PHYSICAL REVIEW D **90,** 084044 (2014)

Teleparallel equivalent of Gauss-Bonnet gravity and its modifications

Georgios Kofinas[1,*] and Emmanuel N. Saridakis[2,3,†]

[1]*Research Group of Geometry, Dynamical Systems and Cosmology, Department of Information and Communication Systems Engineering, University of the Aegean, Karlovassi 83200, Samos, Greece*
[2]*Physics Division, National Technical University of Athens, 15780 Zografou Campus, Athens, Greece*
[3]*Instituto de Física, Pontificia Universidad de Católica de Valparaíso, Casilla 4950 Valparaíso, Chile*
(Received 10 April 2014; published 23 October 2014)

Inspired by the teleparallel formulation of general relativity, whose Lagrangian is the torsion invariant T, we have constructed the teleparallel equivalent of Gauss-Bonnet gravity in arbitrary dimensions. Without imposing the Weitzenböck connection, we have extracted the torsion invariant T_G, equivalent (up to boundary terms) to the Gauss-Bonnet term G. T_G is constructed by the vielbein and the connection, it contains quartic powers of the torsion tensor, it is diffeomorphism and Lorentz invariant, and in four dimensions it reduces to a topological invariant as expected. Imposing the Weitzenböck connection, T_G depends only on the vielbein, and this allows us to consider a novel class of modified gravity theories based on $F(T, T_G)$, which is not spanned by the class of $F(T)$ theories, nor by the $F(R, G)$ class of curvature modified gravity. Finally, varying the action we extract the equations of motion for $F(T, T_G)$ gravity.

DOI: 10.1103/PhysRevD.90.084044 PACS numbers: 04.50.Kd, 95.36.+x, 98.80.-k

I. INTRODUCTION

The central foundation of Einstein's gravitational ideas is that gravity is described through geometry. In his first complete gravitational theory, general relativity (GR), he made the additional *assumption* that geometry should be described only by curvature, setting torsion to zero, along with vanishing nonmetricity [1]. Technically, this is achieved by assuming the connection to be symmetric in coordinate frame, that is using the Levi-Civita connection. In this framework one can construct the curvature (Riemann) tensor which carries all the information of the geometry, and thus of the gravitational field too, and then, by suitable contractions the simplest (Ricci) scalar R can be constructed, which contains up to second-order derivatives in the metric. This Ricci scalar is exactly the Einstein-Hilbert Lagrangian, whose action gives rise to the Einstein field equations through variation in terms of the metric.

However, some years later, it was Einstein himself that realized that the same gravitational equations could arise by a different geometry, characterized not by curvature but by torsion [2]. Technically, this is achieved by *assuming* that the antisymmetric piece of the connection is not vanishing, that is using the Weitzenböck connection. In this framework, one can construct the torsion tensor, which carries all the information of the geometry and therefore of the gravitational field, and then simple scalars can be constructed which contain up to first-order vierbein derivatives. Finally, one can take a specific combination of these scalars and define the "torsion" scalar T, which will be used as the gravitational Lagrangian, demanding its action to give rise to the Einstein gravitational field equations through variation in terms of the vierbein. Since these equations coincide with those of general relativity, Einstein called this alternative formulation "teleparallel equivalent of general relativity" (TEGR).

On the other hand, the nonrenormalizability of general relativity, string theory consequences, and the need to describe the universe acceleration, led a huge amount of research toward the modification of gravity at the classical level. Using general relativity as the starting theory, the simplest modification is to generalize the action using arbitrary functions of the Ricci scalar, resulting to the so-called $F(R)$ modified gravity [3,4], which has the advantage of being ghost free. However, one can construct more complicated generalizations of the Einstein-Hilbert action by introducing higher-curvature corrections, such as the Gauss-Bonnet term G [5–7] or functions of it [7,8], Lovelock combinations [9,10], Weyl combinations [11], or higher spatial-derivatives as in Hořava-Lifshitz gravity [12].

Hence, a question that arises naturally is the following: can we modify gravity starting from TEGR instead of general relativity, that is from its torsional formulation? For the moment, and inspired by the $F(R)$ modification of general relativity, only the simplest such torsional modification exists, namely the $F(T)$ paradigm, in which one extends the teleparallel Lagrangian T to an arbitrary function $F(T)$ [13,14]. Interestingly enough, although TEGR coincides with general relativity at the level of equations of motion, $F(T)$ does not coincide with $F(R)$, so $F(T)$ is a novel class of gravitational modification with no (known) equivalent curvature description. This feature led to a detailed investigation of its cosmological implications [13–16] and black-hole behavior [17].

*gkofinas@aegean.gr
†Emmanuel_Saridakis@baylor.edu

In this work, we are interested in extending the modification of TEGR inserting higher-order torsion invariants. In particular, inspired by the Gauss-Bonnet (GB) modification of general relativity, we first construct the teleparallel equivalent of Gauss-Bonnet term (TEGB) by finding its "torsion" equivalent T_G, which gives the GB field equations. Then, we use it in order to formulate a modification of TEGR. As a result, the modification of TEGR plus the TEGB term does not coincide with the modification of GR plus the GB term, so it is a novel modification of gravity with no (known) curvature formulation.

The plan of the work is as follows: In Sec. II we review the teleparallel formulation of GR in both the coordinate and the differential form language. In Sec. III we find the teleparallel equivalent of GB gravity, while in Sec. IV we derive the equations of motion for the general $F(T, T_G)$ theory. Finally, a summary of the obtained results is given in Sec. V of conclusions.

II. CONSTRUCTION OF TELEPARALLEL EQUIVALENT OF GENERAL RELATIVITY

In this section we present the construction of teleparallel equivalent of general relativity. We follow the detailed and conceptually more enlightening way of construction, starting from an arbitrary connection with vanishing curvature and not restricted to the Weitzenböck one [18–20]. The benefit of this is that the quantities defined are both Lorentz and diffeomorphism invariants. In the same spirit we continue in the next section with the procedure of constructing the teleparallel equivalent of the Gauss-Bonnet combination. As usual, in the end we focus on the Weitzenböck connection.

In the whole manuscript we use the following notation: Greek indices $\mu, \nu, \ldots$ run over all coordinates of D-dimensional space-time $1, 2, \ldots, D$, while Latin indices $a, b, \ldots$ run over the tangent space $1, 2, \ldots, D$. Note that we perform the analysis both in the coordinate and the form languages. Although in the $f(T)$ literature the former is preferred, going to more complicated expressions, such as the Gauss-Bonnet term, the latter proves much more convenient.

A. Construction of TEGR in coordinate language

The dynamical variables in torsional formulation of gravity are the vielbein field $e_a(x^\mu)$, and the connection 1-forms $\omega^a{}_b(x^\mu)$ which defines the parallel transportation. In terms of coordinates, they can be expressed in components as $e_a = e_a{}^\mu \partial_\mu$ and $\omega^a{}_b = \omega^a{}_{b\mu} dx^\mu = \omega^a{}_{bc} e^c$. The dual vielbein is defined as $e^a = e^a{}_\mu dx^\mu$. One can express the commutation relations of the vielbein as

$$[e_a, e_b] = C^c{}_{ab} e_c, \tag{1}$$

where $C^c{}_{ab}$ are the structure coefficients functions given by

$$C^c{}_{ab} = e_a{}^\mu e_b{}^\nu (e^c{}_{\mu,\nu} - e^c{}_{\nu,\mu}), \tag{2}$$

and comma denotes differentiation.

One can now define the torsion tensor, expressed in tangent components as

$$T^a{}_{bc} = \omega^a{}_{cb} - \omega^a{}_{bc} - C^a{}_{bc}, \tag{3}$$

and in "mixed" ones as

$$T^a{}_{\mu\nu} = e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu} + \omega^a{}_{b\mu} e^b{}_\nu - \omega^a{}_{b\nu} e^b{}_\mu. \tag{4}$$

Similarly, one can define the curvature tensor as

$$R^a{}_{bcd} = \omega^a{}_{bd,c} - \omega^a{}_{bc,d} + \omega^e{}_{bd}\omega^a{}_{ec} - \omega^e{}_{bc}\omega^a{}_{ed} - C^e{}_{cd}\omega^a{}_{be},$$
$$R^a{}_{b\mu\nu} = \omega^a{}_{b\nu,\mu} - \omega^a{}_{b\mu,\nu} + \omega^a{}_{c\mu}\omega^c{}_{b\nu} - \omega^a{}_{c\nu}\omega^c{}_{b\mu}. \tag{5}$$

Thus, as one can see from (4) and (5), although the torsion tensor depends on both the vielbein and the connection, that is $T^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu})$, the curvature tensor depends only on the connection, namely $R^a{}_{b\mu\nu}(\omega^a{}_{b\mu})$.

Additionally, there is an independent object which is the metric tensor g. This allows us to make the vielbein orthonormal $g(e_a, e_b) = \eta_{ab}$, where $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots 1)$, and we have the relation

$$g_{\mu\nu} = \eta_{ab} e^a{}_\mu e^b{}_\nu. \tag{6}$$

Indices $a, b, \ldots$ are raised/lowered with the Minkowski metric η_{ab}. Finally, throughout the work we impose zero non-metricity, i.e., $\eta_{ab|c} = 0$, which means $\omega_{abc} = -\omega_{bac}$, where $|$ denotes covariant differentiation with respect to the connection $\omega^a{}_{bc}$.

