

**Laboratory tests of Lorentz and *CPT* symmetry with muons**André H. Gomes,<sup>1</sup> V. Alan Kostelecký,<sup>2</sup> and Arnaldo J. Vargas<sup>2</sup><sup>1</sup>*Departamento de Física, Universidade Federal de Viçosa, 36570-000 Viçosa, MG, Brazil*<sup>2</sup>*Physics Department, Indiana University, Bloomington, Indiana 47405, USA*

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The prospects are explored for testing Lorentz and *CPT* symmetry in the muon sector via the spectroscopy of muonium and various muonic atoms, and via measurements of the anomalous magnetic moments of the muon and antimuon. The effects of Lorentz-violating operators of both renormalizable and nonrenormalizable dimensions are included. We derive observable signals, extract first constraints from existing data on a variety of coefficients for Lorentz and *CPT* violation, and estimate sensitivities attainable in forthcoming experiments. The potential of Lorentz violation to resolve the proton radius puzzle and the muon anomaly discrepancy is discussed.

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**I. INTRODUCTION**

Muons have played a significant role in testing relativity since their discovery in the 1930s [1]. Indeed, the first demonstration of time dilation was the Rossi-Hall experiment studying muons originating from cosmic rays [2]. As another example, the clock hypothesis that acceleration *per se* has no effect on a clock's ticking rate has been verified using muons in a ring accelerator [3].

In recent years, the prospect of tiny deviations from relativity has emerged as a promising candidate signal for new physics coming from the Planck scale  $M_P \approx 10^{19}$  GeV, following the demonstration that Lorentz invariance can naturally be broken in a unified framework of quantum gravity such as string theory [4]. Driven by this prospect, many high-precision tests of relativity in different systems have been performed to search for Lorentz violation [5]. Here, we investigate the role of muons in this context, focusing on laboratory studies using spectroscopy of muonic bound states and measurements of the muon anomalous magnetic moment.

The general theoretical description of Lorentz violation is provided by a realistic effective quantum field theory called the Standard-Model Extension (SME) [6,7]. The SME offers a theoretical framework for analysis based on General Relativity and the Standard Model (SM), and it also characterizes *CPT* violation [6,8]. Its Lagrange density consists of all coordinate-independent scalars built from the contractions of Lorentz-violating operators with coefficients determining the size of the associated effects. The mass dimension of each coefficient is fixed by the dimension  $d$  of the operator, with operators of larger  $d$  often taken as higher-order terms in a low-energy expansion of the fundamental theory. The restriction to terms containing operators of renormalizable dimensions  $d \leq 4$  is called the minimal SME.

The structure of the SME reveals that Lorentz and *CPT* violation can be sector dependent, with coefficients varying

according to the particle species involved. The size of Lorentz and *CPT* violation could conceivably increase with mass, for example, if the Yukawa-type couplings from spontaneous Lorentz violation scale like the conventional Yukawa couplings in the SM. Muon-sector experiments are of particular interest in this context because they offer excellent prospects for a sensitive study of Lorentz and *CPT* violation in second-generation matter. However, given the extensive historical impact of research with muons and their comparatively widespread availability, surprisingly little is known about the SME muon sector on both the theoretical and the experimental fronts. For example, inspection of the *Data Tables* [5] reveals that existing constraints on Lorentz and *CPT* violation involving muons comprise only a small fraction of the available limits. The effects of minimal-SME coefficients on the behavior of muons [9] have been studied at impressive sensitivities in the laboratory via muonium hyperfine spectroscopy [10] and via measurements of the anomalous magnetic moments of the muon and antimuon [11,12]. The latter have also been used to place limits on non-minimal interaction terms with  $d = 5$  [13], while minimal-SME interactions have been studied in the context of muon decay [14]. A few constraints from astrophysical processes have been obtained as well, both for minimal-SME coefficients [15] and for isotropic nonminimal operators with  $d = 5, 6$  [16,17].

In the present paper, we take advantage of the recently developed comprehensive approach to Lorentz-violating operators governing the propagation of a massive Dirac fermion [17] to study a broad range of effects involving muon operators of both renormalizable and nonrenormalizable dimension. Nonminimal terms produce effects that grow with energy, so typically higher-energy experiments have greater sensitivity to these effects. However, observable effects in the nonrelativistic limit involve combinations of operators of arbitrary  $d$ , and so studying these offers a different and powerful measure of sensitivity. We therefore

consider both nonrelativistic and relativistic experiments in what follows.

For definiteness and simplicity, the analysis in this paper is restricted to Lorentz violation in the muon sector. Lorentz violation in other sectors can better be sought with correspondingly dedicated experiments. In the event that a nonzero signal is found, comparison of results among different sectors would be necessary to establish unequivocally the origin of the effect. This approach is justified because no compelling experimental evidence for Lorentz violation exists at present, so the subject is currently in a search phase rather than a model-building phase. Our analysis also focuses on effects originating from muon kinematics, which provide the leading-order corrections from Lorentz violation in the experiments we consider. For example, the leading-order correction in Coulomb gauge is independent of the four-vector potential in the bound states discussed below, while the external magnetic fields used in all experiments are tiny compared to the muon mass and hence their Lorentz-violating contributions are suppressed by many orders of magnitude. Disregarding effects in interactions also implies neglecting the Lorentz-violating flavor-changing operators that mix the charged leptons in the SME, but as these necessarily also entail lepton-number violation they can plausibly be taken as suppressed relative to the effects we consider here.

The analysis that follows can therefore be viewed as primarily an investigation of the Lorentz- and *CPT*-violating kinetic Lagrange density for the muon and antimuon,

$$\mathcal{L} \supset \frac{1}{2} \bar{\psi} (\gamma^\mu i\partial_\mu - m_\mu + \hat{Q}) \psi + \text{H.c.}, \quad (1)$$

where  $\psi(x)$  is the muon quantum field,  $m_\mu$  is the muon mass, and  $\hat{Q}$  is a spinor-matrix operator describing all kinetic effects from Lorentz and *CPT* violation, formed from derivatives  $i\partial_\mu$  and SME coefficients for Lorentz and *CPT* violation. By expanding  $\hat{Q}$  in the basis of Dirac matrices in momentum space and performing a decomposition in spherical coordinates, the SME coefficients at each  $d$  can be classified and enumerated [17]. The freedom to perform field redefinitions in the theory without changing the physics implies that only certain combinations of these coefficients, called effective coefficients, are observable in a given experiment. Under the assumptions made here, the muon sector with  $d = 3$  is found to have six independent effective coefficients controlling *CPT*-even effects. The sector with  $d = 4$  contains 30 independent effective coefficients, of which 20 govern *CPT*-odd effects and the other 10 are *CPT* even but observable only if coordinate choices establishing the Minkowski metric have otherwise been fixed. The nonminimal muon sector with  $d = 5$  has 65 independent effective coefficients of which 20 are associated with *CPT*-odd operators, while for  $d = 6$  we find 119 independent effective coefficients with 84

corresponding to *CPT*-odd operators. Each observable effective coefficient represents a distinct physical way to violate Lorentz symmetry. As described in the sections below, the muon experiments considered in this paper can access only a subset of these coefficients in specific linear combinations, but they nonetheless provide a broad-scope survey of possible muon-sector effects and in many cases yield Planck-scale sensitivity.

Our investigations begin in Sec. II with the spectroscopy of muonic bound states. Following some basics presented in Sec. II A, the effects of Lorentz and *CPT* violation on muonium spectroscopy are considered in Sec. II B, concerning first the hyperfine transitions and then the  $1S$ - $2S$  transition and the Lamb shift. We use existing data to place numerous first constraints on nonrelativistic coefficients for Lorentz violation and estimate possible sensitivities in some future experiments. Section II C contains our discussion of the spectroscopy of muonic atoms, with a focus on muonic hydrogen. After some general considerations, the prospects are investigated for future searches using sidereal variations in Zeeman transitions. In Sec. II C 3 we address the proton radius puzzle in the context of Lorentz violation, outlining the requirements for a resolution and the ensuing predictions for future experiments. Section II C 4 describes a scheme for performing searches for Lorentz violation when the applied magnetic field is effectively negligible, as is the case in current experiments. The spectroscopy of various other muonic atoms, including among others muonic deuterium and muonic helium, is discussed in Sec. II C 5.

Section III focuses on measurements of the muon and antimuon magnetic moments. Some relevant theory is presented in Sec. III A. We then turn to comparisons of the muon and antimuon in Sec. III B, where techniques for extracting constraints on *CPT*-odd and *CPT*-even operators are described and used in conjunction with existing data to place numerous first bounds on nonminimal coefficients for Lorentz and *CPT* violation. Another potential signal is sidereal variations, which are the subject of Sec. III C and also lead to a variety of first bounds. In Sec. III D we consider the potential of future analyses to incorporate signals involving annual variations, some of which are tied to the Earth's changing boost as it orbits the Sun. Existing measurements of the muon anomaly lack full concordance with SM calculations, and in Sec. III E we consider the prospects of accounting for the anomaly discrepancy using Lorentz violation and describe some predicted signals in future experiments. Finally, Sec. IV concludes with a summary and discussion of other possibilities for future exploration of muon-sector Lorentz violation.

With a few exceptions described in the text, the notation and conventions throughout this paper are those of Ref. [17]. For simplicity, the index  $\mu$  indicating the muon sector is omitted from all coefficients.

## II. MUONIC BOUND STATES

This section focuses on searches for Lorentz and *CPT* violation using spectroscopy of exotic bound states having a muon or antimuon as a constituent. Among the many possible systems are various onia, which are bound states of a muon with another lepton of opposite charge; muonic atoms or ions, which are atoms or ions with an electron replaced by a muon; and hadron-muon bound states. Recent high-precision spectroscopy has been performed with muonium [10], which is the bound state of an antimuon  $\mu^+$  and an electron  $e^-$ , and with muonic hydrogen  $H_\mu$  [18], which is the bound state of a proton  $p$  with a muon  $\mu^-$ . Spectroscopy of muonic deuterium  $D_\mu$  and of the muonic-helium ions  ${}^3\text{He}_\mu^+$  and  ${}^4\text{He}_\mu^+$  is in the offing [19,20]. Future studies of other exotic bound states involving muons such as muonic tritium  $T_\mu$  [21] or various muonic ions such as  ${}^6\text{Li}_\mu^{2+}$ ,  ${}^7\text{Li}_\mu^{2+}$ ,  ${}^9\text{Be}_\mu^{3+}$ , or  ${}^{11}\text{B}_\mu^{4+}$  [22] may be of interest as well. Here, we consider the effects of Lorentz and *CPT* violation on the spectroscopy of all these muonic bound states.

With minor notation and interpretational changes, many of the results that follow can be directly transcribed to many other muonic bound states of potential interest. One exception is true muonium, the bound state of a muon with an antimuon, which may be observed and studied in electron-positron colliders [23] or in fixed-target electroproduction [24]. The transcription in this case requires taking into account the equal masses of the two constituents and the Lorentz-violating corrections for both the muon and the antimuon. With specific sign changes as indicated in the text below, our results are also directly applicable to antimuonium, which is a bound state of a muon  $\mu^-$  and a positron  $e^+$ , and to antimuonic antihydrogen  $\bar{H}_\mu$ , which is a bound state of an antiproton  $\bar{p}$  and an antimuon  $\mu^+$ . Should precision spectroscopy of these exotic antiatoms eventually become feasible, direct *CPT* tests comparing muonium with antimuonium and  $H_\mu$  with  $\bar{H}_\mu$  could be performed. However, with current technology the search for *CPT* violation in the muonium and  $H_\mu$  systems is of necessity reliant on either studying sidereal variations or comparing observed transition frequencies with theoretical calculations.

### A. Basics

The bound states of interest here involve two particles of different masses, one of which may be an atomic nucleus. The rotational symmetry of the conventional interactions ensures conservation of total angular momentum  $F$  of the system and implies that energy levels labeled with the corresponding quantum number  $F$  are  $(2F + 1)$ -fold degenerate. The asymmetry of the masses leads to a hierarchy in the angular-momentum couplings, which causes the hyperfine structure to be smaller than the fine structure by the ratio of the lighter to the heavier mass. In

the absence of Lorentz violation, the general features of the spectra of the muonic bound states of interest therefore largely parallel those of hydrogen, with appropriate scalings originating in the mass, charge, and nuclear-spin differences. For example, the ground-state energy of  $H_\mu$  is larger than that of H by a factor of the ratio of the corresponding reduced masses, which is about 186. One notable exception is the Lamb shift in  $H_\mu$ , which is enhanced by a factor of order  $1/\alpha^2$  via radiative corrections in quantum field theory, producing a 2S level lying well below the 2P level [25,26].

Some spectroscopic experiments of interest are performed with the system placed in a magnetic field, which breaks rotational symmetry and hence also conservation of  $F$ , thereby lifting the  $(2F + 1)$ -fold degeneracies of the energy levels. In this paper, we treat the applied magnetic field as uniform and constant. We also assume the induced level shifts are smaller than the fine structure, although possibly smaller or larger than the hyperfine structure so that both the hyperfine Zeeman and the hyperfine Paschen-Back limits can be considered. In this scenario the magnitude of the total angular momentum  $J$  of the lighter particle, which is the sum  $J = L + S$  of its orbital and spin angular momenta, can be approximated as independently conserved. The corresponding quantum number  $J$  can therefore be used to label states even when  $F$  cannot.

Since combinations of Lorentz boosts generate rotations, violations of Lorentz symmetry are accompanied by violations of rotation invariance in generic observer frames. The presence of Lorentz violation can therefore lift some or all of the  $(2F + 1)$ -fold degeneracies in the energy levels of the free system, and it can modify the level splittings arising from an applied magnetic field. Unless otherwise specified, in this paper we assume the effects from Lorentz and *CPT* violation are small compared to those from any magnetic field present. The lifting of the degeneracies by the magnetic field then has the technical advantage of avoiding degenerate perturbation theory in calculations of Lorentz-violating corrections. For consistency, we also assume that the Lorentz violation is sufficiently small to ensure maintenance of the perturbative regime where stability and causality are preserved in concordant frames [27].

The muon is nonrelativistic in all the bound systems considered here, so for small Lorentz and *CPT* violation in the muon sector the dominant perturbations to the spectra arise from the nonrelativistic limit. In the Coulomb gauge, all relevant contributions from the electromagnetic interactions arise from the zero component of the covariant derivative acting on the muon field or equivalently in momentum space from the canonical energy of the muon, which in the nonrelativistic limit reduces to the muon mass. The leading-order perturbation is therefore independent of the electromagnetic potential, so it suffices for calculational purposes to consider only the Lorentz-violating corrections to the nonrelativistic free motion of the muon. This is



physically plausible because the binding energy of the system is small compared to the muon mass, and it also matches established results for related analyses of Lorentz violation in conventional atoms [28].

A complete classification of Lorentz-violating terms of arbitrary mass dimension that can appear in the quadratic Lagrange density for a massive Dirac fermion is given in Ref. [17], along with a derivation of the corresponding nonrelativistic Hamiltonian. To apply this framework in the present context, we can work in the zero-momentum frame of the two-particle atom, which in typical applications can be taken as the laboratory frame. The leading-order corrections due to Lorentz and *CPT* violation for a non-relativistic muon of momentum  $\mathbf{p}$  are then described by an effective Hamiltonian  $\delta h^{\text{NR}}(\mathbf{p})$  that can be split into four types of terms, according to whether the physics is spin independent or dependent and whether the *CPT* effects are even or odd.

For experimental applications, it is convenient to decompose  $\delta h^{\text{NR}}$  in spherical coordinates because sensitivity to rotational symmetry is the key to many searches for Lorentz violation. Given the unit momentum vector  $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ , we can define spherical polar angles  $\theta, \phi$  in momentum space by  $\hat{\mathbf{p}} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$ . A basis of unit vectors can be chosen as  $\hat{\mathbf{e}}_r = \hat{\mathbf{p}}$ ,  $\hat{\mathbf{e}}_{\pm} = (\hat{\boldsymbol{\theta}} \pm i\hat{\boldsymbol{\phi}})/\sqrt{2}$ , where  $\hat{\boldsymbol{\theta}}$  and  $\hat{\boldsymbol{\phi}}$  are the standard unit vectors associated with the polar angle  $\theta$  and azimuthal angle  $\phi$ . The result of decomposing  $\delta h^{\text{NR}}$  can then be expressed as [17]

$$\delta h^{\text{NR}} = h_0 + h_r \boldsymbol{\sigma} \cdot \hat{\mathbf{e}}_r + h_+ \boldsymbol{\sigma} \cdot \hat{\mathbf{e}}_+ + h_- \boldsymbol{\sigma} \cdot \hat{\mathbf{e}}_-, \quad (2)$$

where  $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$  contains the three Pauli matrices.

The expression (2) contains four component Hamiltonians  $h_0, h_r, h_{\pm}$  that depend on the magnitude and direction of the momentum and on SME coefficients for Lorentz and *CPT* violation. The spin-independent component  $h_0$  can be written as

$$h_0 = \sum_{kjm} |\mathbf{p}|^k {}_0Y_{jm}(\hat{\mathbf{p}}) (a_{kjm}^{\text{NR}} - c_{kjm}^{\text{NR}}), \quad (3)$$

while the spin-dependent terms take the form

$$\begin{aligned} h_r &= \sum_{kjm} |\mathbf{p}|^k {}_0Y_{jm}(\hat{\mathbf{p}}) (-g_{kjm}^{\text{NR}(OB)} + H_{kjm}^{\text{NR}(OB)}), \\ h_{\pm} &= \sum_{kjm} |\mathbf{p}|^k {}_{\pm 1}Y_{jm}(\hat{\mathbf{p}}) [i g_{kjm}^{\text{NR}(1E)} - i H_{kjm}^{\text{NR}(1E)} \\ &\quad \pm (g_{kjm}^{\text{NR}(1B)} - H_{kjm}^{\text{NR}(1B)})]. \end{aligned} \quad (4)$$

Here, the eight sets of quantities  $a_{kjm}^{\text{NR}}, c_{kjm}^{\text{NR}}, g_{kjm}^{\text{NR}(OB)}, g_{kjm}^{\text{NR}(1B)}, g_{kjm}^{\text{NR}(1E)}, H_{kjm}^{\text{NR}(OB)}, H_{kjm}^{\text{NR}(1B)}, H_{kjm}^{\text{NR}(1E)}$  are nonrelativistic effective coefficients for Lorentz and *CPT* violation. The relationships between these nonrelativistic coefficients

and the complete set of spherical coefficients governing Lorentz and *CPT* violation in the muon sector are given in Eqs. (111) and (112) of Ref. [17]. For the antimuon, the signs of the *a*- and *g*-type coefficients in  $\delta h^{\text{NR}}$  are reversed. The allowed ranges of indices on all the coefficients and their counting for given  $k$  are summarized in Table IV of Ref. [17]. To avoid potential confusion with the principal quantum number  $n$  of the exotic atoms considered here, we use the notation  $k$  instead of  $n$  for the first index on these coefficients. Also, in the above equations the quantities  ${}_s Y_{jm}(\hat{\mathbf{p}}) \equiv {}_s Y_{jm}(\theta, \phi)$  are spin-weighted spherical harmonics of spin weight  $s$ , with the usual spherical harmonics arising for  $s = 0$  as  ${}_0 Y_{jm}(\hat{\mathbf{p}}) \equiv Y_{jm}(\theta, \phi)$ . Some key features of spin-weighted spherical harmonics can be found in Appendix A of Ref. [29]. Note that the indices  $j, m$  characterize the rotational properties of the spherical harmonics and thereby of the associated operators for Lorentz violation. The reader is cautioned that these indices are distinct from the angular-momentum quantum numbers  $J, M$  of a muonic bound state.

