Black holes as self-sustained quantum states and Hawking radiation

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We employ the recently proposed formalism of the "horizon wave function" to investigate the emergence of a horizon in models of black holes as Bose-Einstein condensates of gravitons. We start from the Klein-Gordon equation for a massless scalar (toy graviton) field coupled to a static matter current. The (spherically symmetric) classical field reproduces the Newtonian potential generated by the matter source, and the corresponding quantum state is given by a coherent superposition of scalar modes with continuous occupation number. Assuming an attractive self-interaction that allows for bound states, one finds that (approximately) only one mode is allowed, and the system can be confined in a region the size of the Schwarzschild radius. This radius is then shown to correspond to a proper horizon, by means of the horizon wave function of the quantum system, with an uncertainty in size naturally related to the expected typical energy of Hawking modes. In particular, this uncertainty decreases for larger black hole mass (with a larger number of light scalar quanta), in agreement with semiclassical expectations, a result which does not hold for a single very massive particle. We finally speculate that a phase transition should occur during the gravitational collapse of a star (ideally represented by a static matter current and Newtonian potential) that leads to a black hole (again ideally represented by the condensate of toy gravitons), and suggest an effective order parameter that could be used to investigate this transition.

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I. INTRODUCTION

Recent works by Dvali and Gomez have offered a new perspective on the quantum aspects of black hole physics [1], and are drawing more and more attention [2–9]. The idea is very simple: a black hole can be modeled as a Bose-Einstein condensate (BEC) of gravitons interacting with one another. Such gravitons superpose in a single small region of space, effectively giving rise to a gravitational well, whose depth is proportional to the total number of gravitons present. Since no other (matter) constituents appear in the model, we could view it as a description of "purely gravitational" black holes, or the approximation of the final state of gravitational collapse in which the initial matter contribution has become subdominant with respect to the gravitons themselves (in agreement with the huge gravitational entropy predicted by Bekenstein for astrophysical black holes [10]). One point which remains unclear in Ref. [1] is whether these systems in fact display a horizon, or trapping surface, as one would expect in a "standard" black hole space-time.

Before we delve into this point, let us briefly review the main argument in Ref. [1]. Note that we shall mostly use units with $c = 1$, the Newton constant $G_\mathrm{N} = \ell_\mathrm{p}/m_\mathrm{p}$ (where ℓ_p and m_p are the Planck length and mass, respectively), and $\hbar = \ell_\mathrm{p} m_\mathrm{p}$. These units make it apparent that G_N converts mass into length, thus providing a natural link between energy and size which we shall assume also holds at the quantum level [11]. In the Newtonian approximation, we can assume that a system of N gravitons has total energy $M = Nm$, and that each graviton interacts with the others via the potential

$$V_\mathrm{N} \simeq -\frac{G_\mathrm{N} M}{r} = -\frac{\ell_\mathrm{p} N m}{r m_\mathrm{p}}, \qquad (1.1)$$

where the effective graviton mass m is related to the characteristic quantum-mechanical size via the Compton/de Broglie wavelength,

$$\lambda_m \simeq \frac{\hbar}{m} = \ell_\mathrm{p} \frac{m_\mathrm{p}}{m}. \qquad (1.2)$$

In fact, since gravitons can superpose, one expects them to give rise to a ball of characteristic radius $r \simeq \lambda_m$ for sufficiently large N. For the following argument, it is then sufficient to assume that the potential (1.1) becomes negligible for $r \gtrsim \lambda_m$ (for improved approximations, see Refs. [5,9]), so that the potential energy of each graviton interacting with the remaining $N - 1$ gravitons is given by

$$U_m(r) \simeq m V_\mathrm{N}(\lambda_m) = -N \frac{\alpha\hbar}{\lambda_m} \Theta(\lambda_m - r), \qquad (1.3)$$

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where

$$\alpha = \frac{\ell_p^2}{\lambda_m^2} = \frac{m^2}{m_p^2} \quad (1.4)$$

is the effective gravitational coupling constant and Θ is the Heaviside step function. One can now see that there exist values of N such that the system is a black hole. This happens when the kinetic energy $E_K \simeq m$ of each graviton is just below the amount needed to escape the potential well, which yields the marginally bound condition

$$E_K + U_m \simeq 0, \quad (1.5)$$

equivalent to the "maximal packing"

$$N\alpha = 1. \quad (1.6)$$

From that point on, the "blob" of gravitons becomes a self-confined object, whose effective boson mass and total mass scale according to[1]

$$m \simeq \frac{m_p}{\sqrt{N}}, \quad (1.7)$$

$$M \simeq Nm \simeq \sqrt{N} m_p. \quad (1.8)$$

Boson excitations can further lead to quantum depletion of the condensate out of the ground state. Such "leaking" of gravitons can be interpreted (at least in a first-order approximation) as the emission of Hawking radiation. This kind of toy model is very intuitive and also gives an elegant quantum-mechanical description of black holes in terms of the graviton number N. However, as we already mentioned, it leaves open the question of whether the causal structure of space-time indeed contains a trapping surface.

We already noted that, in this model of "purely gravitational" black holes, only gravitons are considered and there is no trace of (nor, apparently, need of considering) the matter that initially collapsed and formed the black hole. However, it is clear that, unless the black hole originated from a primordial quantum fluctuation of the vacuum in the very early stages of the Universe, the only known mechanism that could possibly lead to such a final state is the gravitational collapse of a star or other astrophysical source. The question then arises naturally of whether neglecting the role of regular matter in the final state is a reliable approximation. One could rather argue that matter always matters, and the final state of gravitational collapse is not a "pure" black hole like the one in Ref. [1] if it is a black hole (in the strict general-relativistic sense) at all. This is the question one would eventually like to answer, although it appears we are still quite far from that.

In this work, we shall first review the picture and scaling relations (1.7) and (1.8) as proposed in Ref. [1] starting from the relativistic field equation for scalar gravitons, but without assuming any specific form for the necessary binding potential. We shall start from the classical solution ϕ_c of the Klein-Gordon equation for a massless scalar field coupled to a static and spherically symmetric matter source J. Such a classical solution is reproduced, in the quantum theory, by a coherent state obtained by superposing modes belonging to a whole range of momenta $k > 0$. However, if one assumes the source J is determined by the scalar field itself inside a finite spatial volume,[2] (namely, if $J \sim \phi_c$), one finds that (roughly) only one mode $k_c^{-1} \sim M$ is allowed. According to the corpuscular model of Ref. [1], this means the quantum coherent state, representing the Newtonian potential of a star, must have collapsed into a macroscopic quantum object made of a large number N of bosons in the same mode k_c. At this point, we shall finally be able to tackle the main issue of investigating the presence of trapping surfaces by means of the formalism of the horizon wave function [11,16,17] for such quantum states in the mode k_c. Our conclusion will be that there is indeed a horizon and that it is of the expected classical size, for large N. We shall also consider a few generalizations of this state —with diverse forms of "quantum hair"—that could possibly be used to model the Hawking radiation or the approach of collapsing matter towards the horizon. We shall then see that the uncertainty in the horizon radius always turns out to be naturally related to the existence of leaking (or Hawking) modes [18], and is thus determined by the Hawking temperature. For larger black holes, this uncertainty in the horizon size clearly decreases, in agreement with semiclassical expectations. It is then important to highlight that a similar result does not hold if one tries to describe a black hole as a single very massive particle (that is, a system with $N = 1$ and $m \gg m_p$), since in the latter case this uncertainty remains of the order of the horizon size itself, regardless of the value of the black hole mass [16].

In Sec. II, we first review the classical solutions of the Klein-Gordon equation for a static source (which is equivalent to the Poisson equation for Newtonian gravity) and the coherent states that reproduce the classical Newtonian potential. We next sketch how the self-sustained state can generically arise in this context. The causal structure of the latter configuration is then analyzed in

[1] There is a large amount of literature on self-gravitating bosons in general relativity, which address the problem of gravitational stability and black hole formation, and for which no analytical solution is available, but where similar scaling relations can be found (see, e.g., Refs. [12–14]).

[2] Let us remark that this self-sourcing condition appears as a crucial aspect in the "classicalization" of gravity; see Refs. [3,15]. Of course, the existence of bound states localized inside a finite volume requires an attractive self-interaction, of the kind one expects for gravity [1], and an example of this was studied in Ref. [7].

Sec. III by means of the horizon wave function of the system, obtained from the spectral decomposition of a few different quantum states of N bosons distributed around the ground state k_c. One case in particular is discussed in which the distribution is thermal, as is expected for Hawking radiation. We conclude with several comments and speculations in Sec. IV.

II. MASSLESS SCALAR FIELD TOY MODEL

We start by reviewing well-known general features of Newtonian gravity and of the corpuscular model of black holes of Ref. [1] by means of a toy scalar field. This introductory material will allow us to highlight some of the main differences between a (Newtonian) star and a black hole, and provide us with an approximate quantum state for the subsequent analysis of the causal structure of space-time in Sec. III.

