Addendum to: Conformal invariance and near-extreme rotating AdS black holes

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We obtained retarded Green's functions for massless scalar fields in the background of near-extreme, near-horizon rotating charged black hole of five-dimensional minimal gauged supergravity in Phys. Rev. D 84, 044018 (2011). For general nonextreme black holes, we also derived the radial part of the massless Klein-Gordon equation with zero azimuthal-angle eigenvalues, and showed that it is a general Heun's equation with a regular singularity at each horizon u_k ($k = 1, 2, 3$) and at infinity. We derived explicitly that the residuum of a pole at each u_k is associated with the surface gravity there. In this addendum, probing regular singularities at each u_k we complete the derivation of the full radial equation with nonzero azimuthal-angle eigenvalues. The residua now include modifications by the angular velocities at respective horizons. This result completes the analysis of the wave equation for the massless Klein-Gordon equation for the general rotating charged black hole of five-dimensional minimal gauged supergravity.

In [1], we studied the massless Klein-Gordon equation for the general nonextreme rotating black hole in minimal five-dimensional gauged supergravity, given by Chong et al. in [2]. The black hole is specified by the mass, two angular momenta, three equal charges and a cosmological constant, parametrized by m, a, b, q, and g.[1] We showed that the wave equation was separable, and obtained the general polar angle equation. As for the radial part, we obtained the explicit equation only for zero eigenvalues ($m_1 = m_2 = 0$) of the azimuthal angle coordinates ϕ and ψ in the scalar field Ansatz

$$\Phi = e^{-i\omega t + im_1\phi + im_2\psi} R(u)F(y). \qquad (1)$$

It was shown in the original paper that for $m_1 = m_2 = 0$ the radial part of the Klein-Gordon equation could be cast in the form

$$\frac{d}{du}\left(\Delta_u \frac{dR}{du}\right) + \frac{1}{4}\left\{\left[\frac{n_1}{\kappa_1^2(u-u_1)} + \frac{n_2}{\kappa_2^2(u-u_2)}\right.\right.$$
$$\left.\left. + \frac{n_3}{\kappa_3^2(u-u_3)} + Gn_4\right]\omega^2 - c_0\right\}R = 0, \qquad (2)$$

where κ_k ($k = 1, 2, 3$) are the surface gravities associated with the three horizons u_k ($k = 1, 2, 3$), which are the solutions of the horizon equation that can be written in the form: $\Delta_u = G(u-u_1)(u-u_2)(u-u_3)$. It should be remembered that we have changed the radial coordinate as $u = r^2$. $G \equiv g^2$ is related to the cosmological constant $\Lambda = -6G$, c_0 is the separation constant and n_k are the constants given in the original article as

$$n_1 = G(u_1 - u_2)(u_1 - u_3), \qquad (3)$$

$$n_2 = -G(u_1 - u_2)(u_2 - u_3), \qquad (4)$$

$$n_3 = G(u_1 - u_3)(u_2 - u_3), \qquad (5)$$

$$n_4 = \frac{1}{G^2}. \qquad (6)$$

It turned out to be difficult to work with the full radial equation with nonzero azimuthal-angle eigenvalues ($\{m_1, m_2\} \neq 0$), in particular to extract the explicit structure of the residuum of a pole at each u_k ($k = 1, 2, 3$) in terms of the surface gravity κ_k and angular velocity Ω_k which are of the form

$$\kappa_1 = \frac{G(u_1 - u_3)(u_1 - u_2)\sqrt{u_1}}{(u_1 + a^2)(u_1 + b^2) + abq}, \qquad (7)$$

$$\kappa_2 = \frac{G(u_2 - u_3)(u_1 - u_2)\sqrt{u_2}}{(u_2 + a^2)(u_2 + b^2) + abq}, \qquad (8)$$

$$\kappa_3 = \frac{G(u_2 - u_3)(u_1 - u_3)\sqrt{u_3}}{(u_3 + a^2)(u_3 + b^2) + abq}, \qquad (9)$$

and

$$\Omega_{ak} = \frac{(1 - a^2G)[a(u_k + b^2) + bq]}{(u_k + a^2)(u_k + b^2) + abq}, \qquad (10)$$

$$\Omega_{bk} = \frac{(1 - b^2G)[b(u_k + a^2) + aq]}{(u_k + a^2)(u_k + b^2) + abq}. \qquad (11)$$

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[1]The reader should refer to [1] for the explicit form of the metric and its parametrization.