To determine the dominant level shifts arising from Lorentz and *CPT* violation requires calculation of the expectation values of  $\delta h^{\text{NR}}$  in the unperturbed eigenstates of the exotic atom, which are the Schrödinger-Coulomb eigenfunctions for the reduced mass  $m_r$ . Inspection reveals that many of these expectation values vanish. The angular-momentum wave functions are parity eigenstates, so the expectation values of odd-parity Lorentz-violating operators are zero. This implies that only coefficients with even  $k \equiv 2q$  can contribute. Also, in the presence of a magnetic field, only components of the Lorentz-violating operators projected in the direction of the field play a role because the azimuthal pieces average to zero. Choosing laboratory coordinates with the field along the  $z$  direction, this implies that only coefficients with  $m = 0$  are relevant. Moreover, the *E*-type coefficients also fail to contribute because the corresponding operators are proportional to  ${}_+1 Y_{j0}(\hat{\mathbf{p}}) \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\phi}}$ , which precesses about the magnetic field. These results and other more specific ones described below significantly reduce the calculations required to obtain the dominant level perturbations.

Given a nonzero expectation value of  $\delta h^{\text{NR}}$ , evaluation of the part involving a power  $|\mathbf{p}|^k$  of the momentum magnitude can be performed directly because it is independent of the angular-momentum couplings. For  $k = 0, 2$ , and  $4$  we obtain

$$\begin{aligned} \langle |\mathbf{p}|^0 \rangle_{nL} &= 1, \\ \langle |\mathbf{p}|^2 \rangle_{nL} &= \left( \frac{\alpha m_r}{n} \right)^2, \\ \langle |\mathbf{p}|^4 \rangle_{nL} &= \left( \frac{\alpha m_r}{n} \right)^4 \left( \frac{8n}{2L+1} - 3 \right), \end{aligned} \quad (5)$$

where  $\alpha$  is the fine-structure constant,  $n$  is the principal quantum number, and  $L$  is the orbital angular-momentum

quantum number. In this paper, we disregard operators  $|\mathbf{p}|^k$  with  $k > 4$ , which have expectation values diverging for states with small values of  $n$  and  $L$ . Although this technical issue can in principle be avoided by regularizing, the physical scale of the expectation values is set by the factor  $(am_r)^k$ , which for  $k > 4$  typically introduces only unobservable corrections to the transition frequencies. For example, even comparatively large coefficients for Lorentz violation  $a_{6j0}^{\text{NR}} \approx 1 \text{ GeV}^{-5}$  would lead only to frequency shifts of order  $10^{-9}$  Hz in muonium and  $10^5$  Hz in  $\text{H}_\mu$ , due to the appearance of the factor  $(am_r)^6$ .

The calculation of expectation values yields the shifts in the transition frequencies. The sizes of these shifts are set by the coefficients for Lorentz violation, which in the above expression for  $\delta h^{\text{NR}}$  are defined in the zero-momentum frame of the two-particle atom. However, this frame is noninertial due to the rotation of the Earth about its axis and, to a lesser extent, due to the revolution of the Earth about the Sun. A reasonable approximation to an inertial frame on the time scale of experiments is the canonical Sun-centered frame [5,30] widely used to report results of searches for Lorentz and *CPT* violation, which has  $Z$  axis aligned with the Earth's rotation axis,  $X$  axis pointing from the Sun towards the vernal equinox, and time  $T$  with origin chosen by convention at the vernal equinox 2000. In this frame, the coefficients for Lorentz violation can be approximated as constants, so the rotation of the Earth introduces time dependence in some laboratory-frame SME coefficients and hence sidereal variations in physical observables [31]. The spherical decomposition greatly simplifies the calculation of these variations because the two frames are related by a rotation. Indeed, the sidereal dependence of the transition frequencies induced by a particular coefficient is essentially determined by its azimuthal index  $m$ . The general expression relating laboratory-frame coefficients to those in the Sun-centered frame is given by Eq. (139) of Ref. [29]. Results specific to the experiments considered here are presented in the subsections that follow.

## B. Muonium

In this subsection, we consider the effects of Lorentz and *CPT* violation on muonium spectroscopy. The full perturbation Hamiltonian  $\delta h^{\text{NR}}$  is used to determine the shifts in  $1S$  hyperfine transitions, while the shift in the  $1S$ - $2S$  transition and the Lamb shift are calculated using the spin-independent perturbation. Existing experimental data are used to place first constraints on some coefficients for Lorentz violation.

### 1. Hyperfine transitions

In a magnetic field the ground state of muonium splits into four sublevels, labeled 1, 2, 3, 4 in order of decreasing energy. Precision spectroscopy of the muonium  $1S$  hyperfine transitions  $\nu_{12}$  and  $\nu_{34}$  has been performed in a

comparatively strong magnetic field of about 1.7 T [10,32]. In this setup, the four levels can be labeled as  $|m_S, m_I\rangle$ , where  $S$  and  $I$  are the electron and muon spin quantum numbers, respectively. The frequency  $\nu_{12}$  corresponds to the transition  $|1/2, 1/2\rangle \leftrightarrow |1/2, -1/2\rangle$ , while  $\nu_{34}$  corresponds to  $|-1/2, -1/2\rangle \leftrightarrow |-1/2, 1/2\rangle$ .

The dominant shifts  $\delta\nu_{12}$  and  $\delta\nu_{34}$  induced by Lorentz violation in these transition frequencies are given by the expectation values of the part  $\delta h_{nS}^{\text{NR}}$  of the perturbation Hamiltonian (2) for antimuons that affects the  $nS_{1/2}$  levels,

$$\begin{aligned} \delta h_{nS}^{\text{NR}} = & \sum_{q=0}^2 |\mathbf{p}|^{2q} [-(\overset{\circ}{a}_{2q}^{\text{NR}} + \overset{\circ}{c}_{2q}^{\text{NR}}) \\ & + (g_{(2q)10}^{\text{NR}(OB)} + H_{(2q)10}^{\text{NR}(OB)}) Y_{10}(\hat{\mathbf{p}}) \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \\ & - \sqrt{2} (g_{(2q)10}^{\text{NR}(1B)} + H_{(2q)10}^{\text{NR}(1B)}) {}_1Y_{10}(\hat{\mathbf{p}}) \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\theta}}], \quad (6) \end{aligned}$$

where  $\overset{\circ}{a}_{2q}^{\text{NR}} \equiv a_{(2q)00}^{\text{NR}}/\sqrt{4\pi}$  and  $\overset{\circ}{c}_{2q}^{\text{NR}} \equiv c_{(2q)00}^{\text{NR}}/\sqrt{4\pi}$  are isotropic nonrelativistic coefficients [17]. This expression contains only nonrelativistic coefficients with  $j \leq 1$  because the expectation values of operators associated with other coefficients vanish. To illustrate this, suppose  $T_{jm}$  is a spherical-tensor operator and  $|j'm'\rangle$  is an angular-momentum eigenstate. The Wigner-Eckart theorem [33] then implies that the expectation value  $\langle j'm'|T_{jm}|j'm'\rangle$  vanishes if  $2j' < j$ . In the present case the angular momentum of each fermion is  $1/2$ , so only coefficients with  $j \leq 1$  contribute.

The relevant eigenstates  $|m_S, m_I\rangle$  for the perturbative calculation are the products of the Schrödinger-Coulomb ground state and the appropriate Pauli spinors. The comparatively strong magnetic field ensures that the transitions are essentially muon-spin flips, so any effects from Lorentz and *CPT* violation in the electron sector can reasonably be disregarded. The nonzero expectation values with respect to  $|m_S, m_I\rangle$  are given by

$$\begin{aligned} \langle Y_{10}(\hat{\mathbf{p}}) \boldsymbol{\sigma} \cdot \hat{\mathbf{p}} \rangle &= \frac{1}{\sqrt{3\pi}} m_I, \\ \langle {}_1Y_{10}(\hat{\mathbf{p}}) \boldsymbol{\sigma} \cdot \hat{\boldsymbol{\theta}} \rangle &= -\sqrt{\frac{2}{3\pi}} m_I. \quad (7) \end{aligned}$$

Incorporating also the results (5) reveals that the frequency shifts take the form

$$\begin{aligned} \delta\nu_{12} = & -\delta\nu_{34} \\ = & \sum_{q=0}^2 \frac{1}{\sqrt{12\pi^3}} (am_r)^{2q} (1 + 4\delta_{q2}) \\ & \times (g_{(2q)10}^{\text{NR}(OB)} + H_{(2q)10}^{\text{NR}(OB)} + 2g_{(2q)10}^{\text{NR}(1B)} + 2H_{(2q)10}^{\text{NR}(1B)}), \quad (8) \end{aligned}$$

where  $\delta_{q2} = 1$  when  $q = 2$  and vanishes otherwise, as usual. Note that the condition  $\delta\nu_{12} + \delta\nu_{34} = 0$  is a specific

prediction of the present theoretical framework. Note also that the corresponding frequency shifts for antimuonium are given by changing the signs of the  $g$ -type coefficients in this result. If it should become practical to perform antimuonium spectroscopy, then a direct comparison of the hyperfine frequency shifts for muonium and antimuonium would make possible independent measurements of the  $g$ -type coefficients.

The result (8) extends the previous expression reported in Ref. [9],  $\delta\nu_{12} = -\delta\nu_{34} = -\tilde{b}_3^*/\pi$ , which involves the coefficient combination [5]  $\tilde{b}_3^* = b_3 + m_\mu d_{30} + H_{12}$  associated with certain minimal-SME operators of mass dimensions 3 and 4. The present result includes also contributions from the minimal-SME coefficients  $g_{\lambda\mu\nu}$ , along with effects from many operators of nonminimal mass dimensions. In the limit where all nonminimal coefficients are set to zero, the minimal-SME result is recovered via the identity

$$g_{010}^{\text{NR}(0B)} + 2g_{010}^{\text{NR}(1B)} + H_{010}^{\text{NR}(0B)} + 2H_{010}^{\text{NR}(1B)} \supset \sqrt{12\pi}\tilde{b}_3^*, \quad (9)$$

where  $\tilde{b}_3^*$  is now defined to contain contributions from  $g_{\lambda\mu\nu}$  as well.

The connection between the coefficients in the Sun-centered frame and those in the laboratory frame takes the generic form

$$\begin{aligned} \mathcal{K}_{k10}^{\text{lab}} &= \mathcal{K}_{k10}^{\text{Sun}} \cos\chi - \sqrt{2}\text{Re}\mathcal{K}_{k11}^{\text{Sun}} \sin\chi \cos\omega_\oplus T \\ &+ \sqrt{2}\text{Im}\mathcal{K}_{k11}^{\text{Sun}} \sin\chi \sin\omega_\oplus T, \end{aligned} \quad (10)$$

where  $\chi$  is the angle between the laboratory magnetic field and the Earth's rotational axis and  $\omega_\oplus \approx 2\pi/(23\text{ h }56\text{ m})$  is the Earth's sidereal frequency. This explicitly displays the time variations in the laboratory-frame coefficients for Lorentz violation induced by the rotation of the Earth. In general, the variations of a coefficient with index  $m$  occur at the  $m$ th harmonic of  $\omega_\oplus$ , but in the present context the only relevant frequency is  $\omega_\oplus$  itself because only coefficients with  $j \leq 1$  play a role. In general, other frequencies appear that are associated with the revolution of the Earth around the Sun, but the corresponding effects are suppressed by a factor of the Earth's orbital speed  $\beta_\oplus \approx 10^{-4}$ . A detailed analysis of these effects is possible in principle but lies outside our present scope. The treatment would follow a path analogous to that taken in Sec. III D below, which investigates annual variations in experiments on the anomalous magnetic moments of the muon and antimuon. The attainable sensitivities via muonium hyperfine spectroscopy would be some orders of magnitude weaker but would involve different linear combinations of coefficients for Lorentz violation.

The published experimental analysis constraining the sidereal variations of  $\nu_{12}$  and  $\nu_{34}$  from existing data [10] can be combined with the above results to extract constraints on

the coefficients for Lorentz violation. Hughes *et al.* found the data contained no sidereal variation to  $\pm 20$  Hz at one standard deviation. Here, we adopt a limit on time variations of  $\delta\nu_{12}$  at the sidereal frequency  $\omega_\oplus$  corresponding to no signal within  $\pm 32$  Hz at the 90% confidence level. Noting that  $\chi \approx 90^\circ$  in this experiment yields the bound

$$\begin{aligned} &\left[ \sum_{m \in \{1, -1\}} \left| \sum_{q=0}^2 (\alpha m_r)^{2q} (1 + 4\delta_{2q}) (g_{(2q)1m}^{\text{NR}(0B)} + H_{(2q)1m}^{\text{NR}(0B)} \right. \right. \\ &\quad \left. \left. + 2g_{(2q)1m}^{\text{NR}(1B)} + 2H_{(2q)1m}^{\text{NR}(1B)} \right|^2 \right]^{1/2} \\ &< \frac{\sqrt{3\pi^3}}{|\sin\chi|} (32\text{ Hz}) \approx 2 \times 10^{-22}\text{ GeV}. \end{aligned} \quad (11)$$

in the Sun-centered frame.

Insight into the reach of this bound can be obtained by extracting from it the attained sensitivities to individual coefficients under the assumption that only one coefficient is nonzero at a time. These sensitivities are useful for several purposes including, for example, for comparisons between experiments and for constraining models. Using this assumption, Table I provides estimated sensitivities in the Sun-centered frame to nonrelativistic coefficients with  $k \leq 4$  that contribute to sidereal effects in muonium hyperfine splittings. Several of these results represent first constraints in the literature.

Since each nonrelativistic coefficient is a linear combination of spherical coefficients of different mass dimensions [17], the bound (11) also can be interpreted in terms of constraints on spherical coefficients. As a simple illustration of this connection, consider the nonrelativistic coefficient  $g_{010}^{\text{NR}(0B)}$  and suppose that only the spherical coefficients  $g_{n,jm}^{(d)(0B)}$  with  $n = 0, j = 1, m = 0$  are nonzero. Then,  $g_{010}^{\text{NR}(0B)} = \sum_d m_\mu^{d-3} g_{010}^{(d)(0B)}$  is an infinite sum of spherical coefficients of arbitrary even  $d \geq 4$ . This confirms that nonrelativistic experiments can access nonminimal coefficients of arbitrary dimensionality. Table II collects

TABLE I. Constraints on the moduli of the real and imaginary parts of muon nonrelativistic coefficients determined from muonium hyperfine transitions.

| Coefficient $\mathcal{K}$                          | Constraint on $ \text{Re}\mathcal{K} ,  \text{Im}\mathcal{K} $ |
|--|--|
| $H_{011}^{\text{NR}(0B)}, g_{011}^{\text{NR}(0B)}$ | $< 2 \times 10^{-22}\text{ GeV}$                               |
| $H_{011}^{\text{NR}(1B)}, g_{011}^{\text{NR}(1B)}$ | $< 7 \times 10^{-23}\text{ GeV}$                               |
| $H_{211}^{\text{NR}(0B)}, g_{211}^{\text{NR}(0B)}$ | $< 1 \times 10^{-11}\text{ GeV}^{-1}$                          |
| $H_{211}^{\text{NR}(1B)}, g_{211}^{\text{NR}(1B)}$ | $< 6 \times 10^{-12}\text{ GeV}^{-1}$                          |
| $H_{411}^{\text{NR}(0B)}, g_{411}^{\text{NR}(0B)}$ | $< 2 \times 10^{-1}\text{ GeV}^{-3}$                           |
| $H_{411}^{\text{NR}(1B)}, g_{411}^{\text{NR}(1B)}$ | $< 8 \times 10^{-2}\text{ GeV}^{-3}$                           |

the corresponding estimated sensitivities in the Sun-centered frame to spherical coefficients with  $d \leq 8$  under the assumption of only one nonzero coefficient at a time, as before.

Future experiments are likely to improve on the results listed in Tables I and II. For example, the proposed muonium Hyperfine Structure (MuHFS) experiment [34] at J-PARC would be capable of measuring the muonium ground-state hyperfine splitting to a few ppb. This would lead to improvements of about a factor of five over the values listed in the above tables.

## 2. $1S$ - $2S$ transition and Lamb shift

The shifts (8) in the hyperfine transition frequencies are independent of the isotropic coefficients  $\overset{\circ}{a}_{2q}^{\text{NR}}$  and  $\overset{\circ}{c}_{2q}^{\text{NR}}$  appearing in the perturbation Hamiltonian  $\delta h_{nS}^{\text{NR}}$  given in Eq. (6). This is unsurprising because the hyperfine transitions involve spin flips, while  $\overset{\circ}{a}_{2q}^{\text{NR}}$  and  $\overset{\circ}{c}_{2q}^{\text{NR}}$  control spin-independent contributions to the Hamiltonian. Indeed, only transitions with  $\Delta L \neq 0$  or  $\Delta n \neq 0$  acquire shifts depending on these isotropic coefficients.

