Lepton mixing, residual symmetries, and trigonometric Diophantine equations

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In this paper, we study residual symmetries in the lepton sector. Our first concern is the symmetry of the charged lepton mass matrix in the basis where the Majorana neutrino mass matrix is diagonal, which is strongly constrained by the requirement that the symmetry group generated by residual symmetries is finite. In a recent work, R. M. Fonseca and W. Grimus found that there exists a set of constraint equations that can be completely solved, which is essential in their approach to the classification of lepton mixing matrices that are fully determined by residual symmetries. In this paper, a method to handle trigonometric Diophantine equations is introduced. We will show that the constraint equations found by Fonseca and Grimus can also be solved by this method. Detailed derivation and discussion will be presented in a self-contained way. In addition, we will also show that, in the case where residual symmetries satisfy a reality condition, this method can be used to solve the equation constraining parameters in the symmetry assignment that controls the group structure generated by residual symmetries and is directly related to mixing matrix elements.

I. INTRODUCTION

In recent years, the relation between lepton mixing and residual symmetries originating from finite flavor groups has received a fair amount of attention [1–12]. Other aspects of residual symmetries have also been studied [13]. Although the idea that lepton mixing may originate from an underlying discrete symmetry has been discussed extensively in literature (for some recent reviews, see, e.g., Ref. [14]), often only specific residual symmetries are involved, e.g., in lepton flavor models (see, e.g., Refs. [15–18] for models based on various discrete groups including $\mathbf{A}_4$, $\mathbf{S}_4$, $\mathbf{\Delta}(6n^2)$, etc.). With the help of the computer algebra system GAP [19], scans of finite groups have also been performed by several groups to look for viable discrete flavor symmetries [20]. Hence, it is worthwhile to see whether there are any general and model-independent results that can be obtained for residual symmetries and the corresponding flavor symmetries and mixing patterns.

In Refs. [6] and [7], Hernandez and Smirnov showed that, if residual symmetries generate a finite group, then the mixing matrix elements can be related to parameters in the symmetry assignment, including several characteristic parameters of residual symmetries. In Ref. [8], this method was used to construct full-mixing matrices, i.e., the mixing matrices that are fully determined by residual symmetries. In particular, using an algebraic method to solve the unitary condition that must be satisfied by any symmetry assignment that can generate a full-mixing matrix, we found that all the full-mixing patterns can be determined in the minimal scenario in which residual symmetries satisfy a

reality condition. In more general cases, explicit expressions for the mixing matrix elements can be obtained (see, e.g., Refs. [7] and [8]), and, to obtain specific solutions, algebraic methods can still be useful, but the problem of finding the complete set of solutions becomes more involved. Nevertheless, a clear and complete answer to this problem certainly can improve our understanding of the role that residual symmetries might play.

Remarkably, a complete classification of lepton mixing matrices from residual symmetries was obtained by Fonseca and Grimus in a recent work [12]. Their approach is also based on the assumption that residual symmetries are originated from a finite flavor group. Nevertheless, in this approach, mixing matrices are not obtained from symmetry assignments but from T, the generator of the residual symmetry in the charged lepton sector. Therefore, it is crucial to find explicitly all the possible forms that T can assume. In Ref. [12], this is done by first establishing the constraint equations of $|T_{ij}|$ and then showing that these equations can be solved completely by using some mathematical results concerning roots of unity.

In this paper, we will show that those constraint equations of $|T_{ij}|$ can also be solved by a method developed in Sec. III. Briefly speaking, our method is based on the observation that the equations to be solved can be written as trigonometric Diophantine equations that can be transformed to arithmetic equations involving only rational integers. They are not equivalent, but the latter can be used as auxiliary conditions that may lead to great simplification. The transformation formula can be written in a simple form and will be derived in a self-contained way. Some mathematical details are provided in the Appendix.

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The solutions to the constraint equations of $|T_{ij}|$ are derived in Sec. IV by using the method introduced in Sec. III. We will show that these equations admit only a few solutions from which the basic forms of $|T|$ can be obtained. Another use of trigonometric algebraic numbers will also be discussed. In Sec. V, we will show that this method can also be used to find the complete solution of the unitary condition in the minimal scenario. The results were already given and discussed in Ref. [8], but the derivation was omitted in order to avoid being distracted by mathematical details,[1] and hence the derivation presented in Sec. V can complete the discussion of the minimal scenario in Ref. [8]. We hope that the discussions in this paper and those in other works including Refs. [8], [9], and [12] may draw attention to some algebraic notions and techniques that are much less well known than group theoretical techniques but may also be useful in the discussions of discrete flavor symmetries.

Before proceeding to the main discussion, to provide the necessary background and set our notations, in Sec. II, we briefly review and discuss the two approaches mentioned above.

II. LEPTON MIXING AND RESIDUAL SYMMETRIES

We denote the generator of the symmetry of the left-handed charged lepton mass matrix $M_l M_l^\dagger$ by T and those of the neutrino mass matrix M_ν by S_i where $i = 1$, 2, or 3. The symmetry groups generated by T and S_i are denoted by G_e and G_ν, respectively. As usual, we adopt the assumption that neutrinos are Majorana particles and G_ν is the Klein 4-group:

$$G_\nu = \{\mathbf{1}, S_1, S_2, S_3\}.$$

In the basis where M_ν is diagonal, S_i can be written as

$$S_1^d = \mathrm{diag}\{1, -1, -1\},$$
$$S_2^d = \mathrm{diag}\{-1, 1, -1\},$$
$$S_3^d = S_1^d S_2^d. \tag{1}$$

On the other hand, in the basis where M_l is diagonal, T becomes diagonal and will be denoted by T_d.