Let us consider the Klein-Gordon equation for a real and massless scalar field ϕ coupled to a scalar current J in Minkowski space-time,

$$\Box\phi(x) = qJ(x), \tag{2.1}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, the scalar field has the standard canonical dimension of length^{-1}, and the coupling q is dimensionless for simplicity (appropriate dimensional factors will be introduced in the final expressions). We shall also assume that the current is time-independent, $\partial_0 J = 0$. In momentum space, with $k^\mu = (k^0, \mathbf{k})$, this implies that

$$k^0\tilde{J}(k^\mu) = 0, \tag{2.2}$$

which is solved by the distribution

$$\tilde{J}(k^\mu) = 2\pi\delta(k^0)\tilde{J}(\mathbf{k}), \tag{2.3}$$

where $\tilde{J}^*(\mathbf{k}) = \tilde{J}(-\mathbf{k})$. For the spatial part, we further assume exact spherical symmetry, so that our analysis is restricted to functions $f(\mathbf{x}) = f(r)$, with $r = |\mathbf{x}|$. We can then introduce functions in momentum space according to

$$\tilde{f}(k) = 4\pi\int_0^{+\infty} dr r^2 j_0(kr) f(r), \tag{2.4}$$

where

$$j_0(kr) = \frac{\sin(kr)}{kr} \tag{2.5}$$

is a spherical Bessel function of the first kind and $k = |\mathbf{k}|$.

A. Classical solutions

Classical spherically symmetric solutions of Eq. (2.1) can be formally written as

$$\phi_c(r) = q\Box^{-1}J(r), \tag{2.6}$$

and are given in momentum space by

$$\tilde{\phi}_c(k) = -q\frac{\tilde{J}(k)}{k^2}. \tag{2.7}$$

For example, if the current has Gaussian support,

$$J(r) = \frac{e^{-r^2/(2\sigma^2)}}{(2\pi\sigma^2)^{3/2}}, \tag{2.8}$$

so that

$$\tilde{J}(k) = e^{-k^2\sigma^2/2}, \tag{2.9}$$

the corresponding classical scalar solution is given by

$$\begin{aligned}\phi_c(r) &= -\frac{q}{2\pi^2}\int_0^{+\infty} dk j_0(kr) e^{-k^2\sigma^2/2} \\ &= -\frac{q}{4\pi r}\operatorname{erf}\left(\frac{r}{\sqrt{2}\sigma}\right).\end{aligned} \tag{2.10}$$

The above expression outside the source J, or for $r \gg \sigma$, reproduces the classical Newtonian potential (1.1), namely

$$V_N = \frac{4\pi}{q}G_N M\phi_c \simeq -\frac{G_N M}{r}, \tag{2.11}$$

where we introduced the suitably dimensioned factor.

B. Quantum coherent states

In the quantum theory, the classical configurations (2.6) are replaced by coherent states. This can be easily seen from the normal-ordered quantum Hamiltonian density in momentum space,

$$\hat{\mathcal{H}} = k\hat{a}_k^{\prime\dagger}\hat{a}'_k + \tilde{\mathcal{H}}_g, \tag{2.12}$$

where $\tilde{\mathcal{H}}_g$ is the ground-state energy density,

$$\tilde{\mathcal{H}}_g = -q^2\frac{|\tilde{J}(k)|^2}{2k^2}, \tag{2.13}$$

and we shifted the standard ladder operators according to

$$\hat{a}'_k = \hat{a}_k + q\frac{\tilde{J}(k)}{\sqrt{2k^3}}. \tag{2.14}$$

The source-dependent ground state $|g\rangle$ is annihilated by the shifted annihilation operator,

$$\hat{a}'_k|g\rangle = 0, \tag{2.15}$$

and is a coherent state in terms of the standard field vacuum,

$$\hat{a}_k|g\rangle = -q\frac{\tilde{J}(k)}{\sqrt{2k^3}}|g\rangle = g(k)|g\rangle, \tag{2.16}$$

where $g = g(k)$ is thus an eigenvalue of the shifted annihilation operator. This implies

$$|g\rangle = e^{-N/2}\exp\left\{\int\frac{k^2 dk}{2\pi^2}g(k)\hat{a}_k^\dagger\right\}|0\rangle, \tag{2.17}$$

where N denotes the expectation value of the number of quanta in the coherent state,

$$\begin{aligned} N &= \int\frac{k^2 dk}{2\pi^2}\langle g|\hat{a}_k^\dagger\hat{a}_k|g\rangle \\ &= \int\frac{k^2 dk}{2\pi^2}|g(k)|^2 \\ &= \frac{q^2}{(2\pi)^2}\int\frac{dk}{k}|\tilde{J}(k)|^2, \end{aligned} \tag{2.18}$$

from which we can read off the occupation number

$$n_k = \left(\frac{q}{2\pi}\right)^2\frac{|\tilde{J}(k)|^2}{k}. \tag{2.19}$$

It is straightforward to verify that the expectation value of the field in the state $|g\rangle$ coincides with its classical value,

$$\begin{aligned} \langle g|\hat{\phi}_k|g\rangle &= \frac{1}{\sqrt{2k}}\langle g|(\hat{a}_k + \hat{a}_{-k}^\dagger)|g\rangle \\ &= \frac{1}{\sqrt{2k}}\langle g|(\hat{a}'_k + \hat{a}'^\dagger_{-k})|g\rangle - q\frac{\tilde{J}(k)}{k^2} \\ &= \tilde{\phi}_c(k), \end{aligned} \tag{2.20}$$

and thus $|g\rangle$ is a realization of the Ehrenfest theorem.

It is interesting to recall that the state $|g\rangle$ and the number N are not mathematically well defined in general: Eq. (2.18) is UV divergent if the source has infinitely thin support, and IR divergent if the source contains modes of vanishing momenta (which would only be physically consistent with an eternal source). The UV issue can be cured, for example, by using a Gaussian distribution like the one in Eq. (2.8), while the IR divergence can be naturally eliminated if the scalar field is massive or the system is enclosed within a finite volume (so that allowed modes are also quantized). These details are however of little importance for what follows and we shall therefore not indulge in them here.

C. Stars and self-sustained scalar states

The system we have considered so far could represent a "star," that is, a classical lump of ordinary matter with density

$$\rho = MJ, \tag{2.21}$$

where M is the total (proper) energy of the star. The Newtonian potential energy for this star is of course given by $U_M = MV_\mathrm{N}$, so that ϕ_c is accordingly determined by Eq. (2.11) and the quantum state of $\hat{\phi}$ by Eq. (2.20). Note that, for a very large mass M, the scalar gravitons in the coherent state $|g\rangle$ are mostly found with an energy around $m \sim U_M/N$ and their total number is $N \sim M^2$, in agreement with Eq. (1.8) [1]. It is interesting to note that this scaling relation for the total mass of the self-gravitating system holds both for a star and in the black hole regime, whereas Eq. (1.7) for the graviton's energy holds *only* in the black hole regime (since the total Newtonian potential energy $U_M \simeq Nm \ll M$ for a regular star).

Let us instead assume that there exists a regime in which the matter contribution is negligible, and the source J on the rhs of Eq. (2.1) is thus provided by the gravitons themselves [1], (at least) inside a finite spatial volume $\mathcal{V}$. As we mentioned in the Introduction, this confinement is a crucial feature for the "classicalization" of gravity [15], and necessarily requires an attractive self-interaction for the scalar field to admit bound states. Of course, one expects this to happen for full-fledged gravity and, in the semiclassical regime, to lead to a curved space-time metric (inside the volume $\mathcal{V}$). The self-interaction, at least in this regime, could then be effectively described by a modified d'Alembertian in Eq. (2.1). Since the details of the (otherwise necessary) confining mechanism are not relevant for our analysis, we can just assume that the momentum modes in Eq. (2.5) are replaced by those in the appropriate curved space-time and that Eq. (2.7) in momentum space still holds. In other words, we just require that the energy density (2.21) sourcing the evolution of each scalar is equal to minus the average total potential energy $NU_m/\mathcal{V}$,[3] where $\mathcal{V} = 4\pi R^3/3$ is the spatial volume where this source has support. This is just the same as the marginally bound condition (1.5) used in Ref. [1] with $NE_\mathrm{K} \sim J$, and it implies that

$$J \simeq -\frac{3NG_\mathrm{N}m}{qR^3}\phi_c \tag{2.22}$$

inside the volume $\mathcal{V}$, where m is again the energy of each scalar graviton and N is their total number. Upon

[3] The total potential energy of each graviton is proportional to $(N-1)U_m$, but since we are interested in the large-N case, we approximate $N - 1 \simeq N$.

inserting this condition into Eq. (2.7), we straightforwardly obtain

$$\frac{3NG_\mathrm{N}m}{R^3k^2} = \frac{3R_\mathrm{H}}{2R^3k^2} \simeq 1, \tag{2.23}$$

where

$$R_\mathrm{H} = 2G_\mathrm{N}M \tag{2.24}$$

is the classical Schwarzschild radius associated with the total mass $M = Nm$. This suggests that, unlike the Newtonian potential generated by an ordinary matter source, a self-sustained system should contain only the modes with momentum numbers $k = k_\mathrm{c}$ such that

$$Rk_\mathrm{c} \simeq \sqrt{\frac{R_\mathrm{H}}{R}}, \tag{2.25}$$

where we dropped a numerical coefficient of order one given the qualitative nature of our analysis. Ideally, this means that, if the scalar gravitons represent the main gravitating source, the quantum state of the system must be given in terms of just one mode ϕ_{k_c}. A coherent state of the form in Eq. (2.17) cannot thus be built, and the relation (2.18) between N and the source momenta does not apply here. Instead, for $N \gg 1$, all scalars will be in the same state $|k_\mathrm{c}\rangle$.

