PHYSICAL REVIEW D **90**, 065024 (2014)

Constraining subleading soft gluon and graviton theorems

Johannes Broedel,[1,*] Marius de Leeuw,[1,†] Jan Plefka,[1,2,‡] and Matteo Rosso[1,§]

[1]*Institut für Theoretische Physik, Eidgenössische Technische Hochschule Zürich, Wolfgang-Pauli-Strasse 27, 8093 Zürich, Switzerland*

[2]*Institut für Physik and IRIS Adlershof, Humboldt-Universität zu Berlin, Newtonstraße 15, D-12489 Berlin, Germany*

(Received 4 July 2014; published 19 September 2014)

We show that the form of the recently proposed subleading soft graviton and gluon theorems in any dimension are severely constrained by elementary arguments based on Poincaré and gauge invariance as well as a self-consistency condition arising from the distributional nature of scattering amplitudes. Combined with the assumption of a local form as it would arise from a Ward identity the orbital part of the subleading operators is completely fixed by the leading universal Weinberg soft pole behavior. The polarization part of the differential subleading soft operators in turn is determined up to a single numerical factor for each hard leg at every order in the soft momentum expansion. In four dimensions, factorization of the Lorentz group allows us to fix the subleading operators completely.

DOI: 10.1103/PhysRevD.90.065024 PACS numbers: 12.38.-t, 11.10.Kk, 04.60.-m, 11.30.Cp

I. INTRODUCTION

Gluon and graviton scattering amplitudes display a universal factorization behavior when a gluon (respectively photon) [1] or a graviton [2] becomes soft, as was shown more than 50 years ago. This leading soft pole behavior is known as Weinberg's soft theorem [2,3]. Recently an interesting proposal was put forward by Cachazo and Strominger [4] in which they conjectured the extension of this theorem for gravitons to subleading and subsubleading orders in the soft momentum expansion. The proposal was shown to hold at tree level using the Britto-Cachazo-Feng-Witten recursion relations [5]. Tree-level gluon amplitudes exhibit a very similar subleading universal behavior as pointed out in Ref. [6] using a proof identical to that of gravitons. In fact such a subleading gluon relation was argued to exist already in Refs. [1,7]; recent investigations and discussions were performed in Ref. [8]. Similarly, the subleading soft graviton behavior was reported already in 1968 [9]; see also the more recent discussion [10].

Collectively these (new) subleading soft theorems state the existence of certain universal differential operators in momenta and polarizations acting on a hard n-point amplitude, which capture the subleading or even subsubleading terms in the soft limit of the associated $(n+1)$-point amplitude with one leg taken soft. For the case of gravity the subleading soft theorems have been conjectured to be Ward identities of a new symmetry of the quantum gravity S matrix [4,11,12], namely the extension of the Bondi, van der Burg, Metzner and Sachs (BMS) symmetry [13] to a Virasoro symmetry [14] acting on a sphere at past and future infinity. This connection was first established in Ref. [12] for the leading soft Weinberg pole term [3]. Recently a connection of the first subleading graviton theorem to the super-rotation symmetry of extended BMS symmetry [14] was reported [15]. Interesting steps toward a better understanding of such a relation through dual holographic [16] or ambitwistor [17] string models also appeared recently.

Inspired by these results a series of papers appeared [6,18–24]. Very interestingly the validity of the gluon and graviton subleading theorems was shown to hold at *any* dimension for tree-level amplitudes [18,23]. This is puzzling in the context of the conjectured relation between gravity and extended BMS symmetry which is clearly special to four dimensions. Similarly, it has been claimed in Ref. [21] that the subleading soft theorem for gauge theory is related to the conformal symmetry of tree-level gluon amplitudes, which again contradicts the existence of the subleading theorem in general dimensions.

