

**Exciting flavored bound states**E. Rojas, B. El-Bennich,<sup>\*</sup> and J. P. B. C. de Melo*Laboratório de Física Teórica e Computacional, Universidade Cruzeiro do Sul,**Rua Galvão Bueno, 868, 01506-000 São Paulo, São Paulo, Brazil*

(Received 14 July 2014; published 17 October 2014)

We study ground and radial excitations of flavor singlet and flavored pseudoscalar mesons within the framework of the rainbow-ladder truncation using an infrared massive and finite interaction in agreement with recent results for the gluon-dressing function from lattice QCD and Dyson-Schwinger equations. Whereas the ground-state masses and decay constants of the light mesons as well as charmonia are well described, we confirm previous observations that this truncation is inadequate to provide realistic predictions for the spectrum of excited and exotic states. Moreover, we find a complex conjugate pair of eigenvalues for the excited  $D_{(s)}$  mesons, which indicates a non-Hermiticity of the interaction kernel in the case of heavy-light systems and the present truncation. Nevertheless, limiting ourselves to the leading contributions of the Bethe-Salpeter amplitudes, we find a reasonable description of the charmed ground states and their respective decay constants.

DOI: 10.1103/PhysRevD.90.074025

PACS numbers: 12.38.-t, 11.10.St, 11.15.Tk, 14.40.Lb

**I. INTRODUCTION**

Bound states are a fascinating subject as they teach us a great deal about the nature of interactions between constituents of a given field theory. In the case of quantum chromodynamics (QCD), this fascination is augmented by the empirical fact that no asymptotically free states, i.e., colored quarks and gluons, are observed. While studies of the analytic structure of colored Green functions are very instructive and were indeed crucial to get insight on their infrared behavior [1–27] and associated confinement mechanisms, they do not offer direct access to experimental tests. Nonetheless, their precise form may be studied indirectly via electromagnetic probes of the bound states formed by the theory's constituents [28–31] and many dedicated experiments at existing and future accelerator facilities will serve exactly this purpose.

In QCD, the simplest possible bound state is given by the pion which plays a pivotal role in the understanding of the low-energy domain, being the lightest strongly bound antiquark-quark state as well as the Goldstone bosons associated with chiral symmetry breaking. Model-independent properties of the Goldstone boson were derived long ago [32] and express the intimate connection between the pion's Bethe-Salpeter amplitude (BSA) and the quark propagator in the chiral limit.

Heavy mesons, on the other hand, provide a formidable opportunity to study additional nonperturbative features of QCD and can be used to test simultaneously all manifestations of the Standard Model, namely, the interplay between electroweak and strong interactions. In the infinitely heavy quark limit, the heavy-quark velocity becomes a conserved quantity and the momentum exchange with

surrounding light degrees of freedom is predominantly soft. Since the heavy-quark spin decouples in this limit, light quarks are blind to it. In essence, they do not experience any different interactions with a much heavier quark in a pseudoscalar or vector meson. In practice, however, heavy-flavor and heavy-spin breaking effects are important and while dynamical chiral symmetry breaking (DCSB) hardly plays a role in charmonium and bottomonium states, it cannot be ignored in heavy-light systems, such as in  $D$  and  $B$  mesons. A daunting challenge is presented by the disparate energy scales and asymmetric momentum distribution within these flavor nonsinglet mesons, i.e., the simultaneous treatment of heavy-quark symmetry breaking effects and DCSB, as the interactions of a heavy with a light quark are governed by the nonperturbative dynamics of the order  $\Lambda_{\text{QCD}}$ . Incidentally, this is also the heavy meson's size scale.

Numerical solutions of the heavy meson's BSA with renormalization-group improved ladder truncation, following work on the kaon BSA [33,34], proved to be unsuccessful: the truncations do not yield the Dirac equation when one of the quark masses is much larger. To circumvent these problems, the Dyson-Schwinger equation (DSE) for the heavy quark was solved for an infrared-suppressed gluon momentum as described in Ref. [35]; no such infrared suppression was applied to the dressing of the light quark and the binding kernel. This approach reproduces well the masses for ground-state pseudoscalar and vector heavy-light mesons, but overestimates experimental weak decay constants by  $\sim 20\%$  in the case of the  $D$  meson and  $\sim 40\%$  for the  $B$  meson. Alternatively, in the heavy-quark propagation the mass function is approximated by a momentum-independent constituent mass. This is justified for the  $b$  quark and reasonable for the  $c$  quark given the modest variation in its functional behavior over a wide

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momentum range and the constituent-quark mass is fitted to the lightest meson, the  $D$  and  $B$  mesons in the case of pseudoscalars [35,36]. The resulting BSA yields again ground-state pseudoscalar and vector meson masses which compare very well with experimental values but strongly underestimate the leptonic decay constants. The situation improves when one uses a complex conjugate pole representation (with three poles) instead of a heavy-quark constituent propagator to compute the normalization and weak decay constant, though omitting a simultaneous application to the Bethe-Salpeter kernel seems inconsistent [36]. Nevertheless, in the latter case the decay constants are overestimated by about 15% for the  $D$ ,  $D_s$ , and  $B$  mesons compared with experimental averages.

Aside from an unfavorable comparison with experimental data, one may object that the constituent-quark approach neglects valuable information of the quark's dressed mass function, namely, its imaginary part,  $\text{Im } M(p^2) \sim 0$  [37–40]. Thus, in solving the Bethe-Salpeter equation (BSE), one approximates the complex mass function by a flat surface on the complex plane. Our motivation is to verify whether a fully consistent, simultaneous numerical solution of the heavy-quark DSE and heavy-light pseudoscalar meson BSE can be obtained with a modern approach to the rainbow-ladder (RL) truncation based on the interaction proposed by Qin *et al.* [41]. This *Ansatz* produces an infrared behavior of the interaction, commonly described by a “dressing function”  $\mathcal{G}(k^2)$  [32], congruent with the decoupling solution found in DSE and lattice studies of the gluon propagator [7–17,21]. Indeed, the gluon propagator is found to be a bounded and regular function of spacelike momenta with a maximum value at  $k^2 = 0$ .

Obviously, the cause for the difficulties encountered in solving the BSE for heavy-light mesons may not solely lie in the infrared behavior of the interaction but also in the truncation employed. In particular, hadron properties are not much affected by the deep infrared of the interaction; rather, they seem to be sensitive to the support and strength of the interaction at the scales of a few hundred MeV [29,42]. As known from a long series of BSE studies, the RL truncation is successful for equal-mass mesons, for instance light  $\bar{q}q$  mesons, such as the pion, kaon, and  $\rho$  [32], but also for  $\bar{Q}Q$  charmonia and bottomium [43–46] where subtle cancellations between contributions from the quark-gluon and antiquark-gluon vertices are in order. Due to the very different momentum and energy-scale distributions, the dressing of the quark-gluon vertex becomes important for heavy-light systems. Progress to go beyond RL truncation has been made and is under way [47–53] and the consistent inclusion of a dressed quark-gluon vertex which satisfies the relevant Ward-Takahashi identities (WTI) can be realized by a general form of the BSE [54].