In this addendum, we will seek a form of the general radial equation, when $\{m_1, m_2\} \neq 0$, as

$$\frac{d}{du}\left(\Delta_u \frac{dR}{du}\right) + \left(\frac{n_1 \alpha_1^2}{u - u_1} + \frac{n_2 \alpha_2^2}{u - u_2} + \frac{n_3 \alpha_3^2}{u - u_3} + \alpha_4\right) R = 0, \tag{12}$$

where α_k ($k = 1, 2, 3$) will now include the surface gravities and the angular velocities for the associated horizons.

Let us probe the radial equation for the regular singular points. We need to consider the radial equation as an equation in the complex r-plane. Then the solutions will have branch cuts at the regular singular points which can be seen by performing a series solution around these points. The solutions will describe outgoing and ingoing waves at the associated horizon. For a general equation in the form

$$[\partial_r \Delta(r) \partial_r - V(r)] R(r) = 0, \tag{13}$$

we have

$$R_{r_*}^{\text{out}} = (r - r_*)^{i\alpha_*}[1 + \mathcal{O}(r - r_*)], \tag{14}$$

$$R_{r_*}^{\text{in}} = (r - r_*)^{-i\alpha_*}[1 + \mathcal{O}(r - r_*)], \tag{15}$$

for the regular singularity at $r = r_*$ other than infinity. It turns out that the exponent in this expansion has a special form, e.g., for the Kerr case one obtains

$$\alpha_* = \frac{\omega - \Omega_* m}{2\kappa_*}, \tag{16}$$

which is needed for the formal wave equation analysis [2]. This observation is also key to our study of α_* as it is also the exponent that is used to form the monodromy matrix associated with that singularity. The details and applications of this method can be found in [3,4].[2]

We need only to probe for the horizon singularities, namely $u_1, u_2,$ and u_3 for our analysis.

For the regular singular points u_k we will have,

$$R_{u_k}^{\text{out}} = (u - u_k)^{i\alpha_k}[1 + \mathcal{O}(u - u_k)], \tag{17}$$

$$R_{u_k}^{\text{in}} = (u - u_k)^{-i\alpha_k}[1 + \mathcal{O}(u - u_k)]. \tag{18}$$

Using this expansion in the radial equation, and after some algebra we find α_k as

$$\alpha_k = \pm\left(\frac{\omega}{2\kappa_k} - \frac{\Omega_{ak}}{2\kappa_k} m_1 - \frac{\Omega_{bk}}{2\kappa_k} m_2\right). \tag{19}$$

Here, κ_k's are the surface gravities (7)–(9) and Ω_{ak} and Ω_{bk} are the two angular velocities (10),(11) [1,2]. Probing for a single regular singularity makes it possible to determine the structure due to the associated angular velocities, once one has the surface gravities.

Using the original radial equation we can read off the n_k constants and the α_4 term in the full equation as

$$n_1 = G(u_1 - u_2)(u_1 - u_3), \tag{20}$$

$$n_2 = -G(u_1 - u_2)(u_2 - u_3), \tag{21}$$

$$n_3 = G(u_1 - u_3)(u_2 - u_3), \tag{22}$$

and

$$\alpha_4 = \frac{1}{4}\left(\frac{\omega^2}{G} - c_0\right), \tag{23}$$

which are the same in the equation without the angular velocities.

In conclusion, we have obtained the full radial equation with explicit contributions from surface gravities and angular velocities by probing the regular singular points (associated with the three horizons of the black hole) for the monodromy exponents with the method, briefly described above. This result completes the analysis of the full wave equation, initiated in [1].

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[2]This result also applies [5,6] to general rotating multi-charged black holes in maximally supersymmetric ungauged supergravities [7,8].

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