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Instantons on sine-cones over Sasakian manifolds

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We investigate instantons on sine-cones over Sasaki-Einstein and 3-Sasakian manifolds. It is shown that these conical Einstein manifolds are Kähler with torsion (KT) manifolds admitting Hermitian connections with totally antisymmetric torsion. Furthermore, a deformation of the metric on the sine-cone over 3-Sasakian manifolds allows one to introduce a hyper-Kähler with torsion (HKT) structure. In the large-volume limit these KT and HKT spaces become Calabi-Yau and hyper-Kähler conifolds, respectively. We construct gauge connections on complex vector bundles over conical KT and HKT manifolds which solve the instanton equations for Yang-Mills fields in higher dimensions.

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I. INTRODUCTION

Kähler geometry is by now an established mathematical field with strong interrelations to theoretical physics, especially to supersymmetry and string theory. Another important geometry used in string theories and M-theory is Sasakian geometry (see [1–3] for reviews and references), especially in the AdS/CFT correspondence in the physically interesting dimensions five and seven. Here, we shall focus on Sasaki-Einstein manifolds and on 3-Sasakian manifolds, which are of dimension $2n+1$ and $4n+3$, respectively.

On a manifold M of real dimension $2n+1$, Sasakian geometry is sandwiched between Kählerian geometries on particular manifolds of the two neighboring dimensions. In particular, the metric cone over any Sasakian manifold is Kähler, and over any 3-Sasakian manifold it is hyper-Kähler. If the metric on a Sasakian manifold is Einstein, then the metric cone over it is a Calabi-Yau manifold. As examples, some well-known homogeneous and some recently discovered inhomogeneous Sasakian spaces often occur in string compactifications [4–6]. These manifolds admit a connection with nonvanishing torsion and a structure group of $\mathrm{SU}(n)\subset\mathrm{SO}(2n+1)$ or $\mathrm{Sp}(n)\subset\mathrm{SO}(4n+3)$, respectively. There exist brane solutions of ten-dimensional supergravity which interpolate between an $\mathrm{Ad}S_{p+1}\times X_{9-p}$ near-horizon geometry and an asymptotic geometry $M_p\times C(X_{9-p})$, where M_p is p-dimensional Minkowski space and $C(X_{9-p})$ is a metric cone over X_{9-p} (see e.g. [7,8] and references therein). Such kinds of brane solutions in heterotic supergravity with Yang-Mills instantons on the metric cones $C(X_{9-p})$ were considered in [9,10]. We intend to generalize them by considering sine-cones with a Kähler-torsion structure instead of metric cones with a Kähler structure.

In this paper we will consider sine-cones over Sasaki-Einstein and 3-Sasakian manifolds and solve instanton equations for Yang-Mills fields on these conical manifolds. Recall that for any Riemannian metric g on a manifold M, the warped product metric $\tilde{g}$ on $C(M)=\mathbb{R}_+\times M$ is defined as

$$\tilde{g}=\mathrm{d}r^2+f^2(r)g, \tag{1.1}$$

where $r\in\mathbb{R}_+$ and $f(r)$ is a *warping* function. $(C(M),\tilde{g})$ is called the *metric cone* over M if $f(r)=r$, and it is known as the *sine-cone* over M if $f(r)=\sin r$. It is known that the sine-cone metric $\tilde{g}$ over a Sasaki-Einstein or 3-Sasakian manifold is Einstein [2].[1] We show that these conical manifolds admit Kähler with torsion (KT) structures, i.e. on them there exists a Hermitian connection with a totally antisymmetric torsion $\tilde{T}=J\mathrm{d}\tilde{\omega}$. Here, J is an almost complex structure, and $\tilde{\omega}(\cdot,\cdot)=\tilde{g}(J\cdot,\cdot)$ is the fundamental 2-form. For $\tilde{T}=0$ the Hermitian structure is Kähler.

We also show that for any 3-Sasakian manifold one can deform the sine-cone metric $\tilde{g}$ in such a way that $C(M)$ will be a hyper-Kähler with torsion (HKT) manifold. HKT geometry has been described in detail e.g. in [11] and intensively studied since then (see e.g. [12–15] and references therein). In fact, these Hermitian manifolds with three integrable almost complex structures J^α, $\alpha=1,2,3$, are not hyper-Kähler for nonvanishing torsion

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[1] The corresponding metric-cone metric is even Ricci-flat.

$\tilde{T} = J^1 \mathrm{d}\tilde{\omega}^1 = J^2 \mathrm{d}\tilde{\omega}^2 = J^3 \mathrm{d}\tilde{\omega}^3$, but we will follow the established terminology. Here, $\tilde{\omega}^\alpha(\cdot,\cdot) = \tilde{g}(J^\alpha\cdot,\cdot)$ are three Hermitian 2-forms.

After introducing the KT and HKT structures on the sine-cones over Sasaki-Einstein and 3-Sasakian manifolds, we consider the appropriate instanton equations on these conical manifolds. Such first-order gauge equations, which generalize the Yang-Mills anti-self-duality equations in $d = 4$ to higher-dimensional manifolds with special holonomy (or, more generally, G-structure), were introduced and studied both in the physical [16–18] and mathematical [19–21] literature. Some instanton solutions were found e.g. in [22–28]. Due to the foliated structure of the considered conical KT and HKT manifolds, a natural ansatz for the gauge fields reduces the instanton equations to matrix-model equations. We discuss their simplest analytic and numerical solutions.

This article is arranged as follows. In Sec. II we collect various geometric facts concerning Sasaki-Einstein, 3-Sasakian, Kähler-torsion and hyper-Kähler with torsion manifolds. We introduce the KT and HKT structures on sine-cones over Sasaki-Einstein and 3-Sasakian manifolds. In Sec. III we discuss the instanton equations in more than four dimensions and specialize them for KT and HKT manifolds. Then we describe an ansatz reducing these instanton equations to matrix equations and present some solutions.

II. HERMITIAN MANIFOLDS WITH TORSION

A. KT manifolds

A G-structure on a smooth orientable manifold M of dimension m is a reduction of the structure group $\mathrm{GL}(m,\mathbb{R})$ of the tangent bundle TM to a closed subgroup $G \subset \mathrm{GL}(m,\mathbb{R})$. Choosing an orientation and a Riemannian metric g defines an $\mathrm{SO}(m)$-structure on M. We assume that $m = 2n$ is even and that (M,g) is an almost Hermitian manifold. This means that there exists an almost complex structure $J \in \mathrm{End}(TM)$, with $J^2 = -\mathbf{1}_{\mathrm{TM}}$, which is compatible with the metric g, i.e. $g(JX,JY) = g(X,Y)$ for all $X, Y \in TM$. One can introduce the fundamental two-form ω as

$$\omega(X,Y) := g(JX,Y) \quad \text{for } X,Y \in TM, \tag{2.1}$$

and the canonical objects (g,J,ω) define a $\mathrm{U}(n)$-structure on M. The additional existence of a complex n-form Ω reduces the $\mathrm{U}(n)$- to an $\mathrm{SU}(n)$-structure. It implies that the almost Hermitian manifold (M,g,J) has a topologically trivial canonical bundle. If the almost complex structure J is integrable, then (M,g,J) is a Hermitian manifold and (M,g,J,Ω) is Calabi-Yau.

On any almost Hermitian manifold (M,g,J) one has a Hermitian connection[2] with totally antisymmetric torsion T. Two particularly interesting cases arise when the torsion T is the real part of either a (3,0)-form or a (2,1)-form. In the former case, the manifold M is called nearly Kähler [29,30]. The latter case entails a property called Kähler-torsion (KT) [31,32]. Namely, on a Hermitian manifold (M,g,J) the KT connection is a Hermitian connection Γ with antisymmetric torsion T given by

$$T = J\mathrm{d}\omega, \tag{2.2}$$

where J acts on the p-form $e^{a_1} \wedge \cdots \wedge e^{a_p}$ as

$$J(e^{a_1} \wedge e^{a_2} \wedge \cdots \wedge e^{a_p}) = Je^{a_1} \wedge Je^{a_2} \wedge \cdots \wedge Je^{a_p} \quad \text{with} \quad Je^a = J^a_b e^b. \tag{2.3}$$

Here $\{e^a\}$ with $a = 1,\dots,2n$ is a local frame for the cotangent bundle T^*M, and J^a_b are the corresponding components of the almost complex structure J. Note that J is integrable in the KT case and nonintegrable in the nearly Kähler case. For traceless anti-Hermitian Γ the KT manifolds are called Calabi-Yau torsion [33].