An important question is whether the subleading soft theorems receive radiative corrections. Loop-level modifications of the leading soft-gluon theorem are known to arise due to infrared singularities [25]. Whereas the leading Weinberg soft graviton is protected, the subleading operators are shown to be corrected in Refs. [19,20]. This argument, however, was challenged in the recent work [22], where the authors argue for an order-of-limits problem: Taking the soft limit prior to sending the dimensional regulator to zero would not cause any corrections to the soft theorems.

In this paper we hope to shed some light on the above questions from a different point of view. We will show that rather elementary arguments can take one quite far. Beyond the obvious Poincaré and gauge invariance we will assume a certain *local* form of the soft operators (as it would follow from a Ward identity). In conjunction with a self-consistency

*jbroedel@itp.phys.ethz.ch
†deleeuwm@itp.phys.ethz.ch
‡plefka@physik.hu-berlin.de
§mrosso@itp.phys.ethz.ch

condition of the theorems arising from the distributional nature of scattering amplitudes, the form of the subleading operators is strongly constrained. Our argument applies to all dimensions and determines the orbital part of the subleading operators uniquely from the form of the known leading pole functions.

While our argument does not prove the *existence* of a universal subleading soft gluon and graviton theorems, it states that if a such a behavior exists, it is inevitably of the form proposed recently. Therefore, the only input needed from a potential new symmetry of the quantum gravity or gauge theory S matrix is the mere existence of a Ward identity pertaining to subleading orders in the soft limit. The form of the orbital part of the theorems is then fixed—at least at tree level.

This paper is organized as follows. In Sec. II we provide our general arguments and derive the central distributional constraint linking the subleading operator in the soft theorems to the leading one. In Sec. III we apply the established constraints to the subleading soft operators for gluons and gravitons and show that they are capable of fixing the orbital piece while strongly constraining the polarization part. In Sec. IV we apply the same reasoning to the subsubleading soft graviton operator yielding identical results. In Sec. V we specialize to four dimensions and employ the spinor-helicity formalism in order to find that the same line of arguments now entirely determine the subleading soft operators. We end with a discussion in Sec. VI.

II. GENERAL ARGUMENTS

Let us briefly summarize the subleading soft theorems and our central argument. We will consider amplitudes in D-dimensional pure gauge and gravity theories denoted by $\mathcal{A}_n = \delta^{(D)}(P)A_n$, where $P = \sum_{a=1}^{n} p_a$ is the total momentum. The soft momentum of leg $n+1$ is taken to be ϵq^μ, which allows us to control the soft limit by sending ϵ to zero. The subleading soft theorems may be stated as[1]

$$\mathcal{A}_{n+1}(p_1,\ldots,p_n,\epsilon q) = \mathcal{S}^{[l]}(p_1,\ldots,p_n,\epsilon q)\mathcal{A}_n(p_1,\ldots,p_n) + \mathcal{O}(\epsilon^l), \quad (1)$$

where we call $\mathcal{S}^{[l]}$ a soft operator. The integer parameter l controls the expansion in powers of the soft momentum to which the theorem holds.

This theorem has been known to hold at leading order ($l=0$) for more than 50 years. The corresponding soft factors in gauge theory [1] and gravity [2] read

$$\mathcal{S}^{[0]}(\epsilon q) = \begin{cases} \frac{1}{\epsilon}S^{(0)}_{\mathrm{YM}} = \frac{1}{\epsilon}\left(\frac{E\cdot p_1}{p_1\cdot q} - \frac{E\cdot p_n}{p_n\cdot q}\right) & \text{Yang-Mills theory (color ordered)} \\ \frac{1}{\epsilon}S^{(0)}_{\mathrm{G}} = \frac{1}{\epsilon}\sum_{a=1}^{n}\frac{E_{\mu\nu}p_a^\mu p_a^\nu}{p_a\cdot q} & \text{Gravity,} \end{cases} \quad (2)$$

where E_μ and $E_{\mu\nu}$ denote the gluon or graviton polarization of the soft leg respectively and the arguments $\{p_a\}$ of $\mathcal{S}^{[l]}(\epsilon q)$ have been suppressed for brevity. Note that we are working with color ordered gauge theory amplitudes.[2] The soft limit is singular, and the pole terms are universal. The graviton pole function $S^{(0)}_{\mathrm{G}}$ does not receive radiative corrections [3,27].