Nonetheless, we here focus on the interaction proposed in Ref. [41] within the RL approximation to describe the properties of the light, strange, and charmed flavor singlet

and nonsinglet pseudoscalar mesons, such as their mass spectrum, that of the first radial excitation and related weak decay constants. As we demonstrate, recent improved numerical techniques [55,56] and a careful numerical treatment of the flavored DSE and BSE in conjunction with a particular infrared behavior of the interaction is able to satisfactorily reproduce the aforementioned observables. In particular, we compute the decay constants of the radial excitations of the kaon,  $\eta_c$ , and a fictitious  $\bar{s}s$  state. We recall that this was not possible for the heavy-light mesons within the conventional RL framework of Ref. [35]. In the present work, solutions for the  $D$  and  $D_s$  mesons are found, though the truncation leads to complex eigenvalues of their excited states if higher moments of the Chebyshev expansion of the BSA are included. We thus confirm that an accurate and veracious description of the heavy-light flavored mesons requires BSE solutions beyond the RL approximation.

## II. PSEUDOSCALAR BOUND STATES

### A. Dyson-Schwinger equation

We work in a continuum approach of QCD based on the RL truncation of the quark DSE and of the quark-antiquark kernel of the meson's BSE, which is the leading term in a systematic symmetry-preserving truncation scheme. The quark's gap equation is described by the following DSE,<sup>1</sup>

$$S^{-1}(p) = Z_2(i\not{p} + m^{\text{bm}}) + \Sigma(p^2), \quad (1)$$

where the dressed-quark self-energy contribution is ( $q = k - p$ ),

$$\Sigma(p^2) = Z_1 g^2 \int_k^\Lambda D^{\mu\nu}(q) \frac{\lambda^a}{2} \gamma_\mu S(k) \Gamma_\nu^a(k, p). \quad (2)$$

The mnemonic shorthand  $\int_k^\Lambda \equiv \int^\Lambda d^4k / (2\pi)^4$  represents a Poincaré-invariant regularization of the integral with the regularization mass scale,  $\Lambda$ , and where  $Z_{1,2}(\mu, \Lambda)$  are the vertex and quark wave-function renormalization constants. The nonperturbative interactions contribute to the self-energy,  $\Sigma(p^2)$ , which corrects the current-quark bare mass,  $m^{\text{bm}}(\Lambda)$ . The integral is over the dressed gluon propagator,  $D_{\mu\nu}(q)$ , and the dressed quark-gluon vertex,  $\Gamma_\nu^a(k, p)$ , and the color matrices  $\lambda^a$  are in the fundamental representation of SU(3). In Landau gauge the gluon propagator is purely transversal,

$$D_{\mu\nu}^{ab}(q) = \delta^{ab} \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{q^2} \right) \frac{\Delta(q^2)}{q^2}, \quad (3)$$

<sup>1</sup>We employ throughout a Euclidean metric in our notation:  $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ ;  $\gamma_\mu^\dagger = \gamma_\mu$ ;  $\gamma_5 = \gamma_4\gamma_1\gamma_2\gamma_3$ ,  $\text{tr}[\gamma_4\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = -4\epsilon_{\mu\nu\rho\sigma}$ ;  $\sigma_{\mu\nu} = (i/2)[\gamma_\mu, \gamma_\nu]$ ;  $a \cdot b = \sum_{i=1}^4 a_i b_i$ ; and  $P_\mu$  timelike  $\Rightarrow P^2 < 0$ .

where  $\Delta(k^2)$  is the gluon-dressing function. The quark-gluon vertex in this truncation is simply given by

$$\Gamma_\mu^a(k, p) = \frac{\lambda^a}{2} \gamma_\mu. \quad (4)$$

Following Ref. [41] and suppressing color indices, we write

$$Z_1 g^2 D_{\mu\nu}(q) \Gamma_\mu(k, p) \rightarrow Z_2^2 \frac{\mathcal{G}(q^2)}{q^2} T_{\mu\nu}(q) \gamma_\mu, \quad (5)$$

with the transverse projection operator  $T_{\mu\nu}(q) := g_{\mu\nu} - q_\mu q_\nu / q^2$  and the effective coupling is the sum of two terms:

$$\frac{\mathcal{G}(q^2)}{q^2} = \frac{8\pi^2}{\omega^4} D \exp\left(-\frac{q^2}{\omega^2}\right) + \frac{8\pi^2 \gamma_m \mathcal{F}(q^2)}{\ln[\tau + (1 + q^2/\Lambda_{\text{QCD}}^2)]}. \quad (6)$$

In Eq. (6),  $\gamma_m = 12/(33 - 2N_f)$ ,  $N_f = 4$ ,  $\Lambda_{\text{QCD}} = 0.234$  GeV;  $\tau = e^2 - 1$ ; and  $\mathcal{F}(q^2) = [1 - \exp(-q^2/4m_t^2)]/q^2$ ,  $m_t = 0.5$  GeV. The parameters  $\omega$  and  $D$ , respectively, control the width and strength of the interaction and appropriate values will be discussed in Sec. IV. The second term is a bounded, monotonically decreasing and regular continuation of the perturbative-QCD running coupling for all spacelike values of  $k^2$ , whereas the first term is an *Ansatz* for the interaction at infrared momenta. It is crucial that the form of this *Ansatz* provides enough strength to realize DCSB and/or confinement. For momenta  $k^2 \gtrsim 2$  GeV<sup>2</sup>, the perturbative contributions markedly dominate the interaction.

With this, the solutions for spacelike momenta,  $p^2 > 0$ , to the gap equation (1) for a given flavor,  $f$ , are generally

$$S_f(p) = [i\not{p}A_f(p^2) + \not{D}B_f(p^2)]^{-1}, \quad (7)$$

where one imposes the renormalization condition,

$$Z_f(p^2) = 1/A_f(p^2)|_{p^2=\mu^2} = 1, \quad (8)$$

at large spacelike  $\mu^2 \gg \Lambda_{\text{QCD}}^2$ . The mass function,  $M_f(p^2) = B_f(p^2, \mu^2)/A_f(p^2, \mu^2)$ , is independent of the renormalization point  $\mu$ . In order to make quantitative matching with pQCD, another renormalization condition,

$$S_f^{-1}(p)|_{p^2=\mu^2} = i\not{p} + m_f(\mu)\not{D}, \quad (9)$$

is imposed, where  $m_f(\mu)$  is the renormalized running quark mass,

$$Z_m^f(\mu, \Lambda) m_f(\mu) = m_f^{\text{bm}}(\Lambda), \quad (10)$$

where  $Z_m^f(\mu, \Lambda) = Z_4^f(\mu, \Lambda)/Z_2^f(\mu, \Lambda)$  is the flavor dependent mass-renormalization constant and  $Z_4^f(\mu, \Lambda)$  is the

renormalization constant associated with the Lagrangian's mass term. In particular,  $m_f(\mu)$  is nothing else but the dressed-quark mass function evaluated at one particular deep spacelike point,  $p^2 = \mu^2$ , namely,

$$m_f(\mu) = M_f(\mu). \quad (11)$$

The renormalization-group invariant current-quark mass can be inferred via

$$\hat{m}_f = \lim_{p^2 \rightarrow \infty} M_f(p^2) \left[ \frac{1}{2} \ln \left( \frac{p^2}{\Lambda_{\text{QCD}}^2} \right) \right]^{\gamma_m}. \quad (12)$$