One transition of this kind that can be experimentally studied in muonium is the  $1S_{1/2}$ - $2S_{1/2}$  transition. The lesser attainable measurement precision of the corresponding frequency  $\nu_{1S2S}$  compared to the studies of hyperfine transitions implies that it is reasonable to disregard spin-dependent terms in calculating the Lorentz-violating shift  $\delta\nu_{1S2S}$ . As before, we neglect possible contributions from Lorentz violation in the electron sector, which in any case can be investigated at higher precision using other systems such as hydrogen [35].

Taking the appropriate expectation values of Eq. (6) and applying the result (5) yields the frequency shift for the  $1S_{1/2}$ - $2S_{1/2}$  transition as

$$\delta\nu_{1S2S} = \frac{3(m_r\alpha)^2}{8\pi} \left[ \overset{\circ}{a}_2^{\text{NR}} + \overset{\circ}{c}_2^{\text{NR}} + \frac{67}{12}(m_r\alpha)^2(\overset{\circ}{a}_4^{\text{NR}} + \overset{\circ}{c}_4^{\text{NR}}) \right]. \quad (12)$$

TABLE II. Constraints on the moduli of the real and imaginary parts of muon spherical coefficients determined from muonium hyperfine transitions.

| Coefficient<br>$\mathcal{K}$ | Constraint on<br>$ \text{Re}\mathcal{K} ,  \text{Im}\mathcal{K} $ |
|------------------------------|---|
| $H_{011}^{(3)(0B)}$          | $< 5 \times 10^{-23}$ GeV   |
| $g_{011}^{(4)(0B)}$          | $< 5 \times 10^{-22}$   |
| $H_{011}^{(5)(0B)}$          | $< 5 \times 10^{-21}$ GeV <sup>-1</sup>                           |
| $g_{011}^{(6)(0B)}$          | $< 5 \times 10^{-20}$ GeV <sup>-2</sup>                           |
| $H_{011}^{(7)(0B)}$          | $< 4 \times 10^{-19}$ GeV <sup>-3</sup>                           |
| $g_{011}^{(8)(0B)}$          | $< 4 \times 10^{-18}$ GeV <sup>-4</sup>                           |

This result is independent of the presence of a magnetic field and also of the hyperfine sublevel involved in the transition. It represents a rotationally invariant but Lorentz- and *CPT*-violating shift in the transition frequency. Note that the corresponding shift for antimuonium is given by changing the signs of the coefficients  $\overset{\circ}{a}_2^{\text{NR}}$  and  $\overset{\circ}{a}_4^{\text{NR}}$ , so comparisons of muonium and antimuonium would permit direct measurement of the isotropic *a*-type coefficients.

The rotational invariance of the shift (12) ensures that sidereal variations of  $\delta\nu_{1S2S}$  of frequency  $\omega_{\oplus}$  are absent. Also, although the Lorentz violation implies an annual variation in  $\delta\nu_{1S2S}$  induced by the revolution of the Earth about the Sun, this variation is suppressed by the orbital speed  $\beta_{\oplus} \approx 10^{-4}$  and hence attainable constraints are of only limited interest. However, the shift  $\delta\nu_{1S2S}$  does represent a predicted physical effect.

One way to estimate a bound on this effect is to compare the observed experimental value  $\nu_{1S2S}^{\text{expt}}$  with the theoretical value  $\nu_{1S2S}^{\text{th}}$  calculated in conventional quantum electrodynamics, requiring that the Lorentz-violating contribution be no larger than the difference between them. For illustrative purposes, we adopt the experimental value [36]  $\nu_{1S2S}^{\text{expt}} = 2455528941.0(9.8)$  MHz and the theoretical value [26]  $\nu_{1S2S}^{\text{th}} = 2455528935.7(0.3)$  MHz. Some care is required in using the latter as it depends partly on other experimental measurements, including the Rydberg constant, the fine-structure constant, and the muon-electron mass ratio. However, the first two of these are determined by non-muonic experiments [37], so they can reasonably be treated as independent of the nonrelativistic coefficients in the muon sector. The determination of the muon-electron mass ratio does involve experiments with muonium [32], but it is performed using spin-dependent transitions that can be considered independent of the isotropic nonrelativistic coefficients of interest here.

Taking the difference between the experimental and theoretical values gives  $\nu_{1S2S}^{\text{expt}} - \nu_{1S2S}^{\text{th}} = 5.3 \pm 9.8$  MHz. We interpret this conservatively as implying the difference is zero to within  $\pm 20$  MHz, yielding the bound on a combination of isotropic nonrelativistic coefficients given by

$$\left| \overset{\circ}{a}_2^{\text{NR}} + \overset{\circ}{c}_2^{\text{NR}} + (8 \times 10^{-11} \text{ GeV}^2)(\overset{\circ}{a}_4^{\text{NR}} + \overset{\circ}{c}_4^{\text{NR}}) \right| < 8 \times 10^{-6} \text{ GeV}^{-1}. \quad (13)$$

Note that isotropy implies this bound holds unchanged when the coefficients are evaluated in the Sun-centered frame. The resulting constraints on individual coefficients taken one at a time are listed in the first four rows of Table III.

Another interesting option for studying the effects of isotropic nonrelativistic coefficients is the splitting  $2S_{1/2}$ - $2P_{1/2}$ , which is the Lamb shift in muonium. The shift  $\delta\nu_{\text{Lamb}}$  of the Lamb frequency  $\nu_{\text{Lamb}}$  due to Lorentz



TABLE III. Constraints on muon isotropic nonrelativistic coefficients from muonium spectroscopy.

| Transition          | Coefficient         | Constraint                            |
|---------------------|---------------------|---------------------------------------|
| $1S_{1/2}-2S_{1/2}$ | $ a_2^{\text{NR}} $ | $< 8 \times 10^{-6} \text{ GeV}^{-1}$ |
|                     | $ c_2^{\text{NR}} $ | $< 8 \times 10^{-6} \text{ GeV}^{-1}$ |
|                     | $ a_4^{\text{NR}} $ | $< 1 \times 10^5 \text{ GeV}^{-3}$    |
|                     | $ c_4^{\text{NR}} $ | $< 1 \times 10^5 \text{ GeV}^{-3}$    |
| Lamb shift          | $ a_4^{\text{NR}} $ | $< 1 \times 10^6 \text{ GeV}^{-3}$    |
|                     | $ c_4^{\text{NR}} $ | $< 1 \times 10^6 \text{ GeV}^{-3}$    |

and *CPT* violation can be obtained by noting that the perturbative Hamiltonian (6) applies for both the  $2S_{1/2}$  and the  $2P_{1/2}$  levels and by using Eq. (5) to calculate the appropriate expectation values. Restricting attention to the spin-independent terms, we thereby obtain

$$2\pi\delta\nu_{\text{Lamb}} = -\frac{2}{3}(am_r)^4(a_4^{\text{NR}} + c_4^{\text{NR}}). \quad (14)$$

As before, we can estimate a bound on this effect by comparing the experimental and theoretical values,  $\nu_{\text{Lamb}}^{\text{expt}} = 1042_{-23}^{+21}$  MHz and  $\nu_{\text{Lamb}}^{\text{th}} = 1047.490(300)$  MHz [38]. Taking the Lorentz-violating effect as smaller than  $\pm 30$  MHz gives the conservative bound

$$|a_4^{\text{NR}} + c_4^{\text{NR}}| < 1 \times 10^6 \text{ GeV}^{-3}, \quad (15)$$

which also holds in the Sun-centered frame. The resulting constraints on each of the two isotropic nonrelativistic coefficients taken in turn are given in the fifth and sixth rows of Table III.

### C. Muonic atoms and ions

Next, we investigate the use of spectroscopy of muonic atoms and ions to study Lorentz and *CPT* violation. The focus in the first few subsections that follow is primarily on the  $H_\mu$  transitions  $2S_{1/2}^{F-1}-2P_{3/2}^F$  with  $F = 1, 2$ . These have recently been measured at the Paul Scherrer Institute (PSI) [18], leading to the proton radius puzzle [39]. In Sec. II C 5, the analogous transitions in various other muonic atoms and ions are discussed. In what follows, we identify signals for Lorentz violation that can be sought in experiments for all these systems, and we investigate the prospects for resolving the proton radius puzzle via Lorentz and *CPT* violation.

#### 1. Generalities

To enable comparisons between  $H_\mu$  and muonium experiments, it is convenient to consider scenarios with similar relative precision and use scaling arguments to determine the relevant absolute precision. Thus, for

example, if the relative precisions of the  $1S-2S$  frequency in  $H_\mu$  and muonium are roughly comparable, then the energies and hence the absolute precisions are scaled by the ratio  $\approx 187$  of the reduced masses. For instance, a 2 GHz sensitivity in  $H_\mu$  corresponds to roughly the same relative precision as a 10 MHz sensitivity in muonium. This energy scaling affects the reach of searches for Lorentz and *CPT* violation. The experimental sensitivity to coefficients with  $d = 3$  is determined by the absolute frequency resolution and is therefore reduced in  $H_\mu$  experiments, though it can still be of definite interest in the absence of other available results. Also,  $H_\mu$  and muonium experiments with similar relative precision should have comparable sensitivity to dimensionless coefficients. However, for  $d \geq 5$  the reach of  $H_\mu$  experiments can be expected to be superior by a factor of about  $(187)^{d-4}$ .

Many of the derivations for muonium in the previous subsection can be adapted for  $H_\mu$ , and the attainable sensitivities can crudely be estimated using similar reasoning. For example, studies of sidereal variations of the  $1S$  hyperfine splittings in  $H_\mu$  would be of definite interest and can be expected to lead to sensitivities to  $H_{211}^{\text{NR}(OB)}$ ,  $H_{211}^{\text{NR}(1B)}$ ,  $g_{211}^{\text{NR}(OB)}$ ,  $g_{211}^{\text{NR}(1B)}$  improved by about an order of magnitude and sensitivities to  $H_{411}^{\text{NR}(OB)}$ ,  $H_{411}^{\text{NR}(1B)}$ ,  $g_{411}^{\text{NR}(OB)}$ ,  $g_{411}^{\text{NR}(1B)}$  improved by more than five orders of magnitude.

One primary interest is the Lamb shift in  $H_\mu$ , and in particular the transitions  $2S_{1/2}^{F-1}-2P_{3/2}^F$  with  $F = 1, 2$ . Since  $F \leq 2$ , only SME coefficients with  $j \leq 3$  can contribute to spin-dependent effects. However, sensitivity to all such effects is better by many orders of magnitude in experiments studying the muon anomalous magnetic moment, as is verified in Sec. III below. The essential point is that in both types of experiments the unperturbed system has even parity while the relevant spin-dependent Lorentz-violating operators have parity  $(-1)^{j+1}$ , so only coefficients with odd  $j$  contribute. We can therefore reasonably disregard spin-dependent effects in the present context.

For spin-independent effects, only the coefficients with  $j \leq 2$  can contribute because the total angular momentum of the muon is  $J \leq 3/2$ . Neglecting any Lorentz violation in the proton sector, which in any event can be better studied using conventional atoms, the perturbative terms relevant for the Lamb shift can therefore be taken as

$$\delta h_{2S2P}^{\text{NR}} = \sum_{q=0}^2 |\mathbf{p}|^{2q} \left[ (a_{2q}^{\text{NR}} - c_{2q}^{\text{NR}}) + \sum_{m=-2}^2 (a_{(2q)2m}^{\text{NR}} - c_{(2q)2m}^{\text{NR}}) Y_{2m}(\hat{\mathbf{p}}) \right]. \quad (16)$$

The coefficients  $a_{2q}^{\text{NR}}$  and  $c_{2q}^{\text{NR}}$  have  $j = 0$  and so affect both the  $2S_{1/2}^F$  and the  $2P_{3/2}^F$  states isotropically. The other terms have  $j = 2$  and contribute only to the  $2P_{3/2}^F$  levels, with the



shift varying with the orientation of the total angular momentum  $F$ .

## 2. Zeeman transitions

If the Zeeman splittings due to the applied magnetic field are larger than those due to Lorentz violation, then sidereal variations of the Lamb transition frequencies occur. Searching for these effects in  $H_\mu$  requires resolving the Zeeman shift and accumulating sufficient statistics to perform sidereal studies, which is unrealized to date but may be possible in future experiments.

The frequency shift  $\delta\nu(F, m_F)$  of the  $2S_{1/2}^{F-1}-2P_{3/2}^F$  transition at fixed  $m_F$  induced by Lorentz violation involves the expectation value

$$\langle F, m_F | Y_{20}(\hat{\mathbf{p}}) | F, m_F \rangle = \frac{(5-2F)(F(F+1)-3m_F^2)}{12\sqrt{5}\pi}, \quad (17)$$

where the state  $|F, m_F\rangle$  is understood to have  $J = 3/2$  and  $L = 1$ . Using this result and the perturbation Hamiltonian (16), we obtain

$$2\pi\delta\nu(F, m_F) = \frac{2}{3}(am_r)^4(\overset{\circ}{c}_4^{\text{NR}} - \overset{\circ}{a}_4^{\text{NR}}) + \frac{(5-2F)(F(F+1)-3m_F^2)}{12\sqrt{5}\pi}q_{20}, \quad (18)$$

where for convenience we define

$$q_{jm} \equiv \sum_{k=0}^2 \langle |\mathbf{p}|^{2k} \rangle_{21} (a_{(2k)jm}^{\text{NR}} - c_{(2k)jm}^{\text{NR}}). \quad (19)$$

To display explicitly its sidereal time dependence, the frequency shift  $\delta\nu(F, m_F)$  can be expressed in terms of coefficients in the Sun-centered frame. Using the Wigner rotation matrices [40]  $D_{mn}^{(j)}(\alpha, \beta, \gamma)$  to transform between frames shows that the laboratory-frame combination  $q_{20}^{\text{lab}}$  is related to coefficients  $q_{jm}^{\text{Sun}}$  in the Sun-centered frame by [29]

$$q_{20}^{\text{lab}} = \sum_{m=-2}^2 D_{0m}^{(2)}(0, -\chi, -\omega_\oplus T_\oplus) q_{2m}^{\text{Sun}} = \sqrt{\frac{4\pi}{5}} \sum_{m=-2}^2 Y_{2m}(\chi, \omega_\oplus T_\oplus) q_{2m}^{\text{Sun}}, \quad (20)$$

where  $\chi$  is the angle between the magnetic field and the rotational north pole of the Earth,  $\omega_\oplus$  is the Earth sidereal frequency, and  $T_\oplus$  is the sidereal time, as before. This expression implies that in the Sun-centered frame the frequency shift  $\delta\nu(F, m_F)$  takes the form

$$2\pi\delta\nu(F, m_F) = \frac{2}{3}(am_r)^4(\overset{\circ}{c}_4^{\text{NR}} - \overset{\circ}{a}_4^{\text{NR}}) + \frac{(5-2F)(F(F+1)-3m_F^2)}{30} \times \sum_{m=-2}^2 Y_{2m}(\chi, \omega_\oplus T_\oplus) q_{2m}, \quad (21)$$

where all coefficients are now expressed in the Sun-centered frame.

The result (21) reveals that future experiments sensitive to the Zeeman shift in Lamb transitions can be used to search for Lorentz violation through sidereal variations. The coefficients  $a_{221}^{\text{NR}}, a_{421}^{\text{NR}}, c_{221}^{\text{NR}}, c_{421}^{\text{NR}}$  control oscillations at the sidereal frequency  $\omega_\oplus$ , while  $a_{222}^{\text{NR}}, c_{222}^{\text{NR}}, a_{422}^{\text{NR}}, c_{422}^{\text{NR}}$  control ones at  $2\omega_\oplus$ . Suppose, for example, measurements are made for  $F = 1, m_F = 0$ , corresponding to an experiment with a laser polarized in the direction of the magnetic field. Assume the magnetic field is inclined at  $\chi = 45^\circ$  to the Earth's rotation axis and the experiment establishes no sidereal signal at  $\pm 1$  GHz. Then, constraints of order  $10^{-7} \text{ GeV}^{-1}$  could be placed on  $|a_{22m}^{\text{NR}}|, |c_{22m}^{\text{NR}}|$  and ones of order  $1 \text{ GeV}^{-3}$  on  $|a_{42m}^{\text{NR}}|, |c_{42m}^{\text{NR}}|$ . A comparable muonium counterpart experiment would achieve a resolution of about  $\pm 5$  MHz, with corresponding sensitivities some two orders of magnitude weaker on  $|a_{22m}^{\text{NR}}|, |c_{22m}^{\text{NR}}|$  and about seven orders of magnitude weaker on  $|a_{42m}^{\text{NR}}|, |c_{42m}^{\text{NR}}|$ .

## 3. Proton radius puzzle

The result (21) for the shifts in the  $2S_{1/2}^{F-1}-2P_{3/2}^F$  transition frequencies contains terms with  $m = 0$  that are independent of sidereal time. These constant shifts can be expected to appear as a discrepancy between experimental measurements and conventional Lorentz-invariant theoretical predictions. However, the theoretical predictions depend on the value of the proton charge radius  $r_p$ , so in practice the recent PSI measurements of these transitions are used to extract an independent measure of  $r_p$  instead [18]. Surprisingly, this measure is in apparent disagreement by about seven standard deviations with the 2010 CODATA value obtained by combining results from hydrogen spectroscopy and from electron elastic scattering data [37]. The difference  $\Delta r_p \approx -0.037 \pm 0.005 \text{ fm}$  between these values suggests a smaller proton radius measured by  $H_\mu$  spectroscopy, a result called the proton radius puzzle [39].

Since Lorentz violation can induce a constant shift in the Lamb transition frequencies and hence an apparent constant shift in the inferred proton charge radius, we can ask what size Lorentz violation would suffice to resolve the puzzle and what implications this might have for other experiments. For simplicity we assume only muon-sector Lorentz violation as before, so effects arise in  $H_\mu$  spectroscopy but are absent in H spectroscopy and electron elastic scattering. We also disregard any effects from the sidereal variations discussed in the previous subsection.

A more complete analysis could conceivably demonstrate a sidereal-time dependence in the inferred value of the proton charge radius.