B. Remark

Let

$$\theta^j := e^{2j-1} + \mathrm{i}e^{2j} \quad \text{and} \quad \theta^{\bar{j}} := \overline{\theta^j} \quad \text{with} \quad j = 1,\dots,n \tag{2.4}$$

constitute a local frame for the (1,0) and (0,1) parts of the complexified cotangent bundle. Then J, g and ω may be chosen as follows:

$$J\theta^j = \mathrm{i}\theta^j, \qquad g = \sum_{j=1}^n \theta^j \otimes \theta^{\bar{j}} \quad \text{and}$$
$$\omega = \frac{\mathrm{i}}{2}\sum_{j=1}^n \theta^j \wedge \theta^{\bar{j}}. \tag{2.5}$$

Their nonvanishing components are thus given by

$$J_k{}^j = \mathrm{i}\delta_k{}^j, \qquad J^{\bar{j}}_{\bar{k}} = -\mathrm{i}\delta^{\bar{j}}{}_{\bar{k}},$$
$$g_{j\bar{k}} = \frac{1}{2}\delta_{j\bar{k}} \quad \text{and} \quad \omega_{j\bar{k}} = \frac{\mathrm{i}}{2}\delta_{j\bar{k}} \tag{2.6}$$

with respect to the θ-basis. For g^{-1} and ω^{-1} we have

$$g^{j\bar{k}} = 2\delta^{j\bar{k}} \quad \text{and} \quad \omega^{j\bar{k}} = -2\mathrm{i}\delta^{j\bar{k}}. \tag{2.7}$$

C. Cones

For any Riemannian manifold $(\mathcal{M},g)$ we define

$$C_\Lambda(\mathcal{M}) = ((0,\Lambda\pi) \times \mathcal{M}, \tilde{g}) \quad \text{with}$$
$$\tilde{g} = \mathrm{d}r^2 + \Lambda^2 \sin^2\left(\frac{r}{\Lambda}\right) g \tag{2.8}$$

[2] This connection $\nabla = \mathrm{d} + \Gamma$ preserves g, J and ω.

to be the *sine-cone* over $\mathcal{M}$. Here $r \in (0, \Lambda\pi)$ which is an open interval, and so the volume of the sine-cone is $\Lambda \mathrm{vol}\mathcal{M}$. In the infinite-volume limit $\Lambda \to \infty$ the sine-cone becomes the *metric cone*[3]

$$C_\infty(\mathcal{M}) \equiv C(\mathcal{M}) = (\mathbb{R}_+ \times \mathcal{M}, \tilde{g}) \quad \text{with} \quad \tilde{g} = \mathrm{d}r^2 + r^2 g. \tag{2.9}$$

Note that

$$\tilde{g} = \mathrm{d}r^2 + \Lambda^2 \sin^2\left(\frac{r}{\Lambda}\right) g =: \Lambda^2 \sin^2\varphi (\mathrm{d}\tau^2 + g), \tag{2.10}$$

where

$$\varphi = \frac{r}{\Lambda} \quad \text{and} \quad \tau = \log\left(2\frac{\Lambda}{r_0}\tan\frac{\varphi}{2}\right) \Leftrightarrow r = 2\Lambda \arctan\left(\frac{r_0 \mathrm{e}^\tau}{2\Lambda}\right) \tag{2.11}$$

with $\tau \in \mathbb{R}$ and a constant $r_0 \in \mathbb{R}_+$. In the limit $\Lambda \to \infty$, (2.11) simplifies to

$$\tau = \log\left(\frac{r}{r_0}\right) \Leftrightarrow r = r_0 \mathrm{e}^\tau \tag{2.12}$$

which are valid for the metric cone with

$$\tilde{g} = \mathrm{d}r^2 + r^2 g = r^2\left(\frac{\mathrm{d}r^2}{r^2} + g\right) = r_0^2 \mathrm{e}^{2\tau}(\mathrm{d}\tau^2 + g). \tag{2.13}$$

It follows from (2.12) and (2.13) that both cones are conformally equivalent to the cylinder

$$\mathrm{Cyl}(\mathcal{M}) = (\mathbb{R} \times \mathcal{M}, g_{\mathrm{cyl}}) \quad \text{with} \quad g_{\mathrm{cyl}} = \mathrm{d}\tau^2 + g. \tag{2.14}$$

D. Sasaki-Einstein manifolds

A $(2n+1)$-dimensional Riemannian manifold $(\mathcal{M}, g)$ is called *Sasakian* if the metric cone $(C(\mathcal{M}), \tilde{g})$ is Kähler.[4] A Sasakian manifold $(\mathcal{M}, g)$ is *Sasaki-Einstein* if in addition the metric g is Einstein. In this case the metric cone $C(\mathcal{M})$ is a Ricci-flat Kähler manifold (Calabi-Yau), so its holonomy group is reduced from $\mathrm{U}(n)$ to $\mathrm{SU}(n)$. Sasaki-Einstein manifolds have a reduced structure group of $\mathrm{SU}(n) \subset \mathrm{SO}(2n+1)$. They are endowed with 1-, 2-, 3- and 4-forms η, ω, P and Q, which can be defined in a local orthonormal basis $\{e^{\hat{a}}\}$, $\hat{a} = 1, \ldots, 2n+1$, as

$$\eta = -e^{2n+1}, \qquad \omega = \sum_{j=1}^{n} e^{2j-1} \wedge e^{2j},$$
$$P = \eta \wedge \omega \quad \text{and} \quad Q = \frac{1}{2}\omega \wedge \omega. \tag{2.15}$$

These forms satisfy the relations

$$\mathrm{d}e^{2n+1} = -2\omega \quad \text{and} \quad \mathrm{d}P = 4Q. \tag{2.16}$$

Note that, in the above basis, the torsion T of the canonical $su(n)$-valued connection Γ on $\mathcal{M}$ is not totally antisymmetric and has the components (see e.g. [9])

$$T^a = \frac{n+1}{2n} P_{a\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} \quad \text{and} \quad T^{2n+1} = P_{2n+1\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}, \tag{2.17}$$

where $\{\hat{a}\} = \{a, 2n+1\}$ with $a = 1, \ldots, 2n$.

E. 3-Sasakian manifolds

A $(4n+3)$-dimensional Riemannian manifold $(\mathcal{M}, g)$ is called *3-Sasakian* if the metric cone $(C(\mathcal{M}), \tilde{g})$ on $\mathcal{M}$ is hyper-Kähler. The structure group of $\mathcal{M}$ then is $\mathrm{Sp}(n) \subset \mathrm{SO}(4n+3)$, and we let the index a run from 1 to $4n$. Note that any 3-Sasakian manifold is Einstein and can be endowed with three 1-forms η^α, three 2-forms ω^α, a 3-form P and a 4-form Q with $\alpha = 1, 2, 3$ [1,2]. In a local orthonormal co-frame $\{e^{\hat{a}}\}$ where $\{\hat{a}\} = \{a, 4n+\alpha\}$, these forms can be written as

$$\eta^\alpha = -e^{4n+\alpha}, \tag{2.18}$$

$$\omega^1 = \sum_{j=1}^{n} (e^{4j-3} \wedge e^{4j} + e^{4j-2} \wedge e^{4j-1}), \tag{2.19}$$

$$\omega^2 = \sum_{j=1}^{n} (-e^{4j-3} \wedge e^{4j-1} + e^{4j-2} \wedge e^{4j}), \tag{2.20}$$

$$\omega^3 = \sum_{j=1}^{n} (e^{4j-3} \wedge e^{4j-2} + e^{4j-1} \wedge e^{4j}), \tag{2.21}$$

$$P = \frac{1}{3}\left(\sum_\alpha \eta^\alpha \wedge \omega^\alpha + \eta^{123}\right)$$
$$= -\frac{1}{3}\left(\sum_\alpha \omega^\alpha \wedge e^{4n+\alpha} + e^{4n+1} \wedge e^{4n+2} \wedge e^{4n+3}\right), \tag{2.22}$$

$$Q = \frac{1}{6}\sum_\alpha \omega^\alpha \wedge \omega^\alpha. \tag{2.23}$$

The forms $e^{4n+\alpha}$ and ω^α satisfy the differential identities

[3]One usually omits the adjective "metric" and simply says "cone".
[4]This is one of several equivalent definitions of Sasakian manifolds [2].