## B. Bethe-Salpeter equation

The homogeneous BSE for a  $\bar{q}q$  bound state with relative momentum  $p$  and total momentum  $P$  can be written as

$$\Gamma_{mn}^{fg}(p, P) = \int_k^\Lambda \mathcal{K}_{mn}^{kl}(p, k, P) [S_f(k_+) \Gamma^{fg}(k, P) S_g(k_-)]_{lk}, \quad (13)$$

where  $m, n, k, l$  collect Dirac and color indices;  $f, g$  are flavor indices; and  $k_+ = k + \eta_+ P$ ,  $k_- = k - \eta_- P$ ;  $\eta_+ + \eta_- = 1$ . Since we work within the RL truncation, the BSE kernel is given by

$$\mathcal{K}_{mn}^{kl}(p, k, P) = -\frac{Z_2^2 \mathcal{G}(q^2)}{q^2} \left( \frac{\lambda^a}{2} \gamma_\mu \right)_{kn} T_{\mu\nu}(q) \left( \frac{\lambda^a}{2} \gamma_\nu \right)_{ml}, \quad (14)$$

which satisfies the axial-vector WTI [32] and therefore ensures a massless pion in the chiral limit. Equations (13) and (14) define an eigenvalue problem with physical solutions at the mass-shell points,  $P^2 = -M^2$ , where  $M$  is the bound-state mass.

In the following, we consider the spectrum of flavor singlet and nonsinglet pseudoscalar mesons, which are exhibited as a pole contribution to the axial-vector and pseudoscalar vertices (omitting regular terms in the neighborhood of the poles),

$$\Gamma_{5\mu}^{fg}(p, P)|_{P^2+M_n^2=0} = \frac{f_{P_n} P_\mu}{P^2 + M_n^2} \Gamma_{P_n}^{fg}(p, P), \quad (15)$$

$$i\Gamma_5^{fg}(p, P)|_{P^2+M_n^2=0} = \frac{\rho_{P_n}}{P^2 + M_n^2} \Gamma_{P_n}^{fg}(p, P), \quad (16)$$

where  $\Gamma_{P_n}^{fg}(p, P) \equiv [\Gamma_{mn}^{fg}(p, P)]_{P_n}$  is the bound state's BSA. The principal quantum number is  $n = 0$  for the ground state and  $n \geq 1$  for the radial excitations with bound-state mass,  $M_n$ . In Eqs. (15) and (16), the expressions for  $\rho_{P_n}$  and  $f_{P_n}$  are defined in Eqs. (A4) and (A5), respectively. The properties of the excited states are expected to be sensitive to details of the long-range component of the interaction in

Eq. (6), which provides more support at large interquark separation than, e.g., the Maris-Tandy model [34]. The present study thus allows for a test of the interaction of strange- and charm-flavored  $0^-$  mesons and their excited states, in addition to those considered in Ref. [41], via comparison with experimental values where available.

As mentioned in Sec. I, it is most likely that beyond-RL contributions are important in heavy-light mesons. In fact, the RL truncation with the interaction in Eq. (6) should work best for heavy-heavy ( $\bar{Q}Q$ ) systems. Consider for instance the Ball-Chiu *Ansatz* to the nonperturbative quark-gluon vertex,  $\Gamma_\mu(k, p)$ , which is fully determined with the knowledge of  $A(p^2)$  and  $B(p^2)$  [57,58]: for heavier flavors,  $f = c, b$ , the scalar function  $B_c(p^2)$  and even more so  $B_b(p^2)$  remain constant over a wide momentum range, while for the vector functions  $A_{c,b}(k^2) \simeq A_{c,b}(p^2) \simeq 1$ . The heavy-quark gluon vertex can therefore reasonably be approximated by a bare vertex.

Thus, the RL approximation ought to work better for heavy mesons such as the charmonia and bottomium, which is indeed confirmed in numerical studies [29,44,46]. The effects of DSCB and the importance of other tensor structures in the quark-gluon vertex become increasingly more important for lighter quarks, where, for example, mass splittings between parity partners are only satisfactorily described by including the tensor structure which corresponds to the chromomagnetic moment [59]. Nonetheless, a hallmark of the RL is the successful description of the pion, the key point being that the axial-vector WTI be preserved and likewise the Goldstone-boson character of the pion.

### III. NUMERICAL IMPLEMENTATION

#### A. DSE on the complex plane

We follow Refs. [50,55,56] and choose the gluon momentum,  $q$ , to be real. This implies that when the external DSE momentum (1) is complex, so is the internal quark's momentum. It is the case in Eq. (13) which requires DSE solutions for  $S(p_\pm)$  in the complex plane with

$$p_\pm^2 = p^2 - \eta_\pm^2 M_n^2 \pm 2i\sqrt{p^2}\eta_\pm M_n, \quad (17)$$

where  $p$  is collinear with  $P = (\vec{0}, iM_n)$  in the meson's rest frame and in the quark's DSE we employ  $\eta_\pm = 1/2$ . As we add  $P$  to the quark's external momentum,  $p$ , the internal quark momentum becomes  $k_\pm = k \pm \eta_\pm P$ ,  $k \in \mathbb{R}_+$ , and it follows that  $q = k_\pm - p_\pm = k - p$  is real. The internal quark's squared momentum is given by

$$k_\pm^2 = k^2 - \eta_\pm^2 M_n^2 \pm 2i\sqrt{k^2}\eta_\pm M_n z, \quad (18)$$

with the angle,  $-1 \leq z \leq +1$ , and is also bounded by a parabola whose vertex is  $(0, -\eta_\pm^2 M_n^2)$ . Within the parabola, the real integration variable,  $k$ , is limited by  $\Lambda$  as follows.

