

Minimal massive gravity: Conserved charges, excitations, and the chiral gravity limit

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We find the excitations and construct the conserved charges (mass and angular momentum) of the recently found minimal massive gravity (MMG) in $2 + 1$ dimensions in asymptotically anti-de Sitter spacetimes. The field equation of the theory does not come from an action and lacks the required Bianchi identity needed to define conserved charges. But the theory, which also provides a healthy extension of the topologically massive gravity in the bulk and boundary of spacetime, does admit conserved charges for the metric that are solutions. Our construction is based on background Killing vectors and imperative to provide physical meaning to the integration constants in the black hole-type metrics. As an example, we compute the mass and angular momentum of the Bañados-Teitelboim-Zanelli black hole in MMG. We also find the central charges of the boundary field theory and study the chiral gravity limit of MMG.

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I. INTRODUCTION

It is well known that Einstein's gravity in $2 + 1$ dimensions is devoid of any local degrees of freedom even though it has black hole solutions when a negative cosmological constant is added [1]. (We shall work with the mostly plus signature.) On the other hand, the parity-violating topologically massive gravity (TMG) [2] has both a single massive spin-2 local degree of freedom as well as various black hole solutions. But TMG has a bulk-boundary unitarity conflict that makes it rather unsuitable for the AdS/CFT "applications." Namely, either the bulk or the boundary theory is nonunitary as can be seen from the comparison of the unitarity regions defined by the positivity of the two Brown-Henneaux boundary charges [3] and the bulk energies [4]. In a certain parameter choice, one hopes that one has a unitary "chiral gravity" theory [5], but it turns out that at the chiral point there appear log solutions, making the boundary theory a nonunitary logarithmic conformal field theory (CFT) [6–8]. Unfortunately, this state of affairs (the bulk-boundary unitarity conflict) remained intact in the "new massive gravity" (NMG) [9] that provided a nonlinear extension of the three-dimensional Fierz-Pauli theory with two spin-2 degrees of freedom both in flat and anti-de Sitter (AdS) spacetimes [10]. Further modification of NMG with more powers of curvature did not solve the unitarity conflict [11–13]. For example, the Born-Infeld extension of NMG with (in principle) infinitely many powers of curvature with rather remarkable properties is unitary either in the bulk or on the boundary of AdS [12,14].

At this point, one must be quite puzzled: Einstein's gravity is healthy both in the bulk and boundary, but it has no local degrees of freedom and hence in some sense is not a good example in which one can study "quantum gravity."

On the other hand, the above-mentioned many, otherwise healthy, nontrivial theories with one or two massive gravitons fail to be unitary on the boundary. Apparently, one is forced to choose either local-triviality or boundary nonunitarity. In the age of AdS/CFT, the second option is nothing less than heresy, and hence we are back with Einstein's theory. But, recently, a very interesting paper [15] appeared in which the authors showed that there is one more virtue, slightly less dangerous than the previous two, that one can let go; that is the Lagrangian formulation of the theory. Namely, they defined a theory—minimal massive gravity (MMG)—that does not come from the variation of an action with the metric as the independent field and hence also lacks the all-important Bianchi identity and the diffeomorphism invariance. Nevertheless, a consistent restricted version of the theory has a single massive degree of freedom that is unitary in the bulk and gives rise to a unitary CFT on the boundary. [See also the recent work on MMG that couples MMG to matter fields (a nontrivial construction), Refs. [16] and [17], for another derivation of the theory.] All this is quite good since, after all, MMG seems to be the theory without the infamous bulk-boundary unitarity conflict. But, of course, to make sense of the black hole solutions of the theory, to define thermodynamics, etc., one has to have conserved charges, especially energy in the theory. Otherwise, one could not even assign a physical meaning to the parameters (or integration constants) in the solutions of the theory.

This is the task we carry out in this work following the well-established Arnowitt-Deser-Misner [18] charges of general relativity (or in retrospect any purely metric-based gravity theory for asymptotically flat spacetimes) and Abbott-Deser [19] charges of (cosmological) general relativity for asymptotically (A)dS spacetimes and their generalization, the Abbott-Deser-Tekin [20] charges, for quadratic gravity theories and generic f (Riemann) theories

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[21,22]. The details of this Killing charge construction are given elsewhere [20]; hence, the discussion in Sec. II will be just a recapitulation of the essential points.