$$de^{4n+\alpha} = -\varepsilon^{\alpha}_{\beta\gamma} e^{4n+\beta} \wedge e^{4n+\gamma} - 2\omega^{\alpha}, \tag{2.24}$$

$$d\omega^{\alpha} = -2\varepsilon^{\alpha}_{\beta\gamma} e^{4n+\beta} \wedge \omega^{\gamma}, \tag{2.25}$$

where ε is the Levi-Civita tensor. The torsion T of the canonical $sp(n)$-valued connection Γ on any 3-Sasakian manifold takes the form (see e.g. [9])

$$T^{a} = \frac{3}{2} P_{a\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} \quad \text{and} \quad T^{\alpha} = 3P_{4n+\alpha\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}. \tag{2.26}$$

T is not totally antisymmetric for the Einstein metric on $\mathcal{M}$.

F. KT structure on sine-cones

A well-known theorem [2] states that, if $(\mathcal{M}, g)$ is a k-dimensional Einstein manifold with Einstein constant $k-1$, then the sine-cone $(C(\mathcal{M}), \tilde{g})$ over $\mathcal{M}$ with the metric (2.10) for $\Lambda = 1$ is Einstein with Einstein constant k. Here we will show that the sine-cone over any Sasaki-Einstein manifold is not only Einstein but also carries a Kähler-torsion structure.

Consider the cylinder

$$\mathrm{Cyl}(\mathcal{M}) = (\mathbb{R} \times \mathcal{M}, g_{\mathrm{cyl}}) \quad \text{with}$$
$$g_{\mathrm{cyl}} = \delta_{\hat{a}\hat{b}} e^{\hat{a}} \otimes e^{\hat{b}} + e^{2n+2} \otimes e^{2n+2}, \tag{2.27}$$

where $\{\hat{a}\} = \{a, 2n+1\}$ with $a = 1, \ldots, 2n$, and compare it to the sine-cone

$$M^{2n+2} := C_{\Lambda}(\mathcal{M}) = ((0, \Lambda\pi) \times \mathcal{M}, \tilde{g}) \tag{2.28}$$

parametrized via

$$e^{2n+2} = d\tau = \frac{d\varphi}{\sin\varphi}, \quad \varphi = \frac{r}{\Lambda}, \quad \tau \in \mathbb{R}, \quad \varphi \in (0, \pi). \tag{2.29}$$

Then the local basis $\{\tilde{e}^{\hat{a}}, \tilde{e}^{2n+2}\}$ on the latter is defined as

$$\tilde{e}^{\hat{a}} = \Lambda \sin\varphi e^{\hat{a}} \quad \text{and} \quad \tilde{e}^{2n+2} = \Lambda \sin\varphi e^{2n+2} = dr, \tag{2.30}$$

and its metric reads

$$\tilde{g} = \delta_{\hat{a}\hat{b}} \tilde{e}^{\hat{a}} \otimes \tilde{e}^{\hat{b}} + \tilde{e}^{2n+2} \otimes \tilde{e}^{2n+2}. \tag{2.31}$$

Let us also introduce the 2-form

$$\tilde{\omega} := \Lambda^2 \sin^2\varphi (\omega + e^{2n+1} \wedge e^{2n+2}), \tag{2.32}$$

where ω is the 2-form defined in (2.15) and obeying (2.16). It is easy to check that

$$d\tilde{\omega} = \frac{2}{\Lambda} \frac{\cos\varphi - 1}{\sin\varphi} \tilde{\omega} \wedge \tilde{e}^{2n+2} = -\frac{2}{\Lambda} \tan\frac{\varphi}{2} \tilde{\omega} \wedge \tilde{e}^{2n+2}. \tag{2.33}$$

The canonical almost complex structure J on M^{2n+2} is fixed by $\tilde{g}$ and $\tilde{\omega}$ via (2.4)–(2.6) but with the range of j extended to $n+1$.

Finally one can define the torsion

$$\tilde{T} := J d\tilde{\omega} = -\frac{2}{\Lambda} \tan\frac{\varphi}{2} \tilde{\omega} \wedge \tilde{e}^{2n+1} \tag{2.34}$$

which is proportional to P from (2.15). Since $\tilde{T}$ is of type $(2,1) + (1,2)$ with respect to J, the almost complex structure J is integrable, and we obtain a KT structure on the sine-cone (2.28) over any Sasaki-Einstein manifold.

G. Conical HKT structure

The sine-cone

$$M^{4n+4} = C_{\Lambda}(\mathcal{M}) = ((0, \Lambda\pi) \times \mathcal{M}, \tilde{g}) \tag{2.35}$$

over a $(4n+3)$-dimensional 3-Sasakian manifold $(\mathcal{M}, g)$ with a metric (2.10) is an Einstein manifold since $(\mathcal{M}, g)$ is Einstein. Since $(\mathcal{M}, e^{4n+3}, \omega^3 + e^{4n+1} \wedge e^{4n+2}, g)$ is Sasaki-Einstein, the previous subsection applies, and one can introduce a KT structure on M^{4n+4} by choosing the 2-form

$$\tilde{\omega} := \Lambda^2 \sin^2\varphi (\omega^3 + e^{4n+1} \wedge e^{4n+2} + e^{4n+3} \wedge e^{4n+4}) \tag{2.36}$$

as in (2.32)–(2.34), where ω^3 is defined by (2.21) and

$$e^{4n+4} = d\tau, \qquad \tau = \log\left(2\frac{\Lambda}{r_0} \tan\frac{\varphi}{2}\right), \qquad \varphi = \frac{r}{\Lambda}. \tag{2.37}$$

Here $\{e^{\hat{a}}, e^{4n+4}\} = \{e^{a}, e^{4n+\alpha}, e^{4n+4}\}$ with $a = 1, \ldots, 4n$ is a local basis of one-forms on the cylinder $\mathcal{M} \times \mathbb{R}$ with the metric

$$g_{\mathrm{cyl}} = \delta_{ab} e^{a} \otimes e^{b} + \delta_{\mu\nu} e^{4n+\mu} \otimes e^{4n+\nu}, \tag{2.38}$$

where we introduced the index set $\{\mu\} = \{\alpha, 4\}$. Recall that $e^{\hat{a}}$ and ω^{α}, defined in (2.18)–(2.21), satisfy the identities (2.24) and (2.25).

