^a_{ed} - C^e_{cd} \omega^a_{be}. \end{aligned} \quad (4)$$

Furthermore, we use the metric tensor g to make the vielbein orthonormal $g(e_a, e_b) = \eta_{ab}$, where $\eta_{ab} = \text{diag}(-1, 1, \dots, 1)$, and thus we obtain the useful relation

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu, \quad (5)$$

and indices a, b, \dots are raised/lowered with the Minkowski metric η_{ab} . Finally, it proves convenient to define the contorsion tensor as

$$\mathcal{K}_{abc} = \frac{1}{2}(T_{cab} - T_{bca} - T_{abc}) = -\mathcal{K}_{bac}. \quad (6)$$

We now impose the condition of teleparallelism, namely $R^a_{bcd} = 0$, which holds in all frames. One way to realize

¹In this manuscript the notation is as follows: Greek indices μ, ν, \dots run over all space-time coordinates, while Latin indices a, b, \dots run over the tangent space.

this condition is by assuming the Weitzenböck connection $\tilde{\omega}^\lambda_{\mu\nu}$ defined in terms of the vielbein in all coordinate frames as $\tilde{\omega}^\lambda_{\mu\nu} = e_a^\lambda e^a_{\mu,\nu}$, or expressed in the preferred tangent-space components as $\tilde{\omega}^a_{bc} = 0$. The Ricci scalar \bar{R} corresponding to the usual Levi-Civita connection can be expressed as [8,9]

$$e\bar{R} = -eT + 2(eT_\nu^{\nu\mu})_{,\mu}, \quad (7)$$

where we have defined the ‘‘torsion scalar’’ T as

$$T = \frac{1}{4} T^{\mu\nu\lambda} T_{\mu\nu\lambda} + \frac{1}{2} T^{\mu\nu\lambda} T_{\lambda\nu\mu} - T_\nu^{\nu\mu} T^\lambda_{\lambda\mu} \quad (8)$$

and $e = \det(e^a_\mu) = \sqrt{|g|}$.

One can now clearly see that the Lagrangian density $e\bar{R}$ of general relativity, that is the one calculated with the Levi-Civita connection, and the torsional density $-eT$ differ only by a total derivative. Hence, the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{2\kappa_D^2} \int_M d^D x e \bar{R}, \quad (9)$$

up to boundary terms is equivalent to the action

$$S_{\text{tel}}^{(1)} = -\frac{1}{2\kappa_D^2} \int_M d^D x e T \quad (10)$$

in the sense that varying (9) with respect to the metric and varying (10) with respect to the vielbein gives rise to the same equations of motion (κ_D^2 is the D -dimensional gravitational constant) [10]. That is why the above theory, where one uses torsion to describe the gravitational field and imposes the teleparallelism condition, was dubbed by Einstein as TEGR.

The recipe of the construction of TEGR was to express the Ricci scalar R corresponding to a general connection as the Ricci scalar \bar{R} calculated with the Levi-Civita connection, plus terms arising from the torsion tensor. Then, by imposing the teleparallelism condition $R^a_{bcd} = 0$, we obtained that \bar{R} is equal to a torsion scalar plus a total derivative. Hence, we can follow the same steps, but using the Gauss-Bonnet combination $G = R^2 - 4R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda}$ instead of the Ricci scalar. In [28] we have derived the teleparallel equivalent of Gauss-Bonnet gravity characterized by the new torsion scalar T_G , as well as the equations of motion of the modified gravity defined by the function $F(T, T_G)$. In the following, we give the corresponding expressions when restricted to the Weitzenböck connection $\tilde{\omega}^a_{bc} = 0$ (the tildes are withdrawn for notational simplicity). It is

$$e\bar{G} = eT_G + \text{total diverg.}, \quad (11)$$

where \bar{G} is the Gauss-Bonnet term calculated by the Levi-Civita connection, and

$$\begin{aligned}
 T_G = & (\mathcal{K}^{a_1}_{ea} \mathcal{K}^{ea_2}_b \mathcal{K}^{a_3}_{fc} \mathcal{K}^{fa_4}_d - 2\mathcal{K}^{a_1a_2}_a \mathcal{K}^{a_3}_{eb} \mathcal{K}^e_{fc} \mathcal{K}^{fa_4}_d \\
 & + 2\mathcal{K}^{a_1a_2}_a \mathcal{K}^{a_3}_{eb} \mathcal{K}^{ea_4}_f \mathcal{K}^f_{cd} \\
 & + 2\mathcal{K}^{a_1a_2}_a \mathcal{K}^{a_3}_{eb} \mathcal{K}^{ea_4}_{c,d}) \delta^{abcd}_{a_1a_2a_3a_4}, \quad (12)
 \end{aligned}$$

with the generalized δ being the determinant of the Kronecker deltas. Thus, T_G is the teleparallel equivalent of \tilde{G} , in the sense that the action

$$S_{\text{tel}}^{(2)} = \frac{1}{2\kappa_D^2} \int_M d^D x e T_G, \quad (13)$$

varied in terms of the vielbein gives exactly the same equations with the action

$$S_{\text{GB}} = \frac{1}{2\kappa_D^2} \int_M d^D x e \tilde{G}, \quad (14)$$

varied in terms of the metric.