We also study the excitations of the MMG theory in AdS backgrounds and construct the boundary central charges and excitation energies as well as the chiral gravity limit of the theory.

II. CONSERVED CHARGES OF GRAVITY THEORIES ENDOWED WITH BIANCHI IDENTITIES

Given the field equations of the theory as

$$\Phi_{\mu\nu}(g, R, \nabla\text{Riemann}, R^2, \dots) = \kappa\tau_{\mu\nu}, \quad (1)$$

one assumes that for vanishing $\tau_{\mu\nu}$ (A)dS is the background solution, $\Phi_{\mu\nu}(\bar{g}, \bar{R}, \bar{\nabla}\bar{R}, \bar{R}^2, \dots) = 0$, with the metric $\bar{g}_{\mu\nu}$ with vanishing energy and angular momenta (let us stay in generic n dimensions for this brief discussion, so there could be more than one angular momentum). (A)dS has the maximum number of symmetries, so let us denote the Killing vectors as $\bar{\xi}_\mu$. (Note that, so as to not clutter the notation, we do not put another index on the Killing vector that could identify the different Killing vectors.) Given a spacetime with the metric $g_{\mu\nu}$, which asymptotically has the same Killing symmetries as the background space, we can define ‘‘conserved charges’’ by first linearizing the field equations as

$$\mathcal{O}(\bar{g})_{\mu\nu\alpha\beta}h^{\alpha\beta} = \kappa\tau_{\mu\nu}, \quad (2)$$

where $g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + h_{\mu\nu}$. The operator $\mathcal{O}_{\mu\nu\alpha\beta}$ is a (0,4) tensor in AdS which can be easily found when the field equations are given. Note that the right-hand side of (2) has all the terms in the metric perturbation ($h_{\mu\nu}$) save the linear one. If the full equation (1) comes from a diffeomorphism-invariant action, then it satisfies the full Bianchi identity $\nabla_\mu\Phi^{\mu\nu} = 0$ with the covariant derivative taken with respect to the metric compatible $g_{\mu\nu}$. This then leads to the background ‘‘Bianchi identity’’ or background (covariant) conservation of the linearized equation $\bar{\nabla}_\mu\mathcal{O}(\bar{g})^{\mu\nu\alpha\beta}h^{\alpha\beta} = 0$. This is not yet sufficient to define globally conserved charges; one makes use of the Killing vectors to define a partially conserved current via $\sqrt{-\bar{g}}\bar{\nabla}_\mu(\bar{\xi}_\nu T^{\mu\nu}) = \partial_\mu(\sqrt{-\bar{g}}\bar{\xi}_\nu T^{\mu\nu}) = 0$. Using the Stokes theorem, one arrives at the conserved global charges for each Killing vector,

$$Q^\mu(\bar{\xi}) = \int_{\mathcal{M}} d^{n-1}x \sqrt{-\bar{g}}\bar{\xi}_\nu T^{\mu\nu} = \int_\Sigma d\Sigma_i \mathcal{F}^{\mu i}, \quad (3)$$

where \mathcal{M} is our $n - 1$ -dimensional spatial manifold with Σ being its boundary. Here, a crucial step for each theory is to find the antisymmetric tensor $\mathcal{F}^{\mu\nu}$ that satisfies $T^{\mu\nu}\bar{\xi}_\nu = \bar{\nabla}_\nu\mathcal{F}^{\mu\nu}$.