As emphasized in the Introduction, the following discussion is based on the assumption that T and S_i together generate a finite group G, which may be identified with the flavor group G_f. A necessary condition for this assumption to hold is given by the relations [6]

$$S_i^2 = T^m = (W_i)^{p_i} = \mathbb{1}, \tag{2}$$

where m and p_i are integers and $W_i \equiv (S_i T)^{-1}$. Since the lepton mixing matrix $U = U_l^\dagger U_\nu$ where U_l and U_ν are the matrices diagonalizing T and S_i, then from Eq. (2) one finds that $|U_{\rho i}|$ can be determined by parameters in the so-called symmetry assignment including m and p_i. To see that, let us begin with $\mathrm{Tr}[W_i]$. From the definition of W_i, it follows that

$$(\mathrm{Tr}[W_i])^* = \mathrm{Tr}[S_i T] = \mathrm{Tr}[U S_i^d U^\dagger T_d] = 2\mathrm{Tr}[A_i T_d] - \mathrm{Tr}[T_d], \tag{3}$$

where

$$A_i = \mathrm{diag}\{|U_{1i}|^2, |U_{2i}|^2, |U_{3i}|^2\}.$$

In general, because $T^m = (W_i)^{p_i} = \mathbb{1}$, one can write

$$T_d = \mathrm{diag}\{e^{i2\pi k_1/m}, e^{i2\pi k_2/m}, e^{i2\pi k_3/m}\},$$

$$\mathrm{Tr}[W_i] = \sum_{j=1}^{3} e^{i2\pi n_j/p_i}. \tag{4}$$

Then from Eqs. (3) and (4) the relations between $|U_{\rho i}|$ and the parameters in T_d and $\mathrm{Tr}[W_i]$ can be established. More details including the explicit expressions can be found in Refs. [6–8] and will not be repeated here except for the minimal scenario discussed in Sec. V.

To obtain full-mixing matrices, we shall require that all $S_i \in G_f$ because one S_i fixes only a column in $|U|$. Then, from Eq. (3) and $\sum_{i=1}^{3} S_i^d = -\mathbb{1}$, it follows that

$$\sum_{i=1}^{3} (\mathrm{Tr}[W_i])^* = -\mathrm{Tr}[T_d]. \tag{5}$$

This equation will be referred to as the unitarity condition [8], which must be obeyed by any combination of S_i and T that generates a full-mixing matrix[2]

As mentioned in the Introduction, in Ref. [8], attempts were made to find possible full-mixing matrices by solving the unitarity condition given by Eq. (5). It was found that in the minimal scenario in which $\mathrm{Tr}[T_\alpha]$ and $\mathrm{Tr}[W_i]$ are real Eq. (5) can be solved completely, and then all the possible mixing patterns can also be obtained from its solutions. The detailed derivation omitted in Ref. [8] can be found in Sec. V. In nonminimal scenarios, the unitary condition becomes more involved but less restrictive. Nevertheless, solutions to Eq. (5) are still severely constrained by its algebraic nature, as discussed in Ref. [8], which is also implied by the results of numerical calculations. Therefore, it is reasonable to think that it might still be possible to

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[1]Some discussions in this paper can also be found in the first arXiv version of Ref. [8].

[2]Note that if T has degenerate eigenvalues one will need another matrix T' to determine $|U|$ fully because only one row in $|U|$ can be obtained from Eq. (3) even if all $S_i \in G_f$.

obtain a complete picture of full-mixing patterns under assumptions mentioned above, although solving Eq. (5) alone may not be an efficient way because the relations in Eq. (2) are not sufficient for S_i and T to generate a finite group.

This problem is solved by Fonseca and Grimus in a recent work. In Ref. [12], they begin with $Y^{(ij)} = T^\dagger S_i T S_j$, which must also have finite orders. Since one can show that $\det Y^{(ij)} = 1$ and $\text{Tr}[Y^{(ij)}]$ are real, it then follows that $Y^{(ij)}$ have eigenvalues 1, $\lambda^{(ij)}$, and $(\lambda^{(ij)})^*$, where $\lambda^{(ij)}$ are roots of unity and hence can be written as

$$\lambda^{(ij)} = e^{i2\pi k^{(ij)}/m^{(ij)}}, \qquad (6)$$

where $k^{(ij)}$ and $m^{(ij)}$ are coprime. Because $\text{Tr}[Y^{(ij)}]$ are basis independent, from $\sum_{i=1}^{3} S_i^d = -\mathbb{1}$, one finds that

$$\sum_{i=1}^{3} \text{Tr}[Y^{(ij)}] = \sum_{j=1}^{3} \text{Tr}[Y^{(ij)}] = 1,$$

which can be written as

$$\sum_{i=1}^{3} (\lambda^{(ij)} + (\lambda^{(ij)})^*) = -2,$$

$$\sum_{j=1}^{3} (\lambda^{(ij)} + (\lambda^{(ij)})^*) = -2. \qquad (7)$$

In addition, in the basis where S_i are diagonal, one can show that

$$4|T_{ij}|^2 = \text{Tr}[Y^{(ij)}] + 1 = \lambda^{(ij)} + (\lambda^{(ij)})^* + 2. \qquad (8)$$

Therefore, from $\lambda^{(ij)}$ satisfying (7), one can construct the matrix $|T|$ that is defined by $|T|_{ij} = |T_{ij}|$. It was found in Ref. [12] that there exist only five basic forms that $|T|$ can assume. From these basic forms, one can derive all the possible full-mixing patterns because in the basis in which S_i are diagonal, the lepton mixing matrix can be obtained from T alone. For instance, for a given T in the diagonal basis of S_i, one may obtain T_d from $\text{Tr}[T]$ and $\det[T]$ and then solve Eq. (3) for mixing matrix elements.

In Ref. [12], the two equations in (7) are solved by employing a theorem related to roots of unity [21]. Here, we note that substituting Eq. (6) into Eq. (7) leads to

$$\sum_{i=1}^{3} 2\cos\left(\frac{k^{(ij)}}{m^{(ij)}} 2\pi\right) = -2, \qquad (9)$$

$$\sum_{j=1}^{3} 2\cos\left(\frac{k^{(ij)}}{m^{(ij)}} 2\pi\right) = -2. \qquad (10)$$

In addition, in the minimal scenario, the unitary condition in Eq. (5) can also be written in a similar form, as shown in

Sec. V. In the next section, we introduce a method to handle trigonometric Diophantine equations, such as the two equations above. Using this method, we derive the solutions to Eqs. (9) and (10) in Sec. IV.