In Refs. [4,6,18,23] the theorem in Eq. (1) has been demonstrated to extend to $l=1$ in D-dimensional gauge theory and even $l=2$ in D-dimensional gravity at least at tree level,

$$\mathcal{S}^{[l]}(\epsilon q) = \begin{cases} \frac{1}{\epsilon}S^{(0)}_{\mathrm{YM}} + S^{(1)}_{\mathrm{YM}} & \text{Yang-Mills theory } (l=1) \\ \frac{1}{\epsilon}S^{(0)}_{\mathrm{G}} + S^{(1)}_{\mathrm{G}} + \epsilon S^{(2)}_{\mathrm{G}} & \text{Gravity } (l=2). \end{cases} \quad (3)$$

The operators $S^{(1)}_{\mathrm{YM}}$, $S^{(1)}_{\mathrm{G}}$ and $S^{(2)}_{\mathrm{G}}$ are differential operators in the kinematical data of the hard legs and take a local form. Here with *locality* we want to refer to the fact that they are sums over terms depending on a single hard leg and the soft data only, i.e.,

$$S^{(l)} = \sum_a S^{(l)}_a(E,\epsilon q; p_a, \partial_{p_a}, E_a, \partial_{E_a}). \quad (4)$$

This situation is just as one would expect it to arise from a Ward identity.

Naturally, the form of $\mathcal{S}^{[l]}$ is strongly restricted by Poincaré and gauge invariance. While Poincaré invariance implies linearity in the polarization tensors, gauge invariance demands the vanishing of $\mathcal{S}^{[l]}\mathcal{A}_n$ order by order in ϵ upon replacing the polarizations by a gauge transformation.

There is, however, a further less obvious but elementary constraint on $\mathcal{S}^{[l]}$ emerging from the distributional nature of amplitudes. The left-hand and the right-hand sides of the soft theorem Eq. (1) depend on Dirac delta functions which differ in their arguments by the soft momentum ϵq. While this is no issue at leading order ($l=0$), it becomes relevant for the subleading corrections. Therefore, for the subleading soft theorems to be consistent, we need to require that

[1]We only consider amplitudes where the external particles are of the same type.

[2]See, e.g., Ref. [26] for a textbook treatment.

$$\mathcal{S}^{[l]}(\epsilon q)\delta^D(P) = \delta^D(P+\epsilon q)\tilde{\mathcal{S}}^{[l]}(\epsilon q), \tag{5}$$

where the soft operator $\tilde{\mathcal{S}}^{[l]}(\epsilon q)$ acting on the reduced amplitude A_n could differ *a priori* from the soft operator $\mathcal{S}^{[l]}(\epsilon q)$ acting on the full amplitude $\mathcal{A}_n$. Interestingly, the results reported in the literature so far indicate that the $\tilde{\mathcal{S}}^{[l]}(\epsilon q)$ and $\mathcal{S}^{[l]}(\epsilon q)$ are equivalent.[3] We shall show that this has to be the case.

Distinguishing different orders of ϵ, the soft theorem Eq. (1) implies [4] the relations

$$\lim_{\epsilon\to 0}(\epsilon\mathcal{A}_{n+1}(\epsilon)) = S^{(0)}\mathcal{A}_n \tag{6a}$$

$$\lim_{\epsilon\to 0}\left(\mathcal{A}_{n+1}(\epsilon) - \frac{1}{\epsilon}S^{(0)}\mathcal{A}_n\right) = S^{(1)}\mathcal{A}_n \tag{6b}$$

$$\lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\mathcal{A}_{n+1}(\epsilon) - \frac{1}{\epsilon^2}S^{(0)}\mathcal{A}_n - \frac{1}{\epsilon}S^{(1)}\mathcal{A}_n\right) = S^{(2)}\mathcal{A}_n. \tag{6c}$$

Note that these equations simply organize the soft limit expansion in ϵ without touching the expansion in the dimensional regularization parameter $\epsilon_{\rm dim}$ relevant for amplitudes at loop level. In writing these equations we have not committed ourselves to a particular order of the $\epsilon_{\rm dim}\to 0$ and $\epsilon\to 0$ limits.