For an ordinary star, the typical size $R \gg R_\mathrm{H}$ and $k_\mathrm{c} \ll R^{-1}$. The corresponding de Broglie wavelength $\lambda_\mathrm{c} \simeq k_\mathrm{c}^{-1} \gg R$, which would conflict with our assumption that the field is identified as a gravitating source only within a region of size R. However, if we consider the "black hole limit" in which $R \sim R_\mathrm{H}$, and recalling that $m = \hbar k$, we immediately obtain (again, dropping a numerical coefficient of order one)

$$1 \simeq G_\mathrm{N}Mk_\mathrm{c} = N\frac{m^2}{m_\mathrm{p}^2}, \tag{2.26}$$

which leads to the two scaling relations (1.7) and (1.8), namely $m = \hbar k_\mathrm{c} \simeq m_\mathrm{p}/\sqrt{N}$ and a consistent de Broglie wavelength $\lambda_m \simeq \lambda_\mathrm{c} \simeq R_\mathrm{H}$. An ideal system of self-sustained scalars should then be in the quantum state $|k_\mathrm{c}\rangle$[4] with a spatial size that suggests that the system is a black hole [1]. In order to substantiate this last part of the sentence, one should however show that there is a horizon, or at least a trapping surface, in the given space-time.

The standard procedure to show the existence of trapping surfaces requires knowing explicitly the metric that solves the semiclassical Einstein field equations with the prescribed source, the latter being represented by the expectation value of the appropriate stress-energy tensor. One therefore also needs the explicit form of the momentum modes that we implicitly assumed lead to Eq. (2.7). However, the problem of self-gravitating scalar fields in general relativity has been known for decades [12] and analytical solutions have yet to be found [13,14]. Moreover, it is not *a priori* guaranteed that such a semiclassical approach remains valid for the type of quantum sources we are dealing with, since the quantum fluctuations of the source could be large enough to spoil the possibility of employing a curved background geometry to describe the region where the source is located.[5] In fact, one expects that this standard approach in general holds at large distances from the source, and yields the proper (post-)Newtonian approximation [1] in a region far from where trapping surfaces are likely to be located. This is precisely the reason a formalism for describing the gravitational radius of any quantum system was introduced in Refs. [11,16,17], as we shall review shortly.

Before we do so, let us remark that the above argument leading to Eq. (2.26) does not require that the scalar field ϕ vanish (or be negligible) outside the region of radius R_H. This would in fact imply that there is no outer Newtonian potential, which is in contrast with the idea of a gravitational source. All we need in order to recover the classical outer Newtonian potential $V_\mathrm{N} \sim \phi$ is to relax the condition (2.22) for $r \gtrsim R_\mathrm{H}$ (where $J \simeq 0$), and properly match the (expectation) value of $\hat{\phi}$ with the Newtonian ϕ_c from Eq. (2.11) at $r \gtrsim R_\mathrm{H}$. In this matching procedure, at least for $r \gg R_\mathrm{H}$ and $N \gg 1$, one then expects to recover the classical description, for which the only relevant information is the total mass M of the black hole. This can already be seen from the classical analysis of the outer ($r \gg \sigma \sim R_\mathrm{H}$) Newtonian scalar potential in Sec. II A and its quantum counterpart in Sec. II B, as well as in the alternative description of gravitational scattering. It has in fact been known for a long time that geodesic motion in the post-Newtonian expansion of the Schwarzschild metric can be reproduced by tree-level Feynman diagrams with graviton exchanges between a test probe and a (classical) large source [19][6] In this calculation, again for $r \gg R_\mathrm{H}$, the source is just described by its total mass M, and quantum effects should then be suppressed by factors of $1/N$ [1].

III. HORIZON OF SELF-SUSTAINED STATES

In the above review, there is no explicit evidence of a nontrivial causal structure. As we already mentioned, in order to show that a system of $N \gg 1$ scalar gravitons is a black hole in the usual sense, we must be able to identify a radius $r \sim R_\mathrm{H}$ as the actual horizon. In order to do so, we

[4]For more details about the identification of this state as a BEC, see, for example, Ref. [7].

[5]Let us remark that, without a curved background geometry, the very definition of a trapping surface becomes conceptually challenging [11].

[6]For a similarly nongeometric derivation of the action of Einstein gravity, see Ref. [20].

employ the horizon wave function introduced in Ref. [11] (see Refs. [16,17] for more details).

This formalism can be applied to the quantum-mechanical state ψ_S of any system *localized in space* and *at rest* in the chosen reference frame. Having defined suitable Hamiltonian eigenmodes, $\hat{H}|\psi_E\rangle = E|\psi_E\rangle$, where H can be specified depending on the model we wish to consider, the state ψ_S can be decomposed as

$$|\psi_S\rangle = \sum_E C(E)|\psi_E\rangle. \tag{3.1}$$

If we further assume that the system is *spherically symmetric*, we can invert the expression of the Schwarzschild radius,

$$r_H = 2G_N E, \tag{3.2}$$

in order to obtain E as a function of r_H. We then define the *horizon wave function* as

$$\psi_H(r_H) \propto C(m_p r_H / 2\ell_p), \tag{3.3}$$

whose normalization is finally fixed in the inner product

$$\langle\psi_H|\phi_H\rangle = 4\pi \int_0^\infty \psi_H^*(r_H)\phi_H(r_H) r_H^2 dr_H. \tag{3.4}$$

We interpret ψ_H simply as the wave function yielding the probability $P_H(r_H) = 4\pi r_H^2 |\psi_H(r_H)|^2$ that we would detect a gravitational radius $r = r_H$ associated with the given quantum state ψ_S. Such a radius generalizes the classical concept of the Schwarzschild radius of a spherically symmetric distribution of matter and is necessarily "fuzzy," like the position and energy of the particle itself [11,16]. The probability density that the system lies inside its own gravitational radius $r = r_H$ will next be given by the conditional expression

$$P_<(r < r_H) = P_S(r < r_H) P_H(r_H), \tag{3.5}$$

where $P_S(r < r_H) = 4\pi \int_0^{r_H} |\psi_S(r)|^2 r^2 dr$ is the probability that the system is inside a sphere of radius $r = r_H$. Finally, the probability that the system described by the wave function ψ_S is a black hole will be obtained by integrating (3.5) over all possible values of the radius,

$$P_{BH} = \int_0^\infty P_<(r < r_H) dr_H. \tag{3.6}$$

When $P_{BH} \simeq 1$, the system is very likely to be found inside its own gravitational radius, which therefore turns into a trapping surface (or, loosely speaking, a horizon), and can be (at least temporarily) viewed as a black hole. The above general formulation can be easily applied to a particle described by a spherically symmetric Gaussian wave function, for which one obtains a vanishing probability that the particle is a black hole when its mass is much smaller than m_p (and the uncertainty in the position $\lambda_m \gg \ell_p$) [11,16]. However, the uncertainty in the horizon size for a single particle with trans-Planckian mass $M \gg m_p$ turns out to be

$$\Delta r_H \sim \langle \hat{r}_H \rangle \sim R_H, \tag{3.7}$$

which clearly shows that this system cannot represent a large, semiclassical black hole.

The main difference with respect to a single very massive particle [11,16] is that our system is now composed of a very large number N of particles of very small effective mass $m \ll m_p$ (and thus very large de Broglie wavelength, $\lambda_m \gg \ell_p$). According to Ref. [16], they cannot individually form (light) black holes; however, the generalization of the formalism to a system of N such components will enable us to show that the total energy $E = M$ is indeed sufficient to create a proper horizon.

A. Hairless black hole

Let us first consider the highly idealized case in which Eq. (2.25) admits precisely one mode, defined by

$$k_c = \frac{\pi}{R_H} = \frac{\pi}{2\sqrt{N}\ell_p}, \tag{3.8}$$

so that $\phi_{k_c}(R_H) \simeq j_0(k_c R_H) = 0$ and the scalar field vanishes outside of $r = R_H$,

$$\psi_S(r_i) = \langle r_i | k_c \rangle = \begin{cases} \mathcal{N}_c j_0(k_c r_i) & \text{for } r < R_H, \\ 0 & \text{for } r > R_H, \end{cases} \tag{3.9}$$

where $\mathcal{N}_c = \sqrt{\pi / 2R_H^3}$ is a normalization factor such that

$$4\pi \mathcal{N}_c^2 \int_0^{R_H} |j_0(k_c r)|^2 r^2 dr = 1. \tag{3.10}$$

The above approximate mode is plotted in Fig. 1, where it is also compared with a Gaussian distribution of the kind considered in Ref. [5], which appears qualitatively very similar.