Having constructed the teleparallel equivalent of curvature invariants, one can be based on them in order to build modified gravitational theories. Thus, one can start from an action where T is generalized to $F(T)$, resulting in the so-called $F(T)$ gravity [13–19]. Similarly, one can extend T_G to $F(T_G)$ in the action, and since T_G is quartic in torsion

then $F(T_G)$ cannot arise from any $F(T)$. Hence, in [28] we combined both possible extensions and we constructed the $F(T, T_G)$ modified gravity

$$S = \frac{1}{2\kappa_D^2} \int d^D x e F(T, T_G), \quad (15)$$

which is clearly different from both $F(T)$ theory as well as from $F(R, G)$ gravity [22–24], and therefore it is novel gravitational modification. We mention that TEGR (and therefore GR) is obtained for $F(T, T_G) = -T$, while the usual Einstein-Gauss-Bonnet theory arises for $F(T, T_G) = -T + \alpha T_G$, with α the Gauss-Bonnet coupling.

Let us now give the equations of motion in $D = 4$ which is our main interest in the present paper. Varying the action (15) in terms of the vierbein, after various steps, we finally obtain [28]

$$\begin{aligned}
 2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})_{,c} + 2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a}) C^d_{dc} \\
 + (2H^{[ac]d} + H^{dca}) C^b_{cd} + 4H^{[db]c} C_{(dc)}^a + T^a_{cd} H^{cdb} - h^{ab} \\
 + (F - T F_T - T_G F_{T_G}) \eta^{ab} = 0, \quad (16)
 \end{aligned}$$

where

$$\begin{aligned}
 H^{abc} = & F_T (\eta^{ac} \mathcal{K}^{bd}_d - \mathcal{K}^{bca}) + F_{T_G} \left[\epsilon^{cprt} (2\epsilon^a_{dkf} \mathcal{K}^{bk}_p \mathcal{K}^d_{qr} + \epsilon_{qdkf} \mathcal{K}^{ak}_p \mathcal{K}^{bd}_r + \epsilon^{ab}_{kf} \mathcal{K}^k_{dp} \mathcal{K}^d_{qr}) \mathcal{K}^{qf}_t \right. \\
 & + \epsilon^{cprt} \epsilon^{ab}_{kd} \mathcal{K}^{fd}_p \left(\mathcal{K}^k_{fr,t} - \frac{1}{2} \mathcal{K}^k_{fq} C^q_{ir} \right) + \epsilon^{cprt} \epsilon^{ak}_{df} \mathcal{K}^{df}_p \left(\mathcal{K}^b_{kr,t} - \frac{1}{2} \mathcal{K}^b_{kq} C^q_{ir} \right) \left. \right] \\
 & + \epsilon^{cprt} \epsilon^a_{kdf} [(F_{T_G} \mathcal{K}^{bk}_p \mathcal{K}^{df}_r)_{,t} + F_{T_G} C^q_{pi} \mathcal{K}^{bk}_{[q} \mathcal{K}^{df}_{r]}] \quad (17)
 \end{aligned}$$

and

$$h^{ab} = F_T \epsilon^a_{kcd} \epsilon^{bpqd} \mathcal{K}^k_{fb} \mathcal{K}^{fc}_q. \quad (18)$$

We have used the notation $F_T = \partial F / \partial T$, $F_{T_G} = \partial F / \partial T_G$, the (anti)symmetrization symbol contains the factor $1/2$, while the antisymmetric symbol ϵ_{abcd} has $\epsilon_{1234} = 1$, $\epsilon^{1234} = -1$.

We close this section by making some comments on $F(T, T_G)$ modified gravity itself, before proceeding to its cosmological investigation. The first has to do with the Lorentz violation. In particular, as we discussed also in [28], under the use of the Weitzenböck connection the torsion scalar T remains diffeomorphism invariant, however the Lorentz invariance has been lost since we have chosen a specific class of frames, namely the autoparallel orthonormal frames. Nevertheless, the equations of motion of the Lagrangian eT , being the Einstein equations, are still Lorentz covariant. On the contrary, when we replace T by a general function $f(T)$ in the action, the new equations of

motion will not be covariant under Lorentz rotations of the vielbein, although they will indeed be form-invariant, and the same features appear in the $F(T, T_G)$ extension. However, this is not a deficit (it is a sort of analogue of gauge fixing in gauge theories), and the theory, although not Lorentz covariant, is meaningful. Definitely, not all vielbeins will be solutions of the equations of motion, but those which solve the equations will determine the metric uniquely.

The second comment is related to possible acausalities and problems with the Cauchy development of a constant-time hypersurface. Indeed, there are works claiming that a departure from TEGR, as for instance in $f(T)$ gravity, with the subsequent local Lorentz violation, will lead to the above problems [32]. In order to examine whether one also has these problems in the present scenario of $F(T, T_G)$ gravity, he would need to perform a very complicated analysis, extending the characteristics method of [32] for this case, although at first sight one does expect to indeed find them. Nevertheless, even if this proves to be the case, it

does not mean that the theory has to be ruled out, since one could still handle $f(T)$ gravity [and similarly $F(T, T_G)$ one] as an effective theory, in the regime of validity of which the extra degrees of freedom can be removed or be excited in a healthy way (alternatively one could reformulate the theory using Lagrange multipliers) [32]. However, there is a possibility that these problems might be related to the restricting use of the Weitzenböck connection, since the formulation ofTEGR and its modifications using other connections (still in the “teleparallel class”) does not seem to be problematic, and thus, the general formulation of $F(T, T_G)$ gravity that was presented in [28] might be free of the above disadvantages. These issues definitely need further investigation, and the discussion is still open in the literature.