As it is well known, amongst the infinite connection choices there is only one that gives vanishing torsion, namely the Christoffel or Levi-Civita one $\Gamma^a{}_b$, with $\Gamma_{abc} = \frac{1}{2}(C_{cab} - C_{bca} - C_{abc})$, or inversely $C_{abc} = \Gamma_{acb} - \Gamma_{abc}$. For clarity, we denote the curvature tensor corresponding to the Levi-Civita connection as $\bar{R}^a{}_{bcd}$. The arbitrary connection ω_{abc} is then related to the Christoffel connection Γ_{abc} through the relation

$$\omega_{abc} = \Gamma_{abc} + \mathcal{K}_{abc}, \tag{7}$$

where

$$\mathcal{K}_{abc} = \frac{1}{2}(T_{cab} - T_{bca} - T_{abc}) = -\mathcal{K}_{bac} \tag{8}$$

is the contorsion tensor. Inversely, one can straightforwardly find that $T_{abc} = \mathcal{K}_{acb} - \mathcal{K}_{abc}$, while the "mixed" contorsion components write as $\mathcal{K}^a{}_{\mu\nu} = -\frac{1}{2}(T^a{}_{\mu\nu} + T^b{}_{\mu\lambda} e_{b\nu} e^{a\lambda} + T^b{}_{\nu\lambda} e_{b\mu} e^{a\lambda})$, that is $\mathcal{K}^a{}_{\mu\nu}(e^a{}_\mu, \omega^a{}_{b\mu})$.

As long as the vielbein $e^a{}_\mu$ and the connection $\omega^a{}_{b\mu}$ remain independent from each other, the Einstein-Hilbert

Lagrangian density eR (with $R = e^{a\mu}e^{b\nu}R_{ab\mu\nu}$ the Ricci scalar and $e = \det(e^a{}_\mu) = \sqrt{|g|}$) is a function of $e^a{}_\mu, \omega^a{}_{b\mu}$, and thus a first-order formulation is needed.

If we now calculate the Ricci scalar R corresponding to the arbitrary connection, and the Ricci scalar $\bar{R}$ corresponding to the Levi-Civita connection, they are found to be related through

$$\begin{aligned} eR &= e\bar{R} + \frac{1}{4}e(T^{\mu\nu\lambda}T_{\mu\nu\lambda} + 2T^{\mu\nu\lambda}T_{\lambda\nu\mu} - 4T_\nu{}^{\nu\mu}T^\lambda{}_{\lambda\mu}) \\ &\quad - 2(eT_\nu{}^{\nu\mu})_{,\mu} \\ &= e\bar{R} + eT - 2(eT_\nu{}^{\nu\mu})_{,\mu}, \end{aligned} \tag{9}$$

where we have defined

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\lambda\nu\mu} - T_\nu{}^{\nu\mu}T^\lambda{}_{\lambda\mu}. \tag{10}$$

Since $e^{-1}(eT_\nu{}^{\nu\mu})_{,\mu} = T_\nu{}^{\nu\mu}{}_{;\mu}$, where ; denotes covariant differentiation with respect to the Christoffel connection, Eq. (9) is also written as

$$R = \bar{R} + T - 2T_\nu{}^{\nu\mu}{}_{;\mu}. \tag{11}$$

We mention that the quadratic quantity T is diffeomorphism invariant since $T_{\mu\nu\lambda}$ is a tensor under coordinate transformations. Additionally, T is local Lorentz invariant, since T_{abc} is a Lorentz tensor.

One can now introduce the concept of teleparallelism by imposing the condition of vanishing Lorentz curvature

$$R^a{}_{bcd} = 0, \tag{12}$$

which holds in all frames. One way to realize this condition is by assuming the Weitzenböck connection $\tilde{\omega}^\lambda{}_{\mu\nu}$ which is defined in terms of the vielbein $e_a{}^\mu$ in all coordinate frames as

$$\tilde{\omega}^\lambda{}_{\mu\nu} = e_a{}^\lambda e^a{}_{\mu,\nu}. \tag{13}$$

Due to its inhomogeneous transformation law this connection has tangent-space components $\tilde{\omega}^a{}_{bc} = 0$, and then, the corresponding curvature components are indeed $\tilde{R}^a{}_{bcd} = 0$ (tildes denote the quantities calculated using the Weitzenböck connection). Note that $e_a{}^\mu{}_{|\nu} = 0$, and thus the vielbein $e_a{}^\mu$ is autoparallel with respect to the connection $\tilde{\omega}^\lambda{}_{\mu\nu}$. The corresponding torsion tensor is related to the structure coefficients, the contorsion tensor or the Weitzenböck connection, through

$$\tilde{T}^a{}_{\mu\nu} = e^a{}_{\nu,\mu} - e^a{}_{\mu,\nu} = -C^a{}_{bc}e^b{}_\mu e^c{}_\nu \tag{14}$$

$$\tilde{T}^a{}_{bc} = -C^a{}_{bc} = \tilde{\mathcal{K}}^a{}_{cb} - \tilde{\mathcal{K}}^a{}_{bc} \tag{15}$$

$$\tilde{T}^\lambda{}_{\mu\nu} = \tilde{\omega}^\lambda{}_{\nu\mu} - \tilde{\omega}^\lambda{}_{\mu\nu}, \tag{16}$$

while (7) simplifies to

$$\Gamma_{abc} = -\tilde{\mathcal{K}}_{abc}. \tag{17}$$

Now inserting the condition $R^a{}_{bcd} = 0$ into the general expression (9), we obtain

$$e\bar{R} = -eT + 2(eT_\nu{}^{\nu\mu})_{,\mu}, \tag{18}$$

or equivalently

$$\bar{R} = -T + 2T_\nu{}^{\nu\mu}{}_{;\mu}. \tag{19}$$

As we observe the Lagrangian density $e\bar{R}$ of general relativity (that is the one calculated with the Levi-Civita connection) differs from the torsion density $-eT$ only by a total derivative. Therefore, one can immediately deduce that the general relativity action

$$S_{\rm EH} = \frac{1}{2\kappa_D^2}\int_M d^Dx e\bar{R}, \tag{20}$$

is equivalent (up to boundary terms) to the action

$$\begin{aligned} S^{(1)}_{\rm Tel}[e^a{}_\mu, \omega^a{}_{b\mu}] &= -\frac{1}{2\kappa_D^2}\int_M d^Dx eT \\ &= -\frac{1}{8\kappa_D^2}\int_M d^Dx e(T^{abc}T_{abc} + 2T^{abc}T_{cba} \\ &\quad - 4T_b{}^{ba}T^c{}_{ca}) \end{aligned} \tag{21}$$

(κ_D^2 is the D-dimensional gravitational constant). Indeed, varying (21) with respect to the vielbein we get equations which contain up to $e^a{}_{\mu,\nu\lambda}$, $\omega^a{}_{b\mu,\nu}$, and imposing the teleparallel condition these equations coincide with the Einstein field equations as they arise varying (20) with respect to the metric [20].

If the Weitzenböck connection (13) is adopted, then the teleparallel action (21) becomes a functional only of the vielbein, which is denoted for clarity as $S^{(1)}_{\rm tel}[e^a{}_\mu]$ and has the same functional form as (21), but with tilde quantities. Varying $S^{(1)}_{\rm tel}[e^a{}_\mu]$ with respect to the vielbein gives again the Einstein field equations. That is why the constructed theory in which one uses torsion to describe the gravitational field, under the teleparallelism condition, was named by Einstein as teleparallel equivalent of general relativity.[1] Note that now $\tilde{T}$ still remains diffeomorphism invariant, while the Lorentz invariance has been lost since we have chosen specific class of frames. The equations of motion, being the Einstein equations, are still Lorentz covariant. However, when T in the action is replaced by a general

[1]The normalization of the actions $S^{(1)}_{\rm Tel}$, $S^{(1)}_{\rm tel}$ has been defined such that $S^{(1)}_{\rm Tel} = S^{(1)}_{\rm tel} = S_{\rm EH}$.

function $f(T)$, the new equations of motion under Lorentz rotations of the vielbein will not be covariant (although they are form-invariant). This is not a deficit (it is a sort of analogue of gauge fixing in gauge theories), and the theory, although not Lorentz covariant, is meaningful. Not all vielbeins will be solutions of the new equations, and those which solve the equations will determine the metric uniquely.

