

Light propagation in the field of a moving axisymmetric body: Theory and applications to the Juno mission

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Given the extreme accuracy of modern space science, a precise relativistic modeling of observations is required. We use the time transfer function formalism to study light propagation in the field of uniformly moving axisymmetric bodies, which extends the field of application of previous works. We first present a space-time metric adapted to describe the geometry of an ensemble of uniformly moving bodies. Then, we show that the expression of the time transfer functions in the field of a uniformly moving body can be easily derived from its well-known expression in a stationary field by using a change of variables. We also give a general expression of the time transfer function in the case of an ensemble of arbitrarily moving point masses. This result is given in the form of an integral that is easily computable numerically. We also provide the derivatives of the time transfer function in this case, which are mandatory to compute Doppler and astrometric observables. We particularize our results in the case of moving axisymmetric bodies. Finally, we apply our results to study the different relativistic contributions to the range and Doppler tracking for the Juno mission in the Jovian system.

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I. INTRODUCTION

In modern times, the accuracy of spacecraft tracking requires a very detailed modeling of the light propagation in order to compute range and Doppler observables. For example, the Cassini spacecraft reaches the level of few meters accuracy for the range and 3×10^{-6} m/s for the Doppler [1–3], while the future BepiColombo mission should reach an accuracy of 10cm on the range and 10^{-6} m/s on the Doppler [4,5]. Similar accuracies are expected for the Juno mission [6], which will reach the Jovian system by mid-2016.

The computation of radioscience observables, as well as the determination of astrometric observables (very large baseline interferometry tracking [7]) requires determining the propagation of light in a curved space-time. In this context, several approaches exist. Assuming that the metric is known, solving the null geodesic equations [8] or the eikonal equation [9] is the standard method allowing one to get all the information about light propagation between two point events. Many solutions have been proposed in the post-Newtonian (PN) and in the post-Minkowskian (PM) approximations when dealing with the bending effects due to the mass multipole moments of the bodies in the Solar

System [10–17]. On the other hand, the effects of the motion of monopoles on the light propagation have also been studied [10,18–21]. A different approach is also available, initially based on the Synge world function [22–24] and then on the time transfer functions (TTFs) [25,26]. In this formalism, the computation of the coordinate light time, the frequency shift and the light deflection can be computed as integrals of functions of the components of the metric tensor over a straight line joining the emitter and the receiver of the signal [25,26]. This method has already been successfully used to compute the propagation of light in different configurations. For example, the TTF in the field of a stationary axisymmetric body has been determined at the first post-Newtonian (1PN) approximation [22,27]. The light propagation in the field of moving monopoles at 1.5 post-Newtonian order has also been treated [28]. Finally, the TTFs in the field of a static monopole up to the second and third post-Minkowskian (3PM) approximation have also been determined [26,29–31].

In this paper, we use the time transfer function formalism to compute the coordinate propagation time, the frequency shift and the deflection of light in the field of uniformly moving axisymmetric bodies and in the field of arbitrarily moving point masses. In Sec. III, we briefly review how the radioscience and astrometric observables can be determined from the TTF and its derivatives. Then, in Sec. IV, we determine the space-time metric describing the geometry in the field of a uniformly moving axisymmetric body, and we note the metric describing the field of arbitrarily

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moving point masses. In Sec. V, we use these metrics to develop a general expression of the TTF. A general result is given in the form of an integral that is computable numerically. Moreover, an analytical result is developed in the case of a uniform motion. The derivatives of the TTF are also determined. In Sec. VI, we particularize our results in the case of a uniformly moving axisymmetric body by analytically determining the contribution of each multipole to the TTF. Finally, in Sec. VII, we apply our results to determine the different relativistic contributions to the radioscience tracking of the Juno spacecraft in the Jovian system. The contributions of the Sun and Jupiter moving monopoles and of the Jupiter moving J_2 are identified. Finally, we give our conclusions and general remarks in Sec. VIII.

II. NOTATION AND CONVENTIONS

In this paper, c is the speed of light in a vacuum and G is the Newtonian gravitational constant. The Lorentzian metric of space-time V_4 is denoted by g . The signature adopted for g is $(+ - - -)$. We suppose that space-time is covered by some global quasi-Galilean coordinate system $(x^\mu) = (x^0, \mathbf{x})$, where $x^0 = ct$, with t being a time coordinate, and $\mathbf{x} = (x^i)$. We assume that the curves of equations $x^i = \text{const}$ are timelike, which means that $g_{00} > 0$ anywhere. We employ the vector notation \mathbf{a} in order to denote $(a^1, a^2, a^3) = (a^i)$. Considering two such quantities \mathbf{a} and \mathbf{b} , we use $\mathbf{a} \cdot \mathbf{b}$ to denote $a^i b^i$ (Einstein convention on repeated indices is used). The quantity $|\mathbf{a}|$ stands for the ordinary Euclidean norm of \mathbf{a} . For any quantity $f(x^\lambda)$, $f_{,\alpha}$ denotes the partial derivative of f with respect to x^α .

III. TIME TRANSFER FUNCTION AND OBSERVABLES

Let $x_A = (ct_A, \mathbf{x}_A)$ and $x_B = (ct_B, \mathbf{x}_B)$ be two events of space-time that are supposed to be connected by a unique light ray. They denote the emission and reception point of the electromagnetic signal. The coordinate light time of a photon connecting x_A and x_B is given by the TTF [23,25,32,33] as

$$t_B - t_A = \mathcal{T}(\mathbf{x}_A, t_B, \mathbf{x}_B) = \frac{R_{AB}}{c} + \frac{1}{c} \Delta(\mathbf{x}_A, t_B, \mathbf{x}_B), \quad (1)$$

where $\mathcal{T}(\mathbf{x}_A, t_B, \mathbf{x}_B)$ is the TTF,¹ and $R_{AB} = |\mathbf{x}_B - \mathbf{x}_A|$ and $\Delta(\mathbf{x}_A, t_B, \mathbf{x}_B)$ is the so-called ‘‘delay function.’’²

As developed in detail in [26], the range, Doppler and astrometric observables can all be computed from the TTF. The range is directly related to the coordinate time of flight

¹In this paper, we used the reception TTF. Similar results can be obtained using the emission TTF, which depends on t_A instead of t_B [25].

²In this paper, we call for simplicity $\Delta(\mathbf{x}_A, t_B, \mathbf{x}_B)$ a ‘‘delay function’’ even though it has the dimension of a distance.

of the photon through a coordinate transformation (see also [34]).

The frequency shift is given by [24,34,35]

$$\frac{\nu_B}{\nu_A} = \frac{[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j]_A^{1/2}}{[g_{00} + 2g_{0i}\beta^i + g_{ij}\beta^i\beta^j]_B^{1/2}} \times \frac{1 - N_{AB}^i \beta_B^i - \beta_B^i \frac{\partial \Delta}{\partial x_B^i} - \frac{1}{c} \frac{\partial \Delta}{\partial t_B}}{1 - N_{AB}^i \beta_A^i + \beta_A^i \frac{\partial \Delta}{\partial x_A^i}}, \quad (2)$$

where $\beta_{A/B}^i = dx_{A/B}^i / c dt$ is the coordinate velocity.

The astrometric observables are directly related to the TTF through the use of [23]

$$\begin{aligned} (\hat{k}_i)_B &= \left(\frac{k_i}{k_0} \right)_B = -c \frac{\partial \mathcal{T}}{\partial x_B^i} \left[1 - \frac{\partial \mathcal{T}}{\partial t_B} \right]^{-1} \\ &= - \left(N_{AB}^i + \frac{\partial \Delta}{\partial x_B^i} \right) \times \left[1 - \frac{1}{c} \frac{\partial \Delta}{\partial t_B} \right]^{-1}, \end{aligned} \quad (3)$$

where k_μ are the covariant components of the tangent vector to the photon trajectory $(k^\mu)_B = dx^\mu / d\lambda|_B$ (λ being an affine parameter) at x_B and $N_{AB} = \frac{R_{AB}}{R_{AB}} = \frac{x_B - x_A}{R_{AB}}$.

Finally, the angular distance between two light rays coming from two different sources can also be related to $(\hat{k}_i)_B$ [26,36].

Therefore, the computation of the TTF (or equivalently of the delay function) and its derivatives is crucial in order to analyze different effects on observations that are done using light propagation.

IV. METRIC AT FIRST POST-MINKOWSKIAN APPROXIMATION

A. Uniformly moving axisymmetric body

Let us suppose that the gravitational field is generated by an ensemble of axisymmetric bodies. We are interested in calculating the contributions of the mass multipoles and of the motion of the bodies on light propagation. The first step is to consider the metric describing such a space-time. The metric for each of the bodies at 1PM order in its own local reference system is given by $G_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu}$, where $H_{\mu\nu}$ is given by [37]

$$H_{00} = -2 \frac{W(X^\alpha)}{c^2} + \mathcal{O}(G^2), \quad (4a)$$

$$H_{0i} = 0, \quad (4b)$$

$$H_{ab} = -2\delta_{ab} \frac{W(X^\alpha)}{c^2} + \mathcal{O}(G^2), \quad (4c)$$

with the spin multipoles being neglected. Let us stress that the potential W depends on the local coordinate $X^\alpha = (cT, \mathbf{X})$.