III. $F(T, T_G)$ COSMOLOGY

In this section we apply $F(T, T_G)$ gravity in a cosmological framework. First, we add the matter sector along the gravitational one, that is we start by the total action

$$S_{\text{tot}} = \frac{1}{2\kappa^2} \int d^4x e F(T, T_G) + S_m, \quad (19)$$

where S_m corresponds to a matter energy-momentum tensor $\Theta^{\mu\nu}$ and $\kappa^2 = 8\pi G$ is the four-dimensional Newton’s constant. Second, in order to investigate the cosmological implications of the above action, we consider a spatially flat cosmological ansatz

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{\hat{i}\hat{j}}dx^{\hat{i}}dx^{\hat{j}}, \quad (20)$$

where $a(t)$ is the scale factor and $N(t)$ is the lapse function (the hat indices run in the three spatial coordinates). This metric arises from the diagonal vierbein

$$e^a{}_{\mu} = \text{diag}(N(t), a(t), a(t), a(t)) \quad (21)$$

through (5), while the dual vierbein is $e_a{}^{\mu} = \text{diag}(N^{-1}(t), a^{-1}(t), a^{-1}(t), a^{-1}(t))$, and its determinant $e = N(t)a(t)^3$.

Considering as usual $N(t) = 1$ and inserting the vierbein (21) into relations (8) and (12), we find

$$T = 6\frac{\dot{a}^2}{a^2} = 6H^2 \quad (22)$$

$$T_G = 24\frac{\dot{a}^2\ddot{a}}{a^2} = 24H^2(\dot{H} + H^2), \quad (23)$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and dots denote differentiation with respect to t . Additionally, inserting (21) into the general equations of motion (16), after some algebra we obtain the Friedmann equations

$$F - 12H^2F_T - T_GF_{T_G} + 24H^3\dot{F}_{T_G} = 2\kappa^2\rho \quad (24)$$

$$F - 4(\dot{H} + 3H^2)F_T - 4H\dot{F}_T - T_GF_{T_G} + \frac{2}{3H}T_G\dot{F}_{T_G} + 8H^2\ddot{F}_{T_G} = -2\kappa^2p, \quad (25)$$

where the right-hand sides arise from the independent variation of the matter action, considering it to correspond to a perfect fluid with energy density ρ and pressure p (it is $\Theta^t{}^t = \rho$, $\Theta^{\hat{i}\hat{j}} = \frac{p}{a^2}\delta^{\hat{i}\hat{j}}$, $\Theta^{\hat{i}}{}_{\hat{i}} = 3p$). In the above expressions it is $\dot{F}_T = F_{TT}\dot{T} + F_{TT_G}\dot{T}_G$, $\dot{F}_{T_G} = F_{TT_G}\dot{T} + F_{T_GT_G}\dot{T}_G$, $\ddot{F}_{T_G} = F_{TTT_G}\dot{T}^2 + 2F_{TT_GT_G}\dot{T}\dot{T}_G + F_{T_GT_GT_G}\dot{T}_G^2 + F_{TT_G}\ddot{T} + F_{T_GT_G}\ddot{T}_G$, with F_{TT}, F_{TT_G}, \dots denoting multiple partial differentiations of F with respect to T, T_G . Finally, \dot{T}, \ddot{T} and \dot{T}_G, \ddot{T}_G are obtained by differentiating (22) and (23), respectively, with respect to time.

Let us make a comment here on the derivation of the above Friedmann equations. We mention that we followed the robust way, that is we first performed the general variation of the action resulting to the general equations of motion (16), and then we inserted the cosmological ansatz (21), obtaining (24) and (25). This procedure in principle does not give the same results with the shortcut procedure where one first inserts the cosmological ansatz (21) in the action and then performs variation with respect to N and a , since variation and ansatz-insertion do not commute in general, especially in theories with higher-order derivatives [33,34]. This shortcut method is a sort of minisuperspace procedure since the (potential) additional degrees of freedom other than those contained in the scale factor are frozen. However, as we show in the Appendix, following the shortcut procedure in the cosmological application of the scenario at hand leads exactly to the two Friedmann equations (24) and (25) of the robust procedure.

Setting $F(T, T_G) = -T$ in equations (24), (25), we get the standard cosmological equations of general relativity without a cosmological constant. For $F(T, T_G) = F(T)$ we get

$$F - 12H^2F_T = 2\kappa^2\rho, \quad (26)$$

$$F - 4(\dot{H} + 3H^2)F_T - 4H\dot{F}_T = -2\kappa^2p, \quad (27)$$

which are recognized as the standard equations of $F(T)$ gravity. However, note that due to the various conventions adopted in the literature in the definitions of the Riemann tensor, the torsion tensor and the Minkowski metric, which may differ by a total sign, our function $F(T)$ may correspond to $F(-T)$ in some other works (for instance [14]).