An interesting feature of the above analysis is that in (19) the Lagrangian $\bar{R}$ has been expressed in terms of torsion through a splitting into the Lorentz and diffeomorphism invariant term $-T$, containing at most first order derivatives in the fields $e^a{}_\mu$, $\omega^a{}_{b\mu}$, plus a total divergence also Lorentz and diffeomorphism invariant containing the second order derivatives of $e^a{}_\mu$. Note that the Riemann tensor $\bar{R}^\mu{}_{\nu\rho\sigma} = \Gamma^\mu{}_{\nu\sigma,\rho} - \Gamma^\mu{}_{\nu\rho,\sigma} + \Gamma^\tau{}_{\nu\sigma}\Gamma^\mu{}_{\tau\rho} - \Gamma^\tau{}_{\nu\rho}\Gamma^\mu{}_{\tau\sigma}$ is a sum of the first-order in $e^a{}_\mu$ terms $\Gamma^\tau{}_{\nu\sigma}\Gamma^\mu{}_{\tau\rho} - \Gamma^\tau{}_{\nu\rho}\Gamma^\mu{}_{\tau\sigma}$, plus the second-order total divergence terms $\Gamma^\mu{}_{\nu\sigma,\rho} - \Gamma^\mu{}_{\nu\rho,\sigma}$. A similar splitting occurs for the Lagrangian density $e\bar{R}$, known as the "gamma-gamma" form [21], however in that case, the first-order terms as well as the total divergence terms are not diffeomorphism invariant. Hence, the teleparallel splitting provides an advantage since the diffeomorphism invariance is maintained in the separate terms.

B. Construction of TEGR in differential form language

Let us now repeat the presentation of the previous subsection in differential form language. We will need the completely antisymmetric symbol $\epsilon_{a_1\ldots a_D}$, which has $\epsilon_{1\ldots D} = 1$, while the contravariant components $\epsilon^{a_1\cdots a_D} = \eta^{a_1 b_1}\ldots\eta^{a_D b_D}\epsilon_{b_1\ldots b_D}$ have $\epsilon^{1\ldots D} = -1$. The dynamical variables are the vielbein e^a and the connection 1-forms $\omega^a{}_b$, with $\omega_{ab} = -\omega_{ba}$ due to the vanishing nonmetricity. One can express the commutation relations (1) in terms of the dual vielbein as

$$de^a = -\frac{1}{2}C^a{}_{bc}e^b \wedge e^c, \tag{22}$$

where $\wedge$ denotes the wedge product.

One can now define the torsion 2-form as

$$T^a = de^a + \omega^a{}_b \wedge e^b = \frac{1}{2}T^a{}_{bc}e^b \wedge e^c, \tag{23}$$

and the curvature 2-form as

$$\mathcal{R}^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2}R^a{}_{bcd}e^c \wedge e^d. \tag{24}$$

The curvature 2-form corresponding to $\Gamma^a{}_b$ is denoted by $\bar{\mathcal{R}}^a{}_b$. The arbitrary connection $\omega^a{}_b$ is then related to $\Gamma^a{}_b$ through the relation

$$\mathcal{K}_{ab} = -\mathcal{K}_{ba} = \omega_{ab} - \Gamma_{ab} = \mathcal{K}_{abc}e^c, \tag{25}$$

where $\mathcal{K}_{ab}$ is the contorsion 1-form. Inversely, one can straightforwardly find that $T^a = \mathcal{K}^a{}_b \wedge e^b$. Finally, note that under the Weitzenböck connection the previous relation simplifies to

$$\Gamma_{ab} = -\tilde{\mathcal{K}}_{ab}. \tag{26}$$

The action of general relativity is written in terms of the connection $\Gamma^a{}_b$ as

$$S_{\rm EH} = \frac{1}{2\kappa_D^2}\int_M \bar{\mathcal{L}}_1, \tag{27}$$

where

$$\bar{\mathcal{L}}_1 = \frac{1}{(D-2)!}\epsilon_{a_1\ldots a_D}\bar{\mathcal{R}}^{a_1a_2} \wedge e^{a_3} \wedge \ldots \wedge e^{a_D} = \bar{R} * 1, \tag{28}$$

with $*$ denoting the Hodge dual operator. If we now calculate the Lagrangian $\mathcal{L}_1$ corresponding to the arbitrary connection $\omega^a{}_b$, it is related to $\bar{\mathcal{L}}_1$ through

$$\begin{aligned}(D-2)!\mathcal{L}_1 = (D-2)!\bar{\mathcal{L}}_1 &+ d(\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2} \wedge e^{a_3} \wedge \ldots \wedge e^{a_D}) \\ &+ \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2} \wedge d(e^{a_3} \wedge \ldots \wedge e^{a_D}) \\ &+ \epsilon_{a_1\ldots a_D}(\Gamma^{a_1}{}_c \wedge \mathcal{K}^{ca_2} + \mathcal{K}^{a_1}{}_c \wedge \Gamma^{ca_2} \\ &+ \mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{ca_2}) \wedge e^{a_3} \wedge \ldots \wedge e^{a_D},\end{aligned} \tag{29}$$

which after some cancellations provides the analogue of (9)

$$\begin{aligned}\mathcal{L}_1 = \bar{\mathcal{L}}_1 &+ \frac{1}{(D-2)!}\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{ca_2} \wedge e^{a_3} \wedge \ldots \wedge e^{a_D} \\ &+ \frac{1}{(D-2)!}d(\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2} \wedge e^{a_3} \wedge \ldots \wedge e^{a_D}).\end{aligned} \tag{30}$$

Finally, imposing the teleparallel condition $\mathcal{R}^{ab} = 0$, we get the analogue of (18)

$$\bar{\mathcal{L}}_1 = -\mathcal{T} - \frac{1}{(D-2)!}d(\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2} \wedge e^{a_3} \wedge \ldots \wedge e^{a_D}), \tag{31}$$

where

$$\begin{aligned}\mathcal{T} &= \frac{1}{(D-2)!}\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1}{}_c \wedge \mathcal{K}^{ca_2} \wedge e^{a_3} \wedge \ldots \wedge e^{a_D} \\ &= Te^1 \wedge \ldots \wedge e^D\end{aligned} \tag{32}$$

is the TEGR volume form, and

$$T = \mathcal{K}^{abc}\mathcal{K}_{cba} - \mathcal{K}^{ca}{}_a\mathcal{K}_{cb}{}^b \tag{33}$$

the corresponding scalar. Ignoring the boundary term in (31) we obtain again the teleparallel action (21) of Einstein gravity

$$S^{(1)}_{\rm Tel} = -\frac{1}{2\kappa_D^2}\int_M \mathcal{T} = -\frac{1}{2\kappa_D^2}\int_M d^Dx eT. \tag{34}$$

In order to obtain the above results it is much more powerful to introduce the covariant exterior differential D of the connection ω_{ab} acting on a set of p-forms Φ^a_b as $D\Phi^a_b = d\Phi^a_b + \omega^a{}_c\wedge\Phi^c_b - (-1)^p\Phi^a_c\wedge\omega^c{}_b$. Similarly, the differential $\bar{D}$ is defined for the connection Γ_{ab}. Then, $\mathcal{R}^{ab} = \bar{\mathcal{R}}^{ab} + \bar{D}\mathcal{K}^{ab} + \mathcal{K}^a{}_c\wedge\mathcal{K}^{cb}$, $T^a = De^a$, $DT^a = \mathcal{R}^a{}_b\wedge e^b$, $D\mathcal{R}^a{}_b = 0$, $D^2\Phi^a_b = \mathcal{R}^a{}_c\wedge\Phi^c_b - \Phi^a_c\wedge\mathcal{R}^c{}_b$. Since it is $\bar{D}e^a = 0$, we get immediately Eq. (30). This is the method that will be followed in the next section.

III. CONSTRUCTION OF TELEPARALLEL EQUIVALENT OF GAUSS-BONNET TERM

In this section we will construct the teleparallel equivalent of the Gauss-Bonnet gravity. We will follow the procedure of the construction of TEGR described above, based on the corresponding action. The central strategy of the previous section was to express the curvature scalar R corresponding to a general connection as the curvature scalar $\bar{R}$ corresponding to Levi-Civita connection, plus terms arising from the torsion tensor. Then, by imposing the teleparallelism condition $R^a{}_{bcd} = 0$, we acquire that $\bar{R}$ is equal to a torsion scalar plus a total derivative, namely relation (18). This torsion scalar provides the teleparallel equivalent of general relativity, in a sense that if one uses it as a Lagrangian, the same exactly equations with general relativity are obtained.