We can now perform a Poincaré transformation in order to obtain the metric in the case of a uniformly moving body. The procedure is similar to what is developed in [38]. The coordinate transformation is given by

$$x^\mu = b^\mu + \Lambda_\alpha^\mu X^\alpha, \quad (5)$$

where $x^\mu = (ct, \mathbf{x})$ are the coordinates of the global reference system and Λ_α^μ is given by

$$\begin{aligned} \Lambda_0^0 &= \gamma_p, & \Lambda_i^0 &= \Lambda_0^i = \gamma_p \beta_p^i, \\ \Lambda_i^j &= \delta_{ij} + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i \beta_p^j, \end{aligned} \quad (6)$$

where $\beta_p^i = v_p^i/c$, v_p^i is the coordinate velocity of the body and $\gamma_p = 1/\sqrt{1 - \beta_p^2}$ with $\beta_p = |\boldsymbol{\beta}_p|$. Note that b^μ is a constant four-vector that specifies the origin of the coordinate system: it points from the origin of the global reference system to the origin of the comoving frame at $T = 0$ [38]. We have

$$b^i = x_p^i(t_0) \quad \text{and} \quad b^0 = ct_0, \quad (7)$$

and the trajectory of the moving body in the global frame is given by

$$\mathbf{x}_p(t) = \mathbf{x}_p(t_0) + c\boldsymbol{\beta}_p(t - t_0). \quad (8)$$

The inverse coordinate transformation is given by

$$X^\alpha = \tilde{\Lambda}_\mu^\alpha (x^\mu - b^\mu), \quad (9)$$

where $\tilde{\Lambda}_\mu^\alpha$ is the inverse of Λ_α^μ and is given by

$$\begin{aligned} \tilde{\Lambda}_0^0 &= \gamma_p, & \tilde{\Lambda}_i^0 &= \tilde{\Lambda}_0^i = -\gamma_p \beta_p^i, \\ \tilde{\Lambda}_i^j &= \delta_{ij} + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i \beta_p^j. \end{aligned} \quad (10)$$

The metric transformation is given by

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} = \Lambda_\alpha^\mu \Lambda_\beta^\nu G^{\alpha\beta} = \Lambda_\alpha^\mu \Lambda_\beta^\nu (\eta^{\alpha\beta} + H^{\alpha\beta}), \quad (11)$$

which leads to

$$h^{\mu\nu} = \Lambda_\alpha^\mu \Lambda_\beta^\nu H^{\alpha\beta}. \quad (12)$$

From Eq. (4), we have³ $H^{\alpha\beta} = \frac{2W}{c^2} \delta^{ab}$. The introduction of this expression and the expression of Λ_α^μ given by Eq. (6) into Eq. (12) leads to

³Notice that we define $H^{\alpha\beta} = G^{\alpha\beta} - \eta^{\alpha\beta}$, which at the linear order is given by $H^{\alpha\beta} = -\eta^{\alpha\mu} \eta^{\beta\nu} H_{\mu\nu}$.

$$h^{00} = \frac{2W(X^\alpha)}{c^2} \gamma_p^2 (1 + \beta_p^2) + \mathcal{O}(G^2), \quad (13a)$$

$$h^{0i} = \frac{4W(X^\alpha)}{c^2} \beta_p^i \gamma_p^2 + \mathcal{O}(G^2), \quad (13b)$$

$$h^{ij} = \frac{2W(X^\alpha)}{c^2} (\delta_{ij} + 2\beta_p^i \beta_p^j \gamma_p^2) + \mathcal{O}(G^2). \quad (13c)$$

It is worth mentioning that this metric is a generalization of the international astronomical union (IAU) metric [37] which can be recovered in the limit of small β_p . This limit is explicitly developed in Appendix B 1. Let us also stress that W still depends on the local coordinates X^α . Therefore, we still need to use the coordinate transformation (9) to express the potential W as a function of the global coordinates x^α . More precisely, we get

$$W = W(X^\alpha) = W(\tilde{\Lambda}_\mu^\alpha (x^\mu - b^\mu)). \quad (14)$$

The metric (13) describes the geometry generated by a uniformly moving body at 1PM. The metric describing the geometry due to an ensemble of N uniformly moving bodies is then given by

$$h^{00} = \sum_{p=1}^N \frac{2W_p}{c^2} \gamma_p^2 (1 + \beta_p^2) + \mathcal{O}(G^2), \quad (15a)$$

$$h^{0i} = \sum_{p=1}^N \frac{4W_p}{c^2} \beta_p^i \gamma_p^2 + \mathcal{O}(G^2), \quad (15b)$$

$$h^{ij} = \sum_{p=1}^N \frac{2W_p}{c^2} (\delta_{ij} + 2\beta_p^i \beta_p^j \gamma_p^2) + \mathcal{O}(G^2). \quad (15c)$$

In the case of an axisymmetric body, the Newtonian potential can be decomposed in a multipolar expansion

$$W_p(X^i) = \frac{GM_p}{R} \left[1 - \sum_{n=2}^{\infty} J_{np} \left(\frac{r_{pe}}{R} \right)^n P_n \left(\frac{\mathbf{k}_p \cdot \mathbf{X}}{R} \right) \right], \quad (16)$$

where \mathbf{k}_p denotes the unit vector along the symmetry axis of the body p , M_p is the mass of the body p , J_{np} are its mass multipole moments, P_n are the Legendre polynomials, r_{pe} is the equatorial radius of body p and $R = |\mathbf{X}|$. In this paper, we assume that the symmetry axis of the body \mathbf{k}_p is time independent, which means we neglect the precession and nutation of the body.

B. Arbitrarily moving point masses

The determination of the metric describing the geometry around an arbitrarily moving extended body at the

post-Minkowskian approximation is very complex. In particular, one cannot simply use an instantaneous Lorentz transformation, but a local accelerated reference system has to be defined (see the discussion in the conclusion of [38]). This is beyond the scope of this paper. Nevertheless, the metric for arbitrarily moving point masses at the post-Minkowskian approximation has already been determined using the Liénard-Wiechert potentials [18,19,38]. In Appendix A, we briefly note how to compute this metric.

The space-time metric describing the geometry around an arbitrarily moving point mass can be written as [18,19,38] (see also Appendix A)

$$h^{00} = \frac{2W(X^i)}{c^2} \gamma_{pr}^2 (1 + \beta_{pr}^2) + \mathcal{O}(G^2), \quad (17a)$$

$$h^{0i} = \frac{4W(X^i)}{c^2} \beta_{pr}^i \gamma_{pr}^2 + \mathcal{O}(G^2), \quad (17b)$$

$$h^{ij} = \frac{2W(X^i)}{c^2} (\delta_{ij} + 2\beta_{pr}^i \beta_{pr}^j \gamma_{pr}^2) + \mathcal{O}(G^2), \quad (17c)$$

with $W(X^i) = GM/R$ (since this metric is only valid for point masses),

$$X^i = -\beta_{pr}^i \gamma_{pr} r_{pr} + r_{pr}^i + \frac{\gamma_{pr}^2}{1 + \gamma_{pr}} \beta_{pr}^i (\boldsymbol{\beta}_{pr} \cdot \mathbf{r}_{pr}), \quad (18)$$

and where the index r denotes quantities that have to be evaluated at the retarded time t_r defined by

$$t_r = t - \frac{|\mathbf{x} - \mathbf{x}_p(t_r)|}{c} = t - \frac{r_{pr}}{c}, \quad (19)$$

where $r_{pr} = |\mathbf{x} - \mathbf{x}_p(t_r)|$ and $\mathbf{x}_p(t_r)$ is the position of the body p at the retarded time. The expression of the potential $W(X^i)$ can then be explicitly written as

$$W(X^i) = \frac{GM}{|X^i|} = \frac{GM}{\gamma_{pr}(r_{pr} - (\mathbf{r}_{pr} \cdot \boldsymbol{\beta}_{pr}))}. \quad (20)$$

In the limit of small velocities, the expression of the IAU metric is recovered (see Appendix B 2). Finally, the metric for an ensemble of masses is the sum of the metrics generated by each body.

V. TIME TRANSFER FUNCTION AT GENERALIZED 1PM APPROXIMATION

In [24], a PM expansion of the TTF is presented. It develops the TTF in terms of integrals of functions of the metric components over a straight line between the emitter and the receiver of a light signal. At 1PM order, the delay function is given by Ref. [35] as

$$\begin{aligned} \Delta(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{R_{AB}}{2} \int_0^1 [h^{00} - 2N_{AB}^i h^{0i} + N_{AB}^i N_{AB}^j h^{ij}]_{z^\alpha(\lambda)} d\lambda + \mathcal{O}(G^2), \end{aligned} \quad (21)$$

where the integral is taken along a straight line parametrized by

$$z^0(\lambda) = ct = ct_B - \lambda R_{AB}, \quad (22a)$$

$$\mathbf{z}(\lambda) = \mathbf{x}_B - \lambda \mathbf{R}_{AB} = \mathbf{x}_B(1 - \lambda) + \lambda \mathbf{x}_A. \quad (22b)$$

A. General expression in the case of uniform motion

As can be seen from the expression of the metric (15), Δ can be written as a sum of delay functions generated by each individual body $\Delta = \sum_{p=1}^N \Delta_p$. Replacing the expression of the metric (15) in (21) gives

$$\begin{aligned} \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{2R_{AB}}{c^2} \int_0^1 \gamma_p^2 (1 - N_{AB} \cdot \boldsymbol{\beta}_p)^2 W_p(\tilde{\Lambda}_\mu^i(z^\mu(\lambda) - b^\mu)) d\lambda. \end{aligned} \quad (23)$$