We have

$$\omega^{\alpha} = \frac{1}{2} \omega^{\alpha}_{ab} e^{a} \wedge e^{b} \quad \text{for } a = 1, \ldots, 4n, \tag{2.39}$$

where the components ω^{α}_{ab} of the 2-forms ω^{α} can be read off from (2.19)–(2.21). For later use we define three more 2-forms,

$$\omega^{\alpha}_{\perp} := \frac{1}{2} \eta^{\alpha}_{\mu\nu} e^{4n+\mu} \wedge e^{4n+\nu}, \tag{2.40}$$

where $\eta^\alpha_{\mu\nu}$ are the components of the 't Hooft tensor,

$$\eta^\alpha_{\beta\gamma} = \varepsilon^\alpha_{\beta\gamma} \quad \text{and} \quad \eta^\alpha_{\beta 4} = -\eta^\alpha_{4\beta} = \delta^\alpha_\beta. \tag{2.41}$$

Using (2.39)–(2.41), we may introduce on M^{4n+4} three almost complex structures J^α with components

$$J^{\alpha a}_{b} = \omega^\alpha_{bc}\delta^{ca} \quad \text{and} \quad J^{\alpha 4n+\mu}_{4n+\nu} = \eta^\alpha_{\nu\sigma}\delta^{\sigma\mu}. \tag{2.42}$$

It is not difficult to show that

$$J^\alpha J^\beta = -\delta^{\alpha\beta}\mathrm{id} + \varepsilon^{\alpha\beta\gamma}J^\gamma, \tag{2.43}$$

i.e. the three almost complex structures define a quaternionic structure on M^{4n+4}.

The 2-form $\tilde{\omega}$ in (2.36) is of type (1,1) with respect to J^3. Differentiating, we find

$$\tilde{T} = J^3\mathrm{d}\tilde{\omega} = -\frac{2}{\Lambda}\tan\frac{\varphi}{2}\tilde{\omega}\wedge\tilde{e}^{4n+3}, \tag{2.44}$$

where we use local coframe fields

$$\tilde{e}^a = \Lambda\sin\varphi e^a \quad \text{and} \quad \tilde{e}^{4n+\mu} = \Lambda\sin\varphi e^{4n+\mu} \tag{2.45}$$

on $M^{4n+4} = C_\Lambda(\mathcal{M})$. Thus, M^{4n+4} allows for a KT structure.

In order to extend this to a HKT structure on M^{4n+4}, we need three 2-forms

$$\tilde{\omega}^\alpha = \Lambda^2\sin^2\varphi(f_1\omega^\alpha + f_2\omega^\alpha_\perp), \tag{2.46}$$

where $f_1 = f_1(\varphi)$ and $f_2 = f_2(\varphi)$ are yet undefined real functions, and ω^α and $\omega^\alpha_\perp$ are given by (2.39) and (2.40), respectively. Taking the exterior derivative of (2.46), we obtain

$$\mathrm{d}\tilde{\omega}^\alpha = \Lambda^2\sin^2\varphi\Big\{\Big(B_1\omega^\alpha + \frac{1}{2}B_2\varepsilon^\alpha_{\beta\gamma}e^{4n+\beta}\wedge e^{4n+\gamma}\Big)\wedge e^{4n+4} + B_3\varepsilon^\alpha_{\beta\gamma}e^{4n+\beta}\wedge\omega^\gamma\Big\}, \tag{2.47}$$

where

$$B_1 = \dot{f}_1\sin\varphi + 2f_1\cos\varphi - 2f_2, \tag{2.48}$$

$$B_2 = \dot{f}_2\sin\varphi + 2f_2\cos\varphi - 2f_2, \tag{2.49}$$

$$B_3 = -2(f_1 - f_2), \tag{2.50}$$

and the overdot indicates a derivative with respect to φ. The definitions of J^α imply that

$$J^\alpha\omega^\beta = (-1)^{1+\delta_{\alpha\beta}}\omega^\beta \quad \text{and} \tag{2.51}$$

$$\begin{aligned} &J^1e^{4n+1} = -e^{4n+4},\quad J^2e^{4n+1} = e^{4n+3},\quad J^3e^{4n+1} = -e^{4n+2},\\ &J^1e^{4n+2} = -e^{4n+3},\quad J^2e^{4n+2} = -e^{4n+4},\quad J^3e^{4n+2} = e^{4n+1},\\ &J^1e^{4n+3} = e^{4n+2},\quad J^2e^{4n+3} = -e^{4n+1},\quad J^3e^{4n+3} = -e^{4n+4},\\ &J^\alpha e^{4n+4} = e^{4n+\alpha}. \end{aligned} \tag{2.52}$$

In [13] it has been proven that the HKT condition is equivalent to

$$J^1\mathrm{d}\tilde{\omega}^1 = J^2\mathrm{d}\tilde{\omega}^2 = J^3\mathrm{d}\tilde{\omega}^3 = \tilde{T}, \tag{2.53}$$

where $\tilde{T}$ is the torsion of the $sp(n+1)$-valued hyper-Hermitian connection on M^{4n+4}. Using (2.51) and (2.52) as well as demanding that $\tilde{T}$ is proportional to P from (2.22), we obtain from (2.53) the constraints

$$B_1 = B_2 = B_3 \tag{2.54}$$

which are equivalent to the differential equations

$$\begin{aligned} &\dot{f}\sin\varphi + 2f\cos\varphi = 0 \quad \text{and}\\ &\dot{f}_2\sin\varphi + 2f_2(\cos\varphi - 1) + 2f = 0 \quad \text{for } f \coloneqq f_1 - f_2. \end{aligned} \tag{2.55}$$

Solutions of these equations can be chosen in the form

$$f = \frac{c}{\sin^2\varphi} \quad \text{and} \quad f_2 = \frac{2c_2}{\cos^4\frac{\varphi}{2}} + \frac{c}{\sin^2\varphi}$$
$$\Rightarrow f_1 = \frac{2c_2}{\cos^4\frac{\varphi}{2}} + \frac{2c}{\sin^2\varphi}, \tag{2.56}$$

where c and c_2 are yet arbitrary constants of integration. In the limit $\Lambda\to\infty$ we would like our HKT space M^{4n+4} to coincide with the standard hyper-Kähler metric cone $C(\mathcal{M})$ with vanishing torsion $\tilde{T} = 0$. This is achieved for

$$c = \frac{c_1}{\Lambda^3}, \tag{2.57}$$

where c_1 is constant. Then for $\Lambda\to\infty$ we get

$$\tilde{\omega}^\alpha \to 2c_2r^2(\omega^\alpha + \omega^\alpha_\perp) \eqqcolon \hat{\omega}^\alpha \quad \text{with} \quad \mathrm{d}\hat{\omega}^\alpha = 0. \tag{2.58}$$

The metric on the HKT manifold M^{4n+4} with three Hermitian[5] structures (2.46) takes the form

$$\begin{aligned} \tilde{g} &= f_1\delta_{ab}\tilde{e}^a\otimes\tilde{e}^b + f_2\delta_{\mu\nu}\tilde{e}^\mu\otimes\tilde{e}^\nu\\ &= f_2\Big(\frac{f_1}{f_2}\delta_{ab}\tilde{e}^a\otimes\tilde{e}^b + \delta_{\mu\nu}\tilde{e}^\mu\otimes\tilde{e}^\nu\Big). \end{aligned} \tag{2.59}$$

[5]Note that the conditions (2.53) imply the integrability of the almost complex structures (2.42).

It is conformally equivalent, with conformal factor f_2, to the metric on the sine-cone $C_\Lambda(\mathcal{M})$ over a 3-Sasakian manifold $\mathcal{M}$. The simplest case occurs for the choice $c_1 = 0$ when $f_1 = f_2$.

III. INSTANTONS ON CONICAL KT AND HKT MANIFOLDS

A. Instanton equations

Let Σ be a differential form of degree $m-4$ on an m-dimensional Riemannian manifold M, and let $\mathcal{E}$ be a complex vector bundle over M endowed with a connection $\mathcal{A}$. The Σ-anti-self-duality (or instanton) equations are defined as the first-order equations [21]

$$*\mathcal{F} = -\Sigma \wedge \mathcal{F} \tag{3.1}$$

for the connection $\mathcal{A}$ with curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}$. Here $*$ is the Hodge duality operator on M. Taking the exterior derivative of (3.1) and using the Bianchi identity, we obtain

$$\mathrm{d} * \mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - (-1)^m * \mathcal{F} \wedge \mathcal{A} + *\mathcal{H} \wedge \mathcal{F} = 0, \tag{3.2}$$

where the 3-form $\mathcal{H}$ is defined by

$$*\mathcal{H} \coloneqq \mathrm{d}\Sigma. \tag{3.3}$$

The second-order equations (3.2) differ from the standard Yang-Mills equations by the last term involving a 3-form $\mathcal{H}$, which can be identified with a totally antisymmetric torsion on M. This torsion term naturally appears in string-theory compactifications with fluxes [34]. For $\mathrm{d}\Sigma = 0$, the torsion term vanishes and the instanton equations (3.1) imply the ordinary Yang-Mills equations. The latter also holds true when the instanton solution $\mathcal{F}$ satisfies $\mathrm{d}\Sigma \wedge \mathcal{F} = 0$ as well, as e.g. on nearly Kähler 6-manifolds, nearly parallel G_2-manifolds and Sasakian manifolds [9].