We have already commented in the previous section that a scalar field which vanishes everywhere outside $r = R_H$ is actually inconsistent with the presence of an outer Newtonian potential, but let us put this fact aside momentarily. The wave function of the system of N such modes is the (totally symmetrized) product of $N \sim M^2$ equal modes [see the order γ^0 term in Eq. (A2), with $|m\rangle = |k_c\rangle$],

$$\psi_S(r_1, \ldots, r_N) = \frac{\mathcal{N}_c^N}{N!} \sum_{\{\sigma_i\}}^N \prod_{i=1}^N j_0(k_c r_i), \tag{3.11}$$

where the sum is over all the permutations $\{\sigma_i\}$ of the N excitations. This is obviously an energy eigenstate,

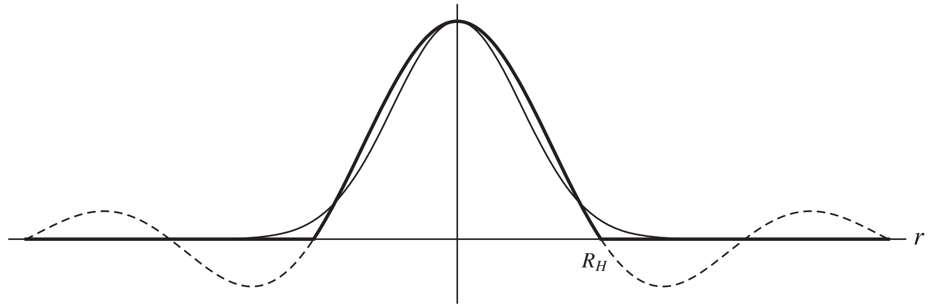

FIG. 1. Scalar field mode of momentum number k_c: the ideal approximation in Eq. (3.9) (thick solid line) is compared to the exact $j_0(k_c r)$ (dashed line). The thin solid line represents a Gaussian distribution of the kind considered in Ref. [5].

$$\hat{H}\psi_S = N\hbar k_c \psi_S = M\psi_S, \tag{3.12}$$

where $\hat{H} = \sum_i \hat{H}_i = \sum_i \hbar \hat{k}_i$ is the total Hamiltonian for N free massless scalars in momentum space. The only nonvanishing coefficient in the spectral decomposition is then given by $C(E) = 1$, for $E = N\hbar k_c = M$, corresponding to a probability density for finding the horizon size between r_H and $r_H + dr_H$,

$$\begin{aligned} dP_H(r_H) &= 4\pi r_H^2 |\psi_H(r_H)|^2 dr_H \\ &= \delta(r_H - R_H) dr_H. \end{aligned} \tag{3.13}$$

This result, along with the fact that all N excitations in the mode k_c are confined within the radius $R_H \simeq \lambda_c$,

$$P_S(r_i < R_H) = 4\pi \mathcal{N}_c^2 \int_0^{R_H} |j_0(k_c r)|^2 r^2 dr = 1, \tag{3.14}$$

immediately leads to the conclusion that the system is indeed a black hole,

$$\begin{aligned} P_{BH} &\simeq 4\pi \mathcal{N}_c^2 \int_0^\infty dr_H \delta(r_H - R_H) \int_0^{r_H} |j_0(k_c r)|^2 r^2 dr \\ &= P_S(r < R_H) = 1. \end{aligned} \tag{3.15}$$

In the above "ideal" approximation (3.9), the horizon would exactly be located at its classical radius,

$$\langle \hat{r}_H \rangle \equiv \langle \psi_H | \hat{r}_H | \psi_H \rangle = R_H, \tag{3.16}$$

with absolutely negligible uncertainty, $\Delta r_H \simeq 0$, where

$$\Delta r_H^2 \equiv \langle \psi_H | (\hat{r}_H^2 - R_H^2) | \psi_H \rangle. \tag{3.17}$$

A zero uncertainty in $\langle \hat{r}_H \rangle$ is not a sound result, which likely parallels the description of a macroscopic black hole as a pure quantum-mechanical state built out of the single-particle wave functions in Eq. (3.9). Moreover, we recall again that a nonvanishing scalar field at $r > R_H$ is also necessary in order to reproduce the expected outer Newtonian potential.

B. Black hole with quantum hair

It is reasonable that a more realistic macroscopic black hole of the kind we consider here (with $N \gg 1$) is not just an energy eigenstate with $k = k_c$ but contains more modes. In particular, we expect that the mode with $k = k_c$ forms a discrete spectrum (which is tantamount to assuming k_c is the minimum allowed momentum, in agreement with the idea of a BEC of gravitons), and must be treated separately. Modes with $k > k_c$ would instead be able to "leak out" (roughly representing the Hawking flux, as we shall see) and form a continuous spectrum, which will turn out to be responsible for the fuzziness in the horizon's location.

For the sake of employing a calculable function, let us assume here that the continuous distribution in momentum space of each of the N scalar states is given by half a Gaussian peaked around k_c (see Fig. 2 for a few modes above k_c),

$$|\psi_S^{(i)}\rangle = \mathcal{N}_\gamma \left(|k_c\rangle + \gamma \int_{k_c}^\infty \frac{\sqrt{2} dk_i}{\sqrt{\Delta_i \sqrt{\pi}}} e^{-\frac{\hbar^2 (k_i - k_c)^2}{2\Delta_i^2}} |k_i\rangle \right), \tag{3.18}$$

where $i = 1, \ldots, N$, the ket $|k\rangle$ denotes the eigenmode of eigenvalue k, and

$$\mathcal{N}_\gamma = (1 + \gamma^2)^{-1/2} \tag{3.19}$$

is a global normalization factor. The parameter γ is a real and dimensionless coefficient that weighs the relative probability of finding the particle in the continuous part of the spectrum with respect to the same particle being in the discrete ground state ϕ_{k_c}, and we shall see later on that it plays a major role in our analysis. In particular, one should note that we are here assuming that γ does not depend on N. We shall also assume that the width $\Delta_i = m \simeq M/N \simeq m_p/\sqrt{N}$, which follows from the typical mode spatial size $k_c^{-1} \sim \sqrt{N} \ell_p$ and is the same for all particles.[7] Since $m = \hbar k_c$ and $E_i = \hbar k_i$, one can also write

[7] This assumption will help to simplify the calculations, although it would be perhaps more realistic to assume a different width for each mode k_i.

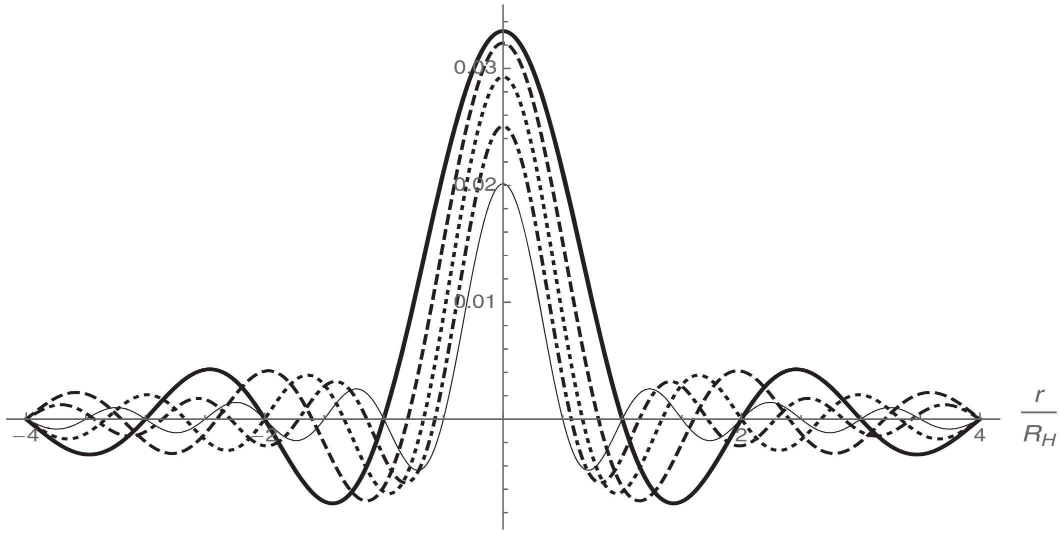

FIG. 2. Modes of momentum number $k = k_c$ (thick solid line), $k = (5/4)k_c$ (dashed line), $k = (3/2)k_c$ (dotted line), $k = (7/4)k_c$ (dash-dotted line), and $k = 2k_c$ (thin solid line). The relative weight is determined according to Eq. (3.18).

$$|\psi_S^{(i)}\rangle = \mathcal{N}_\gamma \left(|m\rangle + \gamma \int_m^\infty \frac{\sqrt{2} dE_i}{\sqrt{m\sqrt{\pi}}} e^{-\frac{(E_i-m)^2}{2m^2}} |E_i\rangle \right). \qquad (3.20)$$

The total wave function will be given by the symmetrized product of N such states [see Eq. (A1)], and one can then identify two regimes, depending on the value of γ (see Appendix A for all the details).