To derive the implications of Eq. (6) for the soft operators $S^{(0)}$ and $S^{(1)}$, it is useful to Laurent expand both the reduced amplitude A_{n+1} as well as its associated delta function,

$$A_{n+1}(\epsilon) = \frac{1}{\epsilon}A^{(-1)}_{n+1} + A^{(0)}_{n+1} + \epsilon A^{(1)}_{n+1} + \mathcal{O}(\epsilon^2) \tag{7}$$

$$\delta^{(D)}(P+\epsilon q) = \delta^{(D)}(P) + \epsilon(q\cdot\partial)\delta^{(D)}(P) + \mathcal{O}(\epsilon), \tag{8}$$

where we introduced the shorthand notation $q\cdot\partial = q^\mu\frac{\partial}{\partial P^\mu}$. Let us now substitute these expansions into Eqs. (6a) and (6b). After noting that

$$[S^{(0)}, \delta^{(D)}(P)] = 0, \tag{9}$$

due to the form of $S^{(0)}$ in Eq. (2) being a mere function, one finds for the *reduced* amplitude from (6b)

$$\lim_{\epsilon\to 0}A_{n+1}(\epsilon) = \frac{1}{\epsilon}S^{(0)}_{\rm YM}A_n + A^{(0)}_{n+1} + \mathcal{O}(\epsilon). \tag{10}$$

For now, we will leave the form of the subleading contribution $A^{(0)}_{n+1}$ undetermined. Equation (6a) then simply implies

$$A^{(-1)}_{n+1} = S^{(0)}A_n. \tag{11}$$

[3] See in particular Refs. [19,22] for a discussion of different prescriptions related to this issue.

Equation (6b) leads to

$$\begin{aligned} S^{(1)}\mathcal{A}_n &= \lim_{\epsilon\to 0}\left[\mathcal{A}_{n+1} - \frac{1}{\epsilon}S^{(0)}\mathcal{A}_n\right], \\ &= \lim_{\epsilon\to 0}\bigg[(\delta^{(D)}(P) + \epsilon(q\cdot\partial)\delta^{(D)}(P)) \\ &\quad \times\left(\frac{1}{\epsilon}A^{(-1)}_{n+1} + A^{(0)}_{n+1}\right) - \frac{1}{\epsilon}S^{(0)}\mathcal{A}_n\bigg], \end{aligned} \tag{12}$$

where we kept only terms not vanishing as $\epsilon\to 0$. Now we can remove the limit on the right-hand side, whereas in the left-hand side we can commute $S^{(1)}$ past the delta function to obtain

$$\begin{aligned} &[S^{(1)}, \delta^{(D)}(P)]A_n + \delta^{(D)}(P)S^{(1)}A_n \\ &\quad = \delta^{(D)}(P)A^{(0)}_{n+1} + S^{(0)}((q\cdot\partial)\delta^{(D)}(P))A_n. \end{aligned} \tag{13}$$