Taking for definiteness the polarization of the laser in the direction of the magnetic field, for which  $\Delta m_F = 0$ , and assuming an equal population of the initial states with  $m_F = -1, 0, 1$ , we find the induced change in the Lamb-shift energy is given by

$$\delta E_{\text{Lamb}} = \frac{2}{3} (am_r)^4 (c_4^{\circ\text{NR}} - a_4^{\circ\text{NR}}) + \frac{3(1 + 3 \cos 2\chi)}{32\sqrt{5}\pi} q_{20}, \quad (22)$$

with

$$q_{20} = \frac{1}{4} (am_r)^2 (a_{220}^{\text{NR}} - c_{220}^{\text{NR}}) + \frac{7}{48} (am_r)^4 (a_{420}^{\text{NR}} - c_{420}^{\text{NR}}). \quad (23)$$

All SME coefficients appearing in these equations have  $m = 0$  and are expressed in the Sun-centered frame.

The theory relating the Lamb shift to the proton charge radius  $r_p$  [18,26,41,42] implies that  $\delta E_{\text{Lamb}}$  can be interpreted as a change  $\delta r_p$  in the determination of  $r_p$  given by

$$\delta r_p (\text{fm}) \simeq -1.1 \times 10^{11} \delta E_{\text{Lamb}} (\text{GeV}). \quad (24)$$

Within the present hypothesis attributing the discrepancy  $\Delta r_p$  to the shift  $\delta r_p$  induced by Lorentz violation, we can impose  $\Delta r_p = \delta r_p$  and thereby establish the requirement for the coefficients for Lorentz violation to resolve the proton radius puzzle. This gives the condition

$$\begin{aligned} & \frac{2}{3} (am_r)^4 (c_4^{\circ\text{NR}} - a_4^{\circ\text{NR}}) + \frac{3(1 + 3 \cos 2\chi)}{32\sqrt{5}\pi} q_{20} \\ & \simeq 3 \times 10^{-13} \text{ GeV}. \end{aligned} \quad (25)$$

Any combination of coefficients satisfying this equation would suffice to resolve the discrepancy in the charge radius. Note that these results depend on the chosen angle  $\chi$  between the magnetic field and the Earth's rotation axis. Using these coefficients to resolve the proton radius puzzle therefore comes with a prediction that the inferred value of  $r_p$  could vary with the orientation of the magnetic field.

The PSI experiment also deduces the  $2S$  hyperfine splitting and hence determines the Zemach magnetic radius  $r_Z$  of the proton [18]. The result is in agreement with data from H spectroscopy and from electron-proton scattering, and the difference  $\Delta r_Z$  between them can conservatively be taken as bounded by  $|\Delta r_Z| < 0.07$  fm. This result places an additional constraint on the coefficients appearing in Eq. (25) because some of them also affect the determination of the hyperfine splitting. In the same scenario as before, we find the Lorentz-violating shift  $\delta E_{\text{HF}}$  in the hyperfine interval to be

$$\delta E_{\text{HF}} = \frac{(1 + 3 \cos 2\chi)}{24\sqrt{5}\pi} q_{20}. \quad (26)$$

The theory relating the hyperfine splitting to the Zemach radius [18,42,43] shows that the shift  $\delta E_{\text{HF}}$  can be understood as a change  $\delta r_Z$  given as

$$\delta r_Z (\text{fm}) \simeq -6.2 \times 10^{12} \delta E_{\text{HF}} (\text{GeV}). \quad (27)$$

Following analogous reasoning as before, we can impose  $\Delta r_Z = \delta r_Z$  to obtain the constraint on SME coefficients required to preserve the agreement between the various experiments determining the Zemach radius. This gives

$$\left| \frac{(1 + 3 \cos 2\chi)}{24\sqrt{5}\pi} q_{20} \right| \lesssim 1 \times 10^{-14} \text{ GeV}. \quad (28)$$

This condition is tighter than the constraint (25) by about an order of magnitude. It depends on the orientation  $\chi$  of the magnetic field in the experiment, which may be worth exploring experimentally. Note that the same linear combination of coefficients  $q_{20}$  appears in both conditions (25) and (28). Within the present scenario, this means that resolving the proton radius puzzle via Eq. (25) requires a nonzero value for the combination  $c_4^{\circ\text{NR}} - a_4^{\circ\text{NR}}$  of isotropic coefficients, a possibility remaining compatible with existing constraints.

Some additional intuition for the implications of these results can be gained by extracting from Eq. (25) the corresponding condition on each coefficient in turn with all others set to zero. This procedure gives  $c_4^{\circ\text{NR}} \simeq 2 \text{ GeV}^{-3}$  and  $a_4^{\circ\text{NR}} \simeq -2 \text{ GeV}^{-3}$  for the isotropic coefficients taken one at a time. For the coefficients with  $j = 2$  and taking  $\chi = 45^\circ$ , we find  $a_{220}^{\text{NR}} = -c_{220}^{\text{NR}} = 10^{-4} \text{ GeV}^{-1}$  and  $a_{420}^{\text{NR}} = -c_{420}^{\text{NR}} = 400 \text{ GeV}^{-3}$ . Comparison with the results in Table III reveals that  $1S$ - $2S$  and Lamb-shift spectroscopy in  $\text{H}_\mu$  is several orders of magnitude more sensitive to the isotropic coefficients than the corresponding measurement in muonium. We remark in passing that the distinction between using  $a_4^{\circ\text{NR}}$  and  $c_4^{\circ\text{NR}}$  in this context is experimentally undetectable as it would require spectroscopic studies of  $\tilde{\text{H}}_\mu$ , for which the coefficient  $a_4^{\circ\text{NR}}$  for  $CPT$ -odd effects would cause an apparent increase of the antiproton charge radius.

#### 4. Negligible magnetic field

For present purposes, the  $\approx 5$  T magnetic field used in the recent PSI experiment [18] can be viewed as negligible because the Zeeman splitting remains unobserved. Disregarding for the moment the laser polarization, the apparatus itself can be idealized as rotationally invariant in the laboratory frame. This implies that observable sidereal variations cannot appear. However, the presence of Lorentz violation acts to break the rotational symmetry of the  $\text{H}_\mu$  atom and partially or wholly lifts the  $(2F + 1)$ -fold

degeneracy with respect to the orientation of its total angular momentum  $\mathbf{F}$ . This offers an alternative route to searching for Lorentz violation, as we describe next.

The energy splittings resulting from Lorentz violation depend on the value of  $F$ , and the corresponding  $2F + 1$  states can be labeled by an effective azimuthal quantum number  $\xi$  taking the values  $-F, -F + 1, \dots, F - 1, F$ , as usual. Note that we use  $\xi$  instead of  $m_F$  here to avoid possible confusion with projection of  $\mathbf{F}$  along the magnetic field. The energy shift  $\delta E(F, \xi)$  for a given state can be obtained from the perturbation (16), which depends on coefficients for Lorentz violation having either  $j = 0$  or  $j = 2$ . The coefficients with  $j = 0$  govern isotropic Lorentz violation, so only those with  $j = 2$  can shift the levels according to the orientation of  $\mathbf{F}$ . As discussed above, only states with  $F = 1$  or  $F = 2$  are affected, and in particular for the  $2S_{1/2}^{F-1} - 2P_{3/2}^F$  transitions only the  $P$  states are relevant. We therefore focus in what follows on the shift of the  $P$  states controlled by coefficients with  $j = 2$ .

For an arbitrary orientation of  $\mathbf{F}$  in the laboratory frame, the explicit form of the level shift  $\delta E(F, \xi)$  is involved, being determined by the solution of a cubic for  $F = 1$  and by a quintic for  $F = 2$ . For example, for  $F = 1$  we obtain

$$\delta E(1, \xi) = \frac{2}{3} (\alpha m_r)^4 (c_4^{\text{NR}} - a_4^{\text{NR}}) + u_\xi D + u_\xi^* \frac{\Delta_0}{D}, \quad (29)$$

where

$$u_\xi = \frac{1 + i\xi\sqrt{3}}{3(1 - 3\xi^2)}, \quad (30)$$

and

$$D = \left( \frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2} \right)^{1/3}. \quad (31)$$

The quantities  $\Delta_0$  and  $\Delta_1$  contain the combinations of coefficients for Lorentz violation defined in Eq. (20) and are given by

$$\begin{aligned} \Delta_0 &= \frac{9}{80\pi} \sum_{m=-2}^2 |q_{2m}|^2, \\ \Delta_1 &= -\frac{27q_{20}}{160\sqrt{5}\pi^3} (6|q_{22}|^2 - (q_{20})^2 - 3|q_{21}|^2) \\ &\quad + \frac{81}{80} \sqrt{\frac{3}{10\pi^3}} \text{Re}[q_{22}^*(q_{21})^2]. \end{aligned} \quad (32)$$

We note in passing that all the quantities appearing in these equations are rotational scalars, so the above expressions hold at leading order both in the laboratory frame and in the Sun-centered frame.

TABLE IV. Spectral shifts  $\delta E(F, \xi)$  for  $j = 2$ .

| $F$ | $\xi$ | $m = 0$                        | $m \neq 0$                                   |
|-----|-------|--------------------------------|--|
| 1   | 1     | $-\frac{q_{20}}{4\sqrt{5}\pi}$ | $\sqrt{\frac{3}{10\pi}} \frac{ q_{2m} }{2}$  |
|     | 0     | $\frac{q_{20}}{2\sqrt{5}\pi}$  | 0  |
|     | -1    | $-\frac{q_{20}}{4\sqrt{5}\pi}$ | $-\sqrt{\frac{3}{10\pi}} \frac{ q_{2m} }{2}$ |
| 2   | 2     | $-\frac{q_{20}}{2\sqrt{5}\pi}$ | $\frac{ q_{2m} }{\sqrt{10\pi}}$              |
|     | 1     | $\frac{q_{20}}{4\sqrt{5}\pi}$  | $\sqrt{\frac{3}{10\pi}} \frac{ q_{2m} }{2}$  |
|     | 0     | $\frac{q_{20}}{2\sqrt{5}\pi}$  | 0  |
|     | -1    | $\frac{q_{20}}{4\sqrt{5}\pi}$  | $-\sqrt{\frac{3}{10\pi}} \frac{ q_{2m} }{2}$ |
|     | -2    | $-\frac{q_{20}}{2\sqrt{5}\pi}$ | $-\frac{ q_{2m} }{\sqrt{10\pi}}$             |

Some insight into the content of the results for both  $F = 1$  and  $F = 2$  can be obtained by extracting the level shifts  $\delta E(F, \xi)$  under the assumption that only one combination of coefficients  $q_{jm}$  is nonzero at a time. Table IV displays the spectral shifts obtained in this scenario. Inspection of the table reveals that the coefficient  $q_{20}$  lifts the degeneracy only partially, while  $q_{2m}$  with  $m \neq 0$  lifts it completely.

To illustrate how the spectral splitting can be used to seek Lorentz violations, suppose that all transitions  $2S_{1/2}^{F=0} - 2P_{3/2}^{F=1, \xi}$  are excited with equal probability, perhaps via an unpolarized laser. An experimental measurement effectively involves a cloud of atoms with random orientations  $\mathbf{F}$ , so the Lorentz violation acts to broaden the observed transition line. The theoretical apparent width  $\Delta E$  of the  $2P$  level due to Lorentz violation is given by

$$(\Delta E)^2 = \frac{1}{3} \sum_{\xi=-1}^1 \delta E(1, \xi)^2 - \frac{1}{9} \left( \sum_{\xi=-1}^1 \delta E(1, \xi) \right)^2, \quad (33)$$

from which the result

$$(\Delta E)^2 = \frac{2}{9} \Delta_0 = \frac{1}{40\pi} \sum_{m=-2}^2 |q_{2m}|^2 \quad (34)$$

is obtained.

Assuming the experiment cannot directly resolve the splitting, it follows that the precision of the measurement can be taken as an upper bound on  $\Delta E$ , thereby yielding upper limits on the combination of coefficients given in Eq. (34). This combination contains 12 independent complex nonrelativistic coefficients. As before, a measure of the attainable sensitivity can be obtained by taking each coefficient nonzero in turn. For example, an experiment achieving a precision of about 1 GHz would place constraints on  $|a_{22m}^{\text{NR}}|, |c_{22m}^{\text{NR}}|$  at the level of about  $10^{-7} \text{ GeV}^{-1}$ , while the constraints on  $|a_{42m}^{\text{NR}}|, |c_{42m}^{\text{NR}}|$  would be at the level of about  $1 \text{ GeV}^{-3}$ . A muonium experiment with comparable fractional sensitivity would have a 5 MHz precision,

but it would yield constraints some two to seven orders of magnitude weaker on the same coefficients.

In realistic applications, the laser polarization is fixed in the laboratory frame. For example, in the PSI experiment [18], the polarization is parallel to the magnetic field. The plane of the polarization rotates with the Earth, inducing sidereal oscillations in the presence of Lorentz violation. A detailed analysis would therefore involve a combination of the line-broadening effects described above with sidereal oscillations associated with the laser polarization.

### 5. Other muonic atoms and ions

A variety of other muonic atoms and ions can be used to search for signals of Lorentz and *CPT* violation. The PSI experiment R98-03 has already measured several transitions in  $D_\mu$  [19], while the Charge Radius Experiment with Muonic Atoms (CREMA), PSI project R10-01, proposes to study the Lamb shift in  ${}^3\text{He}_\mu^+$  and  ${}^4\text{He}_\mu^+$  ions [20]. Possibilities may also exist for muonic tritium  $T_\mu$  [21] and for the heavier ions  ${}^6\text{Li}_\mu^{2+}$ ,  ${}^7\text{Li}_\mu^{2+}$ ,  ${}^9\text{Be}_\mu^{3+}$ , and  ${}^{11}\text{B}_\mu^{4+}$  [22]. The key differences in the hydrogenic spectra of all these muonic systems arise through differences in the nuclear spin  $I$ , the net charge  $Z$ , and the reduced mass  $m_r$ . This makes possible a unified treatment of the Lorentz-violating corrections to their Lamb shifts, following the methods established for  $H_\mu$  in the previous subsections.

Consider first for any one of these systems the analogue of the Zeeman transitions described in Sec. II C 2, which may become accessible to future experiments. For example, in  $D_\mu$  the nucleus has spin  $I = 1$ , so the transitions of interest are  $2S_{1/2}^{F-1}-2P_{3/2}^F$  at fixed  $m_F$  with  $F \in \{3/2, 5/2\}$ . Similarly, for  ${}^3\text{He}_\mu^+$  the spin is  $I = 1/2$ , so the transitions are like those of  $H_\mu$ . For  ${}^4\text{He}_\mu^+$  the spin is  $I = 0$ , so  $F = J$  and the transitions of interest are  $2S_{1/2}-2P_{1/2}$  and  $2S_{1/2}-2P_{3/2}$  at fixed  $m_F$ .

The Lorentz-violating shift in the frequency for any of these systems is given by a generalization of Eq. (18),

$$2\pi\delta\nu(F, m_F) = \frac{2}{3}(Zam_r)^4(c_4^{\text{NR}} - a_4^{\text{NR}}) + \frac{1}{\sqrt{5\pi}}\Lambda(F)(F(F+1) - 3m_F^2)q'_{20}, \quad (35)$$

where  $\Lambda(F)$  is a factor specific to the atom or ion and where

$$q'_{20} = \frac{1}{4}(Zam_r)^2(a_{220}^{\text{NR}} - c_{220}^{\text{NR}}) + \frac{7}{48}(Zam_r)^4(a_{420}^{\text{NR}} - c_{420}^{\text{NR}}). \quad (36)$$

The values taken by  $\Lambda(F)$  are displayed in Table V.

To gain some insight into the implications of Eq. (35), it is useful first to establish a relative measure of spectroscopic precision for the various systems. Since the Lamb shift is proportional to  $Z^4 m_r$ , comparing different

TABLE V. Lamb-shift factors  $\Lambda(F)$  and relative sensitivities of muonic atoms and ions to Lorentz and *CPT* violation. The estimated relative attainable sensitivities to the SME coefficients  $a_{kj0}^{\text{NR}}$  and  $c_{kj0}^{\text{NR}}$  are shown for different values of  $kj0$ , using a normalization relative to  $H_\mu$ .

| System                     | $I$           | $Z$ | $m_r$ (GeV) | $\Lambda(F)$               | $kj0$ |      |      |
|----------------------------|---------------|-----|-------------|----------------------------|-------|------|------|
|                            |               |     |             |                            | 400   | 220  | 420  |
| $H_\mu$                    | $\frac{1}{2}$ | 1   | 0.094       | $\frac{1}{12}(5-2F)$       | 1     | 1    | 1    |
| $D_\mu$                    | 1             | 1   | 0.099       | $\frac{1}{120}(2F+1)$      | 0.85  | 4.7  | 4.3  |
| $T_\mu$                    | $\frac{1}{2}$ | 1   | 0.101       | $\frac{1}{12}(5-2F)$       | 0.81  | 0.93 | 0.81 |
| ${}^3\text{He}_\mu^+$      | $\frac{1}{2}$ | 2   | 0.101       | $\frac{1}{12}(5-2F)$       | 0.81  | 3.7  | 0.81 |
| ${}^4\text{He}_\mu^+$      | 0             | 2   | 0.102       | $\frac{1}{12}(2F-1)$       | 0.79  | 3.7  | 0.79 |
| ${}^6\text{Li}_\mu^{2+}$   | 1             | 3   | 0.103       | $\frac{1}{120}(2F+1)$      | 0.77  | 41   | 3.8  |
| ${}^7\text{Li}_\mu^{2+}$   | $\frac{3}{2}$ | 3   | 0.103       | $\frac{1}{60}(F-2)(17-5F)$ | 0.77  | 10   | 0.95 |
| ${}^9\text{Be}_\mu^{3+}$   | $\frac{3}{2}$ | 4   | 0.104       | $\frac{1}{60}(F-2)(17-5F)$ | 0.75  | 18   | 0.94 |
| ${}^{11}\text{B}_\mu^{4+}$ | $\frac{3}{2}$ | 5   | 0.104       | $\frac{1}{60}(F-2)(17-5F)$ | 0.75  | 28   | 0.94 |

experiments assuming the same relative uncertainty implies comparing ratios of this factor. For example, the ratio of the factors  $Z^4 m_r$  for  ${}^4\text{He}_\mu^+$  and  $H_\mu$  is about 20, so a precision of 1 GHz in  $H_\mu$  is comparable to a precision of 20 GHz in  ${}^4\text{He}_\mu^+$ .