This procedure has been successfully applied to many theories in which the assumed conditions on asymptotic symmetries and the implicit assumption of the proper fall of the perturbation $h_{\mu\nu}$ at the boundary are satisfied. For example, let us recall the conserved charges of TMG that are pertinent to our main discussion of MMG charges. The TMG field equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu} = 0, \quad (4)$$

with the Cotton tensor given as

$$C^{\mu\nu} \equiv \frac{1}{\sqrt{-g}}\epsilon^{\mu\alpha\beta}\nabla_\alpha S^\nu_\beta, \quad (5)$$

where $S^\nu_\beta = R^\nu_\beta - \frac{1}{4}\delta^\nu_\beta R$ is the Schouten tensor in three dimensions. The field equations of TMG come from an action which is diffeomorphism invariant up to a boundary term, and hence the theory is endowed with the Bianchi identity and amenable to our charge definition. This was done in Ref. [4] with the result

$$Q^\mu(\bar{\xi}) = \frac{1}{2\pi G_3} \oint_{\partial\Sigma} dl_i \left(q_E^{\mu i}(\bar{\xi}) + \frac{1}{2\mu} q_E^{\mu i}(\bar{\Xi}) + \frac{1}{2\mu} q_C^{\mu i}(\bar{\xi}) \right), \quad (6)$$

where the parts coming from the Einstein tensor and the Cotton tensor read, respectively, as

$$q_E^{\mu i}(\bar{\xi}) \equiv \sqrt{-\bar{g}}(\bar{\xi}_\nu \bar{\nabla}^\mu h^{i\nu} - \bar{\xi}_\nu \bar{\nabla}^i h^{\mu\nu} + \bar{\xi}^\mu \bar{\nabla}^i h - \bar{\xi}^i \bar{\nabla}^\mu h + h^{\mu\nu} \bar{\nabla}^i \bar{\xi}_\nu - h^{i\nu} \bar{\nabla}^\mu \bar{\xi}_\nu + \bar{\xi}^i \bar{\nabla}_\nu h^{\mu\nu} - \bar{\xi}^\mu \bar{\nabla}_\nu h^{i\nu} + h \bar{\nabla}^\mu \bar{\xi}^i), \quad (7)$$

$$q_C^{\mu i}(\bar{\xi}) \equiv \epsilon^{\mu i \beta} \mathcal{G}_{\nu\beta} \bar{\xi}^\nu + \epsilon^{\nu i \beta} \mathcal{G}^\mu_{\beta\bar{\xi}^\nu} + \epsilon^{\mu\nu\beta} \mathcal{G}^i_{\beta\bar{\xi}^\nu}. \quad (8)$$

Here, interestingly, a new Killing vector built out of the curl of the background Killing vector arises: $\bar{\Xi}^\beta \equiv \epsilon^{\alpha\nu\beta} \bar{\nabla}_\alpha \bar{\xi}_\nu / \sqrt{-\bar{g}}$. All contractions and raising and lowering must be done with the background metric, for example, $h = \bar{g}^{\mu\nu} h_{\mu\nu}$. Even though the background tensors that appeared here were defined in Refs. [4,23], it pays to collect them here as we shall need some of them below for the computation in the MMG theory. The background satisfies

$$\bar{R}_{\mu\nu\alpha\beta} = \Lambda(\bar{g}_{\mu\nu}\bar{g}_{\alpha\beta} - \bar{g}_{\mu\beta}\bar{g}_{\alpha\nu}), \quad \bar{R}_{\mu\nu} = 2\Lambda\bar{g}_{\mu\nu}, \quad \bar{R} = 6\Lambda. \quad (9)$$

In three dimensions, we do not need the linearization of the Riemann tensor; hence, respectively, the linearized Ricci tensor and the Ricci scalar read

$$R_{\mu\nu}^L = \frac{1}{2}(-\bar{\square}h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}^\sigma \bar{\nabla}_\nu h_{\sigma\mu} + \bar{\nabla}^\sigma \bar{\nabla}_\mu h_{\sigma\nu}),$$

$$R^L \equiv (R_{\mu\nu} g^{\mu\nu})^L = R_{\mu\nu}^L \bar{g}^{\mu\nu} - 2\Lambda h = -\bar{\square}h + \bar{\nabla}_\mu \bar{\nabla}_\nu \bar{h}^{\mu\nu} - 2\Lambda h.$$

These can be used to find the linearized cosmological Einstein and the Cotton tensors as

$$\mathcal{G}_{\mu\nu} \equiv (G_{\mu\nu} + \Lambda g_{\mu\nu})^L = R_{\mu\nu}^L - \frac{1}{2}\bar{g}_{\mu\nu}R^L - 2\Lambda h_{\mu\nu},$$

$$C_L^{\mu\nu} = \frac{1}{\sqrt{-\bar{g}}} \epsilon^{\mu\alpha\beta} \bar{g}_{\beta\sigma} \bar{\nabla}_\alpha \left(R_L^{\sigma\nu} - 2\Lambda h^{\sigma\nu} - \frac{1}{4}\bar{g}^{\sigma\nu}R_L \right).$$

Here, as usual, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$.