At this point, several comments are in order. Most importantly, δ and δ' may be treated as independent distributions if one takes partial integration identities into account. Therefore, we will have to match their respective coefficients in order for this equation to be satisfied. Next, $S^{(1)}$ must be a differential operator in the momenta p_a; Eq. (13) implies then that

$$[S^{(1)}, \delta^{(D)}(P)] = S^{(0)}\left(q\cdot\partial\delta^{(D)}(P)\right) + \chi\delta^{(D)}(P), \tag{14}$$

$$A^{(0)}_{n+1} = (S^{(1)} - \chi)A_n, \tag{15}$$

where χ is an undetermined function. Repeating the analysis for Eq. (6c) (extracting the singular behavior from the reduced amplitude, expanding in ϵ and matching coefficients of the delta function and its derivatives) leads to

$$\begin{aligned} [S^{(2)}, \delta^{(D)}(P)] &= \frac{1}{2}S^{(0)}((q\cdot\partial)^2\delta^{(D)}(P)) \\ &\quad + (q\cdot\partial\delta^{(D)}(P))S^{(1)} + \chi'\delta^{(D)}(P), \end{aligned} \tag{16}$$

$$A^{(1)}_{n+1} = (S^{(2)} - \chi')A_n. \tag{17}$$

We see that the above equations constrain the subleading soft terms by relating their form to the leading soft function $S^{(0)}$. We will refer to those equations as *distributional constraints*. Note also that the difference of the soft operators $\mathcal{S}^{[l]}(\epsilon q)$ and $\tilde{\mathcal{S}}^{[l]}(\epsilon q)$ mentioned in (5) is captured by—*a priori*—arbitrary functions χ and χ'.

It is clear that the distributional constraints can only constrain the part of $S^{(l)}$ that contains the derivatives with respect to the hard momenta. We call this piece the orbital part of $S^{(l)}$ and write

$$S^{(l)} = S^{(l)}_{\text{orb}} + S^{(l)}_{\text{polar}} + S^{(l)}_{\text{function}} \tag{18}$$

with the orbital part

$$S^{(l)}_{\text{orb}} = \sum_a S_a^{(l)\mu_1\ldots\mu_l}(E,q;p_a)\frac{\partial}{\partial p_a^{\mu_1}}\cdots\frac{\partial}{\partial p_a^{\mu_l}} \tag{19}$$

and the polarization part $S^{(l)}_{\text{polar}}$ containing derivatives with respect to the polarizations E_a. Finally $S^{(l)}_{\text{function}}$ is a pure function of the soft and hard momenta linear in the soft polarization E. It is not constrained by the distributional constraint as it commutes with the Dirac delta function.

As we are going to show below, distributional constraints, Poincaré, gauge invariance and the assumption of locality of $S^{(l)}$ completely determine the orbital part $S^{(l)}_{\text{orb}}$ of the soft operators in gauge theory and gravity in any dimensions. We now give a simple argument how to constrain also the remaining polarization part.

To treat gluon and graviton polarizations on an equal footing let us agree upon rewriting the graviton polarization of leg a as

$$E_{a\mu\nu} \to E_{a\mu}E_{a\nu} \quad \text{with} \quad E_a\cdot E_a = 0 = p_a\cdot E_a. \tag{20}$$

In four dimensions this is no restriction at all; in general dimensions it is a formal agreement which we can always undo at any stage due to the fact that an amplitude is linear in the polarizations of all its legs. This replacement unifies gauge and gravity theory in the sense that the same operators act on the polarization degrees of freedom in both theories. Using this prescription, the operator representing a gauge transformations on leg a takes the form

$$W_a \coloneqq p_a\cdot\frac{\partial}{\partial E_a}, \tag{21}$$

and the Lorentz generators are represented as

$$J^{\mu\nu} = \sum_a p_a^\mu\frac{\partial}{\partial p_{a\nu}} + E_a^\mu\frac{\partial}{\partial E_{a\nu}} - \mu\leftrightarrow\nu \tag{22}$$

in both theories in any dimension.[4] In this language the polarization part $S^{(l)}_{\text{polar}}$ depends on the differential operators $E_{a\mu}\frac{\partial}{\partial E_{a\nu}}$ in order to preserve linearity of the amplitude in the polarization E_a.