It is useful to express the argument appearing in the expression of the potential W_p in the right-hand side of (23) as

$$\begin{aligned} \tilde{\Lambda}_\mu^i(z^\mu(\lambda) - b^\mu) &= -\beta_p^i \gamma_p c(t - t_0) + z^i(\lambda) - x_{p0}^i \\ &\quad + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i \boldsymbol{\beta}_p \cdot (\mathbf{z}(\lambda) - \mathbf{x}_{p0}) \\ &= x_B^i + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i [\boldsymbol{\beta}_p \cdot (\mathbf{x}_B - \mathbf{x}_{p0})] \\ &\quad - x_{p0}^i - \gamma_p v_p^i (t_B - t_0) - \lambda R_{AB} \\ &\quad \times [N_{AB}^i - \gamma_p \beta_p^i + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i (\boldsymbol{\beta}_p \cdot \mathbf{N}_{AB})], \end{aligned} \quad (24)$$

by using Eqs. (8), (10) and (22) and by denoting $\mathbf{x}_{p0} \equiv \mathbf{x}_p(t_0)$. It is also possible to rewrite Eq. (24) in a more compact form as

$$\tilde{\Lambda}_\mu^i(z^\mu(\lambda) - b^\mu) = R_{pB}^i - \lambda G_{AB}^i \quad (25)$$

by setting

$$\begin{aligned} \mathbf{R}_{pX} &= \mathbf{x}_X + \frac{\gamma_p^2}{1 + \gamma_p} \boldsymbol{\beta}_p [\boldsymbol{\beta}_p \cdot (\mathbf{x}_X - \mathbf{x}_{p0})] \\ &\quad - \mathbf{x}_{p0} - \gamma_p \mathbf{v}_p (t_B - t_0), \end{aligned} \quad (26a)$$

$$\begin{aligned} \mathbf{G}_{AB} &= R_{AB} \left[\mathbf{N}_{AB} - \gamma_p \boldsymbol{\beta}_p + \frac{\gamma_p^2}{1 + \gamma_p} \boldsymbol{\beta}_p (\boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}) \right] \\ &= R_{AB} \mathbf{g}_{pAB} \end{aligned} \quad (26b)$$

and

$$\mathbf{g}_{pAB} = \mathbf{N}_{AB} - \gamma_p \boldsymbol{\beta}_p + \frac{\gamma_p^2}{1 + \gamma_p} \boldsymbol{\beta}_p (\boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}). \quad (26c)$$

Let us denote by I the integral appearing in the TTF expression (23) in the case where the body p is static. Therefore, I is given by

$$I(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = \tilde{I}(\mathbf{R}_{AB}, \mathbf{x}_{pB}) = \int_0^1 W_p(\mathbf{x}_{pB} - \lambda \mathbf{R}_{AB}) d\lambda, \quad (27)$$

where $\mathbf{x}_{pA} = \mathbf{x}_A - \mathbf{x}_p$ and $\mathbf{x}_{pB} = \mathbf{x}_B - \mathbf{x}_p$. Usually, the solution of this integral is given in terms of \mathbf{x}_{pA} and \mathbf{x}_{pB} , but formally, the integral depends on \mathbf{R}_{AB} and \mathbf{x}_{pB} . The transition between the two expressions of I and \tilde{I} in the static case is trivial because $\mathbf{x}_{pA} = \mathbf{x}_{pB} - \mathbf{R}_{AB}$. However, this transition no longer applies in the moving case, and it has to be replaced in Eq. (23) by

$$\tilde{I}(\mathbf{G}_{AB}, \mathbf{R}_{pB}) = \int_0^1 W_p(\mathbf{R}_{pB} - \lambda \mathbf{G}_{AB}) d\lambda \quad (28)$$

with the two variables defined by Eqs. (26) and similarly to what was proposed in [28].

Therefore, all the results in the moving case can be derived from the expressions used in the static case by replacing \mathbf{x}_{pB} by \mathbf{R}_{pB} (26a) and \mathbf{R}_{AB} by \mathbf{G}_{AB} (26b). We can use the conversions given below, where for each ‘‘static case’’ quantity on the left we give the ‘‘moving case’’ equivalent on the right. We get

$$\begin{aligned} \mathbf{x}_{pB} \rightarrow \mathbf{R}_{pB} &= \mathbf{x}_B + \frac{\gamma_p^2}{1 + \gamma_p} \boldsymbol{\beta}_p [\boldsymbol{\beta}_p \cdot (\mathbf{x}_B - \mathbf{x}_{p0})] \\ &\quad - \mathbf{x}_{p0} - \gamma_p \mathbf{v}_p (t_B - t_0), \end{aligned} \quad (29a)$$

$$r_{pB} = |\mathbf{x}_{pB}| \rightarrow R_{pB} = |\mathbf{R}_{pB}|, \quad (29b)$$

$$n_{pB} \rightarrow N_{pB} = \frac{\mathbf{R}_{pB}}{R_{pB}}, \quad (29c)$$

$$\begin{aligned} \mathbf{R}_{AB} \rightarrow \mathbf{G}_{AB} &= R_{AB} \mathbf{g}_{pAB} \\ &= R_{AB} \left[\mathbf{N}_{AB} - \gamma_p \boldsymbol{\beta}_p + \frac{\gamma_p^2}{1 + \gamma_p} \boldsymbol{\beta}_p (\boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}) \right], \end{aligned} \quad (29d)$$

$$R_{AB} \rightarrow R_{AB} \gamma_p (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}), \quad (29e)$$

$$N_{AB} \rightarrow \frac{\mathbf{g}_{pAB}}{g_{pAB}} = \frac{\mathbf{g}_{pAB}}{\gamma_p (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB})}, \quad (29f)$$

$$\mathbf{x}_{pA} = \mathbf{x}_{pB} - \mathbf{R}_{AB} \rightarrow \mathbf{R}_{pB} - \mathbf{G}_{AB} = \mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \quad (29g)$$

$$r_{pA} = |\mathbf{x}_{pA}| \rightarrow R_{pA} = |\mathbf{R}_{pA}|, \quad (29h)$$

$$n_{pA} \rightarrow \frac{\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}}{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}|}, \quad (29i)$$

with

$$g_{pAB} = |\mathbf{g}_{pAB}| = \gamma_p (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}), \quad (30)$$

and \mathbf{R}_{pX} given by Eq. (26a). Therefore, we can rewrite Eq. (23) as

$$\begin{aligned} \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{2R_{AB}}{c^2} \gamma_p^2 (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p)^2 \\ &\quad \times \int_0^1 W_p(R_{pB}^i - \lambda G_{AB}^i) d\lambda. \end{aligned} \quad (31)$$

Then, using the definition of I from Eqs. (27) and (28) and the correspondences (29), we are able to express the exact form of the TTF in the field of moving bodies as

$$\begin{aligned} \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{2R_{AB}}{c^2} \gamma_p^2 (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p)^2 \\ &\quad \times I(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}), \end{aligned} \quad (32)$$

with \mathbf{R}_{pX} given by (26a). The last expression can also be written as

$$\begin{aligned} \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{\gamma_p^2 (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p)^2}{g_{pAB}} \\ &\quad \times \tilde{\Delta}_p(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}) \end{aligned} \quad (33)$$

$$\begin{aligned} &= \gamma_p (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p) \\ &\quad \times \tilde{\Delta}_p(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}), \end{aligned} \quad (34)$$

where $\tilde{\Delta}_p(\mathbf{x}_{pA}, \mathbf{x}_{pB})$ is the expression of the static TTF. This particularly simple equation is very useful since it allows one to determine the TTF of a uniformly moving body from the corresponding static TTF.

The derivatives of the TTF, needed to compute the frequency shift (2) and the astrometric direction (3), can be computed from (34), keeping in mind Eq. (26). In the case of a uniformly moving body, their expressions are given by

$$\begin{aligned} \frac{\partial \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B)}{\partial x_A^i} &= \gamma_p (1 - N_{AB} \cdot \boldsymbol{\beta}_p) \tilde{\Delta}_{p,jA}(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}) \left[\delta_{ij} - \gamma_p \beta_p^j \left(N_{AB}^i - \frac{\gamma_p}{1 + \gamma_p} \beta_p^i \right) \right] \\ &+ \gamma_p \frac{\beta_p^i - N_{AB}^i \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}}{R_{AB}} \tilde{\Delta}_p(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}), \end{aligned} \quad (35a)$$

$$\begin{aligned} \frac{\partial \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B)}{\partial x_B^i} &= \gamma_p (1 - N_{AB} \cdot \boldsymbol{\beta}_p) \left\{ \tilde{\Delta}_{p,jB}(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}) \left[\delta_{ij} + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i \beta_p^j \right] + \gamma_p \beta_p^j N_{AB}^i \tilde{\Delta}_{p,jA}(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}) \right\} \\ &- \gamma_p \frac{\beta_p^i - N_{AB}^i \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}}{R_{AB}} \tilde{\Delta}_p(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}), \end{aligned} \quad (35b)$$

$$\frac{\partial \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B)}{\partial t_B} = -c \gamma_p^2 (1 - N_{AB} \cdot \boldsymbol{\beta}_p) \beta_p^i [\tilde{\Delta}_{p,iA}(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}) + \tilde{\Delta}_{p,iB}(\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB})], \quad (35c)$$

where $\tilde{\Delta}_{p,iX}(\mathbf{x}_A, \mathbf{x}_B)$ is the expression of the derivative of the static TTF with respect to \mathbf{x}_X ,

$$\tilde{\Delta}_{p,iX}(\mathbf{x}_A, \mathbf{x}_B) = \frac{\partial \tilde{\Delta}_p(\mathbf{x}_A, \mathbf{x}_B)}{\partial x_X^i}. \quad (36)$$

It is worth mentioning that in the static case, we have the relation

$$\tilde{\Delta}_p(\mathbf{x}_A, \mathbf{x}_B) = \tilde{\Delta}_p(\mathbf{x}_B, \mathbf{x}_A) \quad (37)$$

and consequently

$$\tilde{\Delta}_{p,iB}(\mathbf{x}_A, \mathbf{x}_B) = \tilde{\Delta}_{p,iA}(\mathbf{x}_B, \mathbf{x}_A). \quad (38)$$

Therefore, the expression of the derivatives of the TTF in the moving case is also obtained by inserting into Eqs. (35) the static TTF and its derivatives. We present an application in the field of moving axisymmetric bodies in Sec. VI.