Before we conclude this section and move on to our main goal, let us note that for a timelike Killing vector $\bar{\xi}^\mu = (-1, 0, 0)$ Q^0 corresponds to the energy that is background diffeomorphism invariant only if the spatial boundary is at infinity [as in the case of flat and (A)dS space]. For $\bar{\xi}^\mu = (0, 0, 1)$ (say in polar coordinates) vectors, Q^0 is the angular momentum. (Please note that the construction is coordinate independent). See some example computations in Refs. [23,24].

III. EXCITATIONS, CONSERVED CHARGES, AND THE CHIRAL GRAVITY LIMIT OF MMG

A. Linearization of the field equations: Excitations

As explained in the Introduction, MMG theory was designed to be free of the bulk-boundary unitarity conflict. But to obtain unitarity everywhere, the authors of Ref. [15] bartered unitarity with the precious Lagrangian formulation, and hence the Bianchi identity that must be valid for any metric is gone. This, of course, is quite worrisome in terms of the conserved charge definition as we stressed above. But there is a resolution as we shall see. The field equations of MMG are

$$E_{\mu\nu} \equiv \Lambda_0 g_{\mu\nu} + \sigma G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} + \frac{\gamma}{\mu^2} J_{\mu\nu} = 0, \quad (10)$$

with two dimensionless parameters σ and γ as well as two dimensional ones μ and Λ_0 . The new ingredient is the J tensor defined as

$$J^{\mu\nu} \equiv \frac{1}{2\det g} \epsilon^{\mu\rho\sigma} \epsilon^{\nu\tau\eta} S_{\rho\tau} S_{\sigma\eta}, \quad (11)$$

and one has a nonvanishing covariant divergence for generic metrics:

$$\sqrt{-\det g} \nabla_\mu J^{\mu\nu} = \epsilon^{\nu\rho\sigma} S_{\rho\tau} C_{\sigma\tau}. \quad (12)$$

This is at the root of the problem, but as noted in Refs. [15,16], for the *solutions* of the theory, this is indeed zero. Therefore, we can define conserved charges. First, let us note that one can rewrite the J tensor as

$$J_{\mu\nu} = G_\mu^\rho G_{\rho\nu} - \frac{1}{2}g_{\mu\nu}G_{\rho\sigma}G^{\rho\sigma} + \frac{1}{4}G_{\mu\nu}R + \frac{1}{16}g_{\mu\nu}R^2. \quad (13)$$

We can now find the effective cosmological constant of the theory (10) by first noting that $\bar{J}^{\mu\nu} = \frac{\Lambda^2}{4}\bar{g}^{\mu\nu}$. So the vacuum field equation reads

$$\Lambda_0 - \sigma\Lambda + \frac{\gamma}{4\mu^2}\Lambda^2 = 0, \quad (14)$$

with solutions

$$\Lambda_\pm = \frac{2\mu^2}{\gamma} \left(\sigma \pm \sqrt{\sigma^2 - \frac{\gamma\Lambda_0}{\mu^2}} \right). \quad (15)$$