III. TRIGONOMETRIC DIOPHANTINE EQUATIONS

In this section, we consider trigonometric Diophantine equations that can be written as

$$\sum_{j=1}^{n} 2\cos\frac{n_j}{p_j} 2\pi = r, \qquad (11)$$

where n_j, p_j, and r are rational integers. Without loss of generality, we require that n_j and p_j are coprime or $\gcd(n_j, p_j) = 1$, where gcd stands for the greatest common divisor. To simplify notations, below we denote rational angles such as $2\pi n/m$ by α_{nm} and $2\cos\alpha_{n_j p_j}$ by β^j. Then, Eq. (11) can be written as

$$\sum_{j=1}^{n} \beta^j = \sum_{j=1}^{n} 2\cos\alpha_{n_j p_j} = r. \qquad (12)$$

We will show that this equation can be transformed to a simple arithmetic equation,[3] i.e.,

$$\sum_{j=1}^{n} \frac{s_j}{d_j} = r, \qquad (13)$$

where d_j is the degree of the minimal polynomial (MP) of β^j, i.e., the polynomial with integer coefficients satisfied by β^j that has the lowest degree,[4] and s_j is the sum of all the roots of that MP. Note that not only d_j but also s_j are rational integers, as explained below. From solutions to Eq. (13), one can find solutions to Eq. (11) because p_j can be determined by d_j and s_j. To derive Eq. (13), we will need some notions and results from algebraic number theory, mostly for convenience. The detailed derivation is presented below in a rather self-contained way, but, for conciseness, some details are relegated to the Appendix. Before proceeding, we should mention that in this paper we only use polynomials with integer coefficients, unless otherwise stated. In addition, the set of all rational numbers and the set of all rational integers are denoted by $\mathbb{Q}$ and $\mathbb{Z}$, respectively.

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[3]Similar results may exist somewhere in literature, but we are not aware of any of them.

[4]In other words, it is irreducible or cannot be written as a product of two monic polynomials with integer coefficients. More can be found in the Appendix.

First, note that from

$$\cos 2\pi n_j = \cos(p_j \cdot \alpha_{n_j p_j}) = 1$$

and the expansion of $\cos(p_j \alpha_{n_j p_j})$ in powers of $\cos(\alpha_{n_j p_j})$, as shown in the Appendix, one can see that $\beta^j = 2\cos\alpha_{n_j p_j}$ are algebraic integers that are defined to be solutions to monic polynomials with integer coefficients. Moreover, all the $\beta^j = 2\cos(2\pi n_j/p_j)$ in Eq. (12) can be expanded in powers of

$$\gamma_1 \equiv 2\cos\frac{2\pi}{q},$$

where $q \equiv \mathrm{LCM}(p_1, \ldots, p_n)$ and LCM stands for the least common multiple. Obviously, γ_1 is also an algebraic integer. Let polynomial $g(x)$ of degree d be the MP for γ_1. It can be written as

$$g(x) = x^d + c_{d-1}x^{d-1} + \cdots + c_0 = \prod_{i=1}^{d}(x - \gamma_i),$$

where γ_i are the roots of $g(x)$. Since the degree of an algebraic integer is defined to be the degree of its MP, we have $\deg(\gamma_1) = d$, where $\deg(\gamma_1)$ denotes the degree of γ_1.

To proceed, we notice that, using $g(\gamma_1) = 0$, one can eliminate any term having a power of γ_1 higher than $d - 1$ from a polynomial in γ_1. Hence, if an algebraic integer θ can be written as a polynomial in γ_1, then one can always write θ as

$$\theta = a_0 + a_1\gamma_1 + \cdots + a_{d-1}\gamma_1^{d-1} = \sum_{k=0}^{d-1} a_k(\gamma_1)^k \equiv p(\gamma_1),$$
$$(14)$$

where $a_i \in \mathbb{Q}$. More importantly, the expression for θ in the form of Eq. (14) is unique. To show that, suppose θ can also be written as

$$\theta = a_0' + a_1'\gamma_1 + \cdots + a_{d-1}'\gamma_1^{d-1}.$$

Then, one has

$$(a_{d-1}' - a_{d-1})\gamma_1^{d-1} + \cdots + (a_1' - a_1)\gamma_1 + (a_0' - a_0) = 0.$$

Since the degree of γ_1 is d and thus γ_1 cannot satisfy any polynomial of degree less than d, it then follows that $a_k' = a_k$ for all k.

According to the discussion above, we can write β^j uniquely as

$$\beta^j = \sum_{k=0}^{d-1} b_k^j(\gamma_1)^k \equiv \tilde{\beta}^j(\gamma_1),$$
$$(15)$$

where $b_k \in \mathbb{Z}$. Then, from Eqs. (12) and (15), one has

$$\sum_{j=1}^{n} \tilde{\beta}^j(\gamma_1) = \sum_{k=0}^{d-1}\left(\sum_{j=1}^{n} b_k^j\right)(\gamma_1)^k = r.$$
$$(16)$$

Now comes a crucial step. Again, since γ_1 cannot satisfy any polynomial of degree less than d, in Eq. (16) one must have

$$\sum_{j=1}^{n} b_k^j = 0$$

for $1 \leq k \leq d - 1$ because r is a rational integer. Hence, in Eq. (16), one may replace γ_1 by any number. For our purpose, we will substitute γ_1 by $\gamma_2, \ldots, \gamma_d$, i.e., the other roots of its MP. Then, besides Eq. (16), we also have

$$\sum_{j=1}^{n} \tilde{\beta}^j(\gamma_i) = r$$

for $2 \leq i \leq d$. Summing over i leads to

$$\sum_{j=1}^{n}\left[\sum_{i=1}^{d} \tilde{\beta}^j(\gamma_i)\right] = rd.$$
$$(17)$$