In this section we follow the same steps to reexpress the Gauss-Bonnet combination

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\kappa\lambda}R^{\mu\nu\kappa\lambda}. \tag{35}$$

However, for convenience we will use the form language which leads to simple expressions compared to the coordinate description. The action of Gauss-Bonnet gravity in terms of the Levi-Civita connection is

$$S_{\rm GB} = \frac{1}{2\kappa_D^2}\int_M \bar{\mathcal{L}}_2, \tag{36}$$

where

$$\bar{\mathcal{L}}_2 = \frac{1}{(D-4)!}\epsilon_{a_1\ldots a_D}\bar{\mathcal{R}}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D} = \bar{G}*1. \tag{37}$$

The corresponding Lagrangian when $\bar{\mathcal{R}}^{ab}$ is replaced by $\mathcal{R}^{ab}$, that is the one that corresponds to an arbitrary connection $\omega^a{}_b$, is denoted by $\mathcal{L}_2$. The relation between $\mathcal{L}_2$ and $\bar{\mathcal{L}}_2$ is found to be

$$(D-4)!\mathcal{L}_2 = (D-4)!\bar{\mathcal{L}}_2 + I_1 + 2I_2 + 2I_3 + 2I_4 + I_5, \tag{38}$$

where

$$\begin{aligned}
I_1 &= \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1}{}_c\wedge\mathcal{K}^{ca_2}\wedge\mathcal{K}^{a_3}{}_d\wedge\mathcal{K}^{da_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},\\
I_2 &= \epsilon_{a_1\ldots a_D}\bar{\mathcal{R}}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},\\
I_3 &= \epsilon_{a_1\ldots a_D}\bar{D}\mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},\\
I_4 &= \epsilon_{a_1\ldots a_D}\bar{D}\mathcal{K}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},\\
I_5 &= \epsilon_{a_1\ldots a_D}\bar{D}\mathcal{K}^{a_1a_2}\wedge\bar{D}\mathcal{K}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}.
\end{aligned} \tag{39}$$

I_1 is an algebraic term quartic in torsion. Since $\bar{D}\bar{\mathcal{R}}^{ab} = 0$ and $\bar{D}e^a = 0$, I_4 is an exact form

$$I_4 = d(\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}). \tag{40}$$

Similarly, since $\bar{D}^2\mathcal{K}^{ab} = \bar{\mathcal{R}}^a{}_c\wedge\mathcal{K}^{cb} + \bar{\mathcal{R}}^b{}_c\wedge\mathcal{K}^{ac}$, we have

$$\begin{aligned}
I_5 &= 2\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}\\
&\quad + d(\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{D}\mathcal{K}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}).
\end{aligned} \tag{41}$$

Therefore,

$$(D-4)!\mathcal{L}_2 = (D-4)!\bar{\mathcal{L}}_2 + I_1 + 2I_3 + 2I_6 + dB, \tag{42}$$

where

$$\begin{aligned}
I_6 &= \epsilon_{a_1\ldots a_D}\bar{\mathcal{R}}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}\\
&\quad + \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},\\
B &= 2\epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}\\
&\quad + \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{D}\mathcal{K}^{a_3a_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}.
\end{aligned} \tag{43}$$

Taking into account that $\bar{\mathcal{R}}^{ab} + \bar{D}\mathcal{K}^{ab} = \mathcal{R}^{ab} - \mathcal{K}^a{}_c\wedge\mathcal{K}^{cb}$, Eq. (42) is written as

$$(D-4)!\mathcal{L}_2 = (D-4)!\bar{\mathcal{L}}_2 + 2J_0 - I_1 + 2J_1 + dB, \tag{44}$$

where

$$\begin{aligned}
J_0 &= \epsilon_{a_1\ldots a_D}\mathcal{R}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},\\
J_1 &= \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\bar{\mathcal{R}}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}.
\end{aligned}$$

To finish, from the identity $\bar{D}\mathcal{K}^a{}_b = D\mathcal{K}^a{}_b - 2\mathcal{K}^a{}_c\wedge\mathcal{K}^c{}_b$ we get $\bar{\mathcal{R}}^a{}_b = \mathcal{R}^a{}_b + \mathcal{K}^a{}_c\wedge\mathcal{K}^c{}_b - D\mathcal{K}^a{}_b$ and substituting into J_1 we obtain

$$(D-4)!\mathcal{L}_2 = (D-4)!\bar{\mathcal{L}}_2 + 2(J_0 + \hat{J}_0) - I_1 + 2J_2 - 2J_3 + dB, \tag{45}$$

where

$$J_2 = \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^c{}_d\wedge\mathcal{K}^{da_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},$$
$$J_3 = \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge D\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}$$
$$= \epsilon_{a_1\ldots a_D}(\mathcal{K}^{a_1a_2}\wedge d\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4} + \mathcal{K}^{a_1a_2}\wedge\omega^{a_3}{}_c\wedge\mathcal{K}^c{}_d\wedge\mathcal{K}^{da_4} + \mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\omega^c{}_d\wedge\mathcal{K}^{da_4})\wedge e^{a_5}\wedge\ldots\wedge e^{a_D},$$
$$\hat{J}_0 = \epsilon_{a_1\ldots a_D}\mathcal{K}^{a_1a_2}\wedge\mathcal{R}^{a_3}{}_c\wedge\mathcal{K}^{ca_4}\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}.$$

In order to extract the teleparallel equivalent of GB gravity we set $\mathcal{R}^{ab}=0$ in (45) obtaining

$$\bar{\mathcal{L}}_2 = \mathcal{T}_G - \frac{1}{(D-4)!}dB, \tag{46}$$

where

$$\mathcal{T}_G = \frac{1}{(D-4)!}\epsilon_{a_1\ldots a_D}(\mathcal{K}^{a_1}{}_c\wedge\mathcal{K}^{ca_2}\wedge\mathcal{K}^{a_3}{}_d\wedge\mathcal{K}^{da_4} - 2\mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^c{}_d\wedge\mathcal{K}^{da_4} + 2\mathcal{K}^{a_1a_2}\wedge D\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^{ca_4})\wedge e^{a_5}\wedge\ldots\wedge e^{a_D} = T_G e^1\wedge\ldots\wedge e^D \tag{47}$$

is the TEGB volume form. The corresponding scalar is

$$T_G = (\mathcal{K}^{a_1}{}_{ea}\mathcal{K}^{ea_2}{}_b\mathcal{K}^{a_3}{}_{fc}\mathcal{K}^{fa_4}{}_d - 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^e{}_{fc}\mathcal{K}^{fa_4}{}_d + 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_f\mathcal{K}^f{}_{cd} + 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_{c|d})\delta^{abcd}_{a_1a_2a_3a_4}. \tag{48}$$

Here, $D\mathcal{K}^a{}_b = d\mathcal{K}^a{}_b + \omega^a{}_c\wedge\mathcal{K}^c{}_b + \mathcal{K}^a{}_c\wedge\omega^c{}_b = (\mathcal{K}^a{}_{bc|d} + \frac{1}{2}\mathcal{K}^a{}_{be}T^e{}_{dc})e^d\wedge e^c$, the covariant derivative of $\mathcal{K}^a{}_{bc}$ with respect to $\omega^a{}_{bc}$ is $\mathcal{K}^a{}_{bc|d} = \mathcal{K}^a{}_{bc,d} + \omega^a{}_{ed}\mathcal{K}^e{}_{bc} - \omega^e{}_{bd}\mathcal{K}^a{}_{ec} - \omega^e{}_{cd}\mathcal{K}^a{}_{be}$, the $C^a{}_{bc}$ is given by Eq. (2), and the generalized δ is the determinant of the Kronecker deltas.

The analogue of Eq. (18) is now

$$e(\bar{R}^2 - 4\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \bar{R}_{\mu\nu\kappa\lambda}\bar{R}^{\mu\nu\kappa\lambda}) = eT_G + \text{total diverg.} \tag{49}$$

Obviously, $\mathcal{T}_G$ is a Lorentz invariant made out of e^a, $\omega^a{}_b$. Since in $D=4$ dimensions the GB term $\bar{\mathcal{L}}_2^{(D=4)}$ is a topological invariant, so must be $\mathcal{T}_G^{(D=4)}$. Indeed, it is

$$\mathcal{T}_G^{(D=4)} = d(32\pi^2\Pi_2 + B), \tag{50}$$

where

$$\Pi_2 = -\frac{1}{8\pi^2}\epsilon_{abcd}n^a\left(\varepsilon\bar{\mathcal{R}}^{bc}\wedge\bar{D}n^d + \frac{2}{3}\bar{D}n^b\wedge\bar{D}n^c\wedge\bar{D}n^d\right) \tag{51}$$

is the second Chern form, n^a is a unit vector with $n^a n_a = \varepsilon = \pm 1$, and $\bar{\mathcal{L}}_2^{(D=4)} = 32\pi^2 d\Pi_2$. *Therefore, we have constructed a new Lorentz invariant $\mathcal{T}_G$ out of e^a, $\omega^a{}_b$, containing quartic powers of the torsion tensor, which in 4 dimensions becomes a topological invariant.*

Ignoring the boundary term B in (46) we obtain the teleparallel action of Gauss-Bonnet gravity

$$S^{(2)}_{\text{Tel}}[e^a,\omega^a{}_b] = \frac{1}{2\kappa_D^2}\int_M \mathcal{T}_G = \frac{1}{2\kappa_D^2}\int_M d^Dx\, eT_G. \tag{52}$$

The action $S^{(2)}_{\text{Tel}}[e^a{}_\mu,\omega^a{}_{b\mu}]$ is diffeomorphism and Lorentz invariant. Beyond $e^a{}_\mu$, $e^a{}_{\mu,\nu}$, $\omega^a{}_{b\mu}$, which exist in $S^{(1)}_{\text{Tel}}$ in (21) too, in $S^{(2)}_{\text{Tel}}$ there appear also $e^a{}_{\mu,\nu\lambda}$, $\omega^a{}_{b\mu,\nu}$, but in a form such that the equations of motion do not contain higher than second derivatives in $e^a{}_\mu$, as expected from the Gauss-Bonnet term.