Let us now consider gauge invariance of a fixed hard leg a for the soft theorem Eq. (1):

$$\begin{aligned}0 &= W_a\mathcal{A}_{n+1}(p_1,\ldots,p_n,\epsilon q)\\ &= W_a(\mathcal{S}^{[l]}(\epsilon q)\mathcal{A}_n(p_1,\ldots,p_n)) = [W_a,\mathcal{S}^{[l]}(\epsilon q)]\mathcal{A}_n = 0.\end{aligned} \tag{23}$$

The orbital part $S^{(l)}_{\text{orb}}$ does not commute with W_a due to the presence of operators $\frac{\partial}{\partial p_a}$. Therefore, it needs to be completed to a gauge invariant structure. Employing the commutators

$$\begin{aligned}\left[W_a, p_a^\mu\frac{\partial}{\partial p_a^\nu}\right] &= -p_a^\mu\frac{\partial}{\partial E_{a\nu}},\\ \left[W_a, E_a^\mu\frac{\partial}{\partial E_a^\nu}\right] &= +p_a^\mu\frac{\partial}{\partial E_{a\nu}},\end{aligned} \tag{24}$$

the unique linear differential operator in p_a and E_a commuting with W_a reads

$$\Lambda_a^{\mu\nu} \coloneqq p_a^\mu\frac{\partial}{\partial p_{a\nu}} + E_a^\mu\frac{\partial}{\partial E_{a\nu}}, \tag{25}$$

which we shall use as building block in constraining $\mathcal{S}^{[l]}$ below.[5] Let us now turn to the explicit analysis.

III. SUBLEADING SOFT OPERATORS

In this section we will apply the general framework outlined in the previous section to determine the subleading soft operators in both gauge theory and gravity. As derived in Sec. II above, the subleading contribution should be fixed upon requiring locality, the distributional constraint and gauge invariance for the soft leg. The last two requirements translate into

$$[S^{(1)},\delta^{(D)}(P)] = S^{(0)}(q\cdot\partial)\delta^{(D)}(P) + \chi\delta^{(D)}(P), \tag{26}$$

$$\left[S^{(1)}, q\cdot\frac{\partial}{\partial E}\right]\cdot\mathcal{A}_n = 0. \tag{27}$$

A. Gauge theory

In gauge theory the leading-order soft factor is given by the universal Weinberg soft gluon function [3]

$$S^{(0)}_{\text{YM}} = \frac{p_1\cdot E}{p_1\cdot q} - \frac{p_n\cdot E}{p_n\cdot q} \tag{28}$$

[4]Strictly speaking this operator does not generate the correct infinitesimal Lorentz transformation rule for the polarizations as these do not transform as vectors; see, e.g., Refs. [2,28]. Next to the vector transformation law there is an additional piece proportional to a gauge transformation in the form of W_a. As this additional piece vanishes acting on amplitudes, the form of (22) is effectively correct.

[5]Note that in fact we only need the weaker condition of $[W_a,\mathcal{S}^{[l]}(\epsilon q)]\sim W_a$ in Eq. (23) as W_a annihilates the amplitudes $\mathcal{A}_n$. This is achieved by the operator $E_a\cdot\frac{\partial}{\partial E_a}$ which obeys $[W_a,E_a\cdot\frac{\partial}{\partial E_a}] = W_a$. However, as any amplitude is an eigenstate of the operator $E_a\cdot\frac{\partial}{\partial E_a}$ with eigenvalue one, including this operator in an Ansatz for $\mathcal{S}^{[l]}$ is tantamount to writing a function. We may therefore discard it in our analysis as functions of the kinematical data cannot be constrained.

with the polarization vector E_μ for the soft particle. We begin with an Ansatz for $S^{(1)}_{\rm YM}$ reflecting the reasoning in Sec. II:

$$S^{(1)}_{\rm YM} = \sum_a E_\mu \Omega_a^{\mu\nu\rho}\left(p_a^\nu \frac{\partial}{\partial p_{a\rho}} + E_a^\nu \frac{\partial}{\partial E_{a\rho}}\right). \tag{29}$$