We agreed that for the solutions of the full theory we have

$$\nabla_\mu J^{\mu\nu} = 0, \quad (16)$$

of which the linearization about the (A)dS vacuum leads to a background conserved tensor,

$$\bar{\nabla}_\mu \mathcal{J}_L^\mu = 0, \quad (17)$$

where $\mathcal{J}_L^{\mu\nu} \equiv (J^{\mu\nu})_L + \frac{\Lambda^2}{4}h^{\mu\nu}$. Note the all-important second term that makes the total expression a background diffeomorphism-invariant expression under transformations $\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$. Let us now compute the linearized form of (13) with the help of the linearized tensors given above to get

$$(J^{\mu\nu})_L = -\frac{\Lambda}{2}\mathcal{G}^{\mu\nu} - \frac{\Lambda^2}{4}h^{\mu\nu}; \quad (18)$$

hence, one has $\mathcal{J}_L^{\mu\nu} = -\frac{\Lambda}{2}\mathcal{G}^{\mu\nu}$, which is needed in defining the conserved charges of the full theory (10) of which the linearization about one of its (A)dS vacua gives

$$\bar{E}_{\mu\nu} + (\Lambda_0 - \sigma\Lambda)h_{\mu\nu} + \sigma\mathcal{G}_{\mu\nu} + \frac{1}{\mu}C_{\mu\nu}^L + \frac{\gamma}{\mu^2}(J_{\mu\nu})^L \equiv \kappa T_{\mu\nu}, \quad (19)$$

where the right-hand side represents all the nonlinear terms in h . We have also introduced a scaled Newton constant (κ) to keep the conventional dimensions of the energy-momentum tensor. Since $(J_{\mu\nu})^L = -\frac{\Lambda}{2}\mathcal{G}_{\mu\nu} + \frac{\Lambda^2}{4}h^{\mu\nu}$, using the vacuum field equation in the first term and the terms multiplying $h_{\mu\nu}$, we arrive at

$$\left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) \mathcal{G}_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu}^L = \kappa T_{\mu\nu}, \quad (20)$$

which is nothing but the linearized field equations of TMG with a modified coefficient in front of the Einsteinian part.

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Before we turn back to the conserved charge issue, let us say a few words about the bulk excitations of the theory: the linearized equation about (A)dS explains why bulk properties of MMG are the same as TMG, albeit with a modified mass; from (20), using the results of the previous works [6,25], we can write the mass of the single spin-2 mode in MMG as

$$M_g^2 = \mu^2 \left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right)^2 + \Lambda, \quad (21)$$

which satisfies the Breitenlohner–Freedman (BF) bound [26] $M_g^2 \geq \Lambda$ in AdS. The graviton mass vanishes at the two “chiral points” (a opposed to the one in TMG) for AdS,

$$\Lambda_0^\pm = \frac{\mu^2}{\gamma^3} (\gamma\sigma(2 + \gamma\sigma) - 2 \pm 2\sqrt{1 - 2\gamma\sigma}). \quad (22)$$

In fact, to actually see the “chirality” of the boundary conformal field theory, let us compute the left and right central charges in the theory. This was done in Ref. [15], but our notation and conventions are quite different, and hence it actually pays to repeat the calculation here since it is quite simple using our formulation. We already know the central charges of TMG, and it is clear that, as far as the central charges of the boundary theory are concerned, linearized MMG (20) with a vanishing left-hand side is sufficient. Hence, the two copies of the Virasoro algebra on the boundary have the central charges

$$\begin{aligned} c_L &= \frac{3\ell}{2G_3} \left(\sigma + \frac{\gamma}{2\mu^2\ell^2} - \frac{1}{\mu\ell} \right), \\ c_R &= \frac{3\ell}{2G_3} \left(\sigma + \frac{\gamma}{2\mu^2\ell^2} + \frac{1}{\mu\ell} \right), \end{aligned} \quad (23)$$

where we have used the AdS radius defined as $\ell^2 = -\frac{1}{\Lambda}$ and normalized the central charges as Brown and Henneaux [3].