With this measure in hand, we can provide estimates of the relative sensitivities of each muonic system to the coefficients for Lorentz violation appearing in Eq. (35). For the coefficients with  $k = 2$ , the relevant factor is  $Z^2/m_r$ , while for coefficients with  $k = 4$  it is  $1/m_r^3$ . Table V shows the resulting estimated sensitivities of the different muonic systems relative to  $H_\mu$  for the SME coefficients  $a_{kj0}^{\text{NR}}$  and  $c_{kj0}^{\text{NR}}$  with  $kj0 = 400, 220, \text{ and } 420$ . The reader is reminded that  $a_{k00}^{\text{NR}} \equiv \sqrt{4\pi}a_k^{\text{NR}}$  and  $c_{k00}^{\text{NR}} \equiv \sqrt{4\pi}c_k^{\text{NR}}$ . The entries in this table assume the smallest possible value of  $F$  allowed in each case. A numerical value less than one implies that Lamb-shift spectroscopy using the corresponding system is estimated to be more sensitive by that value than spectroscopy using  $H_\mu$ .

Table V demonstrates that future experiments studying the Lamb-shift Zeeman transitions in various muonic atoms and ions can achieve interesting sensitivities to nonrelativistic coefficients. In the Sun-centered frame, the transition frequencies acquire a sidereal time dependence following Eq. (20), so experiments can measure the coefficients  $a_{221}^{\text{NR}}, a_{421}^{\text{NR}}, c_{221}^{\text{NR}}, c_{421}^{\text{NR}}$  via the sidereal frequency  $\omega_\oplus$  and the coefficients  $a_{222}^{\text{NR}}, c_{222}^{\text{NR}}, a_{422}^{\text{NR}}, c_{422}^{\text{NR}}$  via  $2\omega_\oplus$ . The sensitivities relative to those of  $H_\mu$  can be obtained from the table. Assuming, for example, that an experiment with  $D_\mu$  detects no sidereal signal at the level of 2 GHz for the transitions with  $F = 3/2, m_F = 1/2$  using a laser polarized along the magnetic field orientation  $\chi = 45^\circ$ , then constraints of order  $10^{-6} \text{ GeV}^{-1}$  could be placed on  $|a_{22m}^{\text{NR}}|, |c_{22m}^{\text{NR}}|$  and ones of order  $1 \text{ GeV}^{-3}$  on  $|a_{42m}^{\text{NR}}|, |c_{42m}^{\text{NR}}|$ . A similar



experiment with  ${}^4\text{He}_\mu^+$  with no sidereal signal at 20 GHz would achieve roughly comparable sensitivities on coefficients with  $k = 2$  and about a factor of 5 improvement on coefficients with  $k = 4$ , with the latter gain being primarily due to the larger reduced mass.

We can also consider the case of negligible magnetic field discussed in Sec. II C 4, where the Zeeman splittings are unresolved. Lorentz and  $CPT$  violation in any of the systems in Table V then again appears as a line broadening resulting from the breaking of rotational symmetry and the associated dependence on the orientation of the total angular momentum  $F$ . The apparent width  $\Delta E$  can be found in each case using the techniques leading to Eq. (34).

Limiting attention to the isotropic muon coefficients, the Lorentz-violating shift in the Lamb energy for any of the muonic atoms or ions is given by

$$\delta E_{\text{Lamb}} = \frac{2}{3} (Zam_r)^4 (\overset{\circ}{c}_4^{\text{NR}} - \overset{\circ}{a}_4^{\text{NR}}). \quad (37)$$

Table V shows that the heavier systems are more sensitive to isotropic coefficients than  $\text{H}_\mu$ . Note that the net effect in each case would be an apparent shift in the nuclear charge radius of the muonic atom or ion relative to the equivalent electron system. In particular, it might be possible with these experiments to exclude or confirm any contribution to the proton radius puzzle arising from isotropic coefficients as discussed in Sec. II C 3.

Finally, we note that another signal of Lorentz violation is a change in the isotope shifts between  $\text{H}_\mu$  and the other muonic atoms or ions relative to the isotope shifts of the corresponding electron systems. Again considering only the isotropic muon coefficients, the 1S-2S transition frequency in a given atom or ion is shifted by

$$2\pi\delta\nu = \frac{3}{4} (Zam_r)^2 (\overset{\circ}{c}_2^{\text{NR}} - \overset{\circ}{a}_2^{\text{NR}}) + \frac{67}{16} (Zam_r)^4 (\overset{\circ}{c}_4^{\text{NR}} - \overset{\circ}{a}_4^{\text{NR}}). \quad (38)$$

This result implies an apparent isotope shift arising from the nuclear charge and the reduced mass, and appearing only in experiments with muonic systems.

### III. MUON MAGNETIC MOMENT

In this section, the effects of Lorentz and  $CPT$  violation on measurements of the muon anomalous magnetic moment are considered. The muon anomaly frequency, which in Lorentz-invariant models is proportional to the muon  $g - 2$  factor, was studied over a 20-year period in a series of experiments at CERN [44]. More recently, it has been measured to an impressive precision of about 0.5 ppm in experiment E821 at the Brookhaven National Laboratory (BNL) [45]. The upcoming experiment E989 at the Fermi National Accelerator Laboratory (Fermilab) [46] and an experiment at the Japan Proton Accelerator Research

Complex (J-PARC) [47] both anticipate roughly a fivefold improvement over this mark. In the presence of Lorentz violation, the muon and antimuon anomaly frequencies  $\omega_a^-$  and  $\omega_a^+$  can acquire a difference, and sidereal and annual variations of  $\omega_a^\pm$  can also appear. In what follows, we outline the underlying basis for these effects and then consider each of the resulting types of signals in turn.

#### A. Basics

In the BNL experiment [45], relativistic polarized  $\mu^+$  or  $\mu^-$  beams were injected into cyclotron orbits in a constant magnetic field  $B \approx 1.45$  T and adjusted to the ‘magic’ momentum  $p \approx 3.094$  GeV with  $\gamma \approx 29.3$  at which the dependence of the anomaly frequency on the electric field is eliminated. Fitting the  $\mu^\pm$  decay spectrum permits inferring the corresponding anomaly frequency  $\omega_a^\pm$ , which is the difference between the spin-precession frequency  $\omega_s^\pm$  and the cyclotron frequency  $\omega_c^\pm$ . The earlier CERN experiment [44] and the upcoming Fermilab experiment [46] involve designs conceptually similar to the BNL one. In contrast, the J-PARC experiment [47] will use ultracold highly polarized  $\mu^+$  beams of momentum  $p \approx 320$  MeV and  $\gamma \approx 3.03$  that can be stored in a magnetic field  $B \approx 3$  T without a focusing electric field.

In all these experimental scenarios, the leading-order corrections to the anomaly frequencies  $\omega_a^\pm$  arising from Lorentz violation can be calculated in perturbation theory. For muon propagation with momentum  $\mathbf{p}$ , the perturbative Hamiltonian  $\delta h(\mathbf{p})$  arising from Lorentz-violating operators of arbitrary mass dimension is derived in Ref. [17], and the motion in the classical limit follows a geodesic in a pseudo-Finsler spacetime [48,49]. For experimental applications, it is convenient to adopt a decomposition of the Hamiltonian using spherical coordinates, which reveals that the perturbative terms are controlled by eight sets of spherical coefficients for Lorentz violation. These are denoted as  $a_{njm}^{(d)}$ ,  $c_{njm}^{(d)}$ ,  $g_{njm}^{(d)(0B)}$ ,  $g_{njm}^{(d)(1B)}$ ,  $g_{njm}^{(d)(1E)}$ ,  $H_{njm}^{(d)(0B)}$ ,  $H_{njm}^{(d)(1B)}$ ,  $H_{njm}^{(d)(1E)}$ , where  $d$  is the mass dimension of the corresponding operator and the allowed ranges of the indices  $n, j, m$  are given in Table III of Ref. [17]. The  $g$ - and  $H$ -type coefficients are associated with spin operators causing birefringence of the muon propagation, which can be interpreted as a Larmor-like precession of the muon spin  $\mathbf{S}$  and affects the spin-precession frequencies  $\omega_s^\pm$ .

Denoting the corresponding pieces of  $\delta h(\mathbf{p})$  as  $h_g = \mathbf{h}_g \cdot \boldsymbol{\sigma}$  and  $h_H = \mathbf{h}_H \cdot \boldsymbol{\sigma}$ , the rate of change of the spin expectation value for the  $\mu^-$  due to Lorentz violation is given by [17]

$$\frac{d\langle \mathbf{S} \rangle}{dt} \approx 2(\mathbf{h}_g + \mathbf{h}_H) \times \langle \mathbf{S} \rangle. \quad (39)$$

The correction to the muon spin-precession frequency can then be identified as  $\delta\omega_s^- = 2(\mathbf{h}_g + \mathbf{h}_H)$ . The result for the

antimuon  $\mu^+$  follows by changing the sign of the  $g$ -type coefficients, which control  $CPT$ -odd operators in  $\delta h$ . Since the cyclotron frequency by definition is produced by level shifts proportional to the magnetic field  $B$ , which is tiny in natural units ( $1 \text{ T} \approx 2 \times 10^{-16} \text{ GeV}^2$ ), Lorentz-violating corrections to this frequency are determined by the product of two small quantities and hence can be neglected. The corrections to the  $\mu^\pm$  anomaly frequencies are therefore given by

$$\delta\omega_a^\pm = \pm 2\mathbf{h}_g + 2\mathbf{h}_H. \quad (40)$$

In experimental applications, the detectors lie in the plane of the storage ring and so only the perpendicular component of  $\delta\omega_a^\pm$  is measured. Moreover, only orbital averages are observed, so the couplings involving both Lorentz violation and the muon momentum can contribute only when cylindrically symmetric about the vertical axis through the storage ring.

The result (40) holds in the laboratory frame. As discussed in Sec. II A in the context of muonic bound states, the rotation of the Earth induces time dependence of some coefficients in the laboratory frame. Disregarding for the moment effects from the revolution of the Earth about the Sun, which are suppressed by about four orders of magnitude, the time dependence of a generic coefficient  $\mathcal{K}_{jm}^{\text{lab}}$  in the frame of a laboratory with  $x$  axis pointing south and  $y$  axis pointing east is given by [29]

$$\mathcal{K}_{jm}^{\text{lab}} = \sum_{m'} e^{im'\omega_\oplus T_\oplus} d_{mm'}^{(j)}(-\chi) \mathcal{K}_{jm'}^{\text{Sun}} \quad (41)$$

in terms of the corresponding coefficients  $\mathcal{K}_{jm}^{\text{Sun}}$  in the canonical Sun-centered frame. As before,  $\omega_\oplus \approx 2\pi/(23 \text{ h } 56 \text{ m})$  is the Earth's sidereal frequency and  $T_\oplus$  is the sidereal time, while the little Wigner matrices  $d_{mm'}^{(j)}$  are taken as defined in Eq. (136) of Ref. [29] and  $\chi$  is the colatitude of the experiment in the northern hemisphere. In what follows, we adopt the values  $\chi \approx 43.7^\circ$  at CERN,  $\chi \approx 49.1^\circ$  at BNL,  $\chi \approx 48.2^\circ$  at Fermilab, and  $\chi \approx 53.5^\circ$  at J-PARC.

Combining the above results yields the experimentally observable perturbative shift  $\delta\omega_a^\pm$  of the anomaly frequency due to Lorentz violation, expressed in terms of spherical coefficients in the Sun-centered frame. The result is

$$\delta\omega_a^\pm = 2 \sum_{dnjm} E_0^{d-3} e^{im\omega_\oplus T_\oplus} G_{jm}(\chi) (\check{H}_{njm}^{(d)} \pm \check{g}_{njm}^{(d)}), \quad (42)$$

where  $E_0$  is the unperturbed muon energy. This is a central result for studying Lorentz and  $CPT$  violation via experiments measuring the muon anomalous magnetic moment.

In Eq. (42), the dimensionless factor

$$G_{jm}(\chi) \equiv \sqrt{j(j+1)} {}_1Y_{j0}(\pi/2, 0) d_{0m}^{(j)}(-\chi) \quad (43)$$

TABLE VI. Some useful values of  $G_{jm}(\chi)$  for the CERN, BNL, Fermilab, and J-PARC experiments.

| $j$ | $m$     | CERN        | BNL         | Fermilab    | J-PARC      |
|-----|---------|-------------|-------------|-------------|-------------|
| 1   | 0       | 0.353       | 0.320       | 0.326       | 0.291       |
|     | $\pm 1$ | $\mp 0.239$ | $\mp 0.261$ | $\mp 0.258$ | $\mp 0.278$ |
| 3   | 0       | 0.156       | 0.314       | 0.291       | 0.410       |
|     | $\pm 1$ | $\pm 0.540$ | $\pm 0.419$ | $\pm 0.441$ | $\pm 0.300$ |
|     | $\pm 2$ | $-0.529$    | $-0.573$    | $-0.568$    | $-0.589$    |
|     | $\pm 3$ | $\pm 0.206$ | $\pm 0.270$ | $\pm 0.259$ | $\pm 0.325$ |
| 5   | 0       | $-0.694$    | $-0.493$    | $-0.536$    | $-0.245$    |
|     | $\pm 1$ | $\pm 0.241$ | $\pm 0.518$ | $\pm 0.481$ | $\pm 0.639$ |
|     | $\pm 2$ | 0.623       | 0.340       | 0.392       | 0.0750      |
|     | $\pm 3$ | $\mp 0.792$ | $\mp 0.801$ | $\mp 0.806$ | $\mp 0.736$ |
|     | $\pm 4$ | 0.453       | 0.588       | 0.566       | 0.684       |
| 7   | $\pm 5$ | $\mp 0.137$ | $\mp 0.215$ | $\mp 0.200$ | $\mp 0.292$ |
|     | 0       | $-0.170$    | $-0.634$    | $-0.576$    | $-0.773$    |

is purely geometrical and involves the spin-weighted spherical harmonics  ${}_1Y_{j0}(\theta, \phi)$  of spin weight 1 defined according to Appendix A of Ref. [29]. The contribution  $\delta\omega_a^\pm$  vanishes for even  $j$  because  ${}_1Y_{j0}(\pi/2, 0)$  does, while for odd  $j = 2k + 1$  we have

$${}_1Y_{(2k+1)0}(\pi/2, 0) = \frac{(-1)^k (2k-1)!!}{2^{k+1} k!} \sqrt{\frac{(1+2k)(3+4k)}{2\pi(1+k)}}. \quad (44)$$

Table VI lists some numerical values of the factor  $G_{jm}(\chi)$  relevant for the CERN, BNL, Fermilab, and J-PARC experiments.

The háček coefficients  $\check{g}_{njm}^{(d)}$ ,  $\check{H}_{njm}^{(d)}$  appearing in the expression (42) represent the linear combinations of spherical coefficients that are observable in the experiments. They are defined as

$$\begin{aligned} \check{g}_{njm}^{(d)} &\equiv \beta^n g_{njm}^{(d)(0B)} + \sqrt{\frac{2}{j(j+1)}} \beta^{n+2} g_{(n+2)jm}^{(d)(1B)}, \\ \check{H}_{njm}^{(d)} &\equiv \beta^n H_{njm}^{(d)(0B)} + \sqrt{\frac{2}{j(j+1)}} \beta^{n+2} H_{(n+2)jm}^{(d)(1B)}, \end{aligned} \quad (45)$$

where the muon velocity is  $\beta = \sqrt{1 - 1/\gamma^2}$  as usual. Note that all coefficients with  $m \neq 0$  are complex, and coefficients with negative  $m$  are related to those with positive  $m$  via expressions of the form  $\mathcal{K}_{jm}^* = (-1)^m \mathcal{K}_{j(-m)}$  [17]. Also, only coefficients with even  $n$  contribute to Eq. (45), as those with odd  $n$  come only with even  $j$  and hence cancel via the geometrical factor (43). For example, we find 32 independent observable combinations can contribute for  $d \leq 6$ , and they are denoted as  $\check{H}_{01m}^{(3)}$ ,  $\check{g}_{01m}^{(4)}$ ,  $\check{H}_{01m}^{(5)}$ ,  $\check{H}_{21m}^{(5)}$ ,  $\check{H}_{23m}^{(5)}$ ,  $\check{g}_{01m}^{(6)}$ ,  $\check{g}_{21m}^{(6)}$ , and  $\check{g}_{23m}^{(6)}$ . Moreover, the coefficients on the right-hand side of Eq. (45) are understood to contribute only if their indices lie in the ranges given in Table III of Ref. [17]. For example, for  $d = 3$  the coefficient  $\check{H}_{njm}^{(3)}$

contains only  $H_{njm}^{(3)(0B)}$  because  $H_{njm}^{(d)(1B)}$  exists only for  $d \geq 5$ . Along similar lines,  $\tilde{H}_{2jm}^{(5)}$  contains only  $H_{2jm}^{(5)(0B)}$  because  $H_{njm}^{(5)(1B)}$  exists only for  $n \leq 2$ .

The expression (42) for  $\delta\omega_a^\pm$  encompasses effects from Lorentz-violating operators of arbitrary mass dimensions. In the appropriate limit, it reduces to the analogous result (11) for  $\delta\omega_a^+$  derived in Ref. [9] in terms of the minimal-SME cartesian coefficients  $b_\mu$ ,  $d_{\mu\nu}$ , and  $H_{\mu\nu}$ . One set of predicted effects for the general case includes sidereal variations of  $\delta\omega_a^\pm$  at harmonics of  $\omega_\oplus$ . Another prediction is a difference  $\Delta\omega_a = \delta\omega_a^+ - \delta\omega_a^-$  between the anomaly frequencies of the muon and antimuon, with both constant and time-varying components. These predictions are discussed in the next two subsections. In addition, the orbital motion of the Earth about the Sun introduces further sensitivities to Lorentz violation beyond those in Eq. (42). Treating these requires a separate analysis, which is the subject of Sec. III D.