B. Case of a nonuniform motion

The previous section gives the exact solution of the TTF in the field of uniformly moving bodies. If the bodies undergo acceleration, it is still possible to use the previous formula, which corresponds to neglecting higher order terms related to the acceleration of the body. In this case, the choice of the parameter t_0 introduced in Eq. (24) becomes critical. It has been shown [10,12,21] that a good choice of t_0 (i.e. which minimizes the approximation error) is given by the time of closest approach of the photon with respect to the body, which is given by

$$t_0 = \max \left(t_A, t_B - \max \left(0, \frac{\mathbf{g} \cdot (\mathbf{x}_B - \mathbf{x}_p(t_B))}{c|\mathbf{g}|^2} \right) \right) \quad (39)$$

with $\mathbf{g} = N_{AB} - \boldsymbol{\beta}_p(t_B)$.

In the case of arbitrarily moving point masses, it is possible to numerically integrate the TTF (21) using the metric (17). This approach has the convenience to be strictly valid at the 1PM order, whatever the motion of the bodies. Inserting (17) in the expression (21) gives

$$\begin{aligned} \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{2R_{AB}}{c^2} \int_0^1 \gamma_{pr}^2 (1 - N_{AB} \cdot \boldsymbol{\beta}_{pr})^2 \\ &\times W_p \left(z^i(\lambda) - x_p^i(t_r) + \frac{\gamma_{pr}^2}{1 + \gamma_{pr}} \beta_{pr}^i \boldsymbol{\beta}_{pr} \right. \\ &\left. \cdot (\mathbf{z}(\lambda) - \mathbf{x}_p(t_r)) - c \beta_{pr}^i \gamma_{pr} (t - t_r) \right) d\lambda, \end{aligned} \quad (40)$$

where γ_{pr} and β_{pr} depend on the retarded time coordinate t_r that is related to t through (19). The integral in Eq. (40) can then be evaluated numerically, whatever the motion of the body $\mathbf{x}_p(t)$.

C. Moving emitter

In the previous sections, we handle the case where the source of the gravitational field is moving. In general, the emitter and the receiver of the electromagnetic signal are also moving. In this case, the determination of the time transfer requires solving Eq. (1), which is now implicit,

$$\begin{aligned} t_B - t_A &= \mathcal{T}(\mathbf{x}_A(t_A), t_B, \mathbf{x}_B) \\ &= \frac{|\mathbf{x}_B - \mathbf{x}_A(t_A)|}{c} + \frac{1}{c} \Delta(\mathbf{x}_A(t_A), t_B, \mathbf{x}_B). \end{aligned}$$

In practice, the solution of this implicit equation can be determined by an iterative procedure to find t_A [for example, see Eq. (7) of [26]]. Another solution consists in a post-Newtonian expansion of t_A from the TTF [for

example, see Eq. (6) of [26]. Let us denote by \bar{t}_A the Minkowskian coordinate time of the emission solution of

$$t_B - \bar{t}_A = \frac{|\mathbf{x}_B - \mathbf{x}_A(\bar{t}_A)|}{c}, \quad (41)$$

and then we can write, at first order in $\bar{\boldsymbol{\beta}}_A = \mathbf{v}_A(\bar{t}_A)/c$,

$$t_B - t_A = \frac{\bar{R}_{AB}}{c} + \Delta(\mathbf{x}_A(\bar{t}_A), t_B, \mathbf{x}_B) - \frac{\bar{R}_{AB}}{c} \bar{\beta}_A^i \frac{\partial \Delta(\mathbf{x}_A(\bar{t}_A), t_B, \mathbf{x}_B)}{\partial x_A^i}, \quad (42)$$

where the ‘‘bar’’ denotes quantities evaluated at \bar{t}_A like $\bar{R}_{AB} = |\mathbf{x}_B - \mathbf{x}_A(\bar{t}_A)|$. The contribution proportional to $\bar{\boldsymbol{\beta}}_A$ is also known as a Sagnac term. It has the same form as the contribution from the velocity of the source of the gravitational field at first post-Newtonian order as can be seen from Eq. (50). The order of magnitude of this contribution can reach a few meters for a Juno-Earth signal, as can be seen from Fig. 4. Therefore, when iteratively solving the light-time equation, one needs to include the relativistic perturbations or take into account the Sagnac terms to avoid the risk of significant errors.

VI. CASE OF UNIFORMLY MOVING AXISYMMETRIC BODIES

We can now use the general procedure presented in the previous section in the case of uniformly moving axisymmetric bodies whose potential is given by the multipole expansion (16). The TTF in the case of a static axisymmetric body has been computed in [27] and is given by

$$\Delta_p(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = \Delta_{Mp}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) + \Delta_{Jpn}(\mathbf{x}_{pA}, \mathbf{x}_{pB}), \quad (43)$$

where Δ_{Mp} represents the mass monopole contribution and Δ_{Jpn} represents the mass multipole contribution.

The TTF corresponding to a static monopole is well known [22] and is given by

$$\tilde{\Delta}_{Mp}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = 2 \frac{GM_p}{c^2} \ln \frac{r_{pA} + r_{pB} + R_{AB}}{r_{pA} + r_{pB} - R_{AB}}. \quad (44)$$

By inserting (44) into (34) and using the substitutions (29), we obtain the TTF in the field of monopoles in uniform motion as

$$\Delta_M(\mathbf{x}_A, t_B, \mathbf{x}_B) = 2 \frac{GM_p}{c^2} \gamma_p (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p) \ln \frac{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}| + R_{pB} + \gamma_p R_{AB} (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB})}{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}| + R_{pB} - \gamma_p R_{AB} (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB})}, \quad (45)$$

with \mathbf{R}_{pX} given by (26a). On the other hand, the mass multipole contribution Δ_{Jpn} has been computed in [27] as

$$\tilde{\Delta}_{Jpn}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = K_{pn} \sum_{m=1}^n \left[\frac{1}{(r_{pA} + r_{pB} - R_{AB})^{n-m+1}} - \frac{1}{(r_{pA} + r_{pB} + R_{AB})^{n-m+1}} \right] \Theta_{nm}(\mathbf{x}_{pA}, \mathbf{x}_{pB}), \quad (46a)$$

with $K_{pn} \equiv 2GM_p J_{np} r_{pe}^n / c^2$ and

$$\Theta_{nm}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = (-1)^{n-m} \sum_{i_1, \dots, i_m} \frac{(n-m)!}{i_1! i_2! \dots i_m!} \prod_{l=1}^m [S_l(\mathbf{x}_{pA}, \mathbf{x}_{pB})]^{i_l}, \quad (46b)$$

where the sum \sum_{i_1, \dots, i_m} denotes the summation over the sets of non-negative integers i_1, i_2, \dots, i_m satisfying the pair of equations

$$\begin{cases} i_1 + 2i_2 + 3i_3 + \dots + mi_m = n \\ i_1 + i_2 + \dots + i_m = n - m + 1 \end{cases} \quad (46c)$$

and where $S_l(\mathbf{x}_{pA}, \mathbf{x}_{pB})$ is defined by

$$S_l(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = \frac{1}{r_{pA}^{l-1}} C_l^{(-1/2)} \left(\frac{\mathbf{k}_p \cdot \mathbf{x}_{pA}}{r_{pA}} \right) + \frac{1}{r_{pB}^{l-1}} C_l^{(-1/2)} \left(\frac{\mathbf{k}_p \cdot \mathbf{x}_{pB}}{r_{pB}} \right), \quad (46d)$$

with $C_l^{(-1/2)}(x)$ the Gegenbauer polynomial of degree l and of parameter $-1/2$.

Therefore, the multipole term of the TTF for the case of moving axisymmetric bodies is given by inserting (46) into the relation (34) and using the substitutions (29),

$$\begin{aligned} \Delta_{J_{np}}(\mathbf{x}_A, t_B, \mathbf{x}_B) &= \frac{2GM_p J_{np} r_{pe}^n}{c^2} \gamma_p (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p) \\ &\times \sum_{m=1}^n \left[\frac{1}{(|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}| + R_{pB} - R_{AB} \gamma_p (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}))^{n-m+1}} \right. \\ &\left. - \frac{1}{(|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}| + R_{pB} + R_{AB} \gamma_p (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}))^{n-m+1}} \right] \Theta_{nm}(\mathbf{R}_{pA} + \boldsymbol{\beta}_p R_{AB}, \mathbf{R}_{pB}), \end{aligned} \quad (47)$$

with \mathbf{R}_{pX} given by (26a).