In order to parametrize the deviation of the theory $F(T, T_G)$ from GR, we write $F(T, T_G) = -T + f(T, T_G)$. Thus, the modification of GR (for instance the effective dark energy component) is included in the function f . Equations (24), (25) are then written as

$$6H^2 + f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G} = 2\kappa^2 \rho, \quad (28)$$

$$2(2\dot{H} + 3H^2) + f - 4(\dot{H} + 3H^2)f_T - 4H\dot{f}_T - T_G f_{T_G} + \frac{2}{3H} T_G \dot{f}_{T_G} + 8H^2 \ddot{f}_{T_G} = -2\kappa^2 p. \quad (29)$$

The Friedmann equations (28), (29) can be rewritten in the usual form

$$H^2 = \frac{\kappa^2}{3} (\rho + \rho_{DE}), \quad (30)$$

$$\dot{H} = -\frac{\kappa^2}{2} (\rho + p + \rho_{DE} + p_{DE}), \quad (31)$$

where the energy density and pressure of the effective dark energy sector are defined as

$$\rho_{DE} = -\frac{1}{2\kappa^2} (f - 12H^2 f_T - T_G f_{T_G} + 24H^3 \dot{f}_{T_G}), \quad (32)$$

$$p_{DE} = \frac{1}{2\kappa^2} \left[f - 4(\dot{H} + 3H^2)f_T - 4H\dot{f}_T - T_G f_{T_G} + \frac{2}{3H} T_G \dot{f}_{T_G} + 8H^2 \ddot{f}_{T_G} \right]. \quad (33)$$

In terms of the initial function F , we can write ρ_{DE} , p_{DE} as

$$\rho_{DE} = \frac{1}{2\kappa^2} (6H^2 - F + 12H^2 F_T + T_G F_{T_G} - 24H^3 \dot{F}_{T_G}), \quad (34)$$

$$p_{DE} = \frac{1}{2\kappa^2} \left[-2(2\dot{H} + 3H^2) + F - 4(\dot{H} + 3H^2)F_T - 4H\dot{F}_T - T_G F_{T_G} + \frac{2}{3H} T_G \dot{F}_{T_G} + 8H^2 \ddot{F}_{T_G} \right]. \quad (35)$$

Since the standard matter is conserved independently, $\dot{\rho} + 3H(\rho + p) = 0$, we obtain from (32), (33) that the dark energy density and pressure also satisfy the usual evolution equation

$$\dot{\rho}_{DE} + 3H(\rho_{DE} + p_{DE}) = 0. \quad (36)$$

Finally, we can define the dark energy equation-of-state parameter as

$$w_{DE} = \frac{p_{DE}}{\rho_{DE}}. \quad (37)$$

IV. SPECIFIC CASES

In the previous section we extracted the Friedmann equations of general $F(T, T_G)$ cosmology, and we defined the effective dark energy sector. Thus, in this section we

proceed to the investigation of some specific $F(T, T_G)$ cases, focusing on the evolution of observables such as the various density parameters $\Omega_i = 8\pi G\rho_i/(3H^2)$ and the dark energy equation-of-state parameter w_{DE} .

$$\mathbf{A. } F(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|}$$

Since T_G contains quartic torsion terms, it will in general and approximately be of the same order with T^2 . Therefore, T and $\sqrt{T^2 + \beta_2 T_G}$ are of the same order, and thus, if one of them contributes during the evolution the other will contribute too. Therefore, it would be very interesting to consider modifications of the form $F(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G}$, which are expected to play an important role at late times. Note that the couplings β_1 , β_2 are dimensionless, and so, no new mass scale enters at late times. Nevertheless, in order to describe the early-times cosmology, one should additionally include higher order corrections like T^2 . Since the scalar T_G is of the same order with T^2 , it should be also included. However, since T_G is topological in four dimensions it cannot be included as it is, and therefore we use the term $T\sqrt{|T_G|}$ which is also of the same order with T^2 and nontrivial. Thus, the total function F is taken to be

$$F(T, T_G) = -T + \beta_1 \sqrt{T^2 + \beta_2 T_G} + \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|}. \quad (38)$$

In summary, when the above function is used as an action, it gives rise to a gravitational theory that can describe both inflation and late-times acceleration in a unified way.

In order to examine the cosmological evolution of a universe governed by the above unified action, we perform a numerical elaboration of the Friedman equations (30), (31), with ρ_{DE} , p_{DE} given by Eqs. (34), (35), under the ansatz (38). In Fig. 1 we present the early-times, inflationary solutions for four parameter choices. As we observe, inflationary, de-Sitter exponential expansions can be easily obtained (with the exponent of the expansion determined by the model parameters), although there is not an explicit cosmological constant term in the action, which is an advantage of the scenario. This was expected, since one can easily extract analytical solutions of the Friedmann equations (30), (31) with $H \approx \text{const}$ (in which case T and T_G as also constants).