Now, choosing the Weitzenböck connection $\omega^a{}_{bc}=0$, the action (52) becomes

$$S^{(2)}_{\text{tel}} = \frac{1}{2(D-4)!\kappa_D^2}\int_M \epsilon_{a_1\ldots a_D}(\mathcal{K}^{a_1}{}_c\wedge\mathcal{K}^{ca_2}\wedge\mathcal{K}^{a_3}{}_d\wedge\mathcal{K}^{da_4} - 2\mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge\mathcal{K}^c{}_d\wedge\mathcal{K}^{da_4} - 2\mathcal{K}^{a_1a_2}\wedge\mathcal{K}^{a_3}{}_c\wedge d\mathcal{K}^{ca_4})\wedge e^{a_5}\wedge\ldots\wedge e^{a_D}. \tag{53}$$

Note that the tildes denoting the quantities corresponding to the Weitzenböck connection are omitted for simplicity. In coordinate language it is

$$S^{(2)}_{\text{tel}} = \frac{1}{2\kappa_D^2}\int_M d^Dx\, e(\mathcal{K}^{a_1}{}_{ea}\mathcal{K}^{ea_2}{}_b\mathcal{K}^{a_3}{}_{fc}\mathcal{K}^{fa_4}{}_d - 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^e{}_{fc}\mathcal{K}^{fa_4}{}_d + 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_f\mathcal{K}^f{}_{cd} + 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_{c,d})\delta^{abcd}_{a_1a_2a_3a_4}, \tag{54}$$

where now

$$T_G = (\mathcal{K}^{a_1}{}_{ea}\mathcal{K}^{ea_2}{}_b\mathcal{K}^{a_3}{}_{fc}\mathcal{K}^{fa_4}{}_d - 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^e{}_{fc}\mathcal{K}^{fa_4}{}_d + 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_f\mathcal{K}^f{}_{cd} + 2\mathcal{K}^{a_1a_2}{}_a\mathcal{K}^{a_3}{}_{eb}\mathcal{K}^{ea_4}{}_{c,d})\delta^{abcd}_{a_1a_2a_3a_4}. \tag{55}$$

The action $S^{(2)}_{\text{tel}}$ is a functional of $e^a{}_\mu$, namely $S^{(2)}_{\text{tel}}[e^a{}_\mu]$, and although T_G in (55) contains $e^a{}_{\mu,\nu\lambda}$, the arising equations of motion contain only $e^a{}_{\mu,\nu\lambda}$ and not higher derivatives, as expected. The quantity T_G in (55) is a diffeomorphism invariant containing quartic scalars of the torsion (or contorsion) tensor. However, Lorentz invariance is lost since preferred autoparallel orthonormal frames have be chosen. As in Einstein gravity, this is not a deficit, it is a sort of analogue of gauge fixing in gauge theories. In four dimensions, as the general T_G of Eq. (48) is a topological invariant, here T_G of Eq. (55) is also a topological invariant constructed out of torsion. This is due to the fact that T_G differs from the Gauss-Bonnet term, which is topological in four dimensions, only by a total derivative. Note that the normalization of the actions $S^{(2)}_{\rm Tel}$, $S^{(2)}_{\rm tel}$ has been defined such that $S^{(2)}_{\rm Tel} = S^{(2)}_{\rm tel} = S_{\rm GB}$. In case of Einstein-Gauss-Bonnet theory the total action is $S_{\rm EGB} = S_{\rm EH} + \alpha S_{\rm GB} = S^{(1)}_{\rm tel} + \alpha S^{(2)}_{\rm tel}$, with α the relevant coupling.

IV. $F(T,T_G)$ GRAVITY AND EQUATIONS OF MOTION

In the previous section, we constructed a new quartic-torsion invariant T_G, arising from the teleparallel equivalent of Gauss-Bonnet gravity. Therefore, in analogue with the $F(T)$ gravitational modifications, we can formulate new modified gravity theories in arbitrary dimensions by considering general functions $F(T_G)$ in the action. Obviously, since T_G is quartic in torsion, $F(T_G)$ cannot arise from any $F(T)$. Supplementing the proposed class of modifications with the usual $F(T)$ term, the total modified gravitational action takes the form

$$S = \frac{1}{2\kappa_D^2}\int d^D x e F(T, T_G), \qquad (56)$$

which is clearly different from $F(R, G)$ gravity [7,8,22] (for other constructions of actions including torsion see [23,24]). Obviously, the usual Einstein-Gauss-Bonnet theory arises in the special case $F(T, T_G) = -T + \alpha T_G$ (with α the Gauss-Bonnet coupling), while TEGR (that is GR) is obtained for $F(T, T_G) = -T$.

In the following, we will extract the equations of motion of $F(T, T_G)$ gravity by varying the action (56). Variation with respect to the vielbein gives

$$2\kappa_D^2 \delta_e S = \int d^D x (e F_T \delta_e T + e F_{T_G} \delta_e T_G + F \delta e), \qquad (57)$$

where $F_T = \partial F/\partial T$, $F_{T_G} = \partial F/\partial T_G$. Since the variation of $\delta_e T_G$ is very complicated, we find it more convenient to make the variations $\delta_e \mathcal{T}_G$ and $\delta_e \mathcal{T}$ using forms. In particular, we have

$$2\kappa_D^2 \delta_e S = \int (F_T \delta_e \mathcal{T} + F_{T_G} \delta_e \mathcal{T}_G) + \int d^D x (F - T F_T - T_G F_{T_G}) \delta e. \qquad (58)$$

Let $i_v \varphi$ denote the inner derivative of a p-form $\varphi = \frac{1}{p!}\varphi_{a_1\ldots a_p} e^{a_1} \wedge \ldots \wedge e^{a_p}$ with respect to the vector field $v = v^a e_a$, i.e., for any $p-1$ vector fields $v_1, \ldots, v_{p-1}$, it holds $(i_v\varphi)(v_1, \ldots, v_{p-1}) = \varphi(v, v_1, \ldots, v_{p-1})$. We are interested in combining this definition with variations. An immediate property is

$$i_{e_a}\delta e^b + i_{\delta e_a} e^b = 0, \qquad (59)$$

which arises from the equations $\delta e^a = e_b{}^\mu \delta e^a{}_\mu e^b$, $\delta e_a = e^b{}_\mu \delta e_a{}^\mu e_b$, $i_{e_a}\delta e^b = e_a{}^\mu \delta e^b{}_\mu$, and $i_{\delta e_a} e^b = e^b{}_\mu \delta e_a{}^\mu$. Using the definition (23) of the torsion, Eq. (59), the linearity of $i_v\varphi$ in both v, φ, and the relations $i_v d + d i_v = \pounds_v$, $i_v(\varphi\wedge\psi) = i_v\varphi\wedge\psi + (-1)^p \varphi\wedge i_v\psi$, we can find

$$\delta_e(i_{e_a} T^b) = \pounds_{e_a}\delta e^b + \pounds_{\delta e_a} e^b + i_{e_a}\omega^b{}_c \wedge \delta e^c + i_{\delta e_a}\omega^b{}_c \wedge e^c, \qquad (60)$$

where $\pounds$ denotes the Lie derivative.

The use of the Lie derivative proves very convenient for the variation procedure. In particular, we use the identity $v(\alpha(w)) = (\pounds_v\alpha)(w) + \alpha(\pounds_v w)$, where α is 1-form and v, w are vector fields, once for $v = \delta e_a$, $w = e_c$, $\alpha = e^b$ to find $(\pounds_{\delta e_a} e^b)(e_c) = e^b(\pounds_{e_c}\delta e_a)$, and once for $v = e_c$, $w = \delta e_a$, $\alpha = e^b$ to find $e^b(\pounds_{e_c}\delta e_a) = e_c(e^b(\delta e_a)) - (\pounds_{e_c} e^b)(\delta e_a)$. Therefore, we obtain

$$\pounds_{\delta e_a} e^b = \pounds_{e_c}(e^b(\delta e_a)) e^c + C^b{}_{cd} e^d(\delta e_a) e^c. \qquad (61)$$

Thus, the quantity appearing in (60) becomes

$$\delta_e(i_{e_a} T^b) = \pounds_{e_a}\delta e^b + \pounds_{e_c}(e^b(\delta e_a)) e^c + C^b{}_{cd} e^d(\delta e_a) e^c + \omega^b{}_{ca}\delta e^c + \omega^b{}_{cd} e^d(\delta e_a) e^c. \qquad (62)$$