In order to compute the derivatives of the TTF in the case of moving bodies from Eq. (35), one also needs the derivatives of the TTF in the static case. The derivative of the TTF in the case of a static monopole is known (see, for example, [39]), and it is given by

$$\tilde{\Delta}_{Mp,iA}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = -\frac{4GM_p N_{AB}^i (r_{pA} + r_{pB}) + R_{AB} n_{pA}^i}{c^2 (r_{pA} + r_{pB})^2 - R_{AB}^2}, \quad (48a)$$

$$\tilde{\Delta}_{Mp,iB}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = +\frac{4GM_p N_{AB}^i (r_{pA} + r_{pB}) - R_{AB} n_{pB}^i}{c^2 (r_{pA} + r_{pB})^2 - R_{AB}^2} = \tilde{\Delta}_{p,iA}(\mathbf{x}_{pB}, \mathbf{x}_{pA}). \quad (48b)$$

Also, the derivatives of Eq. (46) can be computed as

$$\begin{aligned} \tilde{\Delta}_{J_{pn,iA}}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) &= K_{pn} \sum_{m=1}^n \left\{ -(n-m+1) \left[\frac{\mathbf{n}_{pA} + \mathbf{N}_{AB}}{(r_{pA} + r_{pB} - R_{AB})^{n-m+2}} - \frac{\mathbf{n}_{pA} - \mathbf{N}_{AB}}{(r_{pA} + r_{pB} + R_{AB})^{n-m+2}} \right] \Theta(\mathbf{x}_{pA}, \mathbf{x}_{pB}) \right. \\ &\left. + \left[\frac{1}{(r_{pA} + r_{pB} - R_{AB})^{n-m+1}} - \frac{1}{(r_{pA} + r_{pB} + R_{AB})^{n-m+1}} \right] \Upsilon_{A|nm}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) \right\}, \end{aligned} \quad (49a)$$

$$\begin{aligned} \tilde{\Delta}_{J_{pn,iB}}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) &= K_{pn} \sum_{m=1}^n \left\{ -(n-m+1) \left[\frac{\mathbf{n}_{pB} - \mathbf{N}_{AB}}{(r_{pA} + r_{pB} - R_{AB})^{n-m+2}} - \frac{\mathbf{n}_{pB} + \mathbf{N}_{AB}}{(r_{pA} + r_{pB} + R_{AB})^{n-m+2}} \right] \Theta(\mathbf{x}_{pA}, \mathbf{x}_{pB}) \right. \\ &\left. + \left[\frac{1}{(r_{pA} + r_{pB} - R_{AB})^{n-m+1}} - \frac{1}{(r_{pA} + r_{pB} + R_{AB})^{n-m+1}} \right] \Upsilon_{B|nm}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) \right\}, \end{aligned} \quad (49b)$$

where

$$\begin{aligned} \Upsilon_{X|nm}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) &= (-1)^{n-m} \sum_{i_1, \dots, i_m} \frac{(n-m)!}{i_1! i_2! \dots i_m!} \sum_{l=1}^m i_l [S_l(\mathbf{x}_{pA}, \mathbf{x}_{pB})]^{i_l-1} \\ &\times \prod_{q=1, q \neq l}^m [S_q(\mathbf{x}_{pA}, \mathbf{x}_{pB})]^{i_q} \frac{[P_{l-1}(\mathbf{k}_p \cdot \mathbf{n}_{pX}) \mathbf{k}_p - P_l(\mathbf{k}_p \cdot \mathbf{n}_{pX}) \mathbf{n}_{pX}]}{r_{pX}^l}. \end{aligned} \quad (49c)$$

The derivatives of the TTF function in the case of a moving axisymmetric body are then given by combining Eqs. (44) and (49) with Eq. (43) and by using it together with the combination of Eqs. (48) and (49) with Eqs. (35) [using the correspondences (29)].

A. Particular case: Post-Newtonian expansion

Section VA gives a way to compute the TTF in the field of uniformly moving bodies. The obtained expressions are exact at any order in $\boldsymbol{\beta}_p$. Nevertheless, a post-Newtonian expression can sometimes be more practical to use in the case of slowly moving bodies. Therefore, we present here an expansion of the previous results in terms of the small parameter $\boldsymbol{\beta}_p$. An expansion of (32) gives

$$\Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) = (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}) \tilde{\Delta}_p(\mathbf{x}_{pA}, \mathbf{x}_{pB}) + (R_{AB} - c(t_B - t_0)) \boldsymbol{\beta}_p^i \tilde{\Delta}_{p,iA}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) - c(t_B - t_0) \boldsymbol{\beta}_p^i \tilde{\Delta}_{p,iB}(\mathbf{x}_{pA}, \mathbf{x}_{pB}), \quad (50)$$

with $\mathbf{x}_{pX} = \mathbf{x}_X - \mathbf{x}_p(t_0)$. For example, the use of this formula in the case of the moving monopoles leads to

$$\begin{aligned}
 \Delta_p(\mathbf{x}_A, t_B, \mathbf{x}_B) &= 2 \frac{GM_p}{c^2} (1 - \boldsymbol{\beta}_p \cdot \mathbf{N}_{AB}) \ln \frac{r_{pA} + r_{pB} + R_{AB}}{r_{pA} + r_{pB} - R_{AB}} \\
 &\quad - 4 \frac{GM_p R_{AB}}{c^2} \frac{(r_{pA} + r_{pB}) \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p + R_{AB} \mathbf{n}_{pA} \cdot \boldsymbol{\beta}_p}{(r_{pA} + r_{pB})^2 - R_{AB}^2} \\
 &\quad + 4 \frac{GM_p R_{AB}}{c} (t_b - t_0) \frac{\boldsymbol{\beta}_p \cdot (\mathbf{n}_{pA} + \mathbf{n}_{pB})}{(r_{pA} + r_{pB})^2 - R_{AB}^2} + \mathcal{O}(c^{-4}), \tag{51}
 \end{aligned}$$

with $\mathbf{n}_{pX} = \frac{\mathbf{x}_{pX}}{r_{pX}} = \frac{\mathbf{x}_{pX}}{|\mathbf{x}_{pX}|}$. This expression is equivalent to the one given by Eq. (20) of [28]. Obtaining this result in such a straightforward way illustrates the effectiveness of the TTF approach.

B. Particular case: The quadrupolar term

An explicit calculation for each of the multipoles is straightforward given the above formulas. As an example,

let us explicitly develop the expression for the quadrupolar term J_2 . The only sets of integer solutions to Eqs. (46c) are $i_1 = 2$ for $m = 1$ and $\{i_1 = 0, i_2 = 1\}$ for $m = 2$. As shown in [27], we obtain

$$\begin{aligned}
 \tilde{\Delta}_{J_{p2}}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) &= \frac{GM_p J_{p2} r_{pe}^2}{c^2} \frac{R_{AB}}{r_{pA} r_{pB} (1 + \mathbf{n}_{pA} \cdot \mathbf{n}_{pB})} \\
 &\quad \times \left[\frac{1 - (\mathbf{k}_p \cdot \mathbf{n}_{pA})^2}{r_{pA}} + \frac{1 - (\mathbf{k}_p \cdot \mathbf{n}_{pB})^2}{r_{pB}} \right. \\
 &\quad \left. - \left(\frac{1}{r_{pA}} + \frac{1}{r_{pB}} \right) \frac{[\mathbf{k}_p \cdot (\mathbf{n}_{pA} + \mathbf{n}_{pB})]^2}{1 + \mathbf{n}_{pA} \cdot \mathbf{n}_{pB}} \right]. \tag{52}
 \end{aligned}$$

Therefore, inserting (52) into (34) and using the substitutions (29), we obtain

$$\begin{aligned}
 \Delta_{J_{p2}}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) &= \frac{GM_p}{c^2} \gamma_p^2 (1 - \mathbf{N}_{AB} \cdot \boldsymbol{\beta}_p)^2 \frac{J_{p2} r_{pe}^2}{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}| R_{pB} (1 + \tilde{\mathbf{N}}_A \cdot \mathbf{N}_{pB})} \frac{R_{AB}}{R_{pB}} \\
 &\quad \times \left[\frac{1 - (\mathbf{k}_p \cdot \tilde{\mathbf{N}}_A)^2}{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}|} + \frac{1 - (\mathbf{k}_p \cdot \mathbf{N}_{pB})^2}{R_{pB}} - \left(\frac{1}{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}|} + \frac{1}{R_{pB}} \right) \frac{[\mathbf{k}_p \cdot (\tilde{\mathbf{N}}_A + \mathbf{N}_{pB})]^2}{1 + \tilde{\mathbf{N}}_A \cdot \mathbf{N}_{pB}} \right] \tag{53}
 \end{aligned}$$

and

$$\tilde{\mathbf{N}}_A = \frac{\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}}{|\mathbf{R}_{pA} + \gamma_p \boldsymbol{\beta}_p R_{AB}|}. \tag{54}$$

The derivative of (52) with respect to \mathbf{x}_{pA} can be computed using Eq. (49a) and is given by

$$\begin{aligned}
 \tilde{\Delta}_{J_{p2}, iA}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) &= 2 \frac{GM_p}{c^2} J_{p2} r_{pe}^2 \left\{ [\mathbf{k}_p \cdot (\mathbf{n}_{pA} + \mathbf{n}_{pB})]^2 \left[\frac{\mathbf{n}_{pA} + \mathbf{N}_{AB}}{(r_{pA} + r_{pB} - R_{AB})^3} - \frac{\mathbf{n}_{pA} - \mathbf{N}_{AB}}{(r_{pA} + r_{pB} + R_{AB})^3} \right] \right. \\
 &\quad - \frac{1}{2} \left[\frac{1 - (\mathbf{k}_p \cdot \mathbf{n}_{pA})^2}{r_{pA}} + \frac{1 - (\mathbf{k}_p \cdot \mathbf{n}_{pB})^2}{r_{pB}} \right] \left[\frac{\mathbf{n}_{pA} + \mathbf{N}_{AB}}{(r_{pA} + r_{pB} - R_{AB})^2} - \frac{\mathbf{n}_{pA} - \mathbf{N}_{AB}}{(r_{pA} + r_{pB} + R_{AB})^2} \right] \\
 &\quad - \frac{1}{r_{pA}^3} \frac{R_{AB} (r_{pA} + r_{pB})}{r_{pB}^2} \frac{\mathbf{k}_p \cdot (\mathbf{n}_{pA} + \mathbf{n}_{pB})}{(1 + \mathbf{n}_{pA} \cdot \mathbf{n}_{pB})^2} [\mathbf{k}_p - (\mathbf{k}_p \cdot \mathbf{n}_{pA}) \mathbf{n}_{pA}] \\
 &\quad \left. - \frac{1}{2 r_{pA}^3} \frac{R_{AB}}{r_{pB}} \frac{2(\mathbf{k}_p \cdot \mathbf{n}_{pA}) \mathbf{k}_p + [1 - 3(\mathbf{k}_p \cdot \mathbf{n}_{pA})^2] \mathbf{n}_{pA}}{1 + \mathbf{n}_{pA} \cdot \mathbf{n}_{pB}} \right\}, \tag{55a}
 \end{aligned}$$

while the derivatives with respect to \mathbf{x}_{pB} can be obtained by symmetry as

$$\tilde{\Delta}_{J_{p2}, iB}(\mathbf{x}_{pA}, \mathbf{x}_{pB}) = \tilde{\Delta}_{J_{p2}, iA}(\mathbf{x}_{pB}, \mathbf{x}_{pA}). \tag{55b}$$

In order to evaluate the contribution of the moving quadrupole to the derivatives of the time transfer, it is then sufficient to combine Eqs. (55) and (52) as shown in Eqs. (35).