Let us now focus on the late-times evolution. In Fig. 2 we depict the evolution of the matter and effective dark energy density parameters, as well as the behavior of the dark energy equation-of-state parameter, for a specific choice of the model parameters.

As we see, we can obtain the observed behavior, where Ω_m decreases, resulting in its current value of $\Omega_{m0} \approx 0.3$, while $\Omega_{DE} = 1 - \Omega_m$ increases. Concerning w_{DE} , we can see that in this example it lies in the quintessence regime.

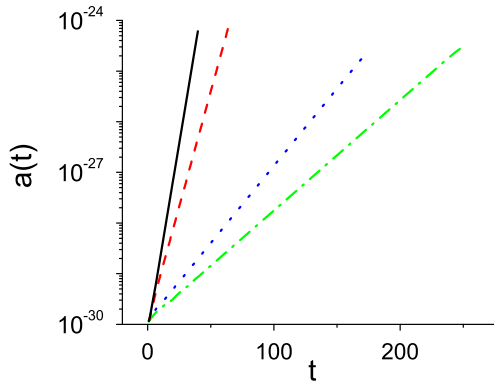


FIG. 1 (color online). Four inflationary solutions for the ansatz $F(T, T_G) = \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|} - T + \beta_1 \sqrt{T^2 + \beta_2 T_G}$, corresponding to (a) $\alpha_1 = -2.8$, $\alpha_2 = 8$, $\beta_1 = 0.001$, $\beta_2 = 1$ (black-solid), (b) $\alpha_1 = -2$, $\alpha_2 = 8$, $\beta_1 = 0.001$, $\beta_2 = 1$ (red-dashed), (c) $\alpha_1 = 8$, $\alpha_2 = 8$, $\beta_1 = 0.001$, $\beta_2 = 1$ (blue-dotted), (d) $\alpha_1 = 20$, $\alpha_2 = 5$, $\beta_1 = 0.001$, $\beta_2 = 1$ (green-dashed-dotted). All parameters are in Planck units.

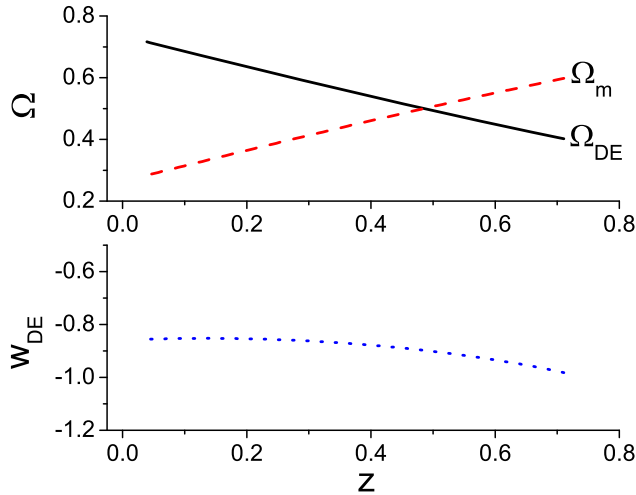


FIG. 2 (color online). Upper graph: The evolution of the dark energy density parameter Ω_{DE} (black-solid) and the matter density parameter Ω_m (red-dashed), as a function of the redshift z , for the ansatz $F(T, T_G) = \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|} - T + \beta_1 \sqrt{T^2 + \beta_2 T_G}$ with $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\beta_1 = 2.5$, $\beta_2 = 1.5$. Lower graph: The evolution of the corresponding dark energy equation-of-state parameter w_{DE} . All parameters are in units where the present Hubble parameter is $H_0 = 1$, and we have imposed $\Omega_{m0} \approx 0.3$, $\Omega_{DE0} \approx 0.7$ at present.

However, as it is usual in modified gravity [35], the model at hand can describe the phantom regime too, for a region of the parameter space, which is an additional advantage. In particular, in Fig. 3 we depict the cosmological evolution for a parameter choice that leads w_{DE} to the phantom regime, while the density parameters maintain their observed behavior. Similarly, note that the scenario can also exhibit the phantom-divide crossing [3] too.

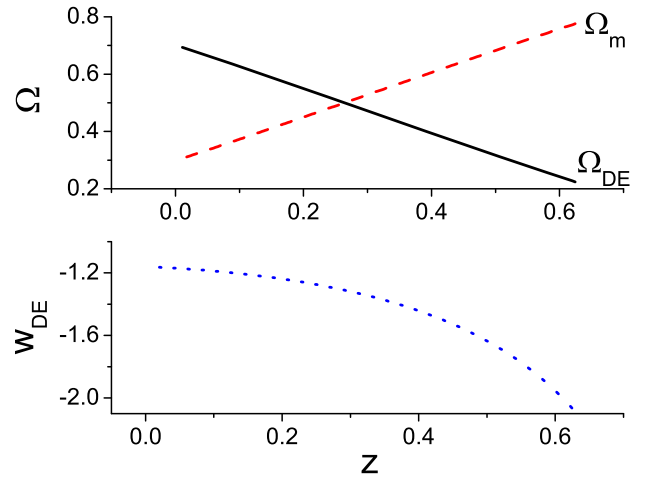


FIG. 3 (color online). Upper graph: The evolution of the dark energy density parameter Ω_{DE} (black-solid) and the matter density parameter Ω_m (red-dashed), as a function of the redshift z , for the ansatz $F(T, T_G) = \alpha_1 T^2 + \alpha_2 T \sqrt{|T_G|} - T + \beta_1 \sqrt{T^2 + \beta_2 T_G}$, with $\alpha_1 = 0.001$, $\alpha_2 = 0.001$, $\beta_1 = 2.6$, $\beta_2 = 2$. Lower graph: The evolution of the corresponding dark energy equation-of-state parameter w_{DE} . All parameters are in units where the present Hubble parameter is $H_0 = 1$, and we have imposed $\Omega_{m0} \approx 0.3$, $\Omega_{DE0} \approx 0.7$ at present.