Let us now consider the energies of the linear excitations following Refs. [5,27]. For this purpose, we need to find the (free) action leading to (20), which is

$$S = \frac{1}{2\pi G_3} \int d^3x \sqrt{-\bar{g}} \left[-\frac{1}{2} \left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) h^{\mu\nu} \mathcal{G}_{\mu\nu} - \frac{1}{2\mu} h^{\mu\nu} C_{\mu\nu}^L \right]. \quad (24)$$

We need to get the Ostrogradsky Hamiltonian; before that, it is a good idea to fix the gauge by choosing the transverse traceless conditions

$$\bar{\nabla}_\mu h^{\mu\nu} = 0, \quad h = 0, \quad (25)$$

which reduces the action to

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$$\begin{aligned} S &= -\frac{1}{4\pi G_3} \int d^3x \sqrt{-\bar{g}} \left[\left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) \nabla_\alpha h_{\mu\nu} \bar{\nabla}^\alpha h^{\mu\nu} \right. \\ &\quad \left. + 2\Lambda h_{\mu\nu} h^{\mu\nu} + \frac{1}{\mu} \epsilon_\mu^{\alpha\beta} \nabla_\alpha h_{\mu\nu} (\bar{\square} - 2\Lambda) h_{\beta\nu} \right]. \end{aligned} \quad (26)$$

Choosing the AdS metric as

$$ds^2 = \frac{1}{-\Lambda} (-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2) \quad (27)$$

and decomposing the metric into massive, left-moving and right-moving fluctuations as was done in Ref. [5],

$$h_{\mu\nu} \equiv h_{\mu\nu}^M + h_{\mu\nu}^L + h_{\mu\nu}^R, \quad (28)$$

the Ostrogradsky Hamiltonian leads to the excitation energies

$$E_M = \frac{M_g^2}{4\pi G_3 \mu T} \int d^3x \sqrt{-\bar{g}} \epsilon_\alpha^{0\mu} h^{\alpha\nu} \partial_t h_{\mu\nu}^M, \quad (29)$$

$$E_L = -\frac{c_L}{6\pi\ell T} \int d^3x \sqrt{-\bar{g}} \bar{\nabla}^0 h_L^{\alpha\nu} \partial_t h_{\mu\nu}^L, \quad (30)$$

$$E_R = -\frac{c_R}{6\pi\ell T} \int d^3x \sqrt{-\bar{g}} \bar{\nabla}^0 h_R^{\alpha\nu} \partial_t h_{\mu\nu}^R, \quad (31)$$

where we have made use of the central charges (23) and defined a (large) time T for which the relevance is explained in Ref. [27]. To judge the positivity or the negativity of all these three energies, we must find all solutions of the linearized theory which split into three equations: one for the massive

$$\epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu}^M + \mu \left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) h_{\mu\nu}^M = 0, \quad (32)$$

and the other two for left- and right-moving modes

$$\epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu}^L + \ell h_{\mu\nu}^L = 0, \quad \epsilon_\mu^{\alpha\beta} \bar{\nabla}_\alpha h_{\beta\nu}^R - \ell h_{\mu\nu}^R = 0. \quad (33)$$

Fortunately, the solutions of these equations were given in Ref. [5] using the $SL(2, R)_L \times SL(2, R)_R$ isometry of the AdS metric. Without going into further details, let us note that all the solutions furnish a representation of this group and can be generated from the three primaries with weights (h, \bar{h}) . The left-moving mode has weights $(2, 0)$, the right-moving mode has weights $(0, 2)$, and the massive mode has weights

$$h = \frac{3}{2} + \frac{\sigma\ell}{2} + \frac{\gamma}{4\ell\mu^2}, \quad \bar{h} = -\frac{1}{2} + \frac{\sigma\ell}{2} + \frac{\gamma}{4\ell\mu^2}, \quad (34)$$

where we assumed $\sigma - \frac{\gamma\Lambda}{2\mu^2} > 0$. We do not depict the explicit solution given in Ref. [5], since we only need the all-important conclusion that once the solutions are plugged in to the energy

expressions (29) to (31) all the integrals yield negative values. This is quite good news for the left- and right-moving modes since then, as noted in Ref. [15], both central charges can be positive, and positivity of central charges is not in conflict with the excitation energies of the left- and right-moving modes. For the massive mode, assuming $\mu > 0$, positivity of the energy demands that the square of the graviton mass is negative. But this is allowed in AdS and does not lead to tachyons as long as the BF condition is satisfied. In this case, as we have shown above, it is indeed satisfied. So we have a stronger condition than the BF condition for positive energy massive modes,