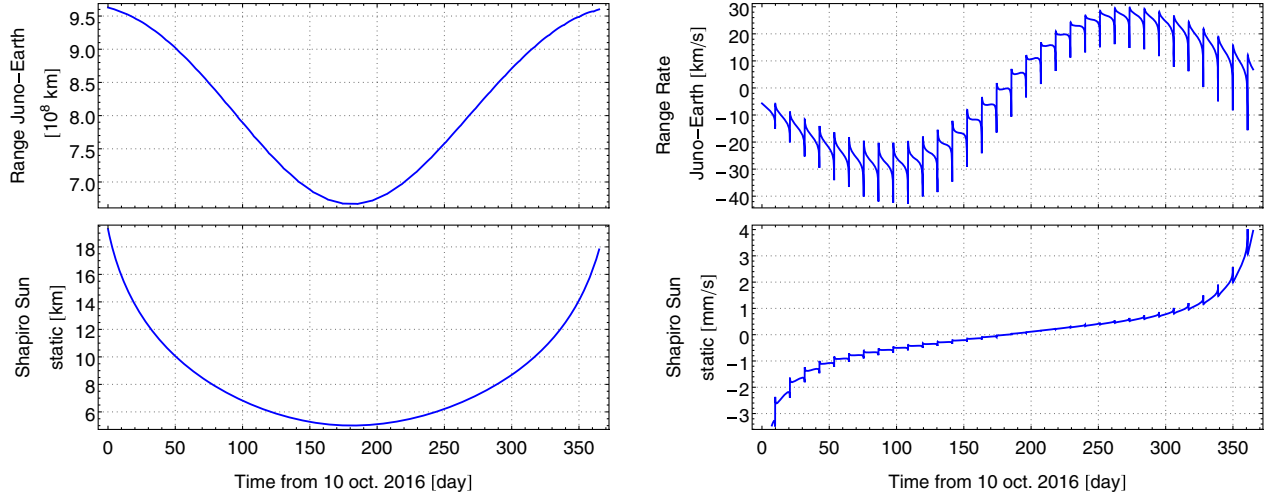


FIG. 1 (color online). Representation of different contributions on the range (left panels) and range rate (right panels) between Juno and Earth over one year. The contributions represented are the lower order contribution (the actual value of the observables) and the corrections produced by the Shapiro due to the monopole of the Sun.

VII. APPLICATION TO JUNO

As an example, we use the equations presented in previous sections to give estimates of the relativistic corrections on the observables for the Juno mission. Juno is currently on its way to Jupiter, which it will reach in 2016. The spacecraft will orbit Jupiter during one year. Some of the relativistic perturbations on the Juno orbit have been studied in [40,41]. The main goal of this section is to assess the order of magnitude produced by different effects due to the Sun and Jupiter on the time transfer. We shall use the nominal orbit of the mission around Jupiter obtained using the Naif SPICE toolkit [42] and kernels, as well as the DE430 planetary ephemeris [43]. The expected accuracy for Juno is of the order of 10 cm on the range and 10^{-6} m/s

on the Doppler [6]. In the following, we present different relativistic contributions to the two-way coordinate light time between Earth and Juno and the corresponding range rate. The range rate has been computed with an integration time of 10 seconds.

In the following figures, all the time scales are given in terms of the coordinate time, which is similar to the barycentric coordinate time introduced in the IAU conventions (see [37]). The observations, done in terms of local time, can be derived by a relativistic coordinate transformation, which is conventional [37,44]. Nevertheless, this transformation will not significantly change the figures presented below.

Figure 1 represents the lower order time transfer and range rate between Juno and Earth, as well as the relativistic

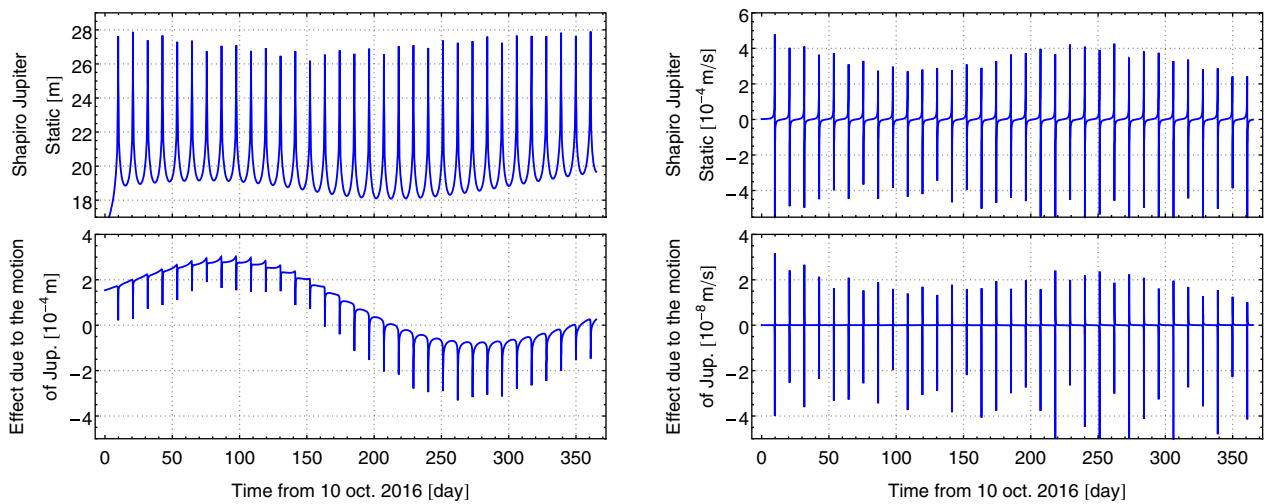


FIG. 2 (color online). Representation of different contributions on the range (left panels) and range rate (right panels) between Juno and Earth over one year. The contributions represented are the corrections produced by the Shapiro due to the monopole of Jupiter (top panels) and the contributions due to the velocity of Jupiter (bottom panels).

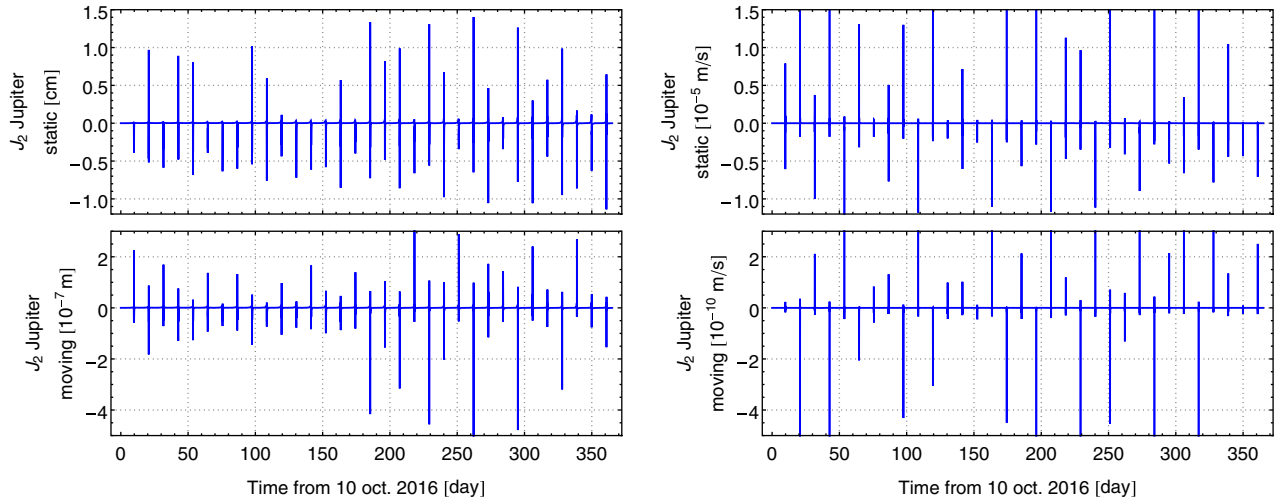


FIG. 3 (color online). Representation of different contributions on the range (left panels) and range rate (right panels) between Juno and Earth over one year. The contributions represented are the corrections produced by the J_2 of Jupiter (top panels) and the contributions produced by the fact that the J_2 is moving (bottom panels).

Shapiro correction from the Sun. These corrections are standard.

Figure 2 represents the contributions of the mass monopole of Jupiter on the range and on the range rate. These contributions have been split into two parts: a part related to the case where Jupiter is static and a contribution proportional to Jupiter’s velocity β_{Jup} . The static part is computed using (44) with the position of Jupiter taken at the critical time t_0 given by Eq. (39). The contribution relative to the velocity is computed by taking the difference between the relations (45) and (44). As one can see, the contributions relative to the motion of Jupiter are 2 orders of magnitude below the expected accuracy of Juno and can

safely be neglected in the modeling of the time transfer. A similar conclusion holds for the motion of the Sun around the Solar System barycenter, which is even smaller. Note that the analytical results presented in these graphs have been checked by numerically integrating the TTF (40).

Figure 3 represents the contributions of the quadrupole of Jupiter (J_2) on the range and on the range rate of Juno. As above, we have split these contributions into two parts: one related to the case where Jupiter is static and one proportional to Jupiter’s velocity β_{Jup} . The static part is computed using (52) with the position of Jupiter taken at the critical time t_0 given by Eq. (39). The contribution relative to the velocity is computed by taking the difference

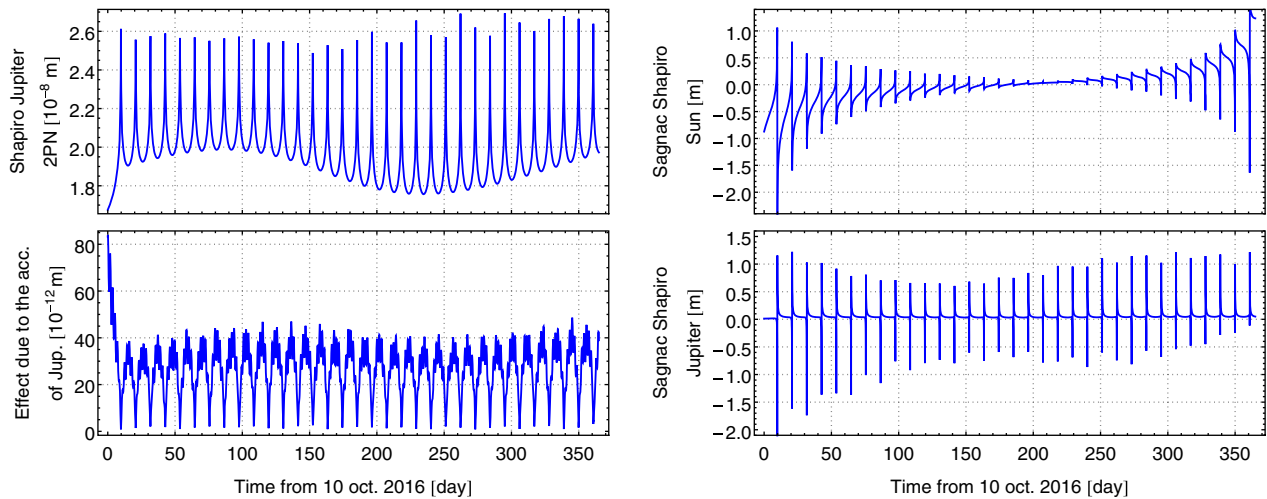


FIG. 4 (color online). Representation of different contributions on the range between Juno and Earth over one year. Top left panel: The 2PN contribution from the monopole of Jupiter (contribution proportional to β_{Jup}^2). Bottom left panel: The contribution proportional to the acceleration of Jupiter. Right panels: The Sagnac contributions (proportional to the Sun mass and to Jupiter mass) due to the motion of Juno.

between the relations (53) and (52). As one can see, the contributions relative to the J_2 of Jupiter are of the same order as the Juno's expected accuracy. Therefore, the effect of the J_2 should be taken into account in the reduction of the tracking data. The contribution related to the velocity of the J_2 is far beyond the current tracking accuracy. Once again, the analytical results presented in these graphs have been checked by numerically integrating the TTF (40). It is important to notice that the curves highly depend on the geometry of the probe orbit. Since Juno has a polar orbit and is never in conjunction with Jupiter; the velocity effects are not detectable. Therefore, the situation can be different for another space mission like JUICE [45].

Figure 4 is given for illustrative purposes, and it shows more effects on the range of Juno. First of all, the effects of the second order in β_{Jup} are represented. They are computed by making the difference between the formula valid at all orders in β_{Jup} (45) and the 1PN expansion (50). This shows that one can safely use the PN expansion presented in Sec. VI A within the Solar System. The effect of the acceleration of Jupiter on the range is also presented. This is computed by making the difference between the numerical integration of the TTF in which we are using the real Jupiter trajectory (40) and the result valid at all orders in the velocity (45). The small rapid oscillations come from oscillations in Jupiter's acceleration, which results from the perturbations due to the Galilean satellites.

Finally, on the right of Fig. 4 the Sagnac effects due to the motion of Juno are represented. The contributions represented are due to the Shapiro of the Sun and Jupiter, which has been computed using (42). These contributions should be included in the analysis of Juno data either as a perturbation or when solving the light-time iterations.

VIII. CONCLUSIONS

In this paper, we computed the TTF and its derivatives in the field of uniformly moving axisymmetric bodies and in the field of arbitrarily moving point masses, which is useful in order to evaluate range, Doppler and astrometric observables. First, in Sec. IV, we computed a metric adapted to describe the space-time geometry due to N bodies in a global reference system by using a Poincaré transformation.

Then, we presented a general method to compute the TTF and its derivatives in the case where the bodies generating the gravitational field are in uniform motion. We showed that the TTF in the case of uniform motion can be directly derived from the static TTF, as can be seen from Eqs. (34) and (35). This result is very powerful and valid for any velocities. Moreover, in Sec. V B, we developed a general expression of the TTF in the case where the gravitational field is generated by arbitrarily moving point masses. The result is given as an integral over a straight line between the emitter and the receiver (40), which can be computed numerically. This general formulation has been used to numerically check our analytical derivations but is

also useful to assess the effects due to the acceleration of the body on the light propagation.

Then, in Sec. VI, we showed how our method can be easily applied to the metric presented in Sec. IV to analytically compute the TTF and its derivatives (and thus the range, frequency shift and astrometric direction) for a light signal propagating in the field of one or more axisymmetric bodies in uniform motion. The results of this paper complete the work of [27,28] and, in general, extend the field of applicability of the TTF formalism [25].

Finally, as an example of our method, we computed the range and Doppler for the Juno mission during its orbit around Jupiter and studied in detail the different perturbations due to the Sun and Jupiter on light propagation. In particular, we showed that in addition to the standard Shapiro contributions due to the mass monopole of Jupiter and of the Sun, the contribution of Jupiter J_2 is also relevant at the level of accuracy expected for Juno. The motion of the Sun and of Jupiter produces effects that are too small compared to Juno accuracy. Nevertheless, this conclusion highly depends on the geometry of Juno orbit and it should be assessed carefully for other space missions (JUICE, for example [45]).

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APPENDIX A: METRIC OF ARBITRARILY MOVING POINT MASSES

The standard way to compute the metric for an arbitrarily moving point mass is to use the Liénard-Wiechert potentials as in [18,19,38]. Based on the analogy between the Maxwell equations and the linearized Einstein field equations [46], the guidelines of classical electromagnetism (see Chapter 8 of [47]) follow. This procedure is described in detail in [38]. According to the formulas for the retarded potentials, the field at the point of observation at time t is determined by the state of motion of the body at the earlier time t_r , which is determined by (19) (in the following, the index r denotes quantities evaluated at the retarded time t_r).

We can introduce a reference system comoving with the body at the retarded time, whose temporal origin coincides with the retarded time. The coordinates with respect to this frame will be denoted by X^α , and they can be derived by the instantaneous Lorentz transformation

$$X^\alpha = \tilde{\Lambda}_{r\mu}^\alpha (x^\mu - x_{pr}^\mu), \quad (\text{A1})$$

where $x_{pr}^0 = ct_r$ and $\mathbf{x}_{pr} = \mathbf{x}_p(t_r)$ are the coordinates of the body at the retarded time. It is important to notice that the Lorentz transformation is done at the retarded time [i.e.

$\tilde{\Lambda}_{r\mu}^\alpha = \tilde{\Lambda}_\mu^\alpha(t_r)$. The four-vector X^α is a null four-vector [$\eta_{\alpha\beta}X^\alpha X^\beta = \eta_{\mu\nu}(x^\mu - x_r^\mu)(x^\nu - x_r^\nu) = c^2(t - t_r)^2 - |\mathbf{x} - \mathbf{x}_p(t_r)|^2 = 0$]. In this frame, the space-time metric is known, $H^{\alpha\beta} = \frac{2W(X^i)}{c^2}\delta^{\alpha\beta}$, which can be written in a manifestly covariant way as $H^{\alpha\beta} = \frac{4W(X^i)}{c^2}(U_{pr}^\alpha U_{pr}^\beta - \frac{1}{2}\eta^{\alpha\beta})$ with U_{pr}^α the four-velocity of the body at the retarded time (in the comoving frame $U_{pr}^\alpha = \delta^{\alpha 0}$). Since the expression of the metric is manifestly covariant, we can express it in the global frame by using the local transformation (A1),

$$h^{\mu\nu} = \frac{4W(X^i)}{c^2} \left(u_{pr}^\alpha u_{pr}^\beta - \frac{1}{2}\eta^{\alpha\beta} \right), \quad (\text{A2})$$

where the X^i are given by the transformation (A1) and where u_{pr}^μ is now given by $u_{pr}^\mu = \gamma_{pr}(1, \boldsymbol{\beta}_{pr})$ with $\gamma_{pr} = \gamma_p(t_r)$ and $\mathbf{v}_{pr} = d\mathbf{x}_p/dt|_{t_r}$.

We can write the space-time metric as (17) with X^i given by (A1), which can be explicitly written as (18). For example, in the case of a point mass ($W = GM/R$), one gets

$$R = |X^i| = \gamma_{pr}(r_{pr} - (\mathbf{r}_{pr} \cdot \boldsymbol{\beta}_{pr}))$$

and the metric can be written as

$$h^{00} = \frac{2GM}{c^2(r_{pr} - (\mathbf{r}_{pr} \cdot \boldsymbol{\beta}_{pr}))} \gamma_{pr}(1 + \beta_{pr}^2) + \mathcal{O}(G^2), \quad (\text{A3a})$$

$$h^{0i} = \frac{4GM}{c^2(r_{pr} - (\mathbf{r}_{pr} \cdot \boldsymbol{\beta}_{pr}))} \beta_{pr}^i \gamma_{pr} + \mathcal{O}(G^2), \quad (\text{A3b})$$

$$h^{ij} = \frac{2GM}{c^2(r_{pr} - (\mathbf{r}_{pr} \cdot \boldsymbol{\beta}_{pr}))\gamma_{pr}} (\delta_{ij} + 2\beta_{pr}^i \beta_{pr}^j \gamma_{pr}^2) + \mathcal{O}(G^2). \quad (\text{A3c})$$

This expression is exactly the same as the one found in [18,19,38]. In the limit of small velocities, the expressions of the IAU metric are recovered (see Appendix B 2).

APPENDIX B: CORRESPONDENCE WITH THE IAU METRIC

1. Case of uniformly moving bodies

In Sec. IV A, we derive the post-Minkowskian metric related to uniformly moving bodies (13). It is interesting to show that the post-Minkowskian limit of the IAU metric [37] is recovered in the limit of the small velocities. In order to show this, we first need to develop the argument appearing in the potential W from Eq. (14) as

$$\begin{aligned} \tilde{\Lambda}_\mu^i(x^\mu - b^\mu) &= -\gamma_p \beta_p^i c(t - t_0) + x^i - x_p^i(t_0) \\ &\quad + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i \boldsymbol{\beta}_p \cdot (\mathbf{x} - \mathbf{x}_p(t_0)). \end{aligned} \quad (\text{B1})$$

Using the fact that the motion of the body is uniform, this expression can also be written as

$$\tilde{\Lambda}_\mu^i(x^\mu - b^\mu) = x^i - x_p^i(t) + \frac{\gamma_p^2}{1 + \gamma_p} \beta_p^i \boldsymbol{\beta}_p \cdot (\mathbf{x} - \mathbf{x}_p(t)). \quad (\text{B2})$$

In the limit of the small velocities ($\beta_p \ll 1$), we have

$$\tilde{\Lambda}_\mu^i(x^\mu - b^\mu) = x^i - x_p^i(t) + \frac{1}{2}\beta_p^i \boldsymbol{\beta}_p \cdot (\mathbf{x} - \mathbf{x}_p(t)) + \mathcal{O}(\beta_p^4). \quad (\text{B3})$$

Using this expansion in the limit of small velocities, the expression of the potential W appearing in the metric (13) becomes

$$\begin{aligned} W[\tilde{\Lambda}_\mu^i(x^\mu - b^\mu)] &= W[x^i - x_p^i(t)] + \frac{1}{2}W_{,j}[x^i - x_p^i(t)]\beta^j \boldsymbol{\beta}_p \\ &\quad \cdot (\mathbf{x} - \mathbf{x}_p(t)) + \mathcal{O}(\beta_p^4), \end{aligned} \quad (\text{B4})$$

where $W_{,j} = \partial W / \partial X^j$. Introducing this expression in the metric (13) leads to

$$\begin{aligned} h^{00} &= \frac{2W(x^i - x_p^i)}{c^2} + 4\frac{W(x^i - x_p^i)}{c^2}\beta_p^2 \\ &\quad + \frac{\boldsymbol{\beta}_p \cdot (\mathbf{x} - \mathbf{x}_p(t))}{c^2}\beta_p^j W_{,j}(x^i - x_p^i(t)) + \mathcal{O}(G^2) \\ &\quad + \mathcal{O}(\beta_p^4/c^2), \end{aligned} \quad (\text{B5a})$$

$$h^{0i} = \frac{4W(x^i - x_p^i)}{c^2}\beta_p^i + \mathcal{O}(G^2) + \mathcal{O}(\beta_p^3/c^2), \quad (\text{B5b})$$

$$h^{ij} = \frac{2W(x^i - x_p^i)}{c^2}\delta_{ij} + \mathcal{O}(G^2) + \mathcal{O}(\beta_p^2/c^2). \quad (\text{B5c})$$

For example, in the case of a point mass ($W = \frac{GM}{R}$), the last expression becomes

$$\begin{aligned} h^{00} &= 2\frac{GM}{r_p c^2} + 4\frac{GM}{r_p c^2}\beta_p^2 - \frac{GM}{c^2 r_p^3}(\mathbf{r} \cdot \boldsymbol{\beta}_p)^2 + \mathcal{O}(G^2) \\ &\quad + \mathcal{O}(\beta_p^4/c^2), \end{aligned} \quad (\text{B6a})$$

$$h^{0i} = 4\frac{GM}{r_p c^2}\beta_p^i + \mathcal{O}(G^2) + \mathcal{O}(\beta_p^3/c^2), \quad (\text{B6b})$$

$$h^{ij} = 2\frac{GM}{r_p c^2}\delta_{ij} + \mathcal{O}(G^2) + \mathcal{O}(\beta_p^2/c^2), \quad (\text{B6c})$$

with $r_p = |\mathbf{r}_p|$ and $\mathbf{r}_p = \mathbf{x} - \mathbf{x}_p(t)$. This expression is exactly the one recommended in the IAU conventions [see Eqs. (8) and (51–55) from [37] or Resolutions B1.5 in the Appendix of the same paper].

2. Case of arbitrarily moving point masses

The metric (17) or (A3) describes the space-time geometry around an arbitrarily moving point mass at the first post-Minkowskian approximation. It is interesting to show that the post-Minkowskian limit of the IAU metric [37] is recovered in the limit of the small velocities.

We need to express the quantities at the retarded time t_r as a function of the quantities at the time t . Since

$$t_r - t = -\frac{r_{pr}}{c}, \quad (\text{B7})$$

we have

$$\begin{aligned} \mathbf{r}_p(t) &= \mathbf{r}_p = \mathbf{x} - \mathbf{x}_p(t) \\ &= \mathbf{x} - \mathbf{x}_p(t_r) - (t - t_r)\mathbf{v}_p(t_r) - \frac{(t - t_r)^2}{2}\mathbf{a}_p(t_r) \\ &\quad + \mathcal{O}((t - t_r)^3) \end{aligned} \quad (\text{B8})$$

$$= \mathbf{r}_{pr} - \frac{r_{pr}}{c}\mathbf{v}_{pr} - \frac{r_{pr}^2}{2c^2}\mathbf{a}_{pr} + \mathcal{O}(c^{-3}). \quad (\text{B9})$$

It is useful to notice that in the last term of this expression, we can replace t_r by t (this will introduce a higher order correction). A simple calculation leads to

$$\begin{aligned} r_p^2 + (\boldsymbol{\beta}_p \cdot \mathbf{r}_p)^2 &= (r_{pr} - (\boldsymbol{\beta}_{pr} \cdot \mathbf{r}_{pr}))^2 + r_{pr}^2 \beta_{pr}^2 \\ &\quad - \frac{r_p^2}{c^2} \mathbf{a}_p \cdot \mathbf{r}_p + \mathcal{O}(c^{-3}). \end{aligned} \quad (\text{B10})$$

This leads to

$$\begin{aligned} r_{pr} - (\boldsymbol{\beta}_{pr} \cdot \mathbf{r}_{pr}) &= r_p \left[1 - \frac{\beta_{pr}^2}{2} + \frac{(\boldsymbol{\beta}_p \cdot \mathbf{r}_p)^2}{2r_p^2} + \frac{\mathbf{a}_p \cdot \mathbf{r}_p}{2c^2} \right] \\ &\quad + \mathcal{O}(c^{-3}). \end{aligned} \quad (\text{B11})$$

Since

$$\gamma_{pr} = \gamma_p + \mathcal{O}(c^{-3}) = 1 + \frac{\beta_p^2}{2} + \mathcal{O}(c^{-3}), \quad (\text{B12})$$

we have

$$\frac{1}{\gamma_{pr}(r_{pr} - (\boldsymbol{\beta}_{pr} \cdot \mathbf{r}_{pr}))} = \frac{1}{r_p} \left[1 - \frac{(\boldsymbol{\beta}_p \cdot \mathbf{r}_p)^2}{2r_p^2} - \frac{\mathbf{a}_p \cdot \mathbf{r}_p}{2c^2} \right]. \quad (\text{B13})$$

Introducing Eq. (B13) in the space-time metric (A3) leads to

$$\begin{aligned} h^{00} &= \frac{2GM}{c^2 r_p} + 4 \frac{GM}{c^2 r_p} \beta_p^2 - \frac{GM}{c^2 r_p^3} (\mathbf{r}_p \cdot \boldsymbol{\beta}_p)^2 \\ &\quad - \frac{GM}{c^4 r_p} (\mathbf{a}_p \cdot \mathbf{r}_p) + \mathcal{O}(G^2) + \mathcal{O}(1/c^5), \end{aligned} \quad (\text{B14a})$$

$$h^{0i} = 4 \frac{GM}{r_p^2 c^2} \beta_p^i + \mathcal{O}(G^2) + \mathcal{O}(1/c^4), \quad (\text{B14b})$$

$$h^{ij} = 2 \frac{GM}{r_p c^2} \delta_{ij} + \mathcal{O}(G^2) + \mathcal{O}(1/c^3). \quad (\text{B14c})$$

The only additional term with respect to the metric (B6) is the term proportional to the acceleration in g_{00} . This term is exactly the one appearing in the IAU metric [see Eq. (54) of [37]] as already noticed in [48].

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