We denote the regularization mass scale in Eq. (13) by  $\Lambda_{\text{BSE}}$  and relate it to the one in Eq. (2) with

$$\Lambda_{\text{DSE}}^2 = \Lambda_{\text{BSE}}^2 + \frac{1}{4}M_f^2 = \Lambda_{\text{BSE}}^2 + \eta_f^2 M_P^2, \quad (19)$$

for  $f = u, d, s, c$ . This procedure is best illustrated with an example, e.g., for a meson with strangeness,  $\bar{u}s$ ; we determine  $\eta_s$  and  $\eta_u$  pushing in each case  $M_{f=u,s}$  to the limit at which we encounter poles while solving the DSE on the complex plane. From the latter equality in Eq. (19), we obtain  $\eta_s$  and  $\eta_u$  and imposing  $\eta_s + \eta_u = 1$  we find

$$\frac{1}{2}M_s = \eta_s M_P; \quad \frac{1}{2}M_u = \eta_u M_P \Rightarrow \frac{M_s + M_u}{2} = M_P, \quad (20)$$

where  $M_P$  is the ‘‘maximal meson mass’’ (not to be confused with  $M_n$ ) for which the DSE can be solved for each flavor and respective parameters,  $\eta_s = M_s/(M_u + M_s)$  and  $\eta_u = M_u/(M_u + M_s)$ . The latter are used as momentum partitioning parameters  $\eta_\pm$  in the BSE. Of course, this is done for computational convenience and we have checked that our results are stable under variations of  $\eta_\pm$ , as expected from a Poincaré-invariant approach. In solving the BSE, Eq. (20) imposes an upper boundary on the mass of the meson and its excited states:  $M_n < M_P$ . For flavor singlet equal-mass  $\bar{q}q$  mesons we simply choose  $\eta_f = \eta_+ = \eta_- = 1/2$ .

The solution of Eq. (1) for a complex value of  $p_\pm^2$  requires the simultaneous iteration on a complex  $2d$  grid bounded by the parabola of Eq. (18). Alternatively, we can exploit the information from the contour of the parabola via the Cauchy theorem [56]. To proceed with the integration, we parametrize the contour (counterclockwise) in the upper complex plane,  $\mathcal{C}_+ : p_+^2$ , with  $p := z + (t) = tp_{\min} + (1-t)\Lambda_{\text{DSE}}$ ; in the lower plane,  $\mathcal{C}_- : p_-^2$ , with  $p := z_-(t) = t\Lambda_{\text{DSE}} + (1-t)p_{\min}$ ; and we close the parabola with the path  $\mathcal{C}_y : p_y^2 := \Lambda_{\text{DSE}}^2 - y^2(t) + 2iy(t)\Lambda_{\text{DSE}}$  with  $y(t) = M_n t - \eta_- M_n$ ,  $t \in [0, 1]$ . More precisely, we use a linear parametrization given by

$$\begin{aligned} dp_+^2 &= 2(z_+(t) + i\eta_+ M_n) dz_+(t); & 0 \leq z_+(t) < \Lambda_{\text{DSE}}, \\ dp_y^2 &= 2(-y(t) + i\Lambda_{\text{DSE}}) dy(t); & -\eta_- M_n \leq y(t) \leq \eta_+ M_n, \\ dp_-^2 &= 2(z_-(t) - i\eta_- M_n) dz_-(t); & 0 \leq z_-(t) < \Lambda_{\text{DSE}}. \end{aligned} \quad (21)$$

The DSE for the quark is solved for 128 complex momenta,  $p_\pm$ , on each contour section using 128 momenta,  $k_\pm$ , and 128 angles,  $z$ , in the interior of the parabola parametrized by Eq. (21), where the values for  $A(k_\pm^2)$  and  $B(k_\pm^2)$  are obtained via the Cauchy theorem and simultaneous iteration of their values on the contour. For numerical precision, the points on the contour are skewed towards

the vertex of the parabola near  $k_{\pm}^2 \sim -\eta_{\pm}^2 M_n^2$ . We use a quadrature method for the integral evaluation and cross-check the results with the numerical integration library CUBA [60]. The BSE (13) is solved for 64 external momenta,  $p$ , 64 internal momenta,  $k$ , and 20 angles.

The renormalization condition in Eqs. (8) and (9) are imposed on the DSE solutions at the spacelike real-axis point,  $\mu = 19$  GeV, a value chosen to match the bulk of extant studies [43,61]. The current-quark masses are fixed by requiring that the pion and kaon BSEs produce  $m_{\pi} \approx 0.138$  GeV and  $m_K \approx 0.493$  GeV. We thus use  $m_{u,d}(\mu) = 3.4$  MeV,  $m_s(\mu) = 82$  MeV [41], and  $m_c(\mu) = 0.905$  GeV [62]. However, in computing the values of  $A(p^2)$  and  $B(p^2)$  for the momenta on the parabola (17), we are numerically limited by the appearance of conjugate poles and cannot use an arbitrarily large cutoff  $\Lambda$  in Eq. (1). This implies that we impose the renormalization conditions, Eqs. (8) and (9), at a lower momentum value,  $\xi < \mu$ , and thus obtain new renormalization constants,  $Z_m^f$  and  $Z_4^f$ . The details of our renormalization procedure are discussed in the Appendix and our complex solutions for  $A(p^2)$  and  $B(p^2)$  with the interaction in Eq. (6) are plotted for the  $u$  quark in Fig. 3 of Ref. [63], where the real parts of  $A(p^2)$  and  $B(p^2)$  on the parabola are described by equipotential lines with decreasing magnitude for increasing values of spacelike  $\text{Re } k^2 > 0$ . The imaginary components of  $A(p^2)$  and  $B(p^2)$  are negative valued and decreasing for  $\text{Im } k^2 > 0$  and positive and increasing for  $\text{Im } k^2 < 0$ , while for real momenta,  $\text{Im } k^2 = 0$  and  $\text{Re } k^2 > 0$ , both functions are naturally real.

Alternatively, we cross-checked our renormalization procedure by setting  $A(\mu^2) = 1$  for  $\mu = 2$  GeV and imposing in the ultraviolet the condition (12) following Refs. [41,61]. We also reproduce our results using a complex pole representation for the quarks with three poles [64] to which we fit the DSE solutions on the real axis renormalized at 19 GeV. The numerical differences with the masses and decay constants obtained with the solutions of the DSE on the complex plane are negligible.

## B. Solving the BSE with Arnoldi factorization

The general Poincaré-invariant form of the solutions of Eq. (13) for the pseudoscalar channel  $J^P = 0^-$  and the trajectory,  $P^2 = -M_n^2$ , in a nonorthogonal base with respect to the Dirac trace,  $\mathcal{A}^\alpha(p, P) = \gamma_5 \{i\mathbb{1}_D, P, \not{p}(p \cdot P), \sigma_{\mu\nu} p_\mu P_\nu\}$ , is given by

$$\begin{aligned} \Gamma_{P_n}(p, P) = & \gamma_5 [i\mathbb{1}_D E_{P_n}(p, P) + \not{P} F_{P_n}(p, P) \\ & + \not{p}(p \cdot P) G_{P_n}(p, P) + \sigma_{\mu\nu} p_\mu P_\nu H_{P_n}(p, P)], \end{aligned} \quad (22)$$

where we have suppressed color, Dirac, and flavor indices for the sake of visibility. The functions  $\mathcal{F}_{P_n}^\alpha(p, P) = \{E_{P_n}(p, P), F_{P_n}(p, P), G_{P_n}(p, P), H_{P_n}(p, P)\}$

are Lorentz-invariant scalar amplitudes. We solve the BSE by projecting on the functions  $\mathcal{F}_{P_n}^\alpha(p, P)$  with the appropriate projectors,

$$\begin{aligned} P^\alpha(p, P) = & \sum_{\beta=1}^4 P^{\alpha\beta}(p, P) \mathcal{A}^\beta(p, P) \\ \frac{1}{4} \sum_{\beta=1}^4 P^{\alpha\beta}(p, P) \text{Tr}_D[\mathcal{A}^\alpha(p, P) \mathcal{A}^\beta(p, P)] = & \delta_{\alpha\beta}, \end{aligned} \quad (23)$$

which allows for an extraction of  $\mathcal{F}_{P_n}^\alpha(p, P)$  from the BSA,

$$\mathcal{F}_{P_n}^\alpha(p, P) = \frac{1}{4} \text{tr}_{\text{CD}}[P^\alpha(p, P) \Gamma_P(p, P)]. \quad (24)$$