Finally, note that one can use dynamical-systems methods in order to examine in a systematic way the late-times cosmological behavior of the scenario at hand, independently of the initial conditions of the universe [36].

B. $F(T, T_G) = -T + f(T^2 + \beta_2 T_G)$

One can go beyond the simple model of the previous paragraph. In particular, since as we already mentioned T_G contains quartic torsion terms, it will in general and approximately be of the same order with T^2 . Therefore, it would be interesting to consider modifications of the form $F(T, T_G) = -T + f(T^2 + \beta_2 T_G)$. The involved building block is an extension of the simple T , and thus, it can significantly improve the detailed cosmological behavior of a suitable reconstructed $F(T)$. The general equations (28), (29) in this case become

$$6H^2 + f - (24H^2T + \beta_2 T_G)f' + 24\beta_2 H^3 (2T\dot{T} + \beta_2 \dot{T}_G)f'' = 2\kappa^2 \rho \quad (39)$$

$$2(2\dot{H} + 3H^2) + f - [8(\dot{H} + 3H^2)T + 8H\dot{T} + \beta_2 T_G]f' + \left\{ \left[\frac{2\beta_2 T_G}{3H} - 8HT \right] (2T\dot{T} + \beta_2 \dot{T}_G) + 8\beta_2 H^2 (2T\dot{T} + \beta_2 \dot{T}_G) \right\} f'' + 8\beta_2 H^2 (2T\dot{T} + \beta_2 \dot{T}_G)^2 f''' = -2\kappa^2 p, \quad (40)$$

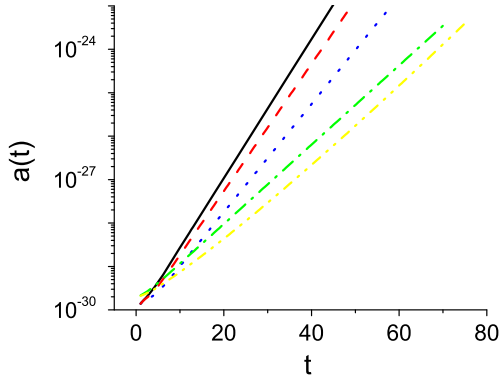


FIG. 4 (color online). Five inflationary solutions for the ansatz $F(T, T_G) = -T + \beta_1(T^2 + \beta_2 T_G) + \beta_3(T^2 + \beta_4 T_G)^2$, corresponding to (a) $\beta_1 = -0.01, \beta_2 = 1, \beta_3 = -1, \beta_4 = -2$ (black-solid), (b) $\beta_1 = -0.1, \beta_2 = 1, \beta_3 = -2, \beta_4 = -2$ (red-dashed), (c) $\beta_1 = -0.01, \beta_2 = 1, \beta_3 = -1, \beta_4 = -5$ (blue-dotted), (d) $\beta_1 = -0.01, \beta_2 = 1, \beta_3 = -6, \beta_4 = -6$ (green-dashed-dotted), (e) $\beta_1 = -0.01, \beta_2 = 1, \beta_3 = -10, \beta_4 = -10$ (yellow-dashed-dotted-dotted). All parameters are in Planck units.

where f', f'', f''' denote the derivatives of the function f and are evaluated at $T^2 + \beta_2 T_G$.

As a representative example we choose the case $F(T, T_G) = -T + \beta_1(T^2 + \beta_2 T_G) + \beta_3(T^2 + \beta_4 T_G)^2$, that is keeping up to fourth-order torsion terms (one could easily proceed to the investigation of other ansatzes in this class and to the detailed description of a unified picture of inflation and late-times acceleration). As expected, the higher-order torsion terms are significant at early times, and thus they can easily drive inflation. In Fig. 4 we show the early-times, inflationary solutions for five parameter choices, changing in particular the value of β_4 in order to see the effect of T_G on the evolution (the value of β_2 is irrelevant since the linear T_G term does not have any effect, since T_G , similarly to G , is topological invariant).

As we observe, inflationary evolution, that is de-Sitter exponential expansions can be easily obtained, and the expansion-exponent is determined by the model parameters. The significant advantage is that the exponential expansion is obtained without an explicit cosmological constant term in the action. Again, this was expected since we can easily extract analytical solutions of the Friedmann equations (39), (40) with $H \approx \text{const}$ (in which case T and T_G are also constants). Additionally, note that in this case the inflation realization is more efficient comparing to the model of the previous subsection, since it leads to more e-foldings in less time, as expected since now higher-order torsion terms are considered.