The form of the correction  $\delta\omega_a^\pm$  given by Eq. (42) reveals that sensitivity to coefficients of larger  $d$  typically increases with the particle energy. This behavior places the planned experiments at Fermilab [46] and J-PARC [47] in distinct positions, as it indicates that each enjoys different sensitivities to some coefficient combinations. Both have a similar overall potential reach for minimal-SME spherical coefficients, but the smaller value of  $\gamma$  to be used at J-PARC implies an improvement of an order of magnitude in sensitivity to certain minimal-SME cartesian coefficients such as  $b_3$ . In contrast, the higher-energy particles to be used at Fermilab lead to greater sensitivity to nonminimal coefficients, and the design of the Fermilab experiment may ultimately permit measurements with both muons and antimuons, a feature unavailable to the currently proposed J-PARC setup. Moreover, the differing colatitudes and energies of the Fermilab and J-PARC experiments suggests that combining results for antimuons would permit constraints on coefficient combinations inaccessible to any single experiment.

## B. Muon-antimuon comparison

### 1. $CPT$ -odd effects

When  $CPT$  violation is present in the muon sector, differences between the anomaly frequencies  $\omega_a^+$  and  $\omega_a^-$  can appear. Simultaneous measurement of the two frequencies is experimentally infeasible, but a comparison of them averaged over many sidereal days can directly isolate the  $CPT$  violation. Denoting the time-averaged anomaly-frequency difference by  $\langle\Delta\omega_a\rangle$ , we obtain

$$\begin{aligned} \langle\Delta\omega_a\rangle &= \langle\delta\omega_a^+\rangle - \langle\delta\omega_a^-\rangle \\ &= 4 \sum_{dnj} E_0^{d-3} G_{j0}(\chi) \hat{g}_{nj0}^{(d)}. \end{aligned} \quad (46)$$

TABLE VII. Constraints on spherical coefficients determined from the anomaly-frequency difference in the BNL experiment. Units are  $\text{GeV}^{4-d}$ .

| $d$ | Coefficient  | Constraint                       |
|-----|--|----------------------------------|
| 4   | $\hat{g}_{010}^{(4)}$  | $(-2.3 \pm 2.4) \times 10^{-25}$ |
| 6   | $\hat{g}_{010}^{(6)}, \hat{g}_{210}^{(6)}$   | $(-2.4 \pm 2.5) \times 10^{-26}$ |
|     | $\hat{g}_{230}^{(6)}$  | $(-2.5 \pm 2.5) \times 10^{-26}$ |
| 8   | $\hat{g}_{010}^{(8)}, \hat{g}_{210}^{(8)}, \hat{g}_{410}^{(8)}$                          | $(-2.5 \pm 2.6) \times 10^{-27}$ |
|     | $\hat{g}_{230}^{(8)}, \hat{g}_{430}^{(8)}$   | $(-2.6 \pm 2.6) \times 10^{-27}$ |
|     | $\hat{g}_{450}^{(8)}$  | $(1.6 \pm 1.7) \times 10^{-27}$  |
| 10  | $\hat{g}_{010}^{(10)}, \hat{g}_{210}^{(10)}, \hat{g}_{410}^{(10)}, \hat{g}_{610}^{(10)}$ | $(-2.6 \pm 2.7) \times 10^{-28}$ |
|     | $\hat{g}_{230}^{(10)}, \hat{g}_{430}^{(10)}, \hat{g}_{630}^{(10)}$                       | $(-2.7 \pm 2.7) \times 10^{-28}$ |
|     | $\hat{g}_{450}^{(10)}, \hat{g}_{650}^{(10)}$   | $(1.7 \pm 1.7) \times 10^{-28}$  |
|     | $\hat{g}_{670}^{(10)}$   | $(1.3 \pm 1.4) \times 10^{-28}$  |

This approach restricts attention to coefficients  $\hat{g}_{nj0}^{(d)}$  for  $CPT$  violation having index  $m = 0$ , which control azimuthally isotropic operators in the Sun-centered frame and hence exhibit no sidereal variations at leading order.

The BNL experiment [12] reported a measurement of the figure of merit  $\langle\Delta\omega_a\rangle/m_\mu$ , which using  $m_\mu = 105.7$  MeV gives the constraint

$$\sum_{dnj} E_0^{d-3} G_{j0}(\chi) \hat{g}_{nj0}^{(d)} = -(2.3 \pm 2.4) \times 10^{-25} \text{ GeV}. \quad (47)$$

Paralleling the above discussion of muonic atoms, it is useful to extract from this expression the attained sensitivities to individual spherical coefficients, taking only one coefficient to be nonzero at a time. The resulting constraints are compiled in Table VII for  $d \leq 10$ . Note that these values correspond closely to limits on the coefficients  $\hat{g}_{nj0}^{(d)(0B)}$  derived using the definition (45) because  $\beta \approx 1$  to an excellent approximation, while limits on the coefficients  $\hat{g}_{nj0}^{(d)(1B)}$  can be obtained by scaling with a factor of  $\sqrt{j(j+1)/2}$ .

Results for the minimal SME are given as a limiting case of the above. Indeed, the minimal-SME coefficient  $\hat{g}_{010}^{(4)}$  can be expressed in terms of cartesian coefficients in the Sun-centered frame as

$$\hat{g}_{010}^{(4)} = \frac{1}{\gamma E_0} \sqrt{\frac{4\pi}{3}} \left( b_Z - m_\mu g_Z^{(A)} + \left( 1 + \frac{3}{2} \beta^2 \gamma^2 \right) m_\mu g_{XYT}^{(M)} \right), \quad (48)$$

where the superscripts (A) and (M) denote the irreducible axial and irreducible mixed-symmetry combinations of the coefficients  $g_{\kappa\lambda}$ , respectively [50,51]. This result extends the one derived in the original theoretical treatment [9], which excludes the coefficients  $g_{\kappa\lambda}$  on the grounds of an expected suppression relative to other minimal-SME

coefficients arising from the breaking of  $SU(2) \times U(1)$  symmetry [6]. It thereby reveals that the measurement of  $b_Z$  reported by the BNL experiment [12] can be extended to

$$\begin{aligned} b_Z - m_\mu g_Z^{(A)} + \left(1 + \frac{3}{2}\beta^2\gamma^2\right) m_\mu g_{XYT}^{(M)} \\ = -(1.0 \pm 1.1) \times 10^{-23} \text{ GeV}. \end{aligned} \quad (49)$$

This represents the first reported constraint containing muon-sector  $g$ -type coefficients in the minimal SME. The comparatively large boost in this experiment provides an enhanced sensitivity to the mixed-symmetry combination taken by itself,  $m_\mu g_{XYT}^{(M)} = -(7.8 \pm 8.5) \times 10^{-27} \text{ GeV}$ , that compares favorably with the Planck-suppressed ratio  $m_\mu^2/M_P \approx 9.2 \times 10^{-22} \text{ GeV}$ .

The existing proposals for the forthcoming Fermilab [46] and J-PARC [47] experiments are focused on measurements of the antimuon anomalous magnetic moment. However, if future studies of muons could also be performed at similar sensitivities, then the constraints given in Table VII could be improved by a factor of roughly 5 from precision alone.

## 2. *CPT-even effects*

The availability of experiments at different latitudes also offers access to isotropic coefficients for *CPT*-even Lorentz violation [12]. The idea is that the predicted time-averaged antimuon anomaly frequency  $\omega_a^+(\chi_1)$  at colatitude  $\chi_1$  differs from the time-averaged muon anomaly frequency  $\omega_a^-(\chi_2)$  at colatitude  $\chi_2 \neq \chi_1$ , and this difference is sensitive also to *CPT*-even effects. Since the two experiments typically also have distinct magnetic fields, it is useful to work in terms of the ratios  $\mathcal{R}^\pm = \omega_a^\pm/\omega_p$  of the muon anomaly frequencies to the proton cyclotron frequency, which removes the dependence on the magnetic field and is widely used in experimental analyses.

The difference  $\langle \Delta\mathcal{R}(1,2) \rangle$  between the time-averaged values of  $\mathcal{R}^+(\chi_1)$  and  $\mathcal{R}^-(\chi_2)$  is found from Eq. (42) to be

$$\begin{aligned} \langle \Delta\mathcal{R}(1,2) \rangle &\equiv \langle \mathcal{R}^+(\chi_1) \rangle - \langle \mathcal{R}^-(\chi_2) \rangle \\ &= 2 \sum_{dnj} E_0^{d-3} \left( \frac{G_{j0}(\chi_1)}{\omega_p(\chi_1)} + \frac{G_{j0}(\chi_2)}{\omega_p(\chi_2)} \right) \check{g}_{nj0}^{(d)} \\ &\quad + 2 \sum_{dnj} E_0^{d-3} \left( \frac{G_{j0}(\chi_1)}{\omega_p(\chi_1)} - \frac{G_{j0}(\chi_2)}{\omega_p(\chi_2)} \right) \check{H}_{nj0}^{(d)}. \end{aligned} \quad (50)$$

As can be seen from Table VI, the dimensionless factor  $G_{j0}(\chi_1) - G_{j0}(\chi_2)$  is small, so the sensitivity to the coefficients  $\check{H}_{nj0}^{(d)}$  is reduced compared with that to *CPT*-odd effects. For practical purposes, we can therefore assume here that the coefficients  $\check{g}_{nj0}^{(d)}$  have been excluded at a sufficient precision so that attention can be focused purely on *CPT*-even effects. Then, only the piece of Eq. (50)

involving the coefficients  $\check{H}_{nj0}^{(d)}$  contributes. Note that these coefficients carry index  $m=0$  and therefore cannot be detected via sidereal variations, so an analysis using Eq. (50) offers an interesting avenue for exploration of *CPT*-even effects that otherwise might escape detection.

The CERN [44] and BNL [45] experiments have each reported values of  $\mathcal{R}^\pm$  and are located at colatitudes differing by about  $5^\circ$ . These results can be used to calculate  $\Delta\mathcal{R}(\text{CERN}, \text{BNL})$ , for example, which involves antimuons at CERN and muons at BNL. The CERN experiment at  $\chi \approx 43.7^\circ$  measured  $\mathcal{R}^+ = 3.707173(36) \times 10^{-3}$  with  $\omega_p/2\pi \approx 6.278302(5) \times 10^7 \text{ Hz}$ , while the BNL experiment at  $\chi \approx 49.1^\circ$  obtained  $\mathcal{R}^- = 3.7072083(26) \times 10^{-3}$  with  $\omega_p/2\pi \approx 6.1791400(11) \times 10^7 \text{ Hz}$ . Using these values, we find  $\Delta\mathcal{R}(\text{CERN}, \text{BNL}) = (-3.5 \pm 3.6) \times 10^{-8}$ . With this value and neglecting as comparatively small the contributions from  $\check{g}_{nj0}^{(d)}$ , we obtain the bound

$$\begin{aligned} \sum_{dnj} E_0^{d-3} \left( \frac{G_{j0}(\chi_1)}{\omega_p(\chi_1)} - \frac{G_{j0}(\chi_2)}{\omega_p(\chi_2)} \right) \check{H}_{nj0}^{(d)} \\ < (-1.8 \pm 1.8) \times 10^{-8}. \end{aligned} \quad (51)$$

As before, we can gain insight by extracting from this bound the attained sensitivities to each individual coefficient at a time. The results for  $d \leq 9$  are displayed in Table VIII. Additional constraints can be obtained by calculating  $\Delta\mathcal{R}(\text{BNL}, \text{CERN})$ , which instead involves muons at CERN and antimuons at BNL. This gives a comparable sensitivity of  $\Delta\mathcal{R}(\text{BNL}, \text{CERN}) = (-5.1 \pm 3.7) \times 10^{-8}$  and slightly weaker constraints on the individual coefficients. Improved results along these lines can be expected once measurements have been made by the forthcoming Fermilab and J-PARC experiments.

TABLE VIII. Constraints on spherical coefficients determined from the difference between CERN antimuon and BNL muon anomaly frequencies. Units are  $\text{GeV}^{4-d}$ .

| $d$ | Coefficient  | Constraint                       |
|-----|--|----------------------------------|
| 3   | $\check{H}_{010}^{(3)}$  | $(-1.6 \pm 1.7) \times 10^{-22}$ |
| 5   | $\check{H}_{010}^{(5)}, \check{H}_{210}^{(5)}$   | $(-1.7 \pm 1.7) \times 10^{-23}$ |
|     | $\check{H}_{230}^{(5)}$  | $(2.9 \pm 3.0) \times 10^{-24}$  |
| 7   | $\check{H}_{010}^{(7)}, \check{H}_{210}^{(7)}, \check{H}_{410}^{(7)}$                        | $(-1.7 \pm 1.8) \times 10^{-24}$ |
|     | $\check{H}_{230}^{(7)}, \check{H}_{430}^{(7)}$   | $(3.0 \pm 3.1) \times 10^{-25}$  |
|     | $\check{H}_{450}^{(7)}$  | $(2.6 \pm 2.6) \times 10^{-25}$  |
| 9   | $\check{H}_{010}^{(9)}, \check{H}_{210}^{(9)}, \check{H}_{410}^{(9)}, \check{H}_{610}^{(9)}$ | $(-1.8 \pm 1.9) \times 10^{-25}$ |
|     | $\check{H}_{230}^{(9)}, \check{H}_{430}^{(9)}, \check{H}_{630}^{(9)}$                        | $(3.2 \pm 3.3) \times 10^{-26}$  |
|     | $\check{H}_{450}^{(9)}, \check{H}_{650}^{(9)}$   | $(2.7 \pm 2.7) \times 10^{-26}$  |
|     | $\check{H}_{670}^{(9)}$  | $(-1.1 \pm 1.1) \times 10^{-26}$ |



### C. Sidereal variations

The general expression (42) for  $\delta\omega_a^\pm$  shows that nonzero coefficients for Lorentz violation with  $m \neq 0$  lead to the variation of  $\omega_a^\pm$  with sidereal time  $T_\oplus$ . The variation is a superposition of oscillations of different frequencies. A given term in the sum has harmonic frequency  $m\omega_\oplus$ , where  $m$  is the index on the corresponding coefficient. The total amplitude of the  $m$ th harmonic is

$$A_m^\pm = \left| 4 \sum_{dnj} E_0^{d-3} G_{jm}(\chi) [\check{H}_{njm}^{(d)} \pm E_0 \check{g}_{njm}^{(d+1)}] \right|, m \neq 0. \quad (52)$$

In evaluating this expression, recall that the argument of the modulus is complex in general because for  $m \neq 0$  the coefficients can have real and imaginary parts.

The above amplitude is valid for any finite range of the operator mass dimension  $d$ . The maximum value of  $d$  in this range determines the highest harmonic variation appearing in the signal. To illustrate this, focus on one specific value of  $d$  at a time, as is typical in data analyses studying the effects of nonminimal Lorentz-violating operators [5]. Since the specific value of  $d$  determines the corresponding range of the coefficient indices  $n$  and  $j$  [17], which in turn controls the largest possible size of  $m$ , it follows that the allowed harmonics for a given  $d$  include values of  $m$  up to a definite maximum  $m_{\max}$ . For odd  $d$  we find  $m_{\max} = d - 2$ , while for even  $d$  we obtain  $m_{\max} = d - 3$ . For example, when  $d = 3$  and 4 only the fundamental sidereal frequency  $\omega_\oplus$  appears, while  $d = 5$  and 6 operators are accompanied also by variations at the frequencies  $2\omega_\oplus$  and  $3\omega_\oplus$ .

The E821 experiment at BNL searched for sidereal variations at the fundamental sidereal frequency  $\omega_\oplus$  [12], obtaining the bounds  $A_1^+ \leq 2.1 \times 10^{-24}$  GeV and  $A_1^- \leq 4.0 \times 10^{-24}$  GeV at the 95% confidence level and placing the tightest limits to date on minimal-SME muon-sector coefficients. In the present context, Eq. (52) reveals that these results also bound some combinations of spherical coefficients associated with Lorentz-violating operators of arbitrarily high mass dimension. We can illustrate this explicitly by determining the sensitivities to individual spherical coefficients for a range of values of  $d$ , with only the real or imaginary part of a single coefficient assumed nonzero at a time. Table IX lists the resulting constraints on all spherical coefficients with  $m = 1$  and  $d \leq 8$ . A reanalysis of the BNL data constraining higher harmonics could yield measurements on all the remaining coefficients with  $m \neq 0$  as well. Indeed, the Lomb power spectrum displayed in Fig. 2 of Ref. [12] suggests no varying signal at the various sidereal harmonics, offering the potential for tight bounds on these coefficients.

In the limiting case of the minimal SME, the amplitudes (52) can be expressed in terms of cartesian coefficients for operators of dimension  $d = 3$  and 4 using the relationships

TABLE IX. Constraints on the moduli of the real and imaginary parts of spherical coefficients determined from sidereal variations of the antimuon anomaly frequency in the BNL experiment. Units are  $\text{GeV}^{4-d}$ .

| $d$ | Coefficient<br>$\check{\mathcal{K}}$                                  | Constraint on<br>$ \text{Re}\check{\mathcal{K}} ,  \text{Im}\check{\mathcal{K}} $ |
|-----|---|---|
| 3   | $\check{H}_{011}^{(3)}$   | $< 2.0 \times 10^{-24}$   |
| 4   | $\check{g}_{011}^{(4)}$   | $< 6.6 \times 10^{-25}$   |
| 5   | $\check{H}_{011}^{(5)}, \check{H}_{211}^{(5)}$                        | $< 2.1 \times 10^{-25}$   |
|     | $\check{H}_{231}^{(5)}$   | $< 1.3 \times 10^{-25}$   |
| 6   | $\check{g}_{011}^{(6)}, \check{g}_{211}^{(6)}$                        | $< 6.8 \times 10^{-26}$   |
|     | $\check{g}_{231}^{(6)}$   | $< 4.3 \times 10^{-26}$   |
| 7   | $\check{H}_{011}^{(7)}, \check{H}_{211}^{(7)}, \check{H}_{411}^{(7)}$ | $< 2.2 \times 10^{-26}$   |
|     | $\check{H}_{231}^{(7)}, \check{H}_{431}^{(7)}$                        | $< 1.4 \times 10^{-26}$   |
|     | $\check{H}_{451}^{(7)}$   | $< 1.1 \times 10^{-26}$   |
| 8   | $\check{g}_{011}^{(8)}, \check{g}_{211}^{(8)}, \check{g}_{411}^{(8)}$ | $< 7.1 \times 10^{-27}$   |
|     | $\check{g}_{231}^{(8)}, \check{g}_{431}^{(8)}$                        | $< 4.5 \times 10^{-27}$   |
|     | $\check{g}_{451}^{(8)}$   | $< 3.6 \times 10^{-27}$   |

$$\begin{aligned} \text{Re}\check{H}_{011}^{(3)} \pm E_0 \text{Re}\check{g}_{011}^{(4)} &= -\sqrt{\frac{2\pi}{3}} b_X^\pm, \\ \text{Im}\check{H}_{011}^{(3)} \pm E_0 \text{Im}\check{g}_{011}^{(4)} &= -\sqrt{\frac{2\pi}{3}} b_Y^\pm \end{aligned} \quad (53)$$

in the Sun-centered frame. Here, the combinations

$$\begin{aligned} b_J^\pm &\equiv \pm \frac{1}{\gamma} (b_J - m_\mu g_J^{(A)}) + \frac{1}{2} \epsilon_{JKL} H_{KL} + m_\mu d_{JT} \\ &\pm \frac{1}{2\gamma} \left( 1 + \frac{3}{2} \beta^2 \gamma^2 \right) m_\mu \epsilon_{JKL} g_{KLT}^{(M)} \end{aligned} \quad (54)$$

generalize the quantities introduced in Ref. [9] to include also contributions from the antisymmetric and mixed-symmetry irreducible combinations of the  $g_{\kappa\lambda}$  coefficients, in analogy with Eq. (48). This shows that the two bounds reported as Eq. (11) of Ref. [12] incorporate also sensitivity to the  $g$ -type coefficients. For example, in a model with only  $g_{KLT}^{(M)}$  nonzero, the constraint  $m_\mu \sqrt{(g_{XZT}^{(M)})^2 + (g_{YZT}^{(M)})^2} < 1.1 \times 10^{-27}$  GeV at the 95% confidence level is obtained. We remark in passing that in the nonrelativistic limit  $\beta \rightarrow 0$ ,  $\gamma \rightarrow 1$  the háček coefficients (54) reduce to combinations of the standard cartesian tilde coefficients [5], yielding the correspondences  $b_J^+ \rightarrow \tilde{b}_J^*$ ,  $b_J^- \rightarrow -\tilde{b}_J$ .

The constraints in Table IX are a consequence of the 0.54 ppm precision attained by the BNL experiment [45]. Future proposals aim to achieve 0.14 ppm at Fermilab [46] and 0.1 ppm at J-PARC [47], which would offer the

opportunity to sharpen the values in Table IX by about a factor of five. The energy dependence of the amplitudes (52) suggests that the Fermilab and J-PARC experiments will achieve approximately the same sidereal reach to the spherical coefficients with  $d = 3$ , with the latter's sensitivity for  $d \geq 4$  suppressed by a factor of about  $10^{d-3}$ . However, the J-PARC experiment enjoys an additional improvement of a factor of about 10 in the sensitivity to certain coefficients that are accompanied by factors of  $1/\gamma$ , such as  $b_J$  in Eq. (54). More broadly, comparing results from experiments at different  $\gamma$  offers the opportunity to disentangle coefficients, as exemplified by the structure of Eq. (54).

#### D. Annual variations

In the presence of Lorentz violation, the motion of the Earth about the Sun can introduce distinct time variations in the anomaly frequencies, offering an opportunity to gain sensitivity to additional coefficients. Comparatively few experimental studies have been performed that take advantage of the changes in the Earth's boost over the course of the solar year, in part due to factors such as the extended period of data collection, the necessary long-term stability of the apparatus, and the statistical power required. Recent analyses accounting in detail for boost effects include ones performed with a dual Xe-He maser [52] and using a spin-torsion pendulum [53]. An analogous investigation is feasible for the muon anomaly frequency, with the added bonus that boost effects for both the muon and the antimuon can be studied, at least in principle.

In this subsection, we consider boost signals arising from minimal-SME operators in the muon sector at leading relativistic order. The spherical decomposition is well suited for analyses of rotational properties but is cumbersome for boosts, so we work instead with cartesian coefficients for Lorentz violation. The nonminimal cartesian coefficients could also be studied, but the corresponding analysis is more involved and lies outside our present scope. The analysis here shows that measurements of the anomaly frequency at existing and planned precisions can yield sensitivities at the Planck-suppressed level to 25 of the 44 independent observables for Lorentz violation in the minimal-SME muon sector. Most of these are unmeasured to date.

In standard coordinates in the laboratory frame [30], the correction to the anomaly frequency due to Lorentz violation is

$$\begin{aligned} \delta\omega_a^\pm &= \pm 2\check{b}_3^\pm \\ &\equiv \pm \frac{1}{\gamma} (b_3 - m_\mu g_3^{(A)}) + m_\mu d_{30} + H_{12} \\ &\quad \pm \frac{1}{\gamma} \left( 1 + \frac{3}{2}\beta^2\gamma^2 \right) m_\mu g_{120}^{(M)}. \end{aligned} \quad (55)$$

As discussed in Sec. III C, expressing the coefficients in the Sun-centered frame reveals that the Earth's rotation introduces dependence on the coefficient combinations  $\check{b}_X^\pm, \check{b}_Y^\pm$  given in Eq. (54). The Earth's boost  $\beta_\oplus \approx 10^{-4}$  about the Sun produces sensitivity to additional coefficient combinations, as does the laboratory boost  $\beta_L \approx 10^{-6}$  due to the surface velocity from the Earth's rotation. Although the sensitivities to additional coefficients are comparatively suppressed by the boost factors, the experimental precision nonetheless would suffice to yield results of definite interest.

The relativistic corrections to the anomaly frequencies at leading order in  $\beta_\oplus$  and  $\beta_L$  can be obtained by transforming from the Sun-centered frame to the laboratory frame. The transformation can be separated into two steps [30]: an instantaneous boost from the Sun-centered frame to a nonrotating frame at the Earth's surface, followed by a rotation to the laboratory frame. The resulting expressions for  $\check{b}_3^\pm$  in terms of coefficients in the Sun-centered frame are given in Table X. In this table, we denote the Earth sidereal rotation frequency by  $\omega_\oplus \approx 2\pi/(23 \text{ h } 56 \text{ m})$  as before, the Earth orbital frequency by  $\Omega_\oplus \approx 2\pi/(365.26 \text{ d})$ , the Earth orbital tilt by  $\eta \approx 23.5^\circ$ , and the colatitude of the laboratory by  $\chi$ . The explicit expressions for  $\check{b}_3^\pm$  are obtained for each particle by multiplying all the factors in a particular row and adding the contributions from all rows. The coefficient factors that appear are expressed in terms of háček coefficients, which are convenient combinations of the basic cartesian coefficients chosen to reduce to the standard tilde combinations [5] in the nonrelativistic limit. Explicit expressions for the háček coefficients in terms of cartesian coefficients are given in Table XI.

The above discussion shows that one goal of a search for Lorentz and *CPT* violation using data from  $g - 2$  experiments is to report sensitivities to the combinations of háček coefficients appearing in Table X. The terms independent of sidereal time can be studied by comparing anomaly frequencies, as described in Sec. III B, while those depending on  $\cos \omega_\oplus T_\oplus$  and  $\sin \omega_\oplus T_\oplus$  but not on  $\cos \Omega_\oplus T$  or  $\sin \Omega_\oplus T$  lead to constraints on  $\check{b}_X^\pm, \check{b}_Y^\pm$  using the Earth's rotation, as in Sec. III C. All other terms represent effects due to boosts. Each anomaly frequency acquires two contributions from the Earth's orbital motion depending on time as  $\cos \Omega_\oplus T$  or  $\sin \Omega_\oplus T$ . Other terms involving the Earth's boost vary as products of  $\cos \omega_\oplus T_\oplus$  or  $\sin \omega_\oplus T_\oplus$  with  $\cos \Omega_\oplus T$  or  $\sin \Omega_\oplus T$  and hence oscillate predominantly at the sidereal frequency with a slow Earth-orbital variation superposed. The remaining terms are suppressed by the laboratory boost  $\beta_L$  and are either constant or vary with sidereal time. Note that some of the latter oscillate at twice the sidereal frequency.

As before, further insight can be gained by considering bounds on individual háček coefficients assuming all others vanish. In this scenario, terms suppressed by  $\beta_L$  can

TABLE X. Factors forming the expansion of the muon and antimuon observables in the Sun-centered frame. For each particle, the complete expression is obtained by multiplying the factors in each row and adding all the relevant rows.

| Particle | Boost Factor   | Sidereal Factor                                    | Colatitude Factor | Coefficient Factor   |
|----------|----------------|--|-------------------|--|
| $\mu^-$  | 1              | 1  | $\cos \chi$       | $\check{b}_Z$  |
|          | 1              | $\cos \omega_\oplus T_\oplus$                      | $\sin \chi$       | $\check{b}_X$  |
|          | 1              | $\sin \omega_\oplus T_\oplus$                      | $\sin \chi$       | $\check{b}_Y$  |
|          | $\beta_\oplus$ | $\cos \Omega_\oplus T$                             | $\cos \chi$       | $-\cos \eta(\check{H}_{TX}) + \sin \eta(-\check{g}_T + 2\check{d}_+ - \check{d}_Q)$  |
|          | $\beta_\oplus$ | $\sin \Omega_\oplus T$                             | $\cos \chi$       | $-\check{d}_{ZX} - \check{H}_{TY}$   |
|          | $\beta_\oplus$ | $\cos \omega_\oplus T_\oplus \cos \Omega_\oplus T$ | $\sin \chi$       | $\cos \eta(\check{d}_{XY} + \check{H}_{TZ}) - \sin \eta \check{H}_{TY}$  |
|          | $\beta_\oplus$ | $\cos \omega_\oplus T_\oplus \sin \Omega_\oplus T$ | $\sin \chi$       | $-\frac{1}{2}\check{b}_T - \frac{1}{2}\check{d}_- + \check{g}_c + \frac{3}{2}\check{g}_T - 2\check{d}_+ + \frac{1}{2}\check{d}_Q$  |
|          | $\beta_\oplus$ | $\sin \omega_\oplus T_\oplus \cos \Omega_\oplus T$ | $\sin \chi$       | $\cos \eta(-\frac{1}{2}\check{b}_T - \frac{1}{2}\check{d}_- + \check{g}_c - \frac{1}{2}\check{g}_T + 2\check{d}_+ - \frac{1}{2}\check{d}_Q) + \sin \eta(\check{d}_{YZ} + \check{H}_{TX})$                    |
|          | $\beta_\oplus$ | $\sin \omega_\oplus T_\oplus \sin \Omega_\oplus T$ | $\sin \chi$       | $\check{H}_{TZ}$   |
|          | $\beta_L$      | 1  | $\sin \chi$       | $-\frac{1}{2}\check{d}_{XY} - \check{H}_{TZ}$  |
|          | $\beta_L$      | $\cos \omega_\oplus T_\oplus$                      | $\cos \chi$       | $\check{H}_{TX}$   |
|          | $\beta_L$      | $\sin \omega_\oplus T_\oplus$                      | $\cos \chi$       | $\check{d}_{ZX} + \check{H}_{TY}$  |
|          | $\beta_L$      | $\cos 2\omega_\oplus T_\oplus$                     | $\sin \chi$       | $-\frac{1}{2}\check{d}_{XY}$   |
|          | $\beta_L$      | $\sin 2\omega_\oplus T_\oplus$                     | $\sin \chi$       | $\frac{1}{2}\check{b}_T + \frac{1}{2}\check{d}_- - \check{g}_c - \frac{1}{2}\check{g}_T$   |
| $\mu^+$  | 1              | 1  | $\cos \chi$       | $\check{b}_Z^*$  |
|          | 1              | $\cos \omega_\oplus T_\oplus$                      | $\sin \chi$       | $\check{b}_X^*$  |
|          | 1              | $\sin \omega_\oplus T_\oplus$                      | $\sin \chi$       | $\check{b}_Y^*$  |
|          | $\beta_\oplus$ | $\cos \Omega_\oplus T$                             | $\cos \chi$       | $\cos \eta(-2\check{g}_{XY} + \check{H}_{TX}) + \sin \eta(-2\check{b}_T + \check{g}_T - 2\check{d}_+ + \check{d}_Q)$   |
|          | $\beta_\oplus$ | $\sin \Omega_\oplus T$                             | $\cos \chi$       | $\check{d}_{ZX} - 2\check{g}_{YX} + \check{H}_{TY}$  |
|          | $\beta_\oplus$ | $\cos \omega_\oplus T_\oplus \cos \Omega_\oplus T$ | $\sin \chi$       | $\cos \eta(-\check{d}_{XY} + 2\check{g}_{ZY} - \check{H}_{TZ}) + \sin \eta(-2\check{g}_{YZ} + \check{H}_{TY})$   |
|          | $\beta_\oplus$ | $\cos \omega_\oplus T_\oplus \sin \Omega_\oplus T$ | $\sin \chi$       | $\frac{1}{2}\check{b}_T + \frac{1}{2}\check{d}_- + \check{g}_c + \frac{1}{2}\check{g}_T + 2\check{d}_+ - \frac{1}{2}\check{d}_Q$   |
|          | $\beta_\oplus$ | $\sin \omega_\oplus T_\oplus \cos \Omega_\oplus T$ | $\sin \chi$       | $\cos \eta(-\frac{3}{2}\check{b}_T + \frac{1}{2}\check{d}_- + \check{g}_c + \frac{1}{2}\check{g}_T - 2\check{d}_+ + \frac{1}{2}\check{d}_Q) + \sin \eta(-\check{d}_{YZ} + 2\check{g}_{XZ} - \check{H}_{TX})$ |
|          | $\beta_\oplus$ | $\sin \omega_\oplus T_\oplus \sin \Omega_\oplus T$ | $\sin \chi$       | $2\check{g}_{ZX} - \check{H}_{TZ}$   |
|          | $\beta_L$      | 1  | $\sin \chi$       | $\frac{1}{2}\check{d}_{XY} - \check{g}_{ZX} - \check{g}_{ZY} + \check{H}_{TZ}$   |
|          | $\beta_L$      | $\cos \omega_\oplus T_\oplus$                      | $\cos \chi$       | $2\check{g}_{XY} - \check{H}_{TX}$   |
|          | $\beta_L$      | $\sin \omega_\oplus T_\oplus$                      | $\cos \chi$       | $-\check{d}_{ZX} + 2\check{g}_{YX} - \check{H}_{TY}$   |
|          | $\beta_L$      | $\cos 2\omega_\oplus T_\oplus$                     | $\sin \chi$       | $\frac{1}{2}\check{d}_{XY} + \check{g}_{ZX} - \check{g}_{ZY}$  |
|          | $\beta_L$      | $\sin 2\omega_\oplus T_\oplus$                     | $\sin \chi$       | $\frac{1}{2}\check{b}_T - \frac{1}{2}\check{d}_- - \check{g}_c - \frac{1}{2}\check{g}_T$   |

reasonably be neglected in the analysis because they are suppressed by a factor of 100 or more relative to all the others and because all individual coefficients appearing in these terms are also present elsewhere in the expressions for the anomaly frequencies. It therefore suffices to analyze experimental data for either the muon or the antimuon to obtain six independent results proportional to  $\beta_\oplus$ , corresponding to the six different component oscillations involving  $\omega_\oplus$  and  $\Omega_\oplus$ . Assuming sufficient statistical power in the data and the design reach of order 0.1 ppm for the forthcoming Fermilab [46] and J-PARC [47] experiments, it appears plausible that sensitivities of order  $10^{-20}$  GeV or better could be attained for each of six measurements on the antimuon anomaly frequency involving the Earth boost  $\beta_\oplus$ . If muons are also available, another six independent constraints can be obtained. Taking one coefficient at a time, these measurements would yield Planck-scale sensitivity to 25 of the 44 observables for

muons in the minimal SME. These 25 observables can be taken as the 25 háček coefficients provided in the first column of Table XI, or equivalently as the 25 independent cartesian coefficients appearing in the combinations listed in the second column.

### E. The apparent anomaly discrepancy

Calculations of the muon anomaly  $a \equiv (g-2)/2$  performed in the context of the SM [54] produce a result lying about three standard deviations below the value measured by the BNL experiment [45,55]. The apparent discrepancy  $\Delta a \equiv a_{\text{expt}} - a_{\text{SM}}$  could originate from comparatively prosaic sources such as a statistical fluctuation in the experiment or uncertainties in the SM theory, or more dramatically from new physics beyond the SM. Typical one-loop corrections arising from Lorentz-invariant new physics with coupling  $g$  and mass scale  $M$  contribute at

order  $g^2(m_\mu/M)^2$ , leading to a variety of predicted signals in existing experiments. As one example, the apparent anomaly discrepancy can be reproduced in unified models with vector-like leptons having couplings  $g \approx 1/2$  and masses  $\approx 150$  GeV, yielding concomitant signals at the LHC [56]. Calculations of one-loop corrections to the anomaly in special Lorentz-violating models have also been performed [57].

Here, we consider a different idea, based on the result (42) showing that the presence of Lorentz violation can shift the measured value of  $\omega_a^\pm$ . An appropriate shift of this type could induce an observed discrepancy in the inferred value of the anomaly. Indeed, the apparent discrepancy  $\Delta a$  would be reproduced by a shift in the anomaly frequency of  $\Delta\omega_a \approx 2 \times 10^{-24}$  GeV. It is then natural to ask whether any coefficients exist that can achieve this shift while remaining compatible with existing constraints and, if so, what predictions this might yield for future experiments.

Since appropriate coefficients for this purpose must of necessity affect the anomaly frequency, inspection of Eq. (42) reveals that they must be a subset of  $\check{g}_{njm}^{(d)}$  and

$\check{H}_{njm}^{(d)}$ . However, the BNL data offer no indication that the discrepancy  $\Delta a$  differs significantly between muons and antimuons [45], so it is reasonable to consider only  $CPT$ -even effects. This limits attention to the coefficients  $\check{H}_{njm}^{(d)}$ . It also has the advantage of bypassing the existing constraints on  $\check{g}_{njm}^{(d)}$  obtained from direct comparisons of  $\omega_a^\pm$  and listed in Table VII. In addition, the Lomb spectrum and power distribution shown in Fig. 2 of Ref. [12] are consistent with no sidereal signal in the anomaly frequency, which suggests restricting attention to the coefficients  $\check{H}_{nj0}^{(d)}$  with  $m = 0$ .