The torsionful Yang-Mills equations (3.2) are the variational equations for the action

$$S = -\int_M \mathrm{Tr}(\mathcal{F} \wedge *\mathcal{F} + \mathcal{F} \wedge \mathcal{F} \wedge \Sigma), \tag{3.4}$$

and the instanton equations (3.1) can be derived from this action using a Bogomol'nyi argument. In the case of a closed form Σ, the second term in (3.4) is topological and the torsion (3.3) disappears from (3.2).

If a manifold is endowed with a 4-form Q, a natural choice for the $(m-4)$-form Σ in the instanton equations (3.1) will be its Hodge dual, $\Sigma \propto Q$. Therefore, on KT manifolds with $m = 2n+2$ one should take

$$\tilde{Q}_{\mathrm{KT}} = \frac{1}{2}\tilde{\omega} \wedge \tilde{\omega} \quad \text{in} \quad *\mathcal{F} = -*\tilde{Q}_{\mathrm{KT}} \wedge \mathcal{F}. \tag{3.5}$$

On HKT manifolds there exist three 2-forms $\tilde{\omega}^\alpha$, from which one can build the 4-form

$$\tilde{Q}_{\mathrm{HKT}} = \frac{1}{6}\tilde{\omega}^\alpha \wedge \tilde{\omega}^\alpha \quad \text{in} \quad *\mathcal{F} = -*\tilde{Q}_{\mathrm{HKT}} \wedge \mathcal{F}. \tag{3.6}$$

B. Reduction to matrix equations

Recall that the instanton equations on the cone $C(\mathcal{M})$ over Sasaki-Einstein or 3-Sasakian manifolds $\mathcal{M}$ are equivalent to the equations on the cylinder $\mathrm{Cyl}(\mathcal{M})$ [9] with the metric

$$\tilde{g} = \mathrm{d}\tau^2 + g, \tag{3.7}$$

where g is the metric on $\mathcal{M}$ and τ is related with r by (2.11). Let us denote by G and H the structure groups of the canonical connection on $C(\mathcal{M})$ and $\mathcal{M}$, respectively. In the KT case we have $(G,H) = (\mathrm{SU}(n+1), \mathrm{SU}(n))$, and in the HKT case $(G,H) = (\mathrm{Sp}(n+1), \mathrm{Sp}(n))$. As a vector space, the Lie algebra $\mathfrak{g} = \mathrm{Lie}G$ decomposes into $\mathfrak{h} = \mathrm{Lie}H$ and its orthogonal complement $\mathfrak{m}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}. \tag{3.8}$$

The vector space $\mathfrak{m}$ can be identified with the linear span of the orthonormal basis $\{e^{\hat{a}}\}$ on $T^*\mathcal{M}$.

In any given irreducible representation ρ of $\mathfrak{g}$, the generators I_i of $\mathfrak{h}$ and $I_{\hat{a}}$ of $\mathfrak{m}$ obeying the commutation relations

$$[I_i, I_j] = f_{ij}^k I_k, \qquad [I_i, I_{\hat{a}}] = f_{i\hat{a}}^{\hat{b}} I_{\hat{b}} \quad \text{and}$$
$$[I_{\hat{a}}, I_{\hat{b}}] = f_{\hat{a}\hat{b}}^i I_i + f_{\hat{a}\hat{b}}^{\hat{c}} I_{\hat{c}} \tag{3.9}$$

act on a representation space $V \cong \mathbb{C}^N$, i.e. $\rho\colon \mathfrak{g} \to \mathrm{End}(V)$. Consider a complex vector bundle $\mathcal{E} \to \mathrm{Cyl}(\mathcal{M})$ such that the fibers are copies of V. Since H is a closed subgroup of G, it also acts on the fibres of $\mathcal{E}$, but the restriction of our $\mathfrak{g}$-representation ρ to the subalgebra $\mathfrak{h}$ in general decomposes into a direct sum of several irreducible $\mathfrak{h}$-representations, with the corresponding invariant subspaces comprising V.

The canonical connection Γ on $T\mathcal{M}$ is always an instanton [9,10]. On the bundle $\mathcal{E}$, it induces the $\rho(\mathfrak{h})$-valued connection (denoted by the same letter)

$$\Gamma \coloneqq \Gamma^i I_i. \tag{3.10}$$

Its curvature

$$R = \mathrm{d}\Gamma + \Gamma \wedge \Gamma = \left(\mathrm{d}\Gamma^i + \frac{1}{2} f_{jk}^i \Gamma^j \wedge \Gamma^k\right) I_i \tag{3.11}$$

satisfies the instanton equations (3.1) [9,10].

Let us consider some matrix-valued functions $X_{\hat{a}}(\tau) \in \mathrm{End}\,(V)$ and introduce on $\mathcal{E}$ a $\rho(\mathfrak{g})$-valued connection

$$\mathcal{A} := \Gamma + X_{\hat{a}} e^{\hat{a}}. \tag{3.12}$$

For $X_{\hat{a}}$ depending on all coordinates of $\mathrm{Cyl}(\mathcal{M})$, this is the general form of a connection on the bundle $\mathcal{E} \to \mathrm{Cyl}(\mathcal{M})$. Below, we shall impose independence of $X_{\hat{a}}$ on the coordinates of $\mathcal{M}$ and certain equivariance conditions, which will reduce (3.1) to ordinary differential equations for $X_{\hat{a}}$.

Recall that

$$\mathrm{d}e^{\hat{a}} = -\Gamma^{\hat{a}}_{\hat{b}} \wedge e^{\hat{b}} + T^{\hat{a}} = -\Gamma^{i} f^{\hat{a}}_{i\hat{b}} \wedge e^{\hat{b}} + \frac{1}{2} T^{\hat{a}}_{\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}}, \tag{3.13}$$

where $f^{\hat{a}}_{i\hat{b}}$ are the structure constants from (3.9). From (3.12) and (3.13) it follows that

$$\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = R + \frac{1}{2}([X_{\hat{a}}, X_{\hat{b}}] + T^{\hat{c}}_{\hat{a}\hat{b}} X_{\hat{c}}) e^{\hat{a}} \wedge e^{\hat{b}} + \dot{X}_{\hat{a}} \mathrm{d}\tau \wedge e^{\hat{a}} + \Gamma^{i} \wedge e^{\hat{a}}([I_i, X_{\hat{a}}] - f^{\hat{b}}_{i\hat{a}} X_{\hat{b}}), \tag{3.14}$$

where R is given in (3.11) and $\dot{X}_{\hat{a}} \equiv \frac{\mathrm{d}}{\mathrm{d}\tau} X_{\hat{a}}$. In [27] it was shown that $\mathcal{F}$ solves the instanton equations (3.1) if the following matrix equations hold:

$$[I_i, X_{\hat{a}}] = f^{\hat{b}}_{i\hat{a}} X_{\hat{b}}, \tag{3.15}$$

$$[X_{\hat{a}}, X_{\hat{b}}] + T^{\hat{c}}_{\hat{a}\hat{b}} X_{\hat{c}} = N^{\hat{c}}_{\hat{a}\hat{b}} \dot{X}_{\hat{c}} + f^{i}_{\hat{a}\hat{b}} N_i(\tau). \tag{3.16}$$

Here $N^{\hat{c}}_{\hat{a}\hat{b}}$ is some constant tensor which we shall specify below for each case, and N_i are some $\rho(\mathfrak{h})$-valued functions defined by (3.16) after resolving the algebraic constraints (3.15) and substituting their solutions $X_{\hat{a}}$ into (3.16). For $X_{\hat{a}}$ satisfying (3.15) and (3.16), we have

$$\mathcal{F} = R + \frac{1}{2} N_i f^{i}_{\hat{a}\hat{b}} e^{\hat{a}} \wedge e^{\hat{b}} + \dot{X}_{\hat{a}} \left(\mathrm{d}\tau \wedge e^{\hat{a}} + \frac{1}{2} N^{\hat{a}}_{\hat{b}\hat{c}} e^{\hat{b}} \wedge e^{\hat{c}} \right), \tag{3.17}$$

where the term with $f^{i}_{\hat{a}\hat{b}}$ satisfies (3.1) all by itself (as does R) due to the properties of the coset G/H, and the term proportional to $\dot{X}_{\hat{a}}$ solves (3.1) after the proper choice of $N^{\hat{a}}_{\hat{b}\hat{c}}$ to be specified below.