For $\gamma \ll 1$, to leading order in γ, one finds that the spectral coefficient for $E \geq M$ is given by the contribution of the discrete quantum state (corresponding to all of the N particles in the mode k_c) plus the contribution with just one particle in the continuum [see Eq. (A3) and the first term on the rhs of Eq. (A5)],

$$C(E \geq M) \simeq \mathcal{N}_\gamma \left[\delta_{E,M} + \gamma \left(\frac{2}{m\sqrt{\pi}} \right)^{1/2} e^{-\frac{(E-M)^2}{2m^2}} \right], \qquad (3.21)$$

where $\delta_{A,B}$ is a Kronecker delta for the discrete part of the spectrum, and the width $m \sim m_p/\sqrt{N}$ is precisely the typical energy of Hawking quanta emitted by a black hole of mass $M \simeq \sqrt{N} m_p$. It is then easy to compute the expectation value of the energy to next-to-leading order for large N and small γ,

$$\begin{aligned} \langle E \rangle &\simeq \mathcal{N}_\gamma^2 \left(M + \int_M^\infty E C^2(E) dE \right) \\ &\simeq \sqrt{N} m_p \left(1 + \frac{\gamma^2/\sqrt{\pi}}{1+\gamma^2} \frac{1}{N} \right) \\ &\simeq \sqrt{N} m_p \left(1 + \frac{\gamma^2}{\sqrt{\pi} N} \right), \end{aligned} \qquad (3.22)$$

and its uncertainty

$$\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2} \simeq \frac{\gamma m_p}{\sqrt{2N}}. \qquad (3.23)$$

Putting the above two expressions together, we obtain the ratio

$$\frac{\Delta E}{\langle E \rangle} \simeq \frac{\gamma}{\sqrt{2} N}, \qquad (3.24)$$

where we have just kept the leading order in the large-N expansion and neglected terms of higher order in γ. From the expression for the Schwarzschild radius (3.2), or $r_H = 2\ell_p E/m_p$, we then immediately obtain $\langle \hat{r}_H \rangle \simeq R_H$, with R_H given in Eq. (2.24), and

$$\frac{\Delta r_H}{\langle \hat{r}_H \rangle} \sim \frac{1}{N}, \qquad (3.25)$$

which vanishes rather fast for large N. This case could hence describe a macroscopic BEC black hole with (very) little quantum hair, in agreement with Ref. [1], thus overcoming the problem of the excessive large fluctuations (3.7) associated with a single massive particle.

Since this "hair" is indeed expected to be the Hawking radiation field, we shall make the connection with thermal Hawking radiation more explicit in Sec. III D and obtain essentially the same estimate for the horizon uncertainty.

C. Quantum hair with no black hole

For $\gamma \gtrsim 1$ and $N \gg 1$, one finds that the distribution in energy is dominated by all of the N particles in the continuum [see the last term on the rhs of Eq. (A5)], so that the ground state ϕ_{k_c} is actually depleted (or was never occupied).

Since the coefficient γ^N (as well as any other overall factors) can be omitted in this case, one finds

$$C(E \geq M) \simeq \int_m^\infty dE_1 \cdots \int_m^\infty dE_N \exp\left\{-\sum_{i=1}^N \frac{(E_i - m)^2}{2m^2}\right\}\delta\left(E - \sum_{i=1}^N E_i\right), \tag{3.26}$$

along with $C(E < M) \simeq 0$. Note that the mass $M = Nm$ should still be viewed as the minimum energy of the system corresponding to the "ideal" black hole with all of the N particles in the ground state $|k_c\rangle$. For $N = M/m \gg 1$, this spectral function is estimated analytically in Appendix B, and is given by[8]

$$C(E \geq M) \simeq \sqrt{\frac{2}{\pi m^3}}(E - M)e^{-\frac{(E-M)^2}{4m^2}}, \tag{3.27}$$

which is peaked slightly above $E \simeq M = Nm$, with a width $\sqrt{2}\Delta_i \sim m$, so that the (normalized) expectation value

$$\langle E\rangle \simeq \int_M^\infty E C^2(E) dE = M + 2\sqrt{\frac{2}{\pi}}m \tag{3.28}$$

is in agreement with the fact that we are now considering a system built out of continuous modes whose energy must be (slightly) larger than m. For $N \gg 1$, however, $\langle E\rangle = M[1 + \mathcal{O}(N^{-1})]$, and the energy quickly approaches the minimum value M, as is confirmed by the uncertainty

$$\Delta E \simeq \sqrt{\frac{3\pi - 8}{\pi}}m, \tag{3.29}$$

or $\Delta E \sim N^{-1/2}$.

The corresponding horizon wave function is again obtained by simply recalling that $r_H = 2\ell_p E/m_p$, and is approximately given by

$$\psi_H(r_H \geq 2\sqrt{N}\ell_p) \simeq (r_H - 2\sqrt{N}\ell_p)e^{-\frac{(r_H - 2\sqrt{N}\ell_p)^2}{16\ell_p^2/N}} \tag{3.30}$$

and $\psi_H(r_H < 2\sqrt{N}\ell_p) \simeq 0$. The probability density of finding the horizon with a radius between r_H and $r_H + dr_H$ is plotted in Fig. 3 for a few values of N. It is clear that for $N \sim 1$, the uncertainty in the horizon location would be large, but it decreases very fast for increasing N. Accordingly, the (unnormalized) expectation value is

$$\langle \hat{r}_H\rangle \simeq 2\sqrt{N}\ell_p\left(1 + \sqrt{\frac{2}{\pi}}\frac{2}{N}\right) = R_H[1 + \mathcal{O}(N^{-1})], \tag{3.31}$$

which approaches the horizon radius of the ideal black hole, $R_H = 2\sqrt{N}\ell_p$, for large N. In agreement with previous comments about the distribution in energy, the position of the horizon has an uncertainty roughly proportional to the energy $m = m_p/\sqrt{N}$, that is,

$$\frac{\Delta r_H}{\langle \hat{r}_H\rangle} = \frac{\sqrt{\langle \hat{r}_H^2 - \langle \hat{r}_H\rangle^2\rangle}}{\langle \hat{r}_H\rangle} \simeq \frac{1}{N}, \tag{3.32}$$

which vanishes as fast as in the previous case for large N, which is expected in a proper semiclassical regime [1].

The numerical analysis of Eq. (3.26) displayed in Appendix C shows that the actual peak of the spectral function $C = C(E)$ is at slightly larger values of E (and the width is narrower) than the ones given by the analytical approximation (3.27) when $N \gg 1$. However, the uncertainties Δr_H obtained so far are very likely just lower bounds. As we show in Appendix A, the spectral coefficient $C = C(E > M)$ contains N contributions, displayed in Eq. (A5), of which we have just tried to include one (at a time). In particular, for $\gamma \simeq 1$, one could argue that all of the N terms in Eq. (A5) are relevant and their sum might yield a larger uncertainty. If each of these terms contributes an uncertainty $\Delta r_H/\langle \hat{r}_H\rangle \sim N^{-1}$ (like the terms already estimated) they could possibly add up to $\Delta r_H \sim \langle \hat{r}_H\rangle$.[9] This would signal that the causal structure of the system is far from being classical.

Let us stress again that cases with $\gamma \lll 1$ cannot be used to model a BEC black hole, since then most or all of the scalars are in some excited mode with $k > k_c$. However, it is not unreasonable to conjecture that these states play a role either at the threshold of black hole formation (before the gravitons condense into the ground state $|k_c\rangle$) or near the end of black hole evaporation (when the black hole is the hottest). We shall further comment about this in the concluding section.

D. Black hole with thermal hair

We have seen in Sec. III B that, for $\gamma \ll 1$, the dominant contribution to the spectral decomposition of the quantum state of N scalars is given by the configuration with just one boson in the continuum of excited states and the remaining $N - 1$ particles in the ground state. In that section, we employed a rather *ad hoc* Gaussian distribution for the continuous part of the spectrum, but we then found that the

[8]See also Appendix C, where we compare Eq. (3.27) with a numerical estimate using a standard Monte Carlo method.

[9]We note in passing that this is the uncertainty one would obtain for a single quantum-mechanical particle of mass $m \gg m_p$ [11,16]. A numerical estimate of all of the N terms in Eq. (A5) is in progress.

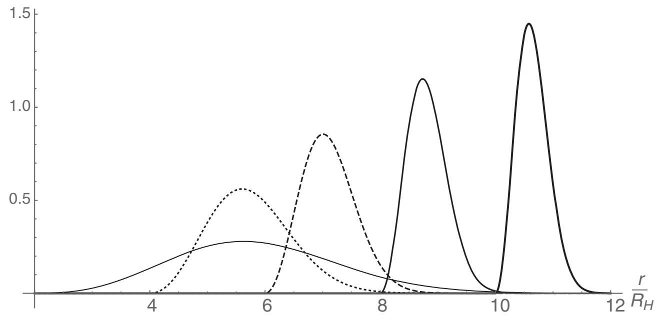

FIG. 3. Probability density of finding the horizon with radius r_{H} for $N = 1$ ($R_{\mathrm{H}} = 2\ell_{\mathrm{p}}$; thin solid line), $N = 4$ ($R_{\mathrm{H}} = 4\ell_{\mathrm{p}}$; dotted line), $N = 9$ ($R_{\mathrm{H}} = 6\ell_{\mathrm{p}}$; dashed line), $N = 16$ ($R_{\mathrm{H}} = 8\ell_{\mathrm{p}}$; solid line), and $N = 25$ ($R_{\mathrm{H}} = 10\ell_{\mathrm{p}}$; thick solid line). The curves clearly become narrower as N becomes larger.

spectral function has a typical width of the order of the Hawking temperature,

$$T_{\mathrm{H}} = \frac{m_{\mathrm{p}}^2}{4\pi M} \simeq \frac{m_{\mathrm{p}}}{\sqrt{N}}, \tag{3.33}$$

or $T_{\mathrm{H}} \simeq m$.