In addition, as discussed in the Appendix, one can show that for any γ_i the corresponding $\tilde{\beta}^j(\gamma_i)$ is a root of the MP of $\tilde{\beta}^j(\gamma_1)$ because γ_i is a root of the MP of γ_1. Now, let d_j be the degree of $\beta^j = \tilde{\beta}^j(\gamma_1)$. Because one can also show that when γ_i in $\tilde{\beta}^j(\gamma_i)$ runs from γ_1 to γ_d each root of the MP of $\tilde{\beta}^j(\gamma_1)$ repeats d/d_j times, we have

$$\sum_{i=1}^{d} \tilde{\beta}^j(\gamma_i) = \frac{d}{d_j} s_j,$$
$$(18)$$

where s_j is the sum of all the roots of the MP of β^j. From Eqs. (17) and (18), it follows that

$$\sum_{j=1}^{n} \frac{s_j}{d_j} = \sum_{j=1}^{n} \frac{1}{\deg(\beta^j)}\sum_c \beta^j = r,$$
$$(19)$$

where $\deg(\beta^j) = d_j$ and $\sum_c \beta^j = s_j$. We would like to emphasize that $\sum_c \beta^j$ should not be confused with $\sum_j \beta^j$. In addition, since the MP of β^j can be written as

$$x^{d_j} - s_j x^{d_j-1} + \cdots,$$

then s_j must be a rational integer.

In the Appendix, the explicit expressions for $d_m = \deg(2\cos\alpha_{nm})$ and $s_m = \sum_c 2\cos\alpha_{nm}$ are given in Eqs. (A2) and (A3). As examples, d_m and s_m for $2 \leq m \leq 20$ are given in the table below, which will also be used in

TABLE I.   The degree of the MP of $2\cos\alpha_{nm}$ and the sum of its roots.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d_m	1	1	1	1	2	1	3	2	3	2	5	2	6	3	4	4	8	3	9	4
s_m	2	−2	−1	0	−1	1	−1	0	0	1	−1	0	−1	1	1	0	−1	0	−1	0

the next two sections. Note that d_m and s_m together can determine the value of m in $\alpha_{nm} = 2\pi n/m$ but not n which can be any integer that is relatively prime to m. For obvious reasons, we can require that $0 < n < m/2$. In addition, the value of m may not be determined uniquely by d_m and s_m, as can be seen from Table I.

The issues mentioned above indicate that Eq. (19) is only a necessary condition for any set of p_j or, more accurately, $\beta^j = 2\cos(2\pi n_j/p_j)$ to satisfy Eq. (11). Although it is not a sufficient condition and hence some solutions to Eq. (19) may not satisfy Eq. (11), because it is not hard to find its solutions in many cases, Eq. (19) can be regarded as an auxiliary condition that may greatly simplify the problem of finding solutions to Eq. (11). As to n_j in Eq. (11), one can find them by trial and error, especially when d_j is small because, as explained in the Appendix, d_j is the number of n_j satisfying $\gcd(n_j, p_j) = 1$ and $0 < n_j < p_j/2$.

Equation (19) is most helpful in the cases in which it can provide enough information for us to solve Eq. (11) completely. Such a situation may occur if the number of terms on the left-hand side or n is not large and $|r|$ is comparable to n. The reason is that $s_j = \sum_c 2\cos(2\pi n_j/p_j)$ can only be ± 1 or 0 for all p_j (except for $p_j = 1$ or 2) and the lower bound on d_j increases with p_j [see the discussion below Eq. (A2) in the Appendix]. Hence, for a small n and a $|r| \sim n$, to satisfy Eq. (19), p_j cannot all be very large in most cases, and thus it might be possible to solve it completely. This is exactly the situation for the two cases discussed in the following two sections. This method may also be useful in some other cases, for example, when one only wants to find certain specific solutions or for some reason p_j can only be chosen from a given set of numbers.

IV. RESIDUAL SYMMETRY IN THE CHARGED LEPTON SECTOR

In this section, we will show that the complete set of solutions to Eqs. (9) and (10) can be easily obtained by the method introduced above. Since these two equations are similar, only Eq. (9) will be discussed in detail. As above, to simplify notations, we write Eq. (9) as

$$\sum_{i=1}^{3}\beta^{(ij)} = \sum_{i=1}^{3}2\cos\alpha^{(ij)} = -2, \qquad (20)$$

where $\alpha^{(ij)} = 2\pi k^{(ij)}/m^{(ij)}$. Then, as shown in the previous section, this equation can be transformed to

$$\sum_{i=1}^{3}y_i = \sum_{i=1}^{3}\frac{s_i}{d_i} = \sum_{i=1}^{3}\frac{1}{\deg(\beta^{(ij)})}\sum_c\beta^{(ij)} = -2, \quad (21)$$

where $y_i = s_i/d_i$, $s_i = \sum_c \beta^{(ij)}$, and $d_i = \deg(\beta^{(ij)})$.

Since, as discussed above, $|s_i| \le 1$ for $m^{(ij)} > 2$ and d_i tend to grow with $m^{(ij)}$, one can show that

$$\left|\frac{s_j}{d_j}\right| \le 2 \qquad (22)$$

for any j. Hence, without loss of generality, we can require that

$$2 \ge |y_1| \ge |y_2| \ge |y_3|. \qquad (23)$$

Then, from Eq. (21), it follows that

$$|y_1| = \left|\frac{s_1}{d_1}\right| \ge \frac{2}{3},$$

and hence in Table I, one finds that

$$m^{(1j)} = 1, 2, 3, \text{ or } 6.$$

If $m^{(1j)} = 1$, then

$$\alpha^{(1j)} = 2\pi, \qquad \beta^{(1j)} = 2\cos\alpha^{(1j)} = 2. \qquad (24)$$

Substituting $\beta^{(1j)} = 2$ into Eq. (20) leads to

$$2\cos\alpha^{(2j)} + 2\cos\alpha^{(3j)} = -4.$$

Therefore, $m^{(2j)} = m^{(3j)} = 2$, and hence

$$\beta^{(2j)} = \beta^{(3j)} = 2\cos\left(\frac{1}{2}\times 2\pi\right) = -2. \qquad (25)$$

In the case in which $m^{(1j)} = 2$, one has $\alpha^{(1j)} = \pi$ and

$$\beta^{(1j)} = 2\cos\alpha^{(1j)} = -2. \qquad (26)$$

Then, from Eq. (20), it follows that

$$\beta^{(2j)} = -\beta^{(3j)} = 2\cos\varphi, \qquad (27)$$

where $\varphi = \alpha^{(2j)}$ is an arbitrary rational angle. Note that no further constraint can be imposed on φ in this case because Eq. (27) can be satisfied if

$$\frac{k^{(2j)}}{m^{(2j)}} + \frac{k^{(3j)}}{m^{(3j)}} = \frac{1}{2},$$

from which one can find a $m^{(3j)}$ for any $m^{(2j)}$.