Additionally, we also need to evaluate the quantity $\delta_e(i_{e_a} i_{e_b} T^c)$. Using the definition (23) of the torsion, Eq. (59), the linearity of $i_v\varphi$ in both v, φ, equations $i_v f = 0$ (f 0-form), $i_v(\varphi\wedge\psi) = i_v\varphi\wedge\psi + (-1)^p\varphi\wedge i_v\psi$, and the relations $i_v d + d i_v = \pounds_v$, $\pounds_v i_w - i_w \pounds_v = i_{[v,w]}$ to transfer the operators $d, \pounds$ on the left, we can find

$$\delta_e(i_{e_a} i_{e_b} T^c) = i_{[e_a,e_b]}\delta e^c + i_{[e_a,\delta e_b]} e^c - i_{[e_b,\delta e_a]} e^c + (i_{e_b}\omega^c{}_d)(i_{e_a}\delta e^d) - (i_{e_a}\omega^c{}_d)(i_{e_b}\delta e^d) + 2\omega^c{}_{[ad]} e^d(\delta e_b) - 2\omega^c{}_{[bd]} e^d(\delta e_a), \qquad (63)$$

where the (anti)symmetrization symbol contains the factor 1/2. Applying the identity $v(\alpha(w)) = (\pounds_v\alpha)(w) + \alpha(\pounds_v w)$ for $v = e_a$, $w = \delta e_b$, $\alpha = e^c$, and since $i_{[e_a,\delta e_b]} e^c = e^c(\pounds_{e_a}\delta e_b)$, we find

$$i_{[e_a,\delta e_b]} e^c = \pounds_{e_a}(e^c(\delta e_b)) + C^c{}_{ad} e^d(\delta e_b). \qquad (64)$$

Finally, using (59), we acquire

$$\delta_e(i_{e_a}i_{e_b}T^c) = \pounds_{e_a}(e^c(\delta e_b)) - \pounds_{e_b}(e^c(\delta e_a)) + C^c{}_{ad}e^d(\delta e_b) - C^c{}_{bd}e^d(\delta e_a) - C^d{}_{ab}e^c(\delta e_d) + \omega^c{}_{ad}e^d(\delta e_b) - \omega^c{}_{bd}e^d(\delta e_a). \quad (65)$$

Now, the contorsion 1-form can be written as

$$2\mathcal{K}_{ab} = i_{e_a}T_b - i_{e_b}T_a - (i_{e_a}i_{e_b}T_c)e^c, \quad (66)$$

therefore we obtain

$$2\delta_e\mathcal{K}_{ab} = \delta_e i_{e_a}T_b - \delta_e i_{e_b}T_a - \delta_e(i_{e_a}i_{e_b}T_c)e^c - T_{cba}\delta e^c. \quad (67)$$

Using (62) and (65) we get

$$2\delta_e\mathcal{K}_{ab} = \pounds_{e_a}\delta\underline{e}_b - \pounds_{e_b}\delta\underline{e}_a + \pounds_{e_c}(i_{e_b}\delta\underline{e}_a)e^c - \pounds_{e_c}(i_{e_a}\delta\underline{e}_b)e^c + \pounds_{e_a}(i_{e_b}\delta\underline{e}_c)e^c - \pounds_{e_b}(i_{e_a}\delta\underline{e}_c)e^c - C^d{}_{ab}(i_{e_d}\delta\underline{e}_c)e^c + 2C_{(ac)d}(i_{e_b}\delta e^d)e^c - 2C_{(bc)d}(i_{e_a}\delta e^d)e^c + T_{cab}\delta e^c + 2\omega_{c[ab]}\delta e^c, \quad (68)$$

where $\underline{e}_a = \eta_{ab}e^b$ are 1-forms and $\underline{e}_b(\delta e_a) = -i_{e_a}\delta\underline{e}_b$.

Varying $\mathcal{T}, \mathcal{T}_G$ from (32) and (47), and due to the fact that $i_{e_a}\delta e^a = -e^a{}_\mu\delta e_a{}^\mu = \frac{\delta e}{e}$, the variation (58) of the action becomes

$$2\kappa_D^2\delta_e S = \int(2\delta_e\mathcal{K}_{ab}\wedge H^{ab} + h_a\wedge\delta e^a) + \int(F - TF_T - T_G F_{T_G})(i_{e_a}\delta e^a)e^1\wedge\ldots\wedge e^D, \quad (69)$$

where

$$H^{ab} = \frac{F_T}{(D-2)!}\epsilon^a{}_{a_1\ldots a_{D-1}}\mathcal{K}^{ba_1}e^{a_2}\ldots e^{a_{D-1}} + \frac{F_{T_G}}{(D-4)!}(2\epsilon^a{}_{a_1\ldots a_{D-1}}\mathcal{K}^{ba_1}\mathcal{K}^{a_2}{}_c\mathcal{K}^{ca_3}e^{a_4}\ldots e^{a_{D-1}} + \epsilon_{a_1\ldots a_D}\mathcal{K}^{aa_1}\mathcal{K}^{ba_2}\mathcal{K}^{a_3a_4}e^{a_5}\ldots e^{a_D} - \epsilon^{ab}{}_{a_1\ldots a_{D-2}}\mathcal{K}^{a_1}{}_c\mathcal{K}^c{}_d\mathcal{K}^{da_2}e^{a_3}\ldots e^{a_{D-2}} + \epsilon^{ab}{}_{a_1\ldots a_{D-2}}D\mathcal{K}^{a_1}{}_c\mathcal{K}^{ca_2}e^{a_3}\ldots e^{a_{D-2}} + \epsilon^a{}_{a_1\ldots a_{D-1}}D\mathcal{K}^{ba_1}\mathcal{K}^{a_2a_3}e^{a_4}\ldots e^{a_{D-1}}) - \frac{1}{(D-4)!}\epsilon^a{}_{a_1\ldots a_{D-1}}D(F_{T_G}\mathcal{K}^{ba_1}\mathcal{K}^{a_2a_3}e^{a_4}\ldots e^{a_{D-1}}) \quad (70)$$

and

$$h_a = \frac{F_T}{(D-3)!}\epsilon_{a_1\ldots a_{D-1}a}\mathcal{K}^{a_1}{}_c\mathcal{K}^{ca_2}e^{a_3}\ldots e^{a_{D-1}} + \frac{F_{T_G}}{(D-5)!}\epsilon_{a_1\ldots a_{D-1}a}(\mathcal{K}^{a_1}{}_c\mathcal{K}^{ca_2}\mathcal{K}^{a_3}{}_d\mathcal{K}^{da_4} - 2\mathcal{K}^{a_1a_2}\mathcal{K}^{a_3}{}_c\mathcal{K}^c{}_d\mathcal{K}^{da_4} + 2\mathcal{K}^{a_1a_2}D\mathcal{K}^{a_3}{}_c\mathcal{K}^{ca_4})e^{a_5}\ldots e^{a_{D-1}}. \quad (71)$$

The quantities H^{ab}, h_a are $D-1$ forms and the $\wedge$ symbols between $\mathcal{K}^{ab}$ and e^a are omitted for safety of space. Moreover, boundary terms have been omitted too. The above relations hold for $D > 4$, while for $D = 4$ all terms exist too, apart from the term containing $(D-5)!$ in h_a, which is absent.

Now, we plague expression (68) in the variation (69), and after use of the identity $\pounds_{e_a} = i_{e_a}d + di_{e_a}$ and the obvious equations $i_{e_b}(\pounds_{e_c}(e^c\wedge H^{[ab]})\wedge\delta\underline{e}_a) = 0$, $i_{e_b}(\pounds_{e_a}(e^c\wedge H^{[ab]})\wedge\delta\underline{e}_c) = 0$, $i_{e_d}(C^d{}_{ab}e^c\wedge H^{ab}\wedge\delta\underline{e}_c) = 0$, $i_{e_b}(C_{(ac)d}e^c\wedge H^{ab}\wedge\delta e^d) = 0$, we obtain (omitting the boundary terms)

$$2\kappa_D^2\delta_e S = \int\delta\underline{e}_a[2\pounds_{e_b}H^{[ab]} - 2i_{e_b}\pounds_{e_c}(e^cH^{[ab]} + e^aH^{[cb]}) - C^d{}_{cb}i_{e_d}(e^aH^{cb}) + 4C_{(dc)}{}^a i_{e_b}(e^cH^{[db]}) + (T^a{}_{bc} + 2\omega^a{}_{[bc]})H^{bc} - (-1)^D h^a + (F - TF_T - T_GF_{T_G})\vartheta^a], \quad (72)$$

where $\vartheta_a = i_{e_a}(e^1\wedge\ldots\wedge e^D)$. Thus, setting $\delta_e S = 0$ we get the equations of motion for $F(T, T_G)$ gravity

$$2\pounds_{e_b}H^{[ab]} - 2i_{e_b}\pounds_{e_c}(e^cH^{[ab]} + e^aH^{[cb]}) - C^d{}_{cb}i_{e_d}(e^aH^{cb}) + 4C_{(dc)}{}^a i_{e_b}(e^cH^{[db]}) + (T^a{}_{bc} + 2\omega^a{}_{[bc]})H^{bc} - (-1)^D h^a + (F - TF_T - T_GF_{T_G})\vartheta^a = 0. \quad (73)$$

The set ϑ_a forms a basis in the subspace of $D-1$ forms, therefore H^{ab}, h^a can be expressed in components as

$$H^{ab} = H^{abc}\vartheta_c, \qquad h^a = h^{ab}\vartheta_b. \quad (74)$$

Hence, the equations of motion (73) for $F(T, T_G)$ gravity is written in components as

$$2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})_{,c} + 2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})C^d{}_{dc} + (2H^{[ac]d} + H^{dca})C^b{}_{cd} + 4H^{[db]c}C_{(dc)}{}^a + (T^a{}_{cd} + 2\omega^a{}_{[cd]})H^{cdb} - (-1)^D h^{ab} + (F - TF_T - T_GF_{T_G})\eta^{ab} = 0. \quad (75)$$