This leads to the eigenvalue equation ( $\alpha, \beta = 1, \dots, 4$ ),

$$\lambda_n(P^2) \mathcal{F}_{P_n}^\alpha(p, P) = \int_k^\Lambda \mathcal{K}^{\alpha\beta}(p, k, P) \mathcal{F}_{P_n}^\beta(k, P), \quad (25)$$

where  $\mathcal{K}^{\alpha\beta}(p, k, P)$  is obtained using Eq. (14):

$$\begin{aligned} \mathcal{K}^{\alpha\beta}(p, k, P) = & -\frac{Z_2^2 \mathcal{G}(q^2)}{4 q^2} T_{\mu\nu}(q) \text{tr}_{\text{CD}}[P^\alpha(p, P) \\ & \times \gamma_\mu \lambda^a S(k_+) \mathcal{A}^\beta(k, P) S(k_-) \gamma_\nu \lambda^a]. \end{aligned} \quad (26)$$

The propagators and functions  $\mathcal{F}_{P_n}^\alpha(p, P)$  in Eq. (25) are expanded in Chebyshev polynomials of the second kind,  $U_m(z_k)$  and  $U_m(z_p)$ , with the angles,  $z_k = P \cdot k / (\sqrt{P^2} \sqrt{k^2})$  and  $z_p = P \cdot p / (\sqrt{P^2} \sqrt{p^2})$ , and the momenta,  $k$  and  $p$ , are discretized, e.g.,

$$\mathcal{F}_{P_n}^\alpha(p_i, P) = \sum_{m=0}^{\infty} \mathcal{F}_{P_n, m}^\alpha(p_i, P) U_m(z_p) \quad (27)$$

where we define  $\mathcal{F}_{mi}^\alpha \equiv \mathcal{F}_{P_n, m}^\alpha(p_i, P)$ . We employ three Chebyshev polynomials for the ground states and five Chebyshev polynomials for the excited states.

We solve the eigenvalue problem posed in Eq. (25) by means of Arnoldi factorization implemented in the ARPACK library [65,66] which computes the eigenvalue spectrum for a given  $N \times N$  matrix. Practical implementation implies a transcription of the BSE kernel  $\mathcal{K}^{\alpha\beta}(p, k, P)$  in Eq. (26) as

$$\mathcal{F}_{mi}^\alpha = (\mathcal{K}^{\alpha\beta})_{nj}^{mi} \mathcal{F}_{nj}^\beta, \quad (28)$$

which depends on six indices. In order to cast Eq. (28) as an  $N \times N$  matrix,

$$\mathcal{F}_I = \mathcal{K}_{IJ} \mathcal{F}_J, \quad I, J = 1, \dots, N, \quad (29)$$

we relate the two sets of indices,  $I, J$  and  $\alpha, \beta, m, n, i, j$ , with the condition,

$$I(\alpha, m, i) = \alpha(m+1)i, \quad (30)$$

$$J(\beta, n, j) = \beta(n+1)j. \quad (31)$$

To recover the components of the BSA,  $\mathcal{F}_p^\alpha(p, P)$ , one simply identifies  $\mathcal{F}_{mi}^\alpha = \mathcal{F}_I$ , where  $I$  is given by Eq. (30).

As mentioned, the factorization with ARPACK yields the eigenvalue spectrum,  $\lambda_n(P^2)$ , of the kernel,  $\mathcal{K}_{IJ}$ , and therefore we obtain eigenvectors,  $\mathcal{F}_I$ , or equivalently the BSA,  $\mathcal{F}_{P_n}^\alpha(p, P)$ , for each  $\lambda_n(P^2 = -M_n^2) = 1$ .<sup>2</sup> These numerical results are presented and discussed in Sec. IV.

For the sake of completeness, we mention that all BSA are normalized canonically as

$$2P_\mu = \int_k^\Lambda \text{Tr}_{\text{CD}} \left[ \bar{\Gamma}_{P_n}(k, -P) \frac{\partial S(k_+)}{\partial P_\mu} \Gamma_{P_n}(k, P) S(k_-) + \bar{\Gamma}_{P_n}(k, -P) S(k_+) \Gamma_{P_n}(k, P) \frac{\partial S(k_-)}{\partial P_\mu} \right], \quad (32)$$

where we have omitted a third term that stems from the derivative of the kernel,  $\partial \mathcal{K}_{mn}^{kl}(p, k, P) / \partial P_\mu$ , as it does not contribute in the RL truncation of Eq. (14).<sup>3</sup> In Eq. (32), the charge-conjugated BSA is defined as  $\bar{\Gamma}(k, -P) := C\Gamma^T(-k, -P)C^T$ , where  $C$  is the charge conjugation operator. We shall discuss the peculiarities of charge symmetry in the case of heavy-light mesons in Sec. IV.

#### IV. DISCUSSION OF RESULTS

We summarize our results for the mass spectrum and weak decay constants of the pseudoscalar flavor singlet and nonsinglet mesons in Tables I, II, and III, where the DSE and BSE are solved for two parameter sets of the interaction in Eq. (6), denoted by models 1 and 2 in all tables, and also discussed in Ref. [61]. In Table I, we list the masses and decay constants, adopting the Particle Data Group (PDG) conventions [69], of the  $\pi$ ,  $K$ ,  $\pi(1300)$ ,  $K(1460)$ ,  $\eta_c(1S)$ , and  $\eta_c(2S)$ . We also include the mass and decay constant of a pure  $\bar{s}s$  state which cannot be related to the  $\eta$  or  $\eta'$  but whose study is worthwhile to elucidate the trajectory of the weak decay constants as a function of the current quark mass. We remind that the  $\eta_c$  and  $\bar{s}s$  are neutral particles which do not undergo a purely leptonic decay as in Eq. (A5), yet they are useful quantities to calculate [70].

<sup>2</sup>To find the root  $M_n$  of the equation  $\lambda_n(P^2 = -M_n^2) - 1 = 0$ , we made use of the numerical recipe subroutines ZBRENT and RTSEC [67]. We cross-checked the ARPACK solutions for the ground states of the pseudoscalar channel with the commonly used iterative procedure and find excellent agreement of the order  $10^{-16}$ .