The next case is the one in which $m^{(1j)} = 3$, which is a bit more complicated. From Table I, one finds that $d_1 = 1$ and $s_1 = -1$ and hence $y_1 = -1$. Then, from Eq. (21), one has $y_2 + y_3 = -1$. Together with Eq. (23) and $|y_1| = 1$, this leads to $1/2 \leq |y_2| \leq 1$ and $|y_3| \leq 1$. From these results, one finds that $-1 \leq y_2 \leq -1/2$. Therefore, $y_2 = -1$ or $-1/2$, as can be seen from Table I. If $y_2 = -1$, one has $m^{(2j)} = 3$. Together with $m^{(1j)} = 3$ and Eq. (20), it leads to $\beta^{(3j)} = 0$ and hence $m^{(3j)} = 4$. In short, if $y_2 = -1$, one has

$$\beta^{(1j)} = \beta^{(2j)} = 2 \cos \frac{2\pi}{3} = -1,$$

$$\beta^{(3j)} = 2 \cos \frac{2\pi}{4} = 0. \tag{28}$$

If $y_2 = -1/2$, then $y_3 = -1/2$. From Table I, one has $m^{(2j)} = m^{(3j)} = 5$. Because $d_2 = d_3 = 2$, in Eq. (20) both $k^{(2j)}$ and $k^{(3j)}$ have two choices. The solution to Eq. (20) is found to be $k^{(2j)} = 1$ and $k^{(3j)} = 2$. Therefore, one has

$$\beta^{(1j)} = 2 \cos \frac{2\pi}{3} = -1,$$

$$\beta^{(2j)} = 2 \cos \frac{2\pi}{5},$$

$$\beta^{(3j)} = 2 \cos \frac{4\pi}{5}. \tag{29}$$

After that, we are left with the case in which $m^{(1j)} = 6$. In Table I, one finds that $d_1 = 1$ and $s_1 = 1$. However, from Eq. (21), it follows that $y_2 + y_3 = -3$, which is not consistent with Eq. (23), since the latter would lead to $|y_2 + y_3| \leq 2|y_1| = 2$. Therefore, no solution satisfying Eq. (23) exists for $m^{(1j)} = 6$.

Now, we have found all the solutions to Eq. (20) that are given by Eqs. (24)–(29). Since, from Eqs. (6) and (8) one has

$$4|T_{ij}|^2 = \beta^{(ij)} + 2 = 2 \cos \left(\frac{k^{(ij)}}{m^{(ij)}} 2\pi \right) + 2, \tag{30}$$

then, for each solution, the corresponding $|T_{ij}|^2$ can be obtained from the relation above and written collectively as $(|T_{1j}|^2, |T_{2j}|^2, |T_{3j}|^2)$. For the four solutions found above, one has the following four possibilities:

$$(1, 0, 0), \qquad \left(\frac{1}{4}, \frac{3 + \sqrt{5}}{8}, \frac{3 - \sqrt{5}}{8} \right),$$

$$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4} \right), \qquad \left(0, \frac{1 + \cos\varphi}{2}, \frac{1 - \cos\varphi}{2} \right).$$

Note that the order of $|T_{ij}|^2$ can be changed because in Eq. (23) the order of y_1, y_2, and y_3 can also be changed, which may lead to solutions with different orderings.

It is easy to see that the solutions to the second equation in (7) are the same as the first one. Hence, in matrix $|T|^2$ where $(|T|^2)_{ij} = |T_{ij}|^2$, every row and column must be one of the four possibilities given above. It is then not hard to find that there exist only five basic forms that $|T|$ can assume, as shown in Ref. [12]. After that, by a thorough and careful analysis, which can also be found in Ref. [12], one can find all the possible full-mixing patterns.

Before proceeding to the next section, we would like to mention that some results used in our discussions might also be useful on other occasions. For example, because a noninteger rational number is not an algebraic integer, as shown in the Appendix, sometimes this can provide a quick way to show that for a given complex number η there does not exist a rational integer m satisfying $\eta^m = 1$. If such an m exists, then η is a root of unity and can be written as $\eta = e^{i2\pi k/m}$. Hence, $\eta + \eta^* = 2 \cos(2\pi k/m)$ must be an algebraic integer. But if $\eta + \eta^*$ is a noninteger rational number, then it cannot be an algebraic integer, and thus η cannot be a root of unity. For instance, $\eta = (1 + i3\sqrt{7})/8$ or $\eta = (-1 + i\sqrt{15})/4$ cannot be a root of unity because $\eta + \eta^*$ is a noninteger rational number.[5]

V. MIXING PATTERNS IN THE MINIMAL SCENARIO

The derivation of lepton mixing patterns from $|T|$ is rather complicated. In this section, we will show that, in the minimal scenario in which both $\text{Tr}[T]$ and $\text{Tr}[W_i]$ are real, the unitarity condition (5) can also be solved completely, and hence full-mixing matrices can be obtained in a rather straightforward way.