Before imposing the constraints (26) and (27) we note that dimensional analysis and soft scaling requires $\Omega^{\mu\nu\rho}$ to be of mass dimension -1 and to be scale invariant with respect to the soft momentum q respectively. In conjunction with the assumption of locality and $E\cdot q = E_a\cdot p_a = p_a^2 = 0$ we are left with the compact Ansatz

$$\Omega_a^{\mu\nu\rho} = c_1^{(a)}\frac{p_a^\mu q^\nu q^\rho}{(q\cdot p_a)^2} + c_2^{(a)}\frac{\eta^{\mu\nu}q^\rho}{q\cdot p_a} + c_3^{(a)}\frac{\eta^{\mu\rho}q^\nu}{q\cdot p_a} \tag{30}$$

where the numbers $c_i^{(a)}$ are to be determined. To do so we first impose gauge invariance via (27) which leads to

$$0 = \sum_a (c_1^{(a)} + c_2^{(a)} + c_3^{(a)})\frac{q^\nu q^\rho}{q\cdot p_a} \times \left(p_a^\nu \frac{\partial}{\partial p_{a\rho}} + E_a^\nu \frac{\partial}{\partial E_{a\rho}}\right)\mathcal{A}_n. \tag{31}$$

There is no way for this term to conspire to yield a Lorentz charge. Hence, we conclude that $c_1^{(a)} + c_2^{(a)} + c_3^{(a)} = 0$. Turning to the distributional constraint (26) one now easily establishes

$$\begin{aligned}[S^{(1)}_{\rm YM}, \delta^{(D)}(P)] &= \left(\sum_a c_3^{(a)}\right) E\cdot\partial\delta^{(D)}(P) \\ &\quad - \left(\sum_a c_3^{(a)}\frac{E\cdot p_a}{q\cdot p_a}\right) q\cdot\partial\delta^{(D)}(P) \\ &\overset{!}{=} \left(\frac{p_1\cdot E}{p_1\cdot q} - \frac{p_n\cdot E}{p_n\cdot q}\right) q\cdot\partial\delta^{(D)}(P) \\ &\quad + \chi\delta^{(D)}(P),\end{aligned} \tag{32}$$

where we have inserted Eq. (28) for $S^{(0)}_{\rm YM}$. Solving for the undetermined coefficients, we find

$$\begin{aligned}&c_3^{(1)} = -1, \qquad c_3^{(n)} = 1, \\ &c_3^{(a)} = 0 \quad \text{for } a = 2,\ldots,n-1,\end{aligned} \tag{33}$$

along with the vanishing of χ, which implies the identity $\mathcal{S}^{[1]} = \tilde{\mathcal{S}}^{[1]}$ [cf. Eq. (5)]. As $c_3^{(a)} = -c_1^{(a)} - c_2^{(a)}$ the differences of the remaining coefficients $c_-^{(a)} \coloneqq c_1^{(a)} - c_2^{(a)}$ remain unconstrained. In fact they only couple to the polarization degrees of freedom, and the orbital part of $S^{(1)}$ is completely determined. In summary we have established that

$$\begin{aligned}S^{(1)}_{\rm YM} = &\sum_{a=1,n,\text{signed}} \frac{E_\mu q_\nu}{p_a\cdot q}\left(p_a^\mu\frac{\partial}{\partial p_{a\nu}} + E_a^\mu \frac{\partial}{\partial E_{a\nu}} - \mu\leftrightarrow\nu\right) \\ &+ \sum_a \tilde{c}^{(a)}\left(\frac{(E\cdot p_a)(E_a\cdot q)}{p_a\cdot q} - E\cdot E_a\right) \\ &\times \frac{1}{p_a\cdot q} q\cdot\frac{\partial}{\partial E_a},\end{aligned} \tag{34}$$

where the undetermined coefficients $\tilde{c}^{(a)}$ are related to the previous ones via $\tilde{c}^{(1)} = c_-^{(1)} + \frac{1}{2}$, $\tilde{c}^{(n)} = c_-^{(n)} - \frac{1}{2}$ and $\tilde{c}^{(a)} = c_-^{(a)}$ for $a = 2,\ldots,n-1$. Note that the second sum is manifestly gauge invariant with respect to the soft and hard legs. Hence, the orbital part of the subleading soft operator is entirely determined by our constraints and coincides with the explicit tree-level computations in the literature. The polarization piece is constrained up to a single numerical factor for every hard leg.