<sup>3</sup>We verify the values obtained with Eq. (32) with the equivalent normalization condition [50,68]:  $(d \ln \lambda_n / dP^2)^{-1} = \text{tr} \int_k^\Lambda 3\bar{\Gamma}(k, -P)S(k_+)\Gamma(k, P)S(k_-)$ .

TABLE I. Mass spectrum and decay constants for flavor singlet and nonsinglet  $J^P = 0^-$  mesons, where we follow Particle Data Group conventions [69]. Both models refer to the interaction Ansatz in Ref. [41], where we use the values  $\omega = 0.4$  GeV and  $\omega D = (0.8 \text{ GeV})^3$  for model 1 and  $\omega = 0.6$  GeV and  $\omega D = (1.1 \text{ GeV})^3$  for model 2. Dimensioned quantities are reported in GeV and reference values in the last column include experimental averages and lattice-QCD predictions when known.

	Model 1	Model 2	Reference
$m_\pi$	0.138	0.153	0.139 [69]
$f_\pi$	0.139	0.189	0.1304 [69]
$m_{\pi(1300)}$	0.990	1.414	$1.30 \pm 0.10$ [69]
$f_{\pi(1300)}$	$-1.1 \times 10^{-3}$	$-8.3 \times 10^{-4}$	...
$f_{\pi(1300)}^{\text{GMOR}}$	$-1.4 \times 10^{-3}$	$-4.0 \times 10^{-4}$	...
$m_K$	0.493	0.541	0.493 [69]
$f_K$	0.164	0.214	0.156 [69]
$f_K^{\text{GMOR}}$	0.162	0.214	...
$m_{K(1460)}$	1.158	1.580	1.460 [69]
$f_{K(1460)}$	-0.018	-0.017	...
$f_{K(1460)}^{\text{GMOR}}$	-0.018	-0.017	...
$m_{\bar{s}s}$	1.287	1.702	...
$f_{\bar{s}s}$	-0.0214	-0.0216	...
$f_{\bar{s}s}^{\text{GMOR}}$	-0.0215	-0.0218	...
$m_{\eta_c(1S)}$	3.065	3.210	2.984 [69]
$f_{\eta_c(1S)}$	0.389	0.464	0.395 [70]
$f_{\eta_c(1S)}^{\text{GMOR}}$	0.380	0.451	...
$m_{\eta_c(2S)}$	3.402	3.784	3.639 [69]
$f_{\eta_c(2S)}$	0.089	0.105	...
$f_{\eta_c(2S)}^{\text{GMOR}}$	0.088	0.103	...

A direct comparison of the mass and decay constant entries in both model columns reveals that the values obtained with model 1 are in much better agreement with experimental values of the  $0^-$  ground states, namely, the  $\pi$ ,  $K$ , and  $\eta_c$ , whose  $\omega$  dependence in the range  $\omega \in [0.4, 0.6]$  GeV is rather weak. Conversely, model 2 reproduces well the masses of the radially excited states,  $\pi(1300)$ ,  $K(1460)$ , and  $\eta_c(2S)$ , where for

TABLE II. List of exotic states with ‘‘unnatural time parity,’’  $J = 0^{--}$ . In the case of the  $\bar{u}s$  meson,  $C = -1$  must be understood as approximate, since  $|\bar{u}s\rangle$  is not an eigenstate of charge conjugation. Models 1 and 2 are as in Table I and dimensioned quantities are in GeV. The decay constants of flavorless states are of the order  $10^{-6}$  GeV.

	Model 1	Model 2
$m_{\bar{u}u}$	0.733	1.049
$m_{\bar{s}s}$	0.856	1.351
$m_{\bar{c}c}$	3.243	3.515
$m_{\bar{u}s}$	0.917	1.225
$f_{\bar{u}s}$	0.0152	0.0098
$f_{\bar{u}s}^{\text{GMOR}}$	0.0150	0.0097

TABLE III. Charmed meson observables computed in the lowest-order Chebyshev moment approximation (see text for explanations). Models 1 and 2 are as in Table I. Dimensioned quantities are reported in GeV and experimental values are averages from the Particle Data Group.

	Model 1	Model 2	Experiment [69]
$m_D$	2.115	2.255	1.869
$f_D$	0.204	0.281	$0.2067 \pm 0.0085 \pm 0.0025$
$m_{D_s}$	2.130	2.284	1.968
$f_{D_s}$	0.249	0.320	$0.260 \pm 0.005$

$\omega D = (1.1 \text{ GeV})^3$  the ground states are no longer insensitive to  $\omega$  variations [61]. In order to obtain  $m_\pi = 0.138 \text{ GeV}$ ,  $\omega$  must increase beyond our reference value,  $\omega = 0.6 \text{ GeV}$ , for the excited spectrum.

Nonetheless, the ground states are noticeably less dependent on the  $\omega D$  values than the radial excitations, where large mass differences are observed between both columns. This agrees with the observations made in Ref. [61] and extends them to the strange and charm sector: the quantity  $r_\omega := 1/\omega$  is a length scale that measures the range of the infrared component of the interaction,  $\mathcal{G}(q^2)$ , in Eq. (6). The radially excited states or exotics are expected to be more sensitive to long-range characteristics of  $\mathcal{G}(q^2)$  than ground states and we confirm that all radially excited states increase in mass if the range, or strength, of the strong piece of the interaction is reduced (note that for  $\omega = 0$  the range is infinite).

In summary, we do not find a parameter set that describes equally well the entire mass spectrum of ground and excited states which hints at the insufficiency of this truncation. However, model 2 uniformly overestimates all masses by 6%–10% in Table I, where the masses are more “inflated” for the pion and the kaon. It is plausible that radiative and hadronic corrections return them to the observed values [48]. In the mass spectrum, we also find states with unnatural time parity,  $J = 0^{--}$ , referred to as exotics and presented in Table II. The mass difference pattern of these states parallels the ones observed in Table I. However, as they are not experimentally observed and expected to have masses above 2 GeV [69], their appearance in the spectrum is likely an artifact of the current truncation; see, e.g., discussion in Ref. [71].

The weak decay constants are obtained via Eq. (A5) and verified by means of the Gell-Mann–Oakes–Renner (GMOR) relation (A3), a relation valid for every  $0^-$  meson irrespective of the magnitude of the current-quark mass,  $m^{f,g}(\mu)$ . In the chiral limit,

$$\rho_{P_n}^0(\mu) := \lim_{\hat{m} \rightarrow 0} \rho_{P_n}(\mu) < \infty, \quad \forall n, \quad (33)$$

owing to the ultraviolet behavior of the quark-antiquark scattering kernel in QCD which guarantees that  $\rho_{P_n}(\mu)$  in Eq. (A4) is cutoff independent. A necessary corollary is that

in the chiral limit the decay constant of excited states vanishes identically [62]:

$$f_{P_n}^0(\mu) \equiv 0, \quad n \geq 1. \quad (34)$$

We provide numerical verification of Eq. (34) in the case of the first radial excitation of the pion, i.e., the parity partner  $\pi(1300)$ , in Table I from which it can also be read that the decay constant of the parity partner of the kaon, the  $K(1460)$ , is strongly suppressed. Analogously, the decay constant value of the  $\bar{s}s$  state is very small. On the other hand, we note that all weak decay constants of the excited states but the  $\eta_c(2s)$  are negative with model 1, slightly less so when using model 2, which is consistent with the discussion in Ref. [61]. As it becomes clear from Fig. 1, for  $\omega \geq 0.5$  and  $p^2 \gtrsim 1 \text{ GeV}^2$ , the leading amplitude’s lowest Chebyshev projection,  ${}^0E_{P_1}(p^2)$ , is negative definite in the case of the first radial excitations,  $\pi(1300)$  and  $\bar{s}s$ , whereas it remains positive for the ground states, which parallels the pattern of wave functions in quantum mechanics. A necessary consequence is then  $f_\pi > 0$  and  $f_{\pi(1300)} < 0$ , also observed for the kaon and its first radial excitation and for the  $\eta(1760)$ . The negative value for  $f_{\pi(1300)}$  is consistent with lattice-QCD simulations [72]. The equality in Eq. (34) is only valid in the chiral limit and for increasing current-quark masses the excited pseudo-scalar meson’s decay constant first remains negative, but after a minimum, which occurs between the strange and charm-quark mass, it steadily increases and becomes positive before reaching the charm quark mass. We therefore find for the  $\eta_c(2S)$  a positive decay constant for both model parameters. We note that our turning point is below that observed in Ref. [73], which is about the charm-quark mass.

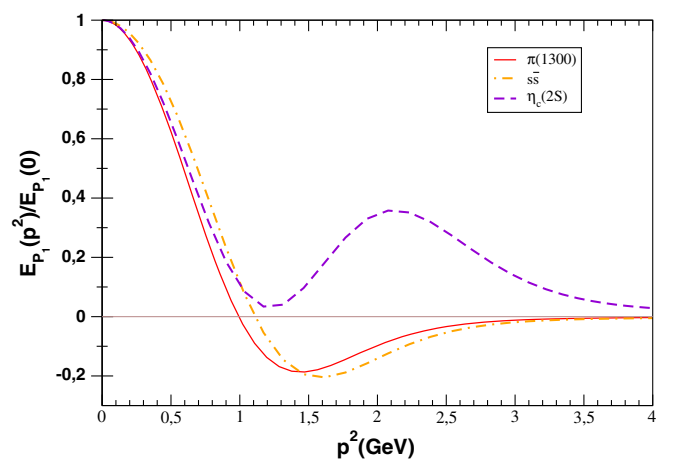


FIG. 1 (color online). Lowest Chebyshev moment,  ${}^0E_{P_1}(p^2)$ , associated with the leading Dirac structure  $E_{P_1}(p^2)$  of the meson’s BSA (22) for the first radial excitations  $\pi(1300)$ ,  $\eta(1760)$ , and  $\eta_c(2S)$ . The interaction parameters are  $\omega = 0.6$  and  $\omega D = (1.1 \text{ GeV})^3$ .

We now turn our attention to the charmed pseudoscalars, in particular the  $D$  and  $D_s$  mesons. Before discussing the results, a few technical comments are in order. Besides being bounded from above, the eigenvalue spectrum of the BSE should be positive definite [74] owing to the Hermiticity requirement of physical operators. However, this is not generally true for flavored mesons with unequal masses, in particular heavy-light systems in the rainbow-ladder truncation. For example, in solving the BSE for the  $D$  and  $D_s$  mesons, we find a complex conjugate pair of eigenvalues for the excited  $D_{(s)}$  states if we include contributions from higher Chebyshev polynomials,  $U_{m>1}(z_p)$ . This occurs for either parameter set of the interaction and indicates a non-Hermiticity of the Hamiltonian in the case of  $|\bar{c}u\rangle$  and  $|\bar{c}s\rangle$  bound states.<sup>4</sup> We have verified that our solutions converge; i.e., they are independent of the amount of Chebyshev moments in the expansion and the ground state and excited  $D$  meson masses do not vary for  $U_m(z_p)$ ,  $m \geq 4$ .

We remind the reader that nonequal mass mesons, such as the  $K$  and  $D$ , are not eigenstates of the charge-conjugation operator and thus  $\bar{\Gamma}(k, P) = \lambda_c \Gamma(k, P)$  does not imply  $\lambda_c = \pm 1$  for the charge parity. On the other hand, for equal-mass ( $\bar{u}u, \bar{d}d, \bar{s}s, \bar{c}c$ ) pseudoscalar mesons with  $J^{PC} = 0^{-+}$ , the constraint that the Dirac base satisfies  $\lambda_c = +1$  requires the dressing functions,  $\mathcal{F}_{p_n}^a(p, P)$ , to be *even* in the angular variable  $z_p$ . In the case of the  $D$  and  $D_s$  mesons, however, we do observe that also *odd* Chebyshev moments contribute and they acquire an imaginary part in the ground and excited states. The ground-state eigenvalues of the  $D$  mesons remain real and though the associated eigenstate is a solution of the BSE, it cannot be interpreted as a physical solution for the  $D$  mesons due to the non-Hermiticity invoked above. If we limit the Chebyshev expansion to the lowest order, the BSA is independent of the angle,  $z_p$ , and therefore remains real in all cases as are the eigenvalues. Hence, it is the angular-dependent higher-order terms of the Chebyshev expansion that lead to complex eigenvalues for the charmed excited states within the RL truncation. On the other hand, this does not occur for the kaon where flavor-symmetry and charge-parity breaking are still negligible.

The mass and decay constant entries in Table III are all obtained in the lowest-order Chebyshev approximation which is independent of the angle,  $z_p$ . Incidentally, this approximation bears similarity with common phenomenological *Ansätze* to the BSA for  $D$  mesons [76–80] and this

<sup>4</sup>A discussion of complex eigenvalues and their origin in the crossing of normal and abnormal (which have vanishing binding energy) eigenstates can be found in Ref. [74]. Note that in quantum mechanics even non-Hermitian operators yield real and positive eigenvalues provided that  $PT$  symmetry is conserved, whereas if it is broken the eigenvalue spectrum is complex [75]. Yet, an analogical observation in quantum field theory and more precisely in nonperturbative QCD is not straightforward.

is exactly what the values in Table III represent: masses and decay constants obtained with a model based on the lowest-order approximation for which the eigenvalues and BSA are real. The mass difference,  $\Delta m \simeq 15\text{--}30$  MeV, between the  $D$  and  $D_s$  mesons is smaller than experimentally observed, i.e.,  $\Delta m \sim 100$  MeV. On the other hand, the weak decay constants are in good agreement with experimental averages for model I which again provides the preferred interaction for the ground states. We also stress that the results in Table III were obtained without any modification of the interaction (6) in the RL truncation, such as suppression of the infrared domain [35] or use of a constituent-quark mass [36]. Nevertheless, as has been realized previously [35], the RL approximation is an inadequate truncation scheme for heavy-light mesons.

## V. CONCLUSION

We computed the BSAs for the ground and first excited states of the flavor singlet and flavored pseudoscalar mesons with an interaction *Ansatz* that is massive and finite in the infrared and massless in the ultraviolet domain. This interaction is qualitatively in accordance with the so-called *decoupling solutions* of the gluon's dressing function and thus represents an improvement on the Maris-Tandy model [34]. In conjunction with the RL truncation, the latter proves to be a successful interaction model for the light meson spectrum,  $M \lesssim 1$  GeV, but fares less well in applications to heavy-light systems where infrared modifications of the interaction are required in order to obtain numerical results which are still not satisfactory [35].

Motivated by the successful application of the interaction *Ansatz* of Eq. (6) to the mass spectrum of light mesons as well as some of their excited states in Ref. [61], we extend this study to the strange and charm sectors and obtain the masses of ground states and resonances as well as decay constants, as presented in Table I. These numerical results are in good agreement with experimental averages for the ground states, yet we confirm the earlier observation that no single parametrization of Eq. (6) is able to reproduce the mass spectrum of both the ground and excited states in the RL truncation. The case of the  $D$  and  $D_s$  mesons remains equally unresolved and clearly shows that the mass asymmetry and disparate scales in heavy-light systems require corrections beyond the leading truncation. We postpone the treatment of heavy-flavored mesons with a BSE *Ansatz* valid for any symmetry-preserving dressed quark-gluon vertex [54] to a future publication. We stress that the merit of BSE solutions for heavy-light systems goes beyond a successful description of the heavy meson's mass spectrum and decay constants and that the light-front projection of the meson's Bethe-Salpeter wave function allows us to extract its light cone distribution amplitudes [64,81]. The latter, commonly expanded in Gegenbauer polynomials, encode the relevant nonperturbative information in QCD factorization of heavy-meson weak decays



and their calculation represents a big step beyond the usually employed asymptotic form,  $\phi(x) \propto 6x(1-x)$ . Indeed, in the case of  $D$  and  $B$  mesons, very little is known about the nonperturbative nature of these distribution amplitudes.

### ACKNOWLEDGMENTS

This work was supported by the São Paulo Research Foundation (FAPESP) and benefitted from access to the computing facility of the Centro Nacional de Supercomputação at the Federal University of Rio Grande do Sul (UFRGS). We appreciated insightful discussions with and comments by Lei Chang, Gernot Eichmann, Andreas Krassnigg, Craig Roberts, and Peter Tandy.

### APPENDIX: RENORMALIZATION CONDITIONS

In order to eliminate the cutoff dependence, it is necessary to impose renormalization conditions. The usual procedure is to relate bare and renormalized masses as in Eq. (10) and for the fermion fields one writes, analogously,

$$q_f^{\text{bm}}(p^2) = (Z_2^f(\mu, \Lambda))^{\frac{1}{2}} q_f(p^2, \mu). \quad (\text{A1})$$

The renormalization constants  $Z_m$  and  $Z_2$  are determined by imposing the conditions,

$$B(\mu, \mu) = m_f(\mu) \quad \text{and} \quad A(\mu, \mu) = 1. \quad (\text{A2})$$

For arbitrary flavors and using Eqs. (10) and (A2), the GMOR expression can be generally written as

$$\begin{aligned} f_{P_n} M_{P_n}^2 &= (m_f^{\text{bm}} + m_g^{\text{bm}}) \rho_{P_n}^{\text{bm}} \\ &= -(m_f^{\text{bm}} + m_g^{\text{bm}}) \langle 0 | \bar{q}_g^{\text{bm}} \gamma_5 q_f^{\text{bm}} | P_n \rangle \\ &= -(Z_m^f m_g(\mu) + Z_m^g m_f(\mu)) (Z_2^f Z_2^g)^{1/2} \langle 0 | \bar{q}_g \gamma_5 q_f | P_n \rangle \\ &= \left( \sqrt{\frac{Z_m^f}{Z_m^g}} m_g(\mu) + \sqrt{\frac{Z_m^g}{Z_m^f}} m_f(\mu) \right) \rho_{P_n}(\mu), \end{aligned} \quad (\text{A3})$$

where we have

$$\begin{aligned} \rho_{P_n}(\mu) &= -(Z_4^f Z_4^g)^{\frac{1}{2}} \langle 0 | \bar{q}_g \gamma_5 q_f | P_n \rangle \\ &= -i (Z_4^f Z_4^g)^{\frac{1}{2}} \text{Tr}_{\text{CD}} \int_k^\Lambda \gamma_5 S_f(k_+) \Gamma_{P_n}^{fg}(k, P) S_g(k_-), \end{aligned} \quad (\text{A4})$$

with  $Z_4^f(\mu, \Lambda) = Z_2^f(\mu, \Lambda) Z_m^f(\mu, \Lambda)$ . The weak decay constant can, of course, be directly inferred from

$$\begin{aligned} f_{P_n} P_\mu &= -i \langle 0 | \bar{q}_g^{\text{bm}} \gamma_5 \gamma_\mu q_f^{\text{bm}} | P_n \rangle \\ &= (Z_2^f Z_2^g)^{\frac{1}{2}} \text{Tr}_{\text{CD}} \int_k^\Lambda \gamma_5 \gamma_\mu S_f(k_+) \Gamma_{P_n}^{fg}(k, P) S_g(k_-). \end{aligned} \quad (\text{A5})$$

In solving the DSE within a complex parabola as discussed in Sec. III A, it is convenient to compute the renormalization constants at a momentum scale,  $\xi$ , different from  $\mu > \xi$  while still demanding that Eq. (A2) be satisfied. Once we know the solutions of  $A(p^2)$  and  $B(p^2)$  renormalized at  $p^2 = \mu^2$ , the values for  $B(\xi, \mu)$  and  $A(\xi, \mu)$  are readily inferred. One may as well calculate  $Z_m^f(\mu, \Lambda)$  and  $Z_2^f(\mu, \Lambda)$  at a point,  $p^2 = \xi^2$ , using the condition,

$$\begin{aligned} B(\xi, \mu) &= Z_4(\mu, \Lambda) m_f(\mu) + \Pi_B(\xi, \Lambda), \\ A(\xi, \mu) &= Z_2(\mu, \Lambda) + \Pi_A(\xi, \Lambda), \end{aligned} \quad (\text{A6})$$

where  $\Pi_B$  and  $\Pi_A$  are the expressions inferred from the appropriate projections of the quark self-energy correction in Eq. (2). Solving the DSE starting by imposing new renormalization conditions, namely,  $B(p^2, \xi)$  and  $A(p^2, \xi)$ , in Eq. (A6), one verifies that the conditions in Eq. (A2) are satisfied with high precision. We check the stability of all results from  $\Lambda = 4$  GeV up to  $\Lambda = 10$  GeV, in which case  $Z_2(\xi, \Lambda)$  and  $Z_4(\xi, \Lambda)$  are modified in order to absorb the cutoff dependence; our results prove to be stable and show no cutoff dependence. The GMOR relation (A3) provides us with an additional cross-check.

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