Let us therefore see what happens if we replace the Gaussian distribution in Eq. (3.18) with a thermal spectrum at the temperature T_{H}, which is predicted according to the Hawking effect, that is,

$$|\psi_{\mathrm{S}}^{(i)}\rangle \simeq \mathcal{N}_\gamma \left(|m\rangle + \gamma \frac{e^{T_{\mathrm{H}}^{-1}\frac{m}{2}}}{\sqrt{T_{\mathrm{H}}}} \int_m^\infty dE_i e^{-T_{\mathrm{H}}^{-1}E_i} |E_i\rangle \right), \tag{3.34}$$

where the arbitrary coefficient γ again weighs the relative probability of having a scalar quantum in the continuous spectrum with respect to it being in the ground state. In the same approximation $\gamma \ll 1$ that was used in Sec. III B, the dominant correction to the ideal black hole is again given by the configuration with just one boson in the continuum, for which

$$\begin{aligned} C(E > M) &\simeq \gamma \left(\frac{e}{m}\right)^{1/2} \int_m^\infty dE_1 e^{-\frac{E_1}{m}} \delta(E - M + m - E_1) \\ &\simeq \gamma \left(\frac{e}{m}\right)^{1/2} e^{-\frac{E-(M-m)}{m}}, \end{aligned} \tag{3.35}$$

where we used $T_{\mathrm{H}} \simeq m$. Adding the contribution from the ground state, we obtain that the (normalized) nonvanishing spectral coefficients are given by

$$C(E \geq M) \simeq \tilde{\mathcal{N}}_\gamma \left[\delta_{E,M} + \gamma \left(\frac{e}{m}\right)^{1/2} e^{-\frac{E-(M-m)}{m}} \right], \tag{3.36}$$

where the new normalization factor is given by

$$\tilde{\mathcal{N}}_\gamma = \left(1 + \frac{\gamma^2}{2e}\right)^{-1/2}. \tag{3.37}$$

It is now easy to compute the expectation value of the total energy, for large N and to leading order in γ,

$$\langle E \rangle \simeq \sqrt{N} m_{\mathrm{p}} \left(1 + \frac{\gamma^2}{4eN}\right), \tag{3.38}$$

and its uncertainty

$$\Delta E \simeq \frac{\gamma m_{\mathrm{p}}}{2\sqrt{eN}}. \tag{3.39}$$

For $N \gg 1$, the above expressions lead to

$$\frac{\Delta E}{\langle E \rangle} \sim \frac{\gamma}{2\sqrt{eN}}. \tag{3.40}$$

Up to an irrelevant numerical factor, this is the same behavior we found in the previous two cases, and one therefore expects the uncertainty in the horizon size to be

$$\frac{\Delta r_{\mathrm{H}}}{\langle \hat{r}_{\mathrm{H}} \rangle} \sim \frac{1}{N}, \tag{3.41}$$

which, for large N, is the same decreasing behavior we found previously in Sec. III B for a Gaussian distribution [1].

This result was expected, since the thermal distribution in Eq. (3.34) gives the excited boson a probability of having an energy $E_i > m$ that is comparable to that of the Gaussian distribution in Eq. (3.20), which was specifically defined with the same width $\Delta_i = m = T_{\mathrm{H}}$. Correspondingly, the uncertainty $\Delta r_{\mathrm{H}} \sim N^{-1/2}$ is approximately the same.

IV. CONCLUDING SPECULATIONS

In this work, we have considered a corpuscular model of black holes of the form conjectured in Ref. [1], and investigated its causal structure by means of a formalism for quantum systems we had previously developed in Refs. [11,16,17]. We thus found the presence of a horizon whose size is in agreement with the classical picture of a Schwarzschild black hole for large N (when the energy of each scalar is much smaller than m_{p}, but the total energy is well above m_{p}). We also found that the uncertainty in the horizon's size is typically of the order of the energy of the expected Hawking quanta (albeit, for a suitably chosen variety of states), the latter being proportional to $1/\sqrt{N}$, as was claimed in Ref. [1]. This result is in striking contrast with Eq. (3.7) for a single particle of mass $m \gg m_{\mathrm{p}}$, for which a proper semiclassical behavior cannot be recovered, and thus further supports the conjecture that black holes must be composite objects made of very light constituents.

Based on the above picture, one may be unsure about what is going on during the gravitational collapse of a star. In this respect, the case considered in Sec. II B represents a simplistic model of a Newtonian lump of ordinary matter: the source $J \sim \rho$ represents the star, with $\langle g|\hat{\phi}|g\rangle \simeq \phi_{\mathrm{c}} \sim V_{\mathrm{N}}$, which reproduces the outer Newtonian potential it generates. In this situation, the energy contribution of the gravitons themselves is negligible. The cases analyzed in Sec. III are then just the opposite; since the contribution of any matter source is neglected therein, the quantum state is self-sustained, roughly confined inside a region whose size is given by its Schwarzschild radius, and (almost) monochromatic (see Sec. II C). This state also satisfies the scaling relations (1.7) and (1.8), which were known to hold for a self-gravitating body near the threshold of black hole formation well before Ref. [1] (see, for example, Refs. [12,14]). In our treatment, all time dependence is frozen, and no connection between the two configurations can be made explicitly. However, one may view these two regimes as approximate descriptions of the beginning and (a possible) end states of the collapse of a proper star, as we briefly suggested in Sec. II C.

If the above picture is to make sense, a phase transition for the graviton state should occur around the time when matter and gravitons have comparable weight, much like what happens in a conductor on the verge of superconductivity. The analysis of this transition might be crucial in order to determine whether the star indeed ever reaches the state of a BEC black hole (according to Refs. [1,7], a black hole is precisely the state at the quantum phase transition), or rather just approaches it asymptotically or (for whatever reason) avoids it. A starting point for tackling this issue in a toy model could be the Klein-Gordon equation with both matter and "graviton" currents,

$$\Box\phi(x) = q_{\mathrm{M}} J_{\mathrm{M}}(x) + q_{\mathrm{G}} J_{\mathrm{G}}(x), \tag{4.1}$$

where J_{M} is the general matter current employed in Sec. II B and J_{G} is the graviton current given in Eq. (2.22). For the configuration corresponding to a star, one could treat the latter as a perturbation by formally expanding for small q_{G}. We can expect that this approximation will lead to a correction for the Newtonian potential at short distances from the star, since the graviton current (2.22) is formally equivalent to a mass term for the "graviton" ϕ. In the opposite regime, ordinary matter becomes a small correction, and one could instead expand for small q_{M}. Outside the black hole, such a correction should then increase the fuzziness of the horizon. Of course, if a phase transition happens, it might be when neither of the sources are negligible, so that perturbative arguments would fail to capture its main features. In any case, an *order parameter* should be identified.

In fact, *close to the phase transition*, and before the black hole forms, one might speculate that the case discussed in Sec. III C with $\gamma \gtrsim 1$ (in which all of the bosons are in a slightly excited state above the BEC energy) hints at the physical processes that could occur there. We recall that the parameter γ is essentially the relative probability of finding each one of the N bosons in an excited state with $k > k_{\mathrm{c}}$ rather than in the ground state with $k = k_{\mathrm{c}}$. Since the case $\gamma \ll 1$ of Sec. III B or—perhaps more accurately—the largely equivalent Hawking case of Sec. III D should represent the quantum state of a formed black hole, one might conjecture that it is the parameter γ (or a function thereof) that can be viewed as an *effective* order parameter for the transition from star to black hole. In a truly dynamical context, γ should furthermore acquire a time dependence, thus decreasing from values of order one or larger to much smaller figures along the collapse. It hence appears to be worth investigating the possible dependence of γ on the physical variables usually employed to define the state of matter along the collapse, in order to identify a *physical* order parameter, although this task will likely be much more difficult.

Let us conclude by stressing the fact that we have worked in the Newtonian approximation for the special-relativistic scalar equation, which basically means that we have relied on solutions of the Poisson equation in order to describe the quantum states of gravity. However, an appropriate description of black holes would naturally call for general relativity. In this respect, the only general-relativistic aspect we have included is the condition for the existence of trapping surfaces, which follows from the Einstein equations for spherically symmetric systems and lies at the foundations of the classical hoop conjecture, as well as that of our horizon wave function (and the generalized uncertainty principle that follows from it [16]). It is important to remark that similarly "fuzzy" descriptions of a black hole's horizon were recently derived from the quantization of spherically symmetric space-time metrics, which do not require any knowledge of the quantum state of the source (see Refs. [21] and [22] and references therein). Other investigations of simple collapsing systems, such as thin shells [23] or thick shells [24], also seem to point towards similar scenarios. We emphasize that our approach is much more general in that it allows one to uniquely relate the causal structure of space-time—encoded by the horizon wave function ψ_{H}—to the presence of any material source in the state ψ_{S}. It is cleat that in order to study the time evolution of the system a "feedback" from ψ_{H} into ψ_{S} must be introduced. These deeply conceptual issues are left for future investigations.

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APPENDIX A: *N*-BOSON SPECTRUM

Starting from the single-particle wave functions (3.20), the total wave function of a system of N bosons will be given by the totally symmetrized product

$$|\psi_{\rm S}\rangle \simeq \frac{1}{N!}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}|\psi_{\rm S}^{(i)}\rangle\right] = \frac{1}{N!}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}\left(|m\rangle + \gamma\int_m^\infty \frac{\sqrt{2}dE_i}{\sqrt{m\sqrt{\pi}}}e^{-\frac{(E_i-m)^2}{2m^2}}|E_i\rangle\right)\right], \tag{A1}$$

where (here and in the following equations) $\sum_{\{\sigma_i\}}^{N}$ always denotes the sum over all of the $N!$ permutations $\{\sigma_i\}$ of the N terms inside the square brackets, and we omit (irrelevant) overall normalization factors of $\mathcal{N}_\gamma$ for the sake of simplicity. Note that we can group equal powers of γ in the above expression and obtain

$$\begin{aligned}|\psi_{\rm S}\rangle \simeq{}& \frac{1}{N!}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}|m\rangle\right]\\
&+\frac{\gamma}{N!}\left(\frac{2}{m\sqrt{\pi}}\right)^{1/2}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=2}^{N}|m\rangle\otimes\int_m^\infty dE_1 e^{-\frac{(E_1-m)^2}{2m^2}}|E_1\rangle\right]\\
&+\frac{\gamma^2}{N!}\left(\frac{2}{m\sqrt{\pi}}\right)\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=3}^{N}|m\rangle\otimes\int_m^\infty dE_1 e^{-\frac{(E_1-m)^2}{2m^2}}|E_1\rangle\otimes\int_m^\infty dE_2 e^{-\frac{(E_2-m)^2}{2m^2}}|E_2\rangle\right]+\dots\\
&+\frac{\gamma^J}{N!}\left(\frac{2}{m\sqrt{\pi}}\right)^{J/2}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=J+1}^{N}|m\rangle\bigotimes_{j=1}^{J}\int_m^\infty dE_j e^{-\frac{(E_j-m)^2}{2m^2}}|E_j\rangle\right]+\dots\\
&+\frac{\gamma^N}{N!}\left(\frac{2}{m\sqrt{\pi}}\right)^{N/2}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}\int_m^\infty dE_i e^{-\frac{(E_i-m)^2}{2m^2}}|E_i\rangle\right],\end{aligned} \tag{A2}$$

where the power of γ clearly equals the number of bosons in a continuous ("excited") mode with $k > k_{\rm c}$. This form will help in obtaining the spectral decomposition.

In fact, one of course finds $C(E < M) = 0$, and

$$C(M) \simeq \langle M|\frac{1}{N!}\sum_{\{\sigma_i\}}^{N}\left[\bigotimes_{i=1}^{N}|m\rangle\right] = N!\langle M|\frac{1}{N!}\bigotimes_{i=1}^{N}|m\rangle = 1, \tag{A3}$$

since the energy E can equal $M = Nm$ only when all of the N bosons are in the ground state of lowest momentum number $k_{\rm c} = m/\hbar$. For $E > M$, the term of order γ^0 instead does not contribute and one obtains

$$\begin{aligned}C(E>M) \simeq{}& \gamma\left(\frac{2}{m\sqrt{\pi}}\right)^{1/2}\int_m^\infty dE_1 e^{-\frac{(E_1-m)^2}{2m^2}}\delta(E-M+m-E_1)\\
&+\gamma^2\left(\frac{2}{m\sqrt{\pi}}\right)\int_m^\infty dE_1\int_m^\infty dE_2 e^{-\frac{(E_1-m)^2}{2m^2}-\frac{(E_2-m)^2}{2m^2}}\delta(E-M+2m-E_1-E_2)+\dots\\
&+\gamma^J\left(\frac{2}{m\sqrt{\pi}}\right)^{J/2}\int_m^\infty dE_1\cdots\int_m^\infty dE_J \exp\left\{-\sum_{j=1}^{J}\frac{(E_j-m)^2}{2m^2}\right\}\delta\left(E-M+Jm-\sum_{j=1}^{J}E_j\right)+\dots\\
&+\gamma^N\left(\frac{2}{m\sqrt{\pi}}\right)^{N/2}\int_m^\infty dE_1\cdots\int_m^\infty dE_N \exp\left\{-\sum_{i=1}^{N}\frac{(E_i-m)^2}{2m^2}\right\}\delta\left(E-\sum_{i=1}^{N}E_i\right),\end{aligned} \tag{A4}$$

which can be further simplified to

$$\begin{aligned}C(E>M) \simeq{}& \gamma\left(\frac{2}{m\sqrt{\pi}}\right)^{1/2}e^{-\frac{(E-M)^2}{2m^2}}\\
&+\gamma^2\left(\frac{2}{m\sqrt{\pi}}\right)\int_m^\infty dE_1 \exp\left\{-\frac{(E_1-m)^2}{2m^2}-\frac{(E-M+m-E_1)^2}{2m^2}\right\}+\dots\\
&+\gamma^N\left(\frac{2}{m\sqrt{\pi}}\right)^{N/2}\int_m^\infty dE_1\cdots\int_m^\infty dE_N \exp\left\{-\sum_{i=1}^{N}\frac{(E_i-m)^2}{2m^2}\right\}\delta\left(E-\sum_{i=1}^{N}E_i\right).\end{aligned} \tag{A5}$$

For fixed m (or N) and $\gamma \ll 1$, one can just keep the first term in Eq. (A5) (this contribution is analyzed in detail in Sec. III B). Conversely, for $\gamma \gtrsim 1$, it is the last term that dominates, which is estimated analytically in Appendix B and leads to the results presented in Sec. III C. For a numerical calculation of this spectral coefficient (3.26), see also Appendix C.

We end this appendix with a word of caution. For $\gamma \simeq 1$, all terms could equally contribute, and their (cumbersome) evaluation is left for future investigations. However, we can already note here that, upon considering $m \simeq m_{\mathrm{p}}/\sqrt{N}$, one can identify the alternative expansion parameter $\tilde{\gamma} = \gamma^4 N$ in front of each contribution in Eq. (A5). The first term, of order $\tilde{\gamma}^{1/4}$, would hence seem to dominate for $\tilde{\gamma} \ll 1$, or $\gamma \ll N^{-1/4}$, which implies a very tight bound on γ for macroscopic black holes. And the last term in Eq. (A5) should likewise dominate for $\gamma \gtrsim N^{-1/4}$. This remark makes it clear that the interplay between the small-γ expansion and the large-N expansion is not trivial, and the validity of the final results can safely be assessed only by checking *a posteriori* that higher-order terms (in each expansion) are smaller than lower-order terms. Indeed, this condition holds true for the cases studied in the main text, provided one expands in γ first and then takes N to be large.

APPENDIX B: ANALYTICAL SPECTRUM FOR $\gamma \gtrsim 1$

We start by noting that the spectral coefficient in Eq. (3.26) can be written as

$$\begin{aligned} C(E \geq M) &\sim \int_m^\infty dE_1 \cdots \int_m^\infty dE_N \exp\left\{-\sum_{i=1}^N \frac{(E_i - m)^2}{2m^2}\right\} \delta\left(E - \sum_{i=1}^N E_i\right) \\ &\sim \int_m^\infty dE_1 \cdots \int_m^\infty dE_{N-1} \exp\left\{-\sum_{i=1}^{N-1} \frac{(E_i - m)^2}{2m^2} - \frac{(E - \sum_{i=1}^{N-1} E_i - m)^2}{2m^2}\right\} \\ &\equiv \int_m^\infty dE_1 \cdots \int_m^\infty dE_{N-1} e^{-\frac{F^2(E,\{\mathcal{E}_i\})}{2m^2}}. \end{aligned} \tag{B1}$$

In order to proceed, we find it convenient to write the argument of the exponential as

$$\begin{aligned} -2m^2 F^2(E, \{\mathcal{E}_i\}) &\equiv \sum_{i=1}^{N-1} (E_i - m)^2 + \left(E - \sum_{i=1}^{N-1} E_i - m\right)^2 \\ &= \sum_{i=1}^{N-1} (E_i - m)^2 + \left[E - \sum_{i=1}^{N-1} (E_i - m) - Nm\right]^2 \\ &= (E - M)^2 + \sum_{i=1}^{N-1} \mathcal{E}_i^2 + \left[\sum_{i=1}^{N-1} \mathcal{E}_i - 2(E - M)\right] \sum_{j=1}^{N-1} \mathcal{E}_j, \end{aligned} \tag{B2}$$

where $\mathcal{E}_i = E_i - m$, so that

$$\begin{aligned} C(E \geq M) &\sim e^{-\frac{(E-M)^2}{2m^2}} \int_0^\infty d\mathcal{E}_1 \cdots \int_0^\infty d\mathcal{E}_{N-1} \\ &\quad \times \exp\left\{-\sum_{i=1}^{N-1} \frac{\mathcal{E}_i^2}{2m^2} - \left[\sum_{i=1}^{N-1} \frac{\mathcal{E}_i}{2m} - \frac{(E-M)}{m}\right] \sum_{j=1}^{N-1} \frac{\mathcal{E}_j}{m}\right\} \\ &\equiv e^{-\frac{(E-M)^2}{2m^2}} I(E, M; m). \end{aligned} \tag{B3}$$