Focusing to the most interesting case of four dimensions we can rewrite the expressions for H^{ab} and h_a as

$$H^{ab} = \frac{F_T}{2}\epsilon^a{}_{cdf}\mathcal{K}^{bc}e^d e^f - \epsilon^a{}_{cdf}D(F_{T_G}\mathcal{K}^{bc}\mathcal{K}^{df}) + F_{T_G}(2\epsilon^a{}_{cdf}\mathcal{K}^{bc}\mathcal{K}^d{}_q\mathcal{K}^{qf} + \epsilon_{cdfq}\mathcal{K}^{ac}\mathcal{K}^{bd}\mathcal{K}^{fq} - \epsilon^{ab}{}_{cd}\mathcal{K}^c{}_f\mathcal{K}^f{}_q\mathcal{K}^{qd} + \epsilon^{ab}{}_{cd}D\mathcal{K}^c{}_f\mathcal{K}^{fd} + \epsilon^{ac}{}_{df}D\mathcal{K}^b{}_c\mathcal{K}^{df}) \tag{76}$$

and

$$h_a = -F_T\epsilon_{abcd}\mathcal{K}^b{}_f\mathcal{K}^{fc}e^d. \tag{77}$$

Since $e^a \wedge e^b \wedge e^c = \epsilon^{abcd}\vartheta_d$, $e^a \wedge e^b = -\frac{1}{2}\epsilon^{abcd}\vartheta_{cd}$, where $\vartheta_{ab} = i_{e_b}\vartheta_a$, the above expressions become

$$\begin{aligned} H^{abc} &= F_T(\eta^{ac}\mathcal{K}^{bd}{}_d - \mathcal{K}^{bca}) + F_{T_G}\bigg[\epsilon^{cprt}(2\epsilon^a{}_{dkf}\mathcal{K}^{bk}{}_p\mathcal{K}^d{}_{qr} + \epsilon_{qdkf}\mathcal{K}^{ak}{}_p\mathcal{K}^{bd}{}_r + \epsilon^{ab}{}_{kf}\mathcal{K}^k{}_{dp}\mathcal{K}^d{}_{qr})\mathcal{K}^{qf}{}_t \\ &\quad + \epsilon^{cprt}\epsilon^{ab}{}_{kd}\mathcal{K}^{fd}{}_p\left(\mathcal{K}^k{}_{fr,t} - \frac{1}{2}\mathcal{K}^k{}_{fq}C^q{}_{tr} + \omega^k{}_{qt}\mathcal{K}^q{}_{fr} + \omega^q{}_{fr}\mathcal{K}^k{}_{qt}\right) \\ &\quad + \epsilon^{cprt}\epsilon^{ak}{}_{df}\mathcal{K}^{df}{}_p\left(\mathcal{K}^b{}_{kr,t} - \frac{1}{2}\mathcal{K}^b{}_{kq}C^q{}_{tr} + \omega^b{}_{qt}\mathcal{K}^q{}_{kr} + \omega^q{}_{kr}\mathcal{K}^b{}_{qt}\right)\bigg] \\ &\quad + F_{T_G}\epsilon^{cprt}\epsilon^a{}_{kdf}\bigg[\frac{1}{F_{T_G}}(F_{T_G}\mathcal{K}^{bk}{}_p\mathcal{K}^{df}{}_r)_{,t} + C^q{}_{pt}\mathcal{K}^{bk}{}_{[q}\mathcal{K}^{df}{}_{r]} + (\omega^b{}_{qp}\mathcal{K}^{qk}{}_r + \omega^k{}_{qp}\mathcal{K}^{bq}{}_r)\mathcal{K}^{df}{}_t \\ &\quad + (\omega^d{}_{qp}\mathcal{K}^{qf}{}_t + \omega^f{}_{qp}\mathcal{K}^{dq}{}_t)\mathcal{K}^{bk}{}_r\bigg] \end{aligned} \tag{78}$$

$$h^{ab} = F_T\epsilon^a{}_{kcd}\epsilon^{bpqd}\mathcal{K}^k{}_{fp}\mathcal{K}^{fc}{}_q. \tag{79}$$

Choosing additionally the Weitzenböck connection $\omega^a{}_{bc} = 0$, for $D = 4$ we finally obtain

$$2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})_{,c} + 2(H^{[ac]b} + H^{[ba]c} - H^{[cb]a})C^d{}_{dc} + (2H^{[ac]d} + H^{dca})C^b{}_{cd} + 4H^{[db]c}C_{(dc)}{}^a + T^a{}_{cd}H^{cdb} - h^{ab} + (F - TF_T - T_GF_{T_G})\eta^{ab} = 0, \tag{80}$$

where

$$\begin{aligned} H^{abc} &= F_T(\eta^{ac}\mathcal{K}^{bd}{}_d - \mathcal{K}^{bca}) + F_{T_G}\bigg[\epsilon^{cprt}(2\epsilon^a{}_{dkf}\mathcal{K}^{bk}p\mathcal{K}^d{}_{qr} + \epsilon_{qdkf}\mathcal{K}^{ak}{}_p\mathcal{K}^{bd}{}_r + \epsilon^{ab}{}_{kf}\mathcal{K}^k{}_{dp}\mathcal{K}^d{}_{qr})\mathcal{K}^{qf}{}_t \\ &\quad + \epsilon^{cprt}\epsilon^{ab}{}_{kd}\mathcal{K}^{fd}{}_p\left(\mathcal{K}^f{}_{fr,t} - \frac{1}{2}\mathcal{K}^k{}_{fq}C^q{}_{tr}\right) + \epsilon^{cprt}\epsilon^{ak}{}_{df}\mathcal{K}^{df}{}_p\left(\mathcal{K}^b{}_{kr,t} - \frac{1}{2}\mathcal{K}^b{}_{kq}C^q{}_{tr}\right)\bigg] \\ &\quad + \epsilon^{cprt}\epsilon^a{}_{kdf}\Big[(F_{T_G}\mathcal{K}^{bk}{}_p\mathcal{K}^{df}{}_r)_{,t} + F_{T_G}C^q{}_{pt}\mathcal{K}^{bk}{}_{[q}\mathcal{K}^{df}{}_{r]}\Big] \end{aligned} \tag{81}$$

and

$$h^{ab} = F_T\epsilon^a{}_{kcd}\epsilon^{bpqd}\mathcal{K}^k{}_{fp}\mathcal{K}^{fc}{}_q. \tag{82}$$

Equations (80) are the equations of motion for $F(T, T_G)$ gravity in four dimensions, for a general vielbein (or equivalently for a general metric) choice. For specific cases, such as the homogeneous and isotropic Friedmann-Robertson-Walker and the spherically symmetric geometries, the above equations are significantly simplified. Thus, one can straightforwardly investigate the application of $F(T, T_G)$ gravity in a cosmological framework. Since this study lies beyond the scope of the present work, it is left for a separate project [25].

V. CONCLUSIONS

Inspired by the teleparallel formulation of general relativity, whose Lagrangian is the torsion invariant T, we have constructed the teleparallel equivalent of Gauss-Bonnet gravity in arbitrary dimensions. Implementing the teleparallel condition, but without imposing the Weitzenböck connection, we have extracted the torsion invariant T_G, equivalent (up to boundary terms) to the Gauss-Bonnet term G. T_G is made out of the vielbein e^a and the connection $\omega^a{}_b$, it contains quartic powers of the torsion tensor, and it is diffeomorphism and Lorentz invariant. In four dimensions it reduces to a topological invariant, as expected. Imposing the Weitzenböck connection, a simpler form for T_G arises containing only the vielbein. This allows us to define a new class of modified gravity theories based on $F(T, T_G)$, which is not spanned by the class of $F(T)$ theories. Moreover, it is also distinct from the $F(R, G)$ class. Hence, $F(T, T_G)$ theory is a novel class of modified gravity. Finally, varying the action with respect to the vielbein, we extracted the equations of motion for a general vielbein (metric) choice. Since $F(T, T_G)$ gravity is a new modified gravitational theory, it would be interesting to study its cosmological applications, and this is performed in a separate publication [25].

ACKNOWLEDGMENTS

The research of E.N.S. is implemented within the framework of the Operational Program "Education and Lifelong Learning" (Actions Beneficiary: General Secretariat for Research and Technology), and is cofinanced by the European Social Fund (ESF) and the Greek State.