In the above analysis we showed that $f(T, T_G)$ cosmology can be very efficient in describing the evolution of the universe at the background level. However, before considering any cosmological model as a candidate for the description of nature it is necessary to perform a detailed investigation of its perturbations, namely to

examine whether the obtained solutions are stable. Furthermore, especially in theories with local Lorentz invariance violation, new degrees of freedom are introduced, the behavior of which is not guaranteed that is stable (for instance this is the case in the initial version of Hořava-Lifshitz gravity [37], in the initial version of de Rham-Gabadadze-Tolley massive gravity [38], etc.), and this makes the perturbation analysis of such theories even more imperative. Although such a detailed and complete analysis of the cosmological perturbations of $f(T, T_G)$ gravity is necessary, its various complications and lengthy calculations make it more convenient to be examined in a separate project [39]. However, for the moment we would like to mention that in the case of simple $f(T)$ gravity, the perturbations of which have been examined in detail [15,17], one does obtain instabilities, but there are many classes of $f(T)$ ansatzes and/or parameter-space regions, where the perturbations are well behaved. This is a good indication that we could expect to find a similar behavior in $f(T, T_G)$ gravity too, although we need to indeed verify this under the detailed perturbation analysis.

V. CONCLUSIONS

In this work we investigated the cosmological applications of $F(T, T_G)$ gravity, which is a modified gravity based on the torsion scalar T and the teleparallel equivalent of the Gauss-Bonnet combination T_G . $F(T, T_G)$ gravity is different from both the simple $F(T)$ theory, as well as from the curvature modification $F(R, G)$, and thus it is a novel class of gravitational modification. First, we extracted the general Friedmann equations and we defined the effective dark energy sector consisting of torsional combinations. Then, choosing specific $F(T, T_G)$ ansatzes we performed a detailed study of various observables, such as the matter and dark energy density parameters and the dark energy equation-of-state parameters.

Amongst the huge number of possible ansatzes, an interesting option is the construction of terms of the same order as T or T^2 using T_G , for instance new combinations of the form $T + \alpha\sqrt{|T_G|}$ or $T^2 + \beta T_G$ can participate in the $F(T, T_G)$ function. The resulting cosmology leads to interesting behaviors. First, the scenario can describe the inflationary regime, without an inflaton field. Second, at late times it provides an effective dark energy sector which can drive the acceleration of the universe, along with the correct evolution of the density parameters, without the need of a cosmological constant. Furthermore, the dark energy equation-of-state parameter can be quintessence or phantom-like, or experiences the phantom-divide crossing, depending on the parameters of the model. Another possible ansatz is the $F(T, T_G) = -T + f(T^2 + \beta_2 T_G)$, the simplest application of which can also easily lead to inflationary behavior. These features make the proposed modified gravity a good candidate for the description of Nature.

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APPENDIX: SHORTCUT PROCEDURE FOR THE EXTRACTION OF THE FIELD EQUATIONS OF $F(T, T_G)$ COSMOLOGY

In this Appendix we follow the shortcut procedure in order to derive the field equations of $F(T, T_G)$ cosmology. In particular, instead of performing the variation of the action (19) in terms of the general vierbein, obtaining the general field equations (16), and then insert into them the cosmological vierbein ansatz (21), we will first insert the cosmological vierbein ansatz into the action and then perform the variation in terms of the scale factor $a(t)$ and the lapse function $N(t)$. Although in principle the shortcut procedure is not guaranteed that it will give the same results as the first robust method [33,34], especially when the action involves higher-order derivatives, in this specific example it proves that we do obtain the same results indeed.

Under the cosmological ansatz (21), namely

$$e^a{}_\mu = \text{diag}(N(t), a(t), a(t), a(t)), \quad (\text{A1})$$

the scalars T and T_G become

$$T = 6 \frac{\dot{a}^2}{N^2 a^2} = 6H^2, \quad (\text{A2})$$

$$\begin{aligned} T_G &= 24 \frac{\dot{a}^2}{N^2 a^2} \left(\frac{\ddot{a}}{N^2 a} - \frac{\dot{N} \dot{a}}{N^3 a} \right) \\ &= 24H^2 \left(\frac{\dot{H}}{N} + H^2 \right). \end{aligned} \quad (\text{A3})$$

Therefore, insertion into the total action (19) gives

$$S_{\text{tot}} = \frac{1}{2\kappa^2} \int dt N a^3 F \left(6H^2, 24H^2 \left(\frac{\dot{H}}{N} + H^2 \right) \right) + S_m. \quad (\text{A4})$$

Let us now perform the variation of (A4) with respect to N and a . Since $\delta_N H = -H \frac{\delta N}{N}$, we obtain

$$\begin{aligned} \delta_N T &= -12H^2 \frac{\delta N}{N}, \\ \delta_N T_G &= -96H^2 \left[\left(\frac{\dot{H}}{N} + H^2 - \frac{H\dot{N}}{4N^2} \right) \frac{\delta N}{N} + \frac{H}{4N^2} (\delta N) \right]. \end{aligned}$$