C. Instantons on conical KT manifolds

Consider the sine-cone $M^{2n+2} = C_\Lambda(\mathcal{M})$ over a Sasaki-Einstein manifold $\mathcal{M}$ of dimension $2n+1$. The geometry of $\mathcal{M}$ and M^{2n+2} has been discussed in Sec. II. In this case, we have $\{\hat{a}\} = \{a, 2n+1\}$ with $a = 1, \ldots, 2n$, $G = \mathrm{SU}(n+1)$, $H = \mathrm{SU}(n)$ and

$$su(n+1) = su(n) \oplus \mathfrak{m}. \tag{3.18}$$

The sine-cone M^{2n+2} is conformally equivalent to the cylinder $\mathrm{Cyl}(\mathcal{M})$ with the local basis 1-forms $e^{\hat{a}}$ and e^{2n+2}. The group $\mathrm{SO}(2n+2)$ acts on the tangent spaces of both $\mathrm{Cyl}(\mathcal{M})$ and $C_\Lambda(\mathcal{M})$. We have

$$so(2n+2) = su(n+1) \oplus u(1) \oplus \mathcal{P}, \tag{3.19}$$

so the space of antisymmetric $(2n+2) \times (2n+2)$ matrices can be split into three mutually orthogonal subspaces, which defines $\mathcal{P}$. The $su(n)$ subspace contains the first two terms in (3.17), which are thus Σ-anti-self-dual. Using the explicit form of the projector from $so(2n+2)$ to $su(n+1)$ [17], one can show that the subspace $\mathfrak{m}$ in (3.18) is spanned by the 2-forms (regarded as antisymmetric matrices)

$$e^{2n+1} \wedge e^{2n+2} - \frac{1}{2n} \omega_{ab} e^a \wedge e^b \quad \text{and} \quad e^a \wedge e^{2n+2} - J^a_b e^b \wedge e^{2n+1}, \tag{3.20}$$

which satisfy the instanton equations (3.1). Here ω_{ab} and J^a_b are as defined in Sec. II. Hence, if we choose $N^{\hat{a}}_{\hat{b}\hat{c}}$ in such a way that the last term in (3.17) becomes a linear combination of the 2-forms (3.20), then $\mathcal{F}$ from (3.17) will also solve the instanton equations (3.1).

From (2.17), (2.29), (3.17) and (3.20) we finally obtain

$$T^{a}_{b2n+1} = -\frac{n+1}{n} J^a_b = -f^{a}_{b2n+1} \quad \text{and} \quad T^{2n+1}_{ab} = 2P_{ab2n+1} = -2\omega_{ab} = -f^{2n+1}_{ab}, \tag{3.21}$$

$$N^{a}_{b2n+1} = J^a_b = \frac{n}{n+1} f^{a}_{b2n+1} \quad \text{and} \quad N^{2n+1}_{ab} = \frac{1}{n}\omega_{ab} = \frac{1}{2n} f^{2n+1}_{ab}. \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.15) and (3.16), we arrive at

$$[I_i, X_a] = f^{b}_{ia} X_b \quad \text{and} \quad [I_i, X_{2n+1}] = 0, \tag{3.23}$$

$$[X_a, X_b] = f^{2n+1}_{ab} \left(X_{2n+1} + \frac{1}{2n} \dot{X}_{2n+1} \right) + f^{j}_{ab} N_j(\tau), \tag{3.24}$$

$$[X_{2n+1}, X_a] = f^{b}_{2n+1a} \left(X_b + \frac{n}{n+1} \dot{X}_b \right). \tag{3.25}$$

The task now is to find solutions to the above matrix equations (3.23)–(3.25). The simplest choice is

$$X_a(\tau) = \psi(\tau) I_a \quad \text{and} \quad X_{2n+1}(\tau) = \chi(\tau) I_{2n+1} \tag{3.26}$$

introducing two functions ψ and χ of τ, which is related with r via (2.11). For this choice, the conditions (3.23) are fulfilled, and we get $N_i = \psi^2 I_i$. This reduces (3.24)–(3.25) to the equations

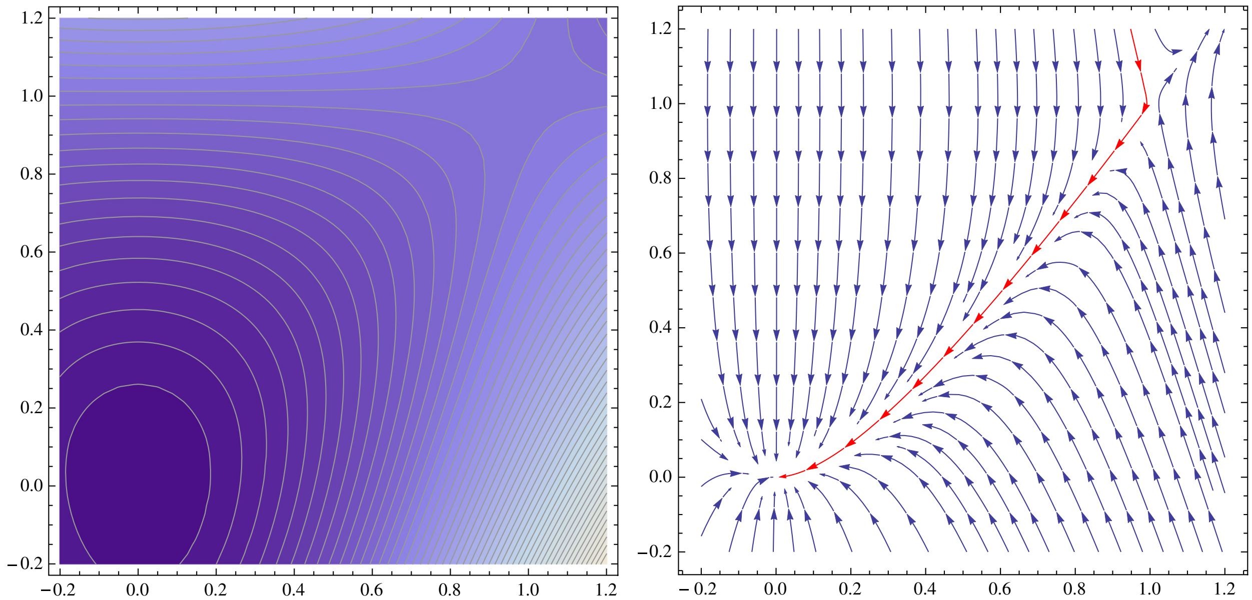

FIG. 1 (color online). Equipotential lines for $W(\psi,\chi)$ (left) and streamlines for $n=2$ (right).