[1] S. Weinberg, *Cosmology* (Oxford University Press Inc., New York, 2008).
[2] A. Einstein, Sitzungsber. Preuss. Akad. Wiss. Phys. Math. Kl., 217 (1928); 224 (1928); A. Unzicker and T. Case, arXiv:physics/0503046.
[3] A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).
[4] S. 'i. Nojiri and S. D. Odintsov, Phys. Rep. **505**, 59 (2011).
[5] D. Boulware and S. Deser, Phys. Rev. Lett. **55**, 2656 (1985); J. T. Wheeler, Nucl. Phys. **B268**, 737 (1986).
[6] I. Antoniadis, J. Rizos, and K. Tamvakis, Nucl. Phys. **B415**, 497 (1994); P. Kanti, J. Rizos, and K. Tamvakis, Phys. Rev. D **59**, 083512 (1999); N. E. Mavromatos and J. Rizos, Phys. Rev. D **62**, 124004 (2000); S. 'i. Nojiri, S. D. Odintsov, and M. Sasaki, Phys. Rev. D **71**, 123509 (2005).
[7] S. 'i. Nojiri and S. D. Odintsov, Phys. Lett. B **631**, 1 (2005).
[8] A. De Felice and S. Tsujikawa, Phys. Lett. B **675**, 1 (2009).
[9] D. Lovelock, J. Math. Phys. (N.Y.) **12**, 498 (1971).
[10] N. Deruelle and L. Farina-Busto, Phys. Rev. D **41**, 3696 (1990).
[11] P. D. Mannheim and D. Kazanas, Astrophys. J. **342**, 635 (1989); E. E. Flanagan, Phys. Rev. D **74**, 023002 (2006); N. Deruelle, M. Sasaki, Y. Sendouda, and A. Youssef, J. Cosmol. Astropart. Phys. 03 (2011) 040; D. Grumiller, M. Irakleidou, I. Lovrekovic, and R. McNees, Phys. Rev. Lett. **112**, 111102 (2014).
[12] P. Horava, J. High Energy Phys. 03 (2009) 020; Phys. Rev. D **79**, 084008 (2009); G. Calcagni, J. High Energy Phys. 09 (2009) 112; E. Kiritsis and G. Kofinas, Nucl. Phys. **B821**, 467 (2009); E. N. Saridakis, Eur. Phys. J. C **67**, 229 (2010).
[13] R. Ferraro and F. Fiorini, Phys. Rev. D **75**, 084031 (2007); **78**, 124019 (2008); G. R. Bengochea and R. Ferraro, Phys. Rev. D **79**, 124019 (2009).
[14] E. V. Linder, Phys. Rev. D **81**, 127301 (2010).
[15] S. H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, Phys. Rev. D **83**, 023508 (2011); J. B. Dent, S. Dutta, and E. N. Saridakis, J. Cosmol. Astropart. Phys. 01 (2011) 009; T. P. Sotiriou, B. Li, and J. D. Barrow, Phys. Rev. D **83**, 104030 (2011); Y. Zhang, H. Li, Y. Gong, and Z.-H. Zhu, J. Cosmol. Astropart. Phys. 07 (2011) 015; Y.-F. Cai, S.-H. Chen, J. B. Dent, S. Dutta, and E. N. Saridakis, Classical Quantum Gravity **28**, 215011 (2011); M. Sharif and S. Rani, Mod. Phys. Lett. A **26**, 1657 (2011); S. Capozziello, V. F. Cardone, H. Farajollahi, and A. Ravanpak, Phys. Rev. D **84**, 043527 (2011); C.-Q. Geng, C.-C. Lee, E. N. Saridakis, and Y.-P. Wu, Phys. Lett. B **704**, 384 (2011); H. Wei, Phys. Lett. B **712**, 430 (2012); C.-Q. Geng, C.-C. Lee, and E. N. Saridakis, J. Cosmol. Astropart. Phys. 01 (2012) 002; Y.-P. Wu and C.-Q. Geng, Phys. Rev. D **86**, 104058 (2012); C. G. Bohmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **85**, 044033 (2012); K. Karami and A. Abdolmaleki, J. Cosmol. Astropart. Phys. 04 (2012) 007; C. Xu, E. N. Saridakis, and G. Leon, J. Cosmol. Astropart. Phys. 07 (2012) 005; K. Bamba, R. Myrzakulov, S. 'i. Nojiri, and S. D. Odintsov, Phys. Rev. D **85**, 104036 (2012); N. Tamanini and C. G. Boehmer, Phys. Rev. D **86**, 044009 (2012); M. E. Rodrigues, M. J. S. Houndjo, D. Saez-Gomez, and F. Rahaman, Phys. Rev. D **86**, 104059 (2012); K. Bamba, J. de Haro, and S. D. Odintsov, J. Cosmol. Astropart. Phys. 02 (2013) 008; M. Jamil, D. Momeni, and R. Myrzakulov, Eur. Phys. J. C **73**, 2267 (2013); J.-T. Li, C.-C. Lee, and C.-Q. Geng, Eur. Phys. J. C **73**, 2315 (2013); A. Aviles, A. Bravetti, S. Capozziello, and O. Luongo, Phys. Rev. D **87**, 064025 (2013); Y. C. Ong, K. Izumi, J. M. Nester, and P. Chen, Phys. Rev. D **88**, 024019 (2013); K. Bamba, S. 'i. Nojiri, and S. D. Odintsov, Phys. Lett. B **725**, 368 (2013); G. Otalora, J. Cosmol. Astropart. Phys. 07 (2013) 044; J. Amoros, J. de Haro, and S. D. Odintsov, Phys. Rev. D **87**, 104037 (2013); G. Otalora, Phys. Rev. D **88**, 063505 (2013); C.-Q. Geng, J.-A. Gu, and C.-C. Lee, Phys. Rev. D **88**, 024030 (2013); M. E. Rodrigues, I. G. Salako, M. J. S. Houndjo, and J. Tossa, Int. J. Mod. Phys. D **23**, 1450004 (2014); K. Bamba, S. Capozziello, M. De Laurentis, S. 'i. Nojiri, and D. Sez-Gmez, Phys. Lett. B **727**, 194 (2013); K. Bamba, S. 'i. Nojiri, and S. D. Odintsov, Phys. Lett. B **731**, 257 (2014); G. Otalora, arXiv:1402.2256; A. Paliathanasis, S. Basilakos, E. N. Saridakis, S. Capozziello, K. Atazadeh, F. Darabi, and M. Tsamparlis, Phys. Rev. D **89**, 104042 (2014); G. G. L. Nashed, arXiv:1403.6937; S. Chattopadhyay, Eur. Phys. J. Plus **129**, 82 (2014); V. Fayaz, H. Hossienkhani, A. Farmany, M. Amirabadi, and N. Azimi, Astrophys. Space Sci. **351**, 299 (2014); T. Harko, F. S. N. Lobo, G. Otalora, and E. N. Saridakis, Phys. Rev. D **89**, 124036 (2014); arXiv:1405.0519.
[16] P. Wu and H. W. Yu, Phys. Lett. B **693**, 415 (2010); G. R. Bengochea, Phys. Lett. B **695**, 405 (2011); L. Iorio and

E. N. Saridakis, Mon. Not. R. Astron. Soc. **427**, 1555 (2012); S. Nesseris, S. Basilakos, E. N. Saridakis, and L. Perivolaropoulos, Phys. Rev. D **88**, 103010 (2013).
[17] T. Wang, Phys. Rev. D **84**, 024042 (2011); R.-X. Miao, M. Li, and Y.-G. Miao, J. Cosmol. Astropart. Phys. 11 (2011) 033; C. G. Boehmer, A. Mussa, and N. Tamanini, Classical Quantum Gravity **28**, 245020 (2011); R. Ferraro and F. Fiorini, Phys. Rev. D **84**, 083518 (2011); M. H. Daouda, M. E. Rodrigues, and M. J. S. Houndjo, Eur. Phys. J. C **71**, 1817 (2011); **72**, 1890 (2012); P. A. Gonzalez, E. N. Saridakis, and Y. Vasquez, J. High Energy Phys. 07 (2012) 053; S. Capozziello, P. A. Gonzalez, E. N. Saridakis, and Y. Vasquez, J. High Energy Phys. 02 (2013) 039; K. Atazadeh and M. Mousavi, Eur. Phys. J. C **72**, 2016 (2012).
[18] R. Aldrovandi and J. G. Pereira, *Teleparallel Gravity: An Introduction* (Springer, Dordrecht, 2013).
[19] J. W. Maluf, J. Math. Phys. (N.Y.) **35**, 335 (1994); H. I. Arcos and J. G. Pereira, Int. J. Mod. Phys. D **13**, 2193 (2004).
[20] J. W. Maluf, Ann. Phys. **525**, 339 (2013).
[21] L. D. Landau and E. M. Lifschitz, *The Classical Theory of Fields* (Addison-Wesley, Reading, MA, 1971).
[22] S. C. Davis, arXiv:0709.4453; B. Eynard and N. Orantin, arXiv:0705.0958; A. De Felice and S. Tsujikawa, Phys. Rev. D **80**, 063516 (2009); A. Jawad, S. Chattopadhyay, and A. Pasqua, Eur. Phys. J. Plus **128**, 88 (2013).
[23] F. Mueller-Hoissen and J. Nitsch, Phys. Rev. D **28**, 718 (1983).
[24] A. Mardones and J. Zanelli, Classical Quantum Gravity **8**, 1545 (1991la); O. Chandia and J. Zanelli, Phys. Rev. D **55**, 7580 (1997).
[25] G. Kofinas and E. N. Saridakis (to be published).