Similarly, since $\delta_a H = \frac{(\delta a)'}{Na} - H \frac{\delta a}{a}$, we acquire

$$\begin{aligned} \delta_a T &= \frac{12H}{Na} [(\delta a)'] - NH\delta a, \\ \delta_a T_G &= \frac{24H}{Na} \left[\frac{H}{N} (\delta a)' + \left(\frac{2\dot{H}}{N} + 2H^2 - \frac{H\dot{N}}{N^2} \right) (\delta a) \right]. \end{aligned}$$

Therefore, variation of the gravitational part of the action (A4) with respect to N and a gives

$$\begin{aligned} \delta_N S &= \frac{1}{2\kappa^2} \int dt a^3 \left[F - 12H^2 F_T + \frac{24}{a^3} \left(\frac{a^3 H^3}{N} F_{T_G} \right) \right. \\ &\quad \left. - 96H^2 \left(\frac{\dot{H}}{N} + H^2 - \frac{H\dot{N}}{4N^2} \right) F_{T_G} \right] \delta N, \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \delta_a S &= \frac{3}{2\kappa^2} \int dt N a^2 \left\{ F - 4H^2 F_T - 24H^2 \left(\frac{\dot{H}}{N} + H^2 \right) F_{T_G} \right. \\ &\quad \left. - \frac{4}{a^2 N} \left[a^2 H F_T + 2a^2 H \left(\frac{2\dot{H}}{N} + 2H^2 - \frac{H\dot{N}}{N^2} \right) F_{T_G} \right] \right. \\ &\quad \left. + \frac{8}{a^2 N} \left(\frac{a^2 H^2}{N} F_{T_G} \right) \right\} \delta a, \end{aligned} \quad (\text{A6})$$

where $F_T \equiv \partial F / \partial T$ and $F_{T_G} \equiv \partial F / \partial T_G$.

Additionally, variation of S_m gives

$$\delta S_m = \frac{1}{2} \int d^4 x e^{\Theta^{\mu\nu}} \delta g_{\mu\nu},$$

and its time-dependent part is

$$\delta S_m = - \int dt N^2 a^3 \Theta^i \delta N + \int dt N a^2 \Theta^i{}_{;i} \delta a, \quad (\text{A7})$$

where hat indices run in the three spatial coordinates.

In summary, taking into account the total action variation, and setting as usual $N = 1$ in the end, the obtained field equations, that is the Friedmann equations, take the form

$$\begin{aligned} F - 12H^2 F_T - 96H^2 \left(\frac{\dot{H}}{N} + H^2 \right) F_{T_G} \\ + \frac{24}{a^3} (a^3 H^3 F_{T_G})' = 2\kappa^2 \Theta^i, \end{aligned} \quad (\text{A8})$$

$$\begin{aligned} F - 4H^2 F_T - 24H^2 \left(\frac{\dot{H}}{N} + H^2 \right) F_{T_G} \\ - \frac{4}{a^2} [a^2 H F_T + 4a^2 H \left(\frac{\dot{H}}{N} + H^2 \right) F_{T_G}] \\ + \frac{8}{a^2} (a^2 H^2 F_{T_G})' = -\frac{2}{3} \kappa^2 \Theta^i{}_{;i}. \end{aligned} \quad (\text{A9})$$

Additionally, if we consider the matter energy-momentum tensor to correspond to a perfect fluid of energy density ρ

and pressure p , we insert in the above field equations $\Theta^{\mu} = \rho$, $\Theta^{\hat{i}\hat{j}} = \frac{p}{a^2} \delta^{\hat{i}\hat{j}}$, $\Theta^{\hat{i}}_{\hat{i}} = 3p$.

Lastly, we can reorganize the terms, performing the involved time derivatives, resulting in the end to

$$F - 12H^2 F_T - T_G F_{T_G} + 24H^3 \dot{F}_{T_G} = 2\kappa^2 \rho, \quad (\text{A10})$$

$$F - 4(\dot{H} + 3H^2) F_T - 4H \dot{F}_T - T_G F_{T_G} + \frac{2}{3H} T_G \dot{F}_{T_G} + 8H^2 \ddot{F}_{T_G} = -2\kappa^2 p, \quad (\text{A11})$$

where $\dot{F}_T = F_{TT} \dot{T} + F_{TT_G} \dot{T}_G$, $\dot{F}_{T_G} = F_{TT_G} \dot{T} + F_{T_G T_G} \dot{T}_G$, $\ddot{F}_{T_G} = F_{TTT_G} \dot{T}^2 + 2F_{TT_G T_G} \dot{T} \dot{T}_G + F_{T_G T_G T_G} \dot{T}_G^2 + F_{TT_G} \ddot{T} + F_{T_G T_G} \ddot{T}_G$, with F_{TT} , F_{TT_G} , ... denoting multiple partial differentiations of F with respect to T , T_G . Here, \dot{T} , \ddot{T} are obtained by differentiating $T = 6H^2$ with respect to time, while \dot{T}_G , \ddot{T}_G by differentiating $T_G = 24H^2(\dot{H} + H^2)$.

As we observe, the two Friedmann equations (A10) and (A11), derived with the above shortcut procedure, coincide with the two Friedmann equations (24) and (25) derived with the robust procedure in Secs. II and III.

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