$$\dot{\psi} = \frac{n+1}{n}\psi(\chi-1) = -\frac{\partial W}{\partial \psi} \quad \text{and}$$
$$\dot{\chi} = 2n(\psi^2-\chi) = -\lambda^2\frac{\partial W}{\partial \chi}, \tag{3.27}$$

which agrees with (4.21) and (4.22) of [9] for the metric cone. Here, $\lambda = 2n/\sqrt{n+1}$, and we introduced the flow potential

$$W(\psi,\chi) = \frac{n+1}{2n}\left(\psi^2 + \frac{1}{2}\chi^2 - \psi^2\chi\right) \tag{3.28}$$

for the variables ψ and $\tilde{\chi} = \chi/\lambda$, so that the second equation in (3.27) reads $\dot{\tilde{\chi}} = -\partial W/\partial\tilde{\chi}$. For an instanton solution, we need ψ and χ to remain bounded for all $\tau \in \mathbb{R}$. This requires the flow to start and end in a critical point of W. Modulo the obvious reflection symmetry $\psi \to -\psi$, the critical points of W are

$$\text{the local minimum } (\psi,\chi) = (0,0) \quad \text{and}$$
$$\text{the saddle point } (\psi,\chi) = (1,1), \tag{3.29}$$

and the flow trajectory connecting them is a separatrix for the vector field ∇W. It is given by

$$2\psi(1-\chi)\mathrm{d}\chi = \lambda^2(\chi-\psi^2)\mathrm{d}\psi, \tag{3.30}$$

which admits analytic solutions only for

$$n=1\colon\ \chi=\psi \quad \text{and} \quad n\to\infty\colon\ \chi=\psi^2. \tag{3.31}$$

These and the numerical solutions for $n = 2, 4, 8$ have been plotted in Fig. 1 of [9]. Here, in our Fig. 1, we display the equipotential lines of the flow potential W and for $n = 2$ the corresponding streamlines. The unique bounded solution to (3.27) with τ defined by (2.11) yields a Yang-Mills instanton after substituting $X_a = \psi I_a$ and $X_{2n+1} = \chi I_{2n+1}$ into (3.12) and (3.17).

D. Instantons on conical HKT manifolds

In Sec. II we have shown that, for $\mathcal{M}$ being 3-Sasakian of dimension $4n+3$, on $M^{4n+4} = C_\Lambda(\mathcal{M})$ one can introduce a HKT structure with a metric conformally equivalent to the metric on the cylinder $\mathrm{Cyl}(\mathcal{M})$. For this reason it suffices to investigate the instanton equation (3.1) on $\mathrm{Cyl}(\mathcal{M})$. In the 3-Sasakian case, $\{\hat{a}\} = \{a, 4n+\alpha\}$ with $a = 1,\ldots,4n$ and $\alpha = 1,2,3$, and we have $G = \mathrm{Sp}(n+1)$ and $H = \mathrm{Sp}(n)$, i.e.

$$sp(n+1) = sp(n) \oplus \mathfrak{m}. \tag{3.32}$$

The group $\mathrm{SO}(4n+4)$ acts on tangent spaces of both $\mathrm{Cyl}(\mathcal{M})$ and M^{4n+4}. One gets

$$so(4n+4) = sp(n+1) \oplus sp(1) \oplus \mathcal{P}',$$

and the Σ-anti-self-dual 2-forms reside in the $sp(n+1)$ subspace of the space of antisymmetric $(4n+4)\times(4n+4)$ matrices [17]. Now the first two terms in (3.17) sit in $sp(n) \subset sp(n+1) \subset so(4n+4)$ and, therefore, satisfy the instanton equation (3.1). Employing the explicit form of the projector from $so(4n+4)$ to $sp(n+1)$ [17], one can show that the 2-forms

$$f_2\bar{\eta}^{\alpha}_{\mu\nu}e^{4n+\mu}\wedge e^{4n+\nu} = f_2(\varepsilon^{\alpha}_{\beta\gamma}e^{4n+\beta}\wedge e^{4n+\gamma} - 2e^{4n+\alpha}\wedge e^{4n+4}) \quad \text{and}$$
$$(f_1f_2)^{1/2}(e^a\wedge e^{4n+4} + J^{\alpha a}_b e^b\wedge e^{4n+\alpha}) \tag{3.33}$$

are Σ-anti-self-dual and form a basis of the subspace $\mathfrak{m}$ in (3.32). Here $e^{4n+4} = \mathrm{d}\tau$, and $J^{\alpha a}_b$ as well as the functions f_1, f_2 have been defined in Sec. II. From (2.26), (3.17) and (3.33) it follows that the nonvanishing components are given by

$$T^{4n+\alpha}_{ab} = -f^{4n+\alpha}_{ab}, \qquad T^b_{a\,4n+\beta} = -f^b_{a\,4n+\beta} \quad \text{and}$$
$$T^{4n+\alpha}_{4n+\beta\,4n+\gamma} = -f^{4n+\alpha}_{4n+\beta\,4n+\gamma}, \tag{3.34}$$

$$N^a_{b\,4n+\alpha} = -J^{\alpha a}_b = -\omega^{\alpha}_{ab} = f^a_{b\,4n+\alpha},$$
$$N^{4n+\alpha}_{4n+\beta\,4n+\gamma} = \varepsilon^{\alpha}_{\beta\gamma} = \frac{1}{2}f^{4n+\alpha}_{4n+\beta\,4n+\gamma}. \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.15) and (3.16), we arrive at

$$[I_i, X_a] = f^b_{ia}X_b \quad \text{and} \quad [I_i, X_{4n+\alpha}] = 0, \tag{3.36}$$

$$[X_a, X_b] = f^{4n+\alpha}_{ab}X_{4n+\alpha} + f^i_{ab}N_i, \tag{3.37}$$

$$[X_a, X_{4n+\beta}] = f^b_{a\,4n+\beta}(X_b + \dot{X}_b), \tag{3.38}$$

$$[X_{4n+\alpha}, X_{4n+\beta}] = f^{4n+\gamma}_{4n+\alpha\,4n+\beta}\left(X_{4n+\gamma} + \frac{1}{2}\dot{X}_{4n+\gamma}\right), \tag{3.39}$$

again independent of the functions f_1 and f_2. If we choose the simplest ansatz

$$X_a(\tau) = \psi(\tau)I_a \quad \text{and} \quad X_{4n+\alpha}(\tau) = \chi(\tau)I_{4n+\alpha}, \tag{3.40}$$

then (3.36) will be satisfied identically. From (3.37) we obtain $N_i = \psi^2 I_i$, and (3.38) and (3.39) reduce to

$$\dot{\psi} = \psi(\chi - 1) \quad \text{and} \quad \dot{\chi} = 2\chi(\chi - 1) \quad \text{as well as} \quad \chi = \psi^2, \tag{3.41}$$

which is the $n\to\infty$ limit of (3.27) and coincides with (4.31)–(4.33) of [9] for the metric cone. The equations decouple to

$$\dot{\psi} = \psi(\psi + 1)(\psi - 1) \quad \text{and} \quad \dot{\chi} = 2\chi(\chi - 1), \tag{3.42}$$

whose only bounded solution is

$$\chi = \psi^2 = \frac{1}{2}(1 - \tanh(\tau - \tau_0)). \tag{3.43}$$

Substituting $X_a = \psi I_a$ and $X_{4n+\alpha} = \chi I_{4n+\alpha}$ into (3.12) and (3.17), one obtains Yang-Mills instanton configurations on the HKT manifold M^{4n+4} after using (2.11) and the relations (2.45) between the co-frame fields on M^{4n+4} and the cylinder. More general instanton solutions may be obtained by considering more general ansätze for the matrices $X_{\hat{a}}$.