We then note that the above integral contains the Gaussian measure,

$$\int_0^\infty d\mathcal{E}_1 \cdots \int_0^\infty d\mathcal{E}_{N-1} \exp\left\{-\sum_{i=1}^{N-1} \frac{\mathcal{E}_i^2}{2m^2}\right\} \sim \int_0^\infty \mathcal{E}^{N-2} d\mathcal{E} \exp\left\{-\frac{\mathcal{E}^2}{2m^2}\right\}, \tag{B4}$$

where $\mathcal{E}^2 = \sum_{i=1}^{N-1} \mathcal{E}_i^2$, which is significantly different from zero only for $\mathcal{E} \lesssim m$. We can therefore approximate

$$\sum_{i=1}^{N-1} \mathcal{E}_i \simeq \mathcal{E}, \tag{B5}$$

from which we obtain

$$I(E,M;m) \sim \int_0^\infty \mathcal{E}^{N-2} d\mathcal{E} \exp\left\{-\frac{\mathcal{E}^2}{m^2} + \frac{(E-M)}{m}\frac{\mathcal{E}}{m}\right\}$$
$$= e^{\frac{(E-M)^2}{4m^2}} \int_0^\infty \mathcal{E}^{N-2} d\mathcal{E} \exp\left\{-\frac{[2\mathcal{E}-(E-M)]^2}{4m^2}\right\}. \tag{B6}$$

This integral can be exactly evaluated in terms of hypergeometric functions, but we can just further approximate it here as

$$I(E,M;m) \sim m(E-M)P_{(N-3)}(E,M)e^{\frac{(E-M)^2}{4m^2}}$$
$$\sim (E-M)e^{\frac{(E-M)^2}{4m^2}}, \tag{B7}$$

where $P_{(N-3)}$ is a polynomial of degree $N-3$. For $N = M/m \gg 1$, we finally obtain

$$C(E \geq M) \sim (E-M)e^{-\frac{(E-M)^2}{4m^2}}, \tag{B8}$$

which is the expression in Eq. (3.27).

APPENDIX C: NUMERICAL SPECTRUM FOR $\gamma \gtrsim 1$

In this appendix we estimate the spectral coefficient in Eq. (B3) [which exactly equals the one in Eq. (3.26)] for various values of N. For this purpose, we have implemented a standard Monte Carlo method in a MATHEMATICA notebook, in which the coefficient is also numerically normalized, so that

$$\int_M^\infty C^2(E)dE = 1. \tag{C1}$$

The dependence of the spectral coefficient on the total energy E is then compared with the analytical approximation (3.27).

Figure 4 shows this comparison for $N = 10$ and $N = 50$. For the former value, the analytical approximation overestimates both the location of the peak and (slightly) the width of the curve (thus underestimating the height of the

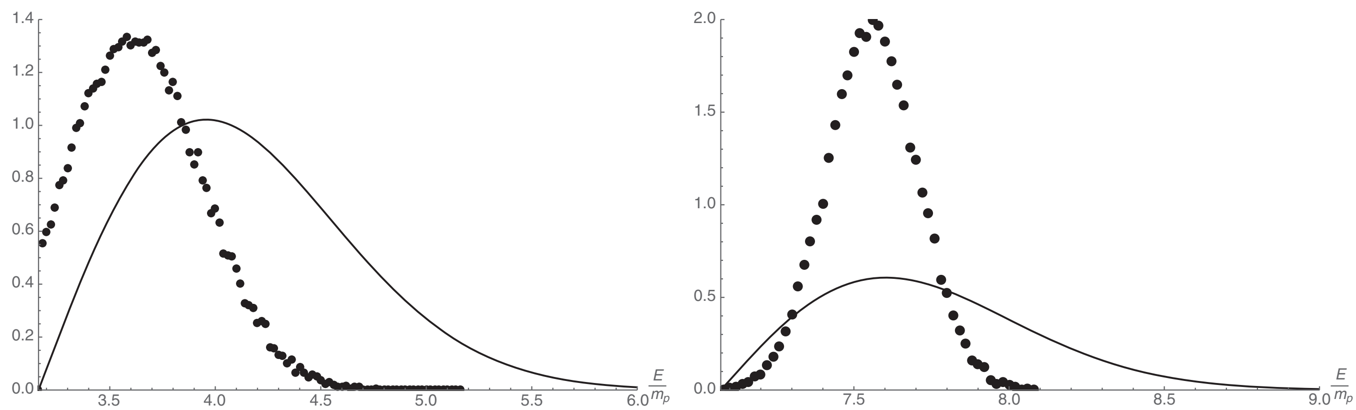

FIG. 4. Monte Carlo estimate of the spectral coefficient in Eq. (B3) (dots) compared to its analytical approximation (3.27) (solid line), both normalized according to Eq. (C1), for $N = 10$ (left panel) and $N = 50$ (right panel).

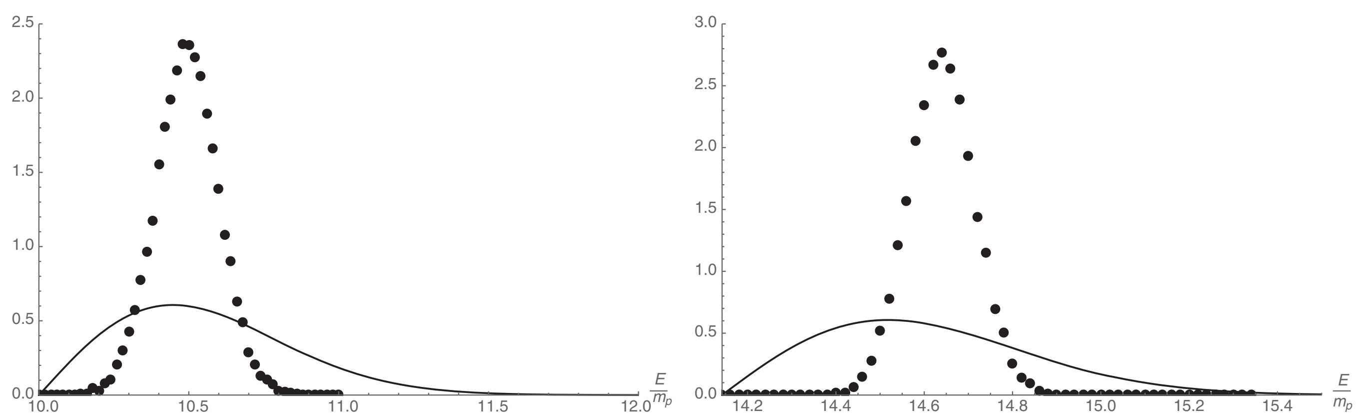

FIG. 5. Monte Carlo estimate of the spectral coefficient in Eq. (B3) (dots) compared to its analytical approximation (3.27) (solid line), both normalized according to Eq. (C1), for $N = 100$ (left panel) and $N = 200$ (right panel).

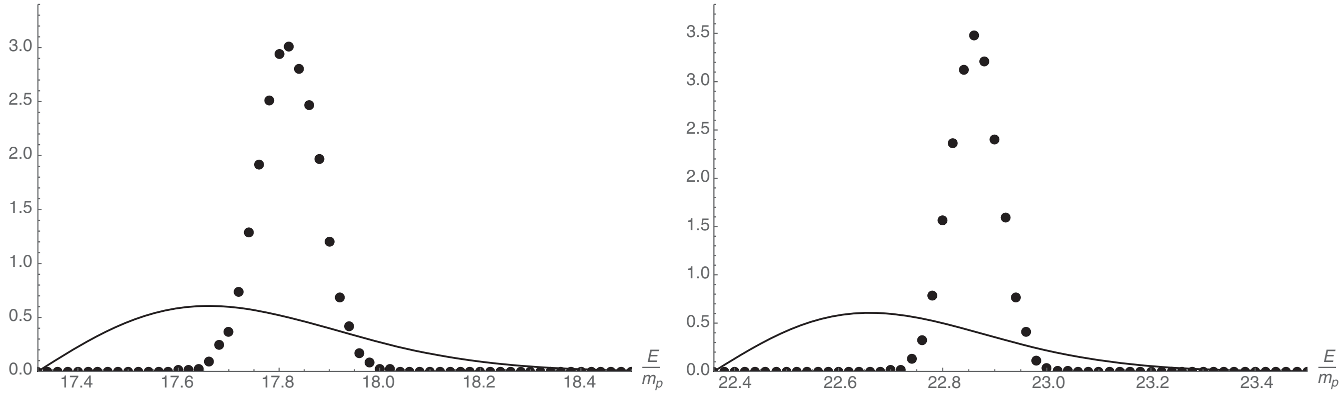

FIG. 6. Monte Carlo estimate of the spectral coefficient in Eq. (B3) (dots) compared to its analytical approximation (3.27) (solid line), both normalized according to Eq. (C1), for $N = 300$ (left panel) and $N = 500$ (right panel).

peak). For $N = 50$, the location of the peak is instead very well identified by Eq. (3.27), but the actual width remains narrower than its analytical approximation (resulting in a large discrepancy in the peak values).

Figure 5 shows the comparison for $N = 100$ and $N = 200$. From these plots, it is clear that the analytical approximation progressively underestimates the value of the energy at which the spectral coefficients peak, and at the same time overestimates more and more the width of the curve (by about a factor of 3 in these two plots), and consequently underestimates the height of the curve at peak values.

All of these trends are further confirmed in Fig. 6, which shows the comparison for $N = 300$ and $N = 500$. The peaks of the numerical estimate and the analytical approximation continue to move further apart, whereas the numerical width becomes narrower than the analytical width for increasing N.

The overall conclusion is that the analytical approximation (3.27) is fairly good at estimating the energy at the peak of the spectral coefficients, but it significantly overestimates (underestimates) the width (peak value) for $N \gtrsim 100$.

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