When $\text{Tr}[T]$ is real and T belongs to $SU(3)$, from Eq. (2) it follows that T_d can be written as one of the following three matrices:

$$T_e \equiv T_1 = \text{diag}\{1, e^{2\pi i n_4/p_4}, e^{-2\pi i n_4/p_4}\},$$

$$T_\mu \equiv T_2 = \text{diag}\{e^{2\pi i n_4/p_4}, 1, e^{-2\pi i n_4/p_4}\},$$

$$T_\tau \equiv T_3 = \text{diag}\{e^{2\pi i n_4/p_4}, e^{-2\pi i n_4/p_4}, 1\}, \tag{31}$$

where n_4 and p_4 are coprime. It then follows that

$$\text{Tr}[T] = 1 + 2 \cos 2\pi n_4/p_4. \tag{32}$$

In this section, we will denote T_d by T_α where $\alpha = 1, 2,$ or 3. In addition, we require that $m > 2$, because otherwise the lepton mixing matrix cannot be fully determined. Similarly, if $\text{Tr}[W_i]$ is real, because $(W_i)^{p_i} = \mathbb{1}$, the eigenvalues of W_i can always be written as 1, $e^{2\pi i n_i/p_i}$, or $e^{-2\pi i n_i/p_i}$ with $\gcd(n_i, p_i) = 1$. Hence,

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[5]These two examples are taken from Ref. [12] where a somewhat different argument is used.

$$\mathrm{Tr}[W_i] = 1 + 2\cos 2\pi n_i/p_i. \qquad (33)$$

Then, from Eqs. (3), (32), and (33) and $\sum_{\rho=1}^{3}|U_{\rho i}|^2 = 1$, one finds that

$$|U_{\alpha i}|^2 = \frac{1 + \mathrm{Tr}[W_i]}{3 - \mathrm{Tr}[T_\alpha]} = \frac{1 + \cos(2\pi n_i/p_i)}{2\sin^2(\pi n_4/p_4)}, \qquad (34)$$

$$|U_{\beta i}|^2 = |U_{\gamma i}|^2 = \frac{1}{2}(1 - |U_{\alpha i}|^2), \qquad (35)$$

where $\beta, \gamma \neq \alpha$, and $\beta < \gamma$. For detailed derivation, see Ref. [6].

As discussed in Sec. II, to generate a full-mixing matrix, W_i and T_α must satisfy the unitary condition in Eq. (5). In the minimal scenario, substituting Eqs. (32) and (33) into Eq. (5) leads to

$$\sum_{j=1}^{4} \beta^j = \sum_{j=1}^{4} 2\cos\alpha_{n_j p_j} = -4, \qquad (36)$$

where $\beta^j = 2\cos\alpha_{n_j p_j}$ and $\alpha_{n_j p_j} = 2\pi n_j/p_j$. As in the previous section, this equation can be transformed to

$$\sum_{j=1}^{4} y_j = \sum_{j=1}^{4} \frac{s_j}{d_j} = \sum_{j=1}^{4} \frac{1}{\deg(\beta^j)} \sum_{c} \beta^j = -4, \qquad (37)$$

which can also be easily solved in a way similar to the previous section.

First, note that Eq. (37) cannot be satisfied if all $y_j > -1$; then, we can require that $y_1 \leq -1$ and $y_1 \leq y_j$ for $j \geq 2$. From Table I, one finds that $y_1 = -1$ or -2 corresponding to $d_1 = 1$ and $s_1 = -1$ or -2.

If $d_1 = 1$ and $s_1 = -1$, from Eq. (37) it follows that $\sum_{j=2}^{4} y_j = -3$. Since $y_j \geq y_1 = -1$ for $j \geq 2$, it cannot be satisfied unless all $y_j = -1$. Therefore, from Table I one finds that $s_j = -1$ and $d_j = 1$, and hence $p_j = 3$ for all j.

If $d_1 = 1$ and $s_1 = -2$, from Table I one finds that $p_1 = 2$. Since one can only let $n_1 = 1$, from Eq. (36) it follows that $\sum_{j=2}^{4} \beta^j = -2$. Besides that, from Eq. (37) and $y_1 = -2$, one has $\sum_{j=2}^{4} y_j = -2$. These equations are identical to Eqs. (20) and (21), and hence their solutions can be obtained from those found in Sec. IV.

Here, we denote the solutions to Eq. (36) by $\{p_1, p_2, p_3, p_4\}$ from which the corresponding mixing matrix can be obtained from Eqs. (34) and (35). In this notation, the solutions we found can be written as $\{3,3,3,3\}$, $\{2,1,2,2\}$, $\{2,2,p_3,p_4\}$, $\{2,3,3,4\}$, and $\{2,3,5,5\}$. As explained in the previous section, in the third solution, p_3 and p_4 can be any integers that satisfy the relation $n_3/p_3 + n_4/p_4 = 1/2$, where n_3 and n_4 are arbitrary integers satisfying $0 < n_j/p_j \leq 1/2$ and $\gcd(n_j, p_j) = 1$. In Eq. (37), one may change the order of y_j, and hence the order of p_j can also be changed in these solutions. Nevertheless, as discussed in Ref. [8], only the

last two solutions can lead to phenomenologically interesting mixing patterns including tribimaximal mixing, bimaximal mixing, or the golden ratio mixings. For more discussion, see Ref. [8].

VI. DISCUSSIONS

In the previous two sections, under the assumption that residual symmetries are originated from a finite group, we discussed the condition constraining the symmetry of the charged lepton mass matrix in the basis in which the neutrino mass matrix is diagonal and the unitary condition in the minimal scenario. Both of them can be completely solved by the algebraic method presented in Sec. III, and their solutions can be used to construct full-mixing matrices. This method can also be used to solve trigonometric Diophantine equations similar to those discussed in this paper.

It is interesting to see that, besides group theoretical techniques, other algebraic techniques can also be useful and even the key to some important results including the basic forms of $|T|$ discussed in Sec. IV. As another example, in Ref. [12] it was found that, under the same assumptions adopted in this paper, only particular trimaximal mixing patterns can survive the current neutrino oscillation data. The corresponding symmetry groups include, e.g., $\mathbf{\Delta}(600)$, $\mathbf{\Delta}(1536)$, and $(\mathbf{Z}_{18} \times \mathbf{Z}_6) \times \mathbf{S}_3$, which can also be obtained from solutions to Eq. (5) in nonminimal scenarios. To find those solutions, notions and tools from algebraic number theory can also be helpful, as discussed in Ref. [8]. In addition, from Eq. (4) one finds that the unitary condition in Eq. (5) can be written as a vanishing sum of 12 roots of unity. The classification of its solutions can be found in Ref. [22] from which one may find another approach to the classification of full-mixing patterns. Hopefully, the works presented in some papers including Refs. [8,9,12], and this one may draw attention to those less well-known techniques.