The simplest terms for  $CPT$ -even effects without sidereal variations are associated with isotropic Lorentz violation,  $j = m = 0$ , and for  $d \leq 6$  only one  $H$ -type coefficient of this kind exists [17]. However, as described in Sec. III A, nonzero contributions to the anomaly frequency appear only for coefficients with odd  $j$ , so purely isotropic terms cannot reproduce the apparent discrepancy. Instead, the available coefficients satisfying the above criteria with  $d \leq 6$  turn out to include one with  $d = 3$ ,  $\check{H}_{010}^{(3)}$ , and three with  $d = 5$ ,  $\check{H}_{010}^{(5)}$ ,  $\check{H}_{210}^{(5)}$ , and  $\check{H}_{230}^{(5)}$ .

TABLE XI. Definitions of háček coefficients.

| Háček coefficient                    | Combination  | Number |
|--------------------------------------|--|--------|
| $\check{b}_J^* \equiv \check{b}_J^+$ | $\frac{1}{\gamma}(b_J - m_\mu g_J^{(A)}) + \frac{1}{2}\epsilon_{JKL}H_{KL} + m_\mu d_{JT} + \frac{1}{2\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu\epsilon_{JKL}g_{KLT}^{(M)}$ | 3      |
| $\check{b}_J \equiv -\check{b}_J^-$  | $\frac{1}{\gamma}(b_J - m_\mu g_J^{(A)}) - \frac{1}{2}\epsilon_{JKL}H_{KL} - m_\mu d_{JT} + \frac{1}{2\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu\epsilon_{JKL}g_{KLT}^{(M)}$ | 3      |
| $\check{b}_T$                        | $\frac{1}{\gamma}(b_T - m_\mu g_T^{(A)}) + \frac{1}{\gamma}(1 - \frac{3}{2}\beta^2\gamma^2)m_\mu g_{XYZ}^{(M)}$  | 1      |
| $\check{g}_T$                        | $\frac{1}{\gamma}(b_T - m_\mu g_T^{(A)}) - \frac{2}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu g_{XYZ}^{(M)}$  | 1      |
| $\check{H}_{TX}$                     | $H_{TX} - m_\mu d_{ZY} - \frac{1}{\gamma}(1 - \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TXT}^{(M)} - g_{XYT}^{(M)})$   |        |
| $\check{H}_{TY}$                     | $H_{TY} - m_\mu d_{XZ} - \frac{1}{\gamma}(1 - \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TYT}^{(M)} - g_{YZZ}^{(M)})$   |        |
| $\check{H}_{TZ}$                     | $H_{TZ} - m_\mu d_{YX} - \frac{1}{\gamma}(1 - \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TZZ}^{(M)} - g_{ZXX}^{(M)})$   | 3      |
| $\check{d}_\pm$                      | $m_\mu(d_{XX} \pm d_{YY})$   | 2      |
| $\check{d}_Q$                        | $m_\mu(d_{XX} + d_{YY} - 2d_{ZZ}) + \frac{3}{\gamma}(1 - \frac{1}{2}\beta^2\gamma^2)m_\mu g_{XYZ}^{(M)}$   | 1      |
| $\check{d}_{XY}$                     | $m_\mu(d_{XY} + d_{YX}) - \frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TZZ}^{(M)} + 2g_{ZXX}^{(M)})$   |        |
| $\check{d}_{YZ}$                     | $m_\mu(d_{YZ} + d_{ZY}) - \frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TXX}^{(M)} + 2g_{XYY}^{(M)})$   |        |
| $\check{d}_{ZX}$                     | $m_\mu(d_{ZX} + d_{XZ}) - \frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TTY}^{(M)} + 2g_{YZZ}^{(M)})$   | 3      |
| $\check{g}_c$                        | $\frac{1}{\gamma}m_\mu(2g_{XYZ}^{(M)} + g_{YZX}^{(M)} - \frac{3}{2}\beta^2\gamma^2 g_{ZXY}^{(M)})$   | 1      |
| $\check{g}_Q$                        | $-\frac{3}{\gamma}m_\mu(g_{TXX}^{(M)} + g_{TTY}^{(M)})$  | 1      |
| $\check{g}_{XZ}$                     | $-\frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(2g_{TXT}^{(M)} + g_{XYY}^{(M)})$   |        |
| $\check{g}_{XY}$                     | $-\frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TXX}^{(M)} - g_{XYY}^{(M)})$  |        |
| $\check{g}_{YX}$                     | $-\frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(2g_{TYT}^{(M)} + g_{YZZ}^{(M)})$   |        |
| $\check{g}_{YZ}$                     | $-\frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TYT}^{(M)} - g_{YZZ}^{(M)})$  |        |
| $\check{g}_{ZY}$                     | $-\frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(2g_{TZZ}^{(M)} + g_{ZXX}^{(M)})$   |        |
| $\check{g}_{ZX}$                     | $-\frac{1}{\gamma}(1 + \frac{3}{2}\beta^2\gamma^2)m_\mu(g_{TZZ}^{(M)} - g_{ZXX}^{(M)})$  | 6      |
| Total: 25                            |  |        |



Assuming only one coefficient is nonzero at a time, we find the approximate values needed to generate the required shift  $\Delta\omega_a$  in the anomaly frequency are

$$\begin{aligned}\check{H}_{010}^{(3)} &\approx 3 \times 10^{-24} \text{ GeV}, \\ \check{H}_{010}^{(5)} &\approx \check{H}_{210}^{(5)} \approx \check{H}_{230}^{(5)} \approx 3 \times 10^{-25} \text{ GeV}^{-1}.\end{aligned}\quad (56)$$

Any one of these four values therefore suffices to reproduce the discrepancy  $\Delta a$ , while somewhat smaller values are required if more than one coefficient is nonzero.

Some experimental bounds already exist on these coefficients, obtained from comparisons of anomaly-frequency measurements at the differing colatitudes of BNL and CERN and presented in Table VIII. A related constraint on  $H_{XY} \equiv \sqrt{3/4\pi}\check{H}_{010}^{(3)}$  is reported in Ref. [12]. All these limits are compatible with any of the four nonzero values (56) needed to reproduce the discrepancy  $\Delta a$ . Moreover, no other relevant constraints exist from muonium spectroscopy or astrophysical observations. As described in Sec. II, limits from muonium hyperfine transitions involve sidereal variations and hence involve contributions only from coefficients with  $m \neq 0$ , while other muonium spectroscopy lacks sufficient sensitivity. Also, at present astrophysical limits have been placed only on isotropic coefficients, and no sensitivity to  $H$ -type coefficients has been identified [17].

Overall, the nonzero values (56) appear largely acceptable on theoretical grounds as well. They are sufficiently small to be plausible as Planck-suppressed contributions from an underlying theory. For example, the required value of  $\check{H}_{010}^{(3)}$  is more than two orders of magnitude below the ratio  $m_\mu^2/M_P \approx 9.2 \times 10^{-22} \text{ GeV}$ . Also, *CPT*-even Lorentz-violating operators arise naturally in some frameworks. For example, noncommutative quantum field theories [58] intrinsically involve Lorentz violation because the commutator of coordinates in the spacetime manifold introduces an antisymmetric two-index object  $\theta^{\mu\nu}$  that provides an orientation to spacetime in a given inertial frame, and in realistic models this generates naturally a subset of *CPT*-even Lorentz-violating operators in the SME [59].

One open theoretical issue beyond our present scope concerns radiative corrections, which could reasonably be expected to mix these coefficients with others and perhaps contribute to *CPT*-even Lorentz-violating effects in other species. Most and possibly all such effects can be expected to lie beyond current sensitivities, but a complete investigation of this would be of definite interest. We also note a potential philosophical disadvantage to the choice (56): the absence of sidereal effects arises because all four coefficients are aligned relative to the  $Z$  axis in the Sun-centered frame, which implies the low-probability scenario that the effects producing the anomaly discrepancy are aligned with the Earth's rotation axis. In a realistic model, at least some

nonzero off-axis components might be expected in addition to the values (56), in which case the sidereal constraints of Table IX would come into play. These additional components could plausibly come with trigonometric factors of order 0.1, in which case any of the choices (56) would remain viable. Nonetheless, it is reasonable to suppose that if Lorentz violation is indeed the origin of the anomaly discrepancy, then sidereal signals can be expected near the present limits.

One distinctive prediction of the choices (56) is a variation of the shift  $\Delta\omega_a$  with the experimental colatitude. Applying Eq. (42) and using Table VI for the relevant  $G_{j0}$  values, the model with nonzero  $\check{H}_{010}^{(3)}$  can be expected to shift the anomaly frequency measured in the forthcoming Fermilab [46] and J-PARC [47] experiments away from the SM prediction by  $\Delta\omega_a = 2G_{10}(\chi)\check{H}_{010}^{(3)} \approx 2 \times 10^{-24} \text{ GeV} \approx 0.5 \text{ rad Hz}$ , with the predicted J-PARC value being about 10% smaller due to the differing colatitudes and  $G_{10}$  values. For any of the three  $d = 5$  coefficients, we find  $\Delta\omega_a = 2E_0^2 G_{j0}(\chi)\check{H}_{nj0}^{(5)} \approx 2 \times 10^{-24} \text{ GeV} \approx 0.5 \text{ rad Hz}$  at Fermilab again, but due primarily to the lower antimuon energy the J-PARC value is predicted to be about 100 times smaller for  $j = 1$  and about 70 times smaller for  $j = 3$ . Observation of this effect would represent a striking signal in favor of these models.

As a final remark, we note that it may seem tempting to try to relate the muon anomaly discrepancy to the proton radius puzzle. However, in the context of Lorentz violation this appears difficult to achieve at best. As described in Sec. II C 3, the proton radius puzzle represents a comparatively large low-energy effect of order  $\Delta E_{\text{Lamb}} \approx 3 \times 10^{-13} \text{ GeV}$ , while the anomaly discrepancy is a much smaller high-energy effect of order  $\Delta\omega_a \approx 2 \times 10^{-24} \text{ GeV}$ . Although nonminimal Lorentz violation can naturally introduce an energy dependence, the corresponding effects typically grow with energy rather than decreasing. It therefore appears challenging to reproduce both observed phenomena with a single SME coefficient, even without considering more detailed issues such as the spin dependence of the effects.

#### IV. SUMMARY AND DISCUSSION

This paper has explored some prospects for using laboratory experiments with muons and antimuons to search for Lorentz and *CPT* violation. The first part of the paper concerns spectroscopic measurements on muonic bound states. Following a discussion of general features in Sec. II A, we begin by considering muonium transitions in Sec. II B. Signals of Lorentz and *CPT* violation in muonium hyperfine transitions are given by Eq. (8), and using published experimental results we compile constraints on various nonrelativistic and spherical coefficients in Tables I and II. We next consider the  $1S$ - $2S$  transition and the Lamb shift in muonium. These offer interesting options

for exploring isotropic Lorentz and *CPT* violation, and by comparing experimental and theoretical values we extract the constraints on isotropic nonrelativistic coefficients shown in Table III.

In Sec. II C, we turn to an investigation of the spectroscopy of muonic atoms and ions. Following some general considerations, we begin by examining possible future searches using sidereal variations in  $H_\mu$  Zeeman transitions. The frequency shift of the  $2S_{1/2}^{F-1}-2P_{3/2}^F$  transitions induced by Lorentz violation is given by Eq. (18), and our analysis shows that interesting sensitivities in future experiments can be achieved. Next, we consider the hypothesis that Lorentz violation could be the origin of the proton radius puzzle, which arises from an apparent disagreement in the value of the proton charge radius obtained from  $H_\mu$  spectroscopy and from other experiments. Nonzero SME coefficients obeying Eq. (25) would generate a frequency shift matching the observed effect while remaining consistent with existing constraints. We then turn to the issue of searching for Lorentz and *CPT* violation when the Zeeman transitions are unresolved. A method is proposed to constrain possible effects by using the apparent broadening of the spectral lines resulting from the breaking of rotational symmetry. Finally, the prospects are investigated for studying Lorentz and *CPT* violation using other muonic atoms and ions including  $D_\mu$ ,  $T_\mu$ ,  ${}^3\text{He}_\mu^+$ ,  ${}^4\text{He}_\mu^+$ ,  ${}^6\text{Li}_\mu^{2+}$ ,  ${}^7\text{Li}_\mu^{2+}$ ,  ${}^9\text{Be}_\mu^{3+}$ , and  ${}^{11}\text{B}_\mu^{4+}$ . The expression (35) governs the frequency shifts in all these systems, and Table V provides a comparative measure of the attainable sensitivities.

Section III of this paper focuses on Lorentz and *CPT* tests using measurements of the anomalous magnetic moments of the muon and antimuon. We begin in Sec. III A with some basic theory, which shows that the observable shifts of the anomaly frequencies  $\omega_a^\pm$  of the muon and antimuon are given by Eq. (42). Several methods are available to place interesting constraints from existing and future data. We first consider comparisons of the muon and antimuon anomaly frequencies, using different schemes to separate constraints on *CPT*-odd and *CPT*-even effects. Existing data are used to extract limits on various spherical coefficients, including numerous first bounds on nonminimal operators. The results are compiled in Tables VII and Table VIII.

Next, we address the information available in the time domain. Sidereal variations are considered in Sec. III C. Existing data are used to place a variety of limits, which are tabulated in Table IX. We then investigate signals associated with the Earth's changing boost as it revolves about the Sun. The resulting modulations in the anomaly frequency include harmonics with both annual and sidereal periodicities, which are gathered in Table X. Estimates for the attainable sensitivities to Lorentz violation from studies of annual variations are given. Finally, we consider the prospects of accounting for the anomaly discrepancy between existing experimental data and SM calculations

using Lorentz violation. This would require nonzero coefficients, as shown in Eq. (56), that are compatible with present constraints and lead to striking predictions for signals in forthcoming experiments.

Except for partial overlap with published results for the minimal SME, the constraints displayed in the various tables in this paper represent first limits on the dominant effects of muon-sector Lorentz and *CPT* violation. Many of the constraints achieved lie at or beyond the level that might be expected from Planck-suppressed effects, and numerous interesting options remain open for further experimental study along the lines suggested here. We note in passing that a comparable phenomenological treatment of the nonminimal sectors for electrons and other first-generation particles is lacking in the literature to date. This means, for example, that the results in the present paper are currently the best available constraints on nonminimal Lorentz and *CPT* violation for charged leptons.

Despite the substantial broadening of the scope of tests of Lorentz symmetry with muons presented in this paper, the techniques presented span only a comparatively small fraction of the theoretically available possibilities for Lorentz violation. Considerable room remains for investigation, including both uncovering additional methods to measure effects from unconstrained terms in the kinematic Lagrange density and also developing tools to study Lorentz-violating interactions with muons. Possibilities along the latter lines include, for example, studying the effects of minimal and nonminimal Lorentz violation on various muon decays, which in general are affected at the level of both muon kinematics and muon interactions [60]. More extensive studies of muon propagation and interactions may be feasible if a muon collider is eventually realized, perhaps to serve as a factory for Higgs bosons [61].

Gravitational interactions of muons also offer an intriguing avenue for exploration. The gravitational sector of the SME includes Lorentz-violating muon couplings with a variety of signals that are in principle accessible to experiment [7]. For example, the old issue of whether antiparticles can gravitate differently from particles [62] can be directly approached using the general matter-gravity couplings in the SME framework [63]. An experiment has been proposed to address this question for muons using muonium interferometry [64], and  $H_\mu$  interferometry may also be an option [65].

In the context of the minimal SME, the signals for the muonium-interferometry experiment are considered in Sec. IX C of Ref. [63]. The gravitational acceleration of muonium is affected differently from that of other matter and also has a component varying with time as the Earth revolves about the Sun. The former effect, which can be understood as a violation of the weak equivalence principle induced by Lorentz violation, is the most natural candidate signal for muonium interferometry. A detailed investigation including also nonminimal gravitational couplings of the

muon is infeasible at present, but we can use dimensional arguments to estimate the attainable sensitivity to the corresponding coefficients for Lorentz violation. The phase shift  $\delta\phi$  in the muonium interferometer takes the form  $\delta\phi \approx \phi_0 m_\mu^{d-4} \mathcal{K}^{(d)}$ , where  $\mathcal{K}^{(d)}$  is a generic coefficient controlling a Lorentz-violating operator of mass dimension  $d$  in the muon-gravity sector. The muon mass  $m_\mu$  enters because the proposed experiment would use nonrelativistic muonium, so the relevant energy is effectively the muon mass. The phase  $\phi_0 = 2\pi g\tau^2/d$  depends on the gravitational acceleration  $g$ , the time of flight  $\tau$ , and the grating separation  $d$ . Assuming the muonium experiment achieves a precision of 10%, then we can estimate sensitivities to  $\mathcal{K}^{(d)}$  of order  $|\mathcal{K}^{(d)}| \lesssim 10^{d-5} \text{ GeV}^{4-d}$ . Note that in general we can expect accompanying sidereal and annual signals as well.

Another promising subject awaiting careful investigation is flavor-changing effects involving muons, which are natural in the SME context [6]. Planned experiments searching for decays such as  $\mu^\pm \rightarrow e^\pm\gamma$ , which are forbidden in the SM but for which Lorentz-violating operators

appear in the SME, are projected to attain sensitivities of a few parts in  $10^{17}$  [66] and hence could be of interest in the context of Planck-suppressed signals. The flavor-changing operators in the SME also predict signals in searches for muonium-antimuonium oscillations, for which the current sensitivity lies at the level of parts in  $10^{11}$  [67]. Although a comprehensive treatment of nonminimal interactions is unavailable to date, dominant effects in flavor-changing processes may well appear in the kinematics, for which general tools are in hand [17]. Evidently, the unexplored territory in the muon sector remains large, and there is considerable promise for future discovery in a wide variety of experiments.

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