$$0 \geq M^2 \geq \Lambda. \quad (35)$$

Therefore, unlike the case of the TMG, there is no conflict between the bulk and boundary unitarity in MMG as noted in Ref. [15]. Let us now look at the chiral point at which $M_{\xi}^2 = 0$ and hence the bulk graviton disappears as well as the left-moving modes with $c_L = 0$. Right-moving modes survive with positive energy and a central charge $c_R = \frac{3}{G_3\mu}$ that differs from that of the chiral gravity limit of TMG. One might worry about the existence of the parameter region that could allow a positive μ under these conditions. It is easy to see that for $\gamma > 0$ the region $\sigma\gamma \leq 1/2$ yields such a μ . There is another problem that we do not deal with here; that is whether the chiral gravity point is “really” unitary in the sense that there will be logarithmic solutions that could lead to a nonunitary logarithmic field theory as in the case of the chiral gravity limit of TMG [7]. For new massive gravity, see the analogous discussions in Ref. [28].

B. Conserved charges

Having found the mass of the bulk spin-2 excitation, let us now return to our construction of conserved charges. With the background knowledge given in the previous section, we can now write down the conserved charges for the MMG theory as

$$Q^\mu(\bar{\xi}) = \frac{1}{2\pi G_3} \oint_{\Sigma} dl_i \left(\left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) q_E^{\mu i}(\bar{\xi}) + \frac{1}{2\mu} q_E^{\mu i}(\bar{\Xi}) + \frac{1}{2\mu} q_C^{\mu i}(\bar{\xi}) \right), \quad (36)$$

where we have chosen the normalization factor to conform to our earlier conventions in TMG. The integral is to be evaluated on a circle at infinity. Let us apply this to the rotating BTZ black hole or a spacetime that is asymptotically a rotating BTZ black hole with the metric

$$ds^2 = (mG_3 + \Lambda r^2) dt^2 - a dt d\phi + r^2 d\phi^2 + \frac{dr^2}{-mG_3 - \Lambda r^2 + \frac{a^2}{4r^2}}, \quad (37)$$

where a is the rotation parameter [1]. Choosing $m = 0$ and $a = 0$ to be the background, we obtain the mass (energy) corresponding to the Killing vector $\bar{\xi}^\mu = -(\frac{\partial}{\partial t})^\mu$ and the angular momentum corresponding to the Killing vector $\bar{\xi}^\mu = (\frac{\partial}{\partial \phi})^\mu$ of the asymptotically BTZ black hole in MMG, respectively, as

$$E = \frac{1}{G_3} \left(\left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) m + \frac{a\Lambda}{\mu} \right), \\ J = \frac{1}{G_3} \left(\left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) a - \frac{m}{\mu} \right). \quad (38)$$

Note that these expressions reduce to the TMG forms when $\gamma = 0$ [24]. Let us see the chiral gravity limit of MMG from these expressions. The angular momentum vanishes when

$$a = \frac{m}{\left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right) \mu}, \quad (39)$$

and at this point, the energy becomes

$$E = \frac{m}{G_3\mu \left(\sigma - \frac{\gamma\Lambda}{2\mu^2} \right)} M_{\text{graviton}}^2, \quad (40)$$

which also vanishes at the point at which the bulk graviton is massless. Note also that positivity of black hole energy is not in conflict with the positivity of the energy of excitations.

One might wonder how conserved charges will be defined for spacetimes that are not asymptotically AdS. We shall not go into that discussion here, since it was carried out in Refs. [29,30] and the expressions are valid for MMG with small adjustments of the coefficients.

IV. CONCLUSIONS

Following the Killing charge techniques, we have constructed the conserved mass and angular momentum of the recently found minimal massive gravity that only has an on-shell Bianchi identity. But that is sufficient to define conserved quantities. We have applied our formulation to the rotating BTZ black hole. We have also studied excitations of the theory and found the left and right central charges of the boundary algebra as well as the energies of massive and massless left- and right-moving modes. There is no conflict between the boundary and bulk unitarity. We also constructed the chiral gravity limit of the MMG theory. As of now, it is not clear if this chiral theory has a unitary boundary CFT or not.

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