IV. CONCLUSIONS

A Killing spinor on a Riemannian manifold $\mathcal{M}$ is a spinor field ϵ obeying the equation $\nabla_{\hat{a}}\epsilon = \mathrm{i}\lambda\gamma_{\hat{a}}\epsilon$, where $\nabla_{\hat{a}}$ is the spinor covariant derivative, $\gamma_{\hat{a}}$ are Clifford γ-matrices and λ is a constant. Manifolds with real Killing spinors often occur in string-theory compactifications. All these manifolds feature connections with non-vanishing torsion and admit a non-integrable H-structure, i.e. a reduction of the structure group $\mathrm{SO}(m)$ of the tangent bundle $T\mathcal{M}$ to $H\subset \mathrm{SO}(m)$. The metric cone $C(\mathcal{M})$ over any such manifold $\mathcal{M}$ has a special (reduced) holonomy group $G\subset \mathrm{SO}(m+1)$ and a Killing spinor ϵ with $\lambda = 0$ (called parallel spinor). These manifolds were classified in [35], and, besides the round spheres, they are the

(i) nearly Kähler 6-manifolds, with $H = \mathrm{SU}(3)$ and $G = G_2$
(ii) nearly parallel 7-manifolds, with $H = G_2$ and $G = \mathrm{Spin}(7)$
(iii) Sasaki-Einstein $(2n+1)$-manifolds, with $H = \mathrm{SU}(n)$ and $G = \mathrm{SU}(n+1)$
(iv) 3-Sasakian $(4n+3)$-manifolds, with $H = \mathrm{Sp}(n)$ and $G = \mathrm{Sp}(n+1)$.

Instantons on metric cones $C(\mathcal{M})$ over the above manifolds $\mathcal{M}$ were described in [9,23,24,27]. Instantons on sine-cones over nearly Kähler 6-manifolds and nearly parallel 7-manifolds with G_2-structure were investigated in [26]. Here, we completed this study by describing Yang-Mills instantons on sine-cones over Sasaki-Einstein and 3-Sasakian manifolds. In [9,10] instantons on metric cones were extended to brane-type solutions of heterotic supergravity. It would be of interest to perform a similar lift of instantons on sine-cones.

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[1] C. P. Boyer and K. Galicki, Surveys Diff. Geom. **7**, 123 (1999).
[2] C. P. Boyer and K. Galicki, *Handbook of Pseudo-Riemannian Geometry and Supersymmetry* (European Mathematical Society Publishing House, Zürich, 2010), p. 39.
[3] C. P. Boyer and K. Galicki, *Sasakian Geometry* (Oxford University Press, New York, 2008).
[4] J. P. Gauntlett, D. Martelli, J. Sparks, and D. Waldram, Adv. Theor. Math. Phys. **8**, 711 (2004); **8**, 987 (2004); D. Martelli and J. Sparks, Phys. Lett. B **621**, 208 (2005).
[5] M. Cvetic, H. Lü, D. N. Page, and C. N. Pope, Phys. Rev. Lett. **95**, 071101 (2005); J. High Energy Phys. 07 (2009) 082.
[6] J. Sparks, Surveys Diff. Geom. **16**, 265 (2011).
[7] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull, and B. J. Spence, Adv. Theor. Math. Phys. **2**, 1249 (1999).
[8] P. Koerber, D. Lüst, and D. Tsimpis, J. High Energy Phys. 07 (2008) 017.
[9] D. Harland and C. Nölle, J. High Energy Phys. 03 (2012) 082.
[10] K.-P. Gemmer, A. S. Haupt, O. Lechtenfeld, C. Nölle, and A. D. Popov, Adv. Theor. Math. Phys. **17**, 771 (2013); C. Nölle, arXiv:1207.7268.
[11] P. S. Howe and G. Papadopoulos, Phys. Lett. B **379**, 80 (1996).
[12] G. W. Gibbons, G. Papadopoulos, and K. S. Stelle, Nucl. Phys. **B508**, 623 (1997); A. Opfermann and G. Papadopoulos, arXiv:math-ph/9807026; J. Gutowski and G. Papadopoulos, Phys. Lett. B **432**, 97 (1998).
[13] G. Grantcharov and Y. S. Poon, Commun. Math. Phys. **213**, 19 (2000).
[14] G. Grantcharov, G. Papadopoulos, and Y. S. Poon, J. Math. Phys. (N.Y.) **43**, 3766 (2002).
[15] M. L. Barberis, Revista de la Union Math. Argentina **49**, 121 (2008); M. Fernandes, A. Fino, L. Ugarte, and R. Villacampa, AIP Conf. Proc. **1360**, 27 (2011).
[16] E. Corrigan, C. Devchand, D. B. Fairlie, and J. Nuyts, Nucl. Phys. **B214**, 452 (1983); R. Ward, Nucl. Phys. **B236**, 381 (1984).
[17] T. A. Ivanova and A. D. Popov, Theor. Math. Phys. **94**, 225 (1993).
[18] L. Baulieu, H. Kanno, and I. M. Singer, Commun. Math. Phys. **194**, 149 (1998); M. Blau and G. Thompson, Phys. Lett. B **415**, 242 (1997); B. S. Acharya, J. M. Figueroa-O'Farrill, B. J. Spence, and M. O'Loughlin, Nucl. Phys. **B514**, 583 (1998).
[19] S. K. Donaldson, Duke Math. J. **54**, 231 (1987); K. K. Uhlenbeck and S.-T. Yau, Commun. Pure Appl. Math. **39**, S257 (1986).
[20] S. K. Donaldson and R. P. Thomas, in *The Geometric Universe* (Oxford University Press, New York, 1998); S. K. Donaldson and E. Segal, Surveys Diff. Geom. **16**, 1 (2011).
[21] G. Tian, Ann. Math. **151**, 193 (2000).
[22] D. B. Fairlie and J. Nuyts, J. Phys. A **17**, 2867 (1984);
S. Fubini and H. Nicolai, Phys. Lett. **155B**, 369 (1985); T. A. Ivanova and A. D. Popov, Lett. Math. Phys. **24**, 85 (1992).
[23] T. A. Ivanova, O. Lechtenfeld, A. D. Popov, and T. Rahn, Lett. Math. Phys. **89**, 231 (2009); D. Harland, T. A. Ivanova, O. Lechtenfeld, and A. D. Popov, Commun. Math. Phys. **300**, 185 (2010).
[24] I. Bauer, T. A. Ivanova, O. Lechtenfeld, and F. Lubbe, J. High Energy Phys. 10 (2010) 044; A. S. Haupt, T. A. Ivanova, O. Lechtenfeld, and A. D. Popov, Phys. Rev. D **83**, 105028 (2011).
[25] F. P. Correia, J. High Energy Phys. 12 (2009) 004; 02 (2011) 054.
[26] K.-P. Gemmer, O. Lechtenfeld, C. Nölle, and A. D. Popov, J. High Energy Phys. 09 (2011) 103.
[27] T. A. Ivanova and A. D. Popov, Phys. Rev. D **85**, 105012 (2012).
[28] T. Walpuski, Geom. Topol. **17**, 2345 (2013); A. Clarke, J. Geom. Phys. **82**, 84 (2014).
[29] J. A. Wolf, *Spaces of Constant Scalar Curvature* (McGraw-Hill, New York, 1967); J. A. Wolf and A. Gray, J. Diff. Geom. **2**, 77 (1968); **2**, 115 (1968).
[30] A. Gray, J. Diff. Geom. **4**, 283 (1970).
[31] P. Gauduchon, Bollettino U.M.I. **11B**, 257 (1997); J. M. Bismut, Math. Ann. **284**, 681 (1989).
[32] A. Fino, M. Parton, and S. Salamon, Commentarii mathematici Helvetici **79**, 317 (2004); M. Fernandez, A. Fino, L. Ugarte, and R. Villacampa, Pac. J. Math. **249**, 49 (2011).
[33] J. Gutowski, S. Ivanov, and G. Papadopoulos, Asian J. Math. **7**, 39 (2003); D. Grantcharov, G. Grantcharov, and Y. S. Poon, J. Diff. Geom. **78**, 13 (2008).
[34] M. Graña, Phys. Rep. **423**, 91 (2006); M. R. Douglas and S. Kachru, Rev. Mod. Phys. **79**, 733 (2007); R. Blumenhagen, B. Kors, D. Lüst, and S. Stieberger, Phys. Rep. **445**, 1 (2007).
[35] C. Bär, Commun. Math. Phys. **154**, 509 (1993).