ACKNOWLEDGMENTS

APPENDIX: TRIGONOMETRIC ALGEBRAIC NUMBERS

In this Appendix, some mathematical details concerning trigonometric algebraic numbers will be provided. Below, we first quickly recall some basic notions from algebraic number theory [23]:

(1) An algebraic number θ over $\mathbb{Q}$, the set (or field) of all rational numbers, is a root of a monic polynomial over $\mathbb{Q}$ that can be written as

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad (A1)$$

where all $a_i \in \mathbb{Q}$ and n is called the degree of the polynomial. Moreover, if all a_i are rational integers, θ can also be called an algebraic integer. If α and β are algebraic numbers (integers), then $\alpha + \beta$ and $\alpha\beta$ are also algebraic numbers (integers).

Note that, unless otherwise mentioned, here we consider only algebraic numbers (integers) and polynomials over $\mathbb{Q}$, as defined above. Hence, from now on, we will omit the phrase "over $\mathbb{Q}$" for the sake of conciseness.

(2) The MP of θ is the polynomial of lowest degree that θ satisfies. One can show that the MP of θ is unique; otherwise, one can construct a polynomial satisfied by θ of lower degree than the MP. For the same reason, the MP is clearly irreducible or cannot be written as a product of two polynomials.

(3) The degree of θ, denoted by $\deg(\theta)$, is defined to be the degree of its MP. All the roots of its MP are also called the conjugates of θ, which are denoted by θ_k, where $k = 1, 2, \ldots, \deg(\theta)$ and $\theta_1 = \theta$. An important and frequently used result is that any polynomial satisfied by θ must be divisible by its MP and hence satisfied by all θ_k because the remainder of the division, if not vanishing, is also satisfied by θ, but of lower degree than its MP. As a consequence, the MP of θ is also the MP of θ_k since it is not reducible.

(4) Since the MP of an algebraic number (integer) θ has rational (integer) coefficients and can always be written as $\prod_k (x - \theta_k)$, it is obvious that $\sum_k \theta_k$ is a rational number (integer). One can also show that θ_k are distinct because, if $\theta_i = \theta_j$, then from the MP one can construct a polynomial satisfied by θ_i or θ_j of lower degree than the MP by taking derivative with respect to x.

Mostly for convenience, simple algebraic extension of $\mathbb{Q}$ is introduced below. No deep result concerning field extensions is needed for our discussions.

(5) By adjoining to $\mathbb{Q}$ an algebraic number θ, the field of rational numbers can be extended to another field denoted by $\mathbb{Q}(\theta)$. As shown in Sec. III, every element λ of $\mathbb{Q}(\theta)$ can be written uniquely in the form

$$\lambda = a_0 + a_1\theta + \cdots + a_{v-1}\theta^{d-1} \equiv p(\theta),$$

where $a_i \in \mathbb{Q}$ and $d = \deg(\theta)$. The conjugates of λ *for* $\mathbb{Q}(\theta)$ are defined to be $p(\theta_k) \equiv \bar{\lambda}_k$, where θ_k are the conjugates of θ and $\bar{\lambda}_1 = \lambda$.

We should emphasize that $\bar{\lambda}_i$ defined above are the conjugates of λ *for* $\mathbb{Q}(\theta)$ but not the conjugates of λ *over* $\mathbb{Q}$ (defined in item 3), which are denoted by λ_k. Nevertheless, they are closely related:

(6) The set $\{\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_d\}$ is not necessarily identical to the set of the conjugates of λ *over* $\mathbb{Q}$, i.e., $\{\lambda_1, \lambda_2, \ldots, \lambda_{\deg(\lambda)}\}$, where $\lambda_1 = \lambda = \bar{\lambda}_1$. However, one can show that in $\{\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_d\}$ each λ_k repeats $d/\deg(\lambda)$ times.

It might be instructive to go over the proof of the result given in item 6, which is very important in the derivation of Eq. (13). Below, we follow Ref. [23]. To begin with, one consider the polynomial[6]

$$f(x) = \prod_{i=1}^{d}(x - \bar{\lambda}_i) = \prod_{i=1}^{d}(x - p(\theta_i)).$$

Let the MP of λ be $g(x)$ of degree $\deg(\lambda)$. Since $\lambda = \bar{\lambda}_1$ is a root of $f(x)$, then, for the reason explained in item 3, $f(x)$ must be divisible by $g(x)$. Therefore, one can write

$$f(x) = [g(x)]^n h(x),$$

where $g(x) \nmid h(x)$. Then, one can show that $h(x)$ must be a constant. Otherwise, for some $\bar{\lambda}_m$, one must have $h(\bar{\lambda}_m) = h(p(\theta_m)) = 0$. Since θ_m satisfies $h(p(x))$, then $h(p(x))$ must also be satisfied by θ_1 (see item 3). Therefore, one has $h(p(\theta_1)) = h(\bar{\lambda}_1) = h(\lambda) = 0$ from which it follows that $g(x)|h(x)$, which contradicts the requirement $g(x) \nmid h(x)$. After that, it is not hard to see that $h(x) = 1$, and hence $f(x) = [g(x)]^n$ and $n = d/\deg(\lambda)$.

Below, we will concentrate on trigonometric algebraic numbers, especially the cosines of rational angles, i.e., $2\cos\alpha_{nm}$, where $\alpha_{nm} \equiv 2\pi n/m$ and $\gcd(n,m) = 1$. As in the main text, below we denote $2\cos\alpha_{nm}$ by β_{nm}.

First of all, recall that [24]

$$2\cos m\alpha = (2\cos\alpha)^m - m(2\cos\alpha)^{m-2}$$
$$+ \frac{m(m-3)}{2}(2\cos\alpha)^{m-4} + \cdots.$$

Replacing $2\cos\alpha$ by x in the expression on the right-hand side, we can define polynomials $c_m(x)$ as[7]

$$c_m(x) \equiv \left[x^m - mx^{m-2} + \frac{m(m-3)}{2}x^{m-4} + \cdots\right] - 2.$$

From $\cos m\alpha_{nm} = 1$, one has

$$c_m(\beta_{nm}) = 2\cos m\alpha_{nm} - 2 = 0,$$

from which it follows that $\beta_{nm} \equiv 2\cos\alpha_{nm}$ is an algebraic integer. An immediate consequence is that, if β_{nm} is a rational number, it must be a rational integer. Note that its

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[6]One can show that the coefficients of $f(x)$ are rational numbers or integers because they can be written as polynomials of the elementary symmetric functions in $\theta_1, \ldots \theta_k$ such as $\sum_k \theta_k$, $\sum_{k,j} \theta_k \theta_j$, etc., which can be shown to be rational numbers or integers since they are the coefficients of the MP of θ_k, i.e., $\prod_k(x - \theta_k)$. A complete proof of this point would take us too far afield. One may find the details in Ref. [23].

[7]If more details are needed, note that $c_m(x)$ can also be written as $2T_n(x/2) - 2$, where $T_n(x)$ are Chebyshev polynomials of the first kind [24].

MP is not $c_m(x)$ but a factor of $c_m(x)$, as discussed above in item 3. Hence, any conjugate of β_{nm} must also be a root of $c_m(x)$.

Next, except for $m = 1$ or 2, the algebraic degree of β_{nm} is given by $\varphi(m)/2$, where $\varphi(m)$ is Euler's φ function [25], i.e.,

$$\deg(2\cos\alpha_{nm}) = \frac{\varphi(m)}{2} = \frac{m}{2}\prod_{P|m}(1 - P^{-1}), \quad (A2)$$

where P are prime numbers and the product involves all the *distinct* prime factors of m. Recall that, by definition, $\varphi(m)$ is equal to the number of positive integers that are less than m and are relatively prime to m. A useful property of $\varphi(m)$ is that $\varphi(m)$ tends to increase with m. In fact, various lower bounds on $\varphi(m)$ can be established [26]. For instance, $\varphi(m) \geq \sqrt{m}$ for $m \neq 2$ and $m \neq 6$, and $\varphi(m) > m^{2/3}$ for $m > 30$.

A rigorous proof of Eq. (A2) can be found in Ref. [27]. A short but less elementary proof can be found in Ref. [28]. Some examples are given in Table I of Sec. III. Below, we give a simple argument showing that the conjugates of β_{nm}, i.e., the roots of its MP, can include only those $2\cos\alpha_{qm}$ with $\gcd(q, m) = 1$, the number of which is exactly $\varphi(m)/2$ after the equality between $2\cos\alpha_{qm}$ and $2\cos\alpha_{(m-q)m}$ is taken into account. To show that, let us consider $\beta_{kp} = 2\cos\alpha_{kp}$ and $\beta_{nm} = 2\cos\alpha_{nm}$ with $\gcd(k, p) = \gcd(n, m) = 1$ and $p < m$. From $c_p(\beta_{kp}) = 0$ and

$$c_p(\beta_{nm}) = 2\cos p\alpha_{nm} - 2 \neq 0,$$

one finds that β_{nm} cannot be a root of $c_p(x)$ that contains the MP of β_{kp}. Therefore, β_{nm} is not a root of the MP of any $2\cos\alpha_{qm}$ if q is *not* relatively prime to m. In other words, the former is not a conjugate of the latter. Since all the conjugates of β_{nm} are among the roots of $c_m(x)$ that can be written as $2\cos\alpha_{qm}$ with $1 \leq q \leq m$, we are left with $2\cos\alpha_{qm}$ with $\gcd(q, m) = 1$, as promised.

Now, we turn to the sum of the conjugates of β_{nm}, including all the distinct $2\cos\alpha_{qm}$ with $\gcd(q, m) = 1$. The results are given by

$$\sum_c 2\cos\alpha_{nm} = \begin{cases} 2, & m = 1 \\ -2, & m = 2 \\ \mu(m), & m \geq 3, \end{cases} \quad (A3)$$

where $\sum_c \theta$ denotes the sum of the conjugates of an algebraic number θ and $\mu(m)$ is the Möbius function [25] defined as

$$\mu(m) = \begin{cases} 1, & m = 1 \\ 0, & \text{if } m \text{ has a square factor} \\ (-1)^k, & \text{if } m = p_1 p_2 \ldots p_k \text{ with } p_i \\ & \text{being different prime numbers.} \end{cases}$$

For some examples, see Table I in Sec. III. Below, we give an example to demonstrate how Eq. (A3) is derived. First, one has

$$\sum_{k=1}^{m-1} \cos\frac{k}{m}2\pi = \text{Re}\left[\sum_{k=1}^{m-1} e^{i\frac{k}{m}2\pi}\right] = -1 \quad (A4)$$

for any $m \geq 2$. Then, one can divide the terms on the left-hand side into groups of conjugates. For example, for $m = 3 \times 7$,

$$\sum_{k=1}^{20} \cos\frac{k}{21}2\pi = \sum_{i=1}^{2} \cos\frac{i}{3}2\pi + \sum_{i=1}^{6} \cos\frac{i}{7}2\pi + 2\sum_{\substack{j\leq 10, \\ (j,21)=1}} \cos\frac{j}{21}2\pi, \quad (A5)$$

where $(j, 21)$ is short for $\gcd(j, 21)$. As explained above, the last term can be written as $\sum_c 2\cos(2\pi n/21)$ where n can be any integer satisfying $\gcd(n, 21) = 1$. From Eqs. (A4) and (A5), one finds that

$$\sum_c 2\cos\frac{n}{21}2\pi = 1.$$

By induction, the result given by Eq. (A3) can also be proved for arbitrary m in a similar way. The detailed derivation is somewhat lengthy and hence will not be presented here. In addition, for a result like Eq. (A3), it seems to us that numerical verification might be a simple way to convince oneself. We have done that for $m \leq 5 \times 10^4$, which should be sufficient for us because in our discussions m represents the order of a group element.

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