Finally, let us briefly comment on the possible functional contribution $S^{(1)}_{\rm YM\,function}$. In our locality assumption $S^{(1)}_{\rm YM\,function}$ must be a sum of terms depending only on the scalars $q\cdot p_a$ and $E\cdot p_a$, while being linear in the latter. This, together with correct dimensionality and the fact that $S^{(1)}_{\rm YM\,function}$ must not scale with q immediately tells us that $S^{(1)}_{\rm YM\,function} = 0$.

B. Gravity

The analysis of the graviton soft operator is almost a carbon copy of the gauge theory one. The leading universal soft function for gravitons reads [3]

$$S^{(0)}_{\rm G} = \sum_{a=1}^n \frac{E_{\mu\nu}p_a^\mu p_a^\nu}{q\cdot p_a}. \tag{35}$$

We again start with an Ansatz for $S^{(1)}_{\rm G}$ of the form

$$S^{(1)}_{\rm G} = \sum_a E_{\mu\nu}\Omega_a^{\mu\nu\rho\sigma}\left(p_a^\rho\frac{\partial}{\partial p_{a\sigma}} + E_a^\rho\frac{\partial}{\partial E_{a\sigma}}\right). \tag{36}$$

Dimensional analysis requires $\Omega_a^{\mu\nu\rho\sigma}$ to be of mass dimension zero and to be scale invariant with respect to the soft momentum q. This together with the assumption of locality and the relations $E_{\mu\nu}q^\nu = E_a\cdot p_a = p_a^2 = 0$ leads us to the most general Ansatz

$$\begin{aligned}\Omega_a^{\mu\nu\rho\sigma} = &c_1^{(a)}\frac{p_a^\mu p_a^\nu q^\rho q^\sigma}{(q\cdot p_a)^2} + c_2^{(a)}\frac{\eta^{\rho(\mu}p_a^{\nu)}q^\sigma}{q\cdot p_a} + c_3^{(a)}\frac{\eta^{\sigma(\mu}p_a^{\nu)}q^\rho}{q\cdot p_a} \\ &+ c_4^{(a)}\eta^{\rho(\mu}\eta^{\nu)\sigma},\end{aligned} \tag{37}$$

with four undetermined numerical coefficients for each hard leg a. Imposing gauge invariance for the soft leg amounts to the replacement $E_{\mu\nu}\to\Lambda_{(\mu}q_{\nu)}$ in (36). We then obtain the condition

$$0 = \frac{1}{2}\sum_a \Bigg[(2c_1^{(a)} + c_2^{(a)} + c_3^{(a)})q^\rho q^\sigma \frac{\Lambda \cdot p_a}{q \cdot p_a} + (c_3^{(a)} + c_4^{(a)})q^\rho \Lambda^\sigma + (c_2^{(a)} + c_4^{(a)})q^\sigma \Lambda^\rho\Bigg] \times \left(p_a^\rho \frac{\partial}{\partial p_{a\sigma}} + E_a^\rho \frac{\partial}{\partial E_{a\sigma}}\right)\mathcal{A}_n. \quad (38)$$

The first term in the above requires $2c_1^{(a)} + c_2^{(a)} + c_3^{(a)} = 0$. For the second and third terms we have to be somewhat more careful. Here we have the possibility of these two terms conspiring to build up the total Lorentz generator $J^{\rho\sigma}$ of (22) which annihilates $\mathcal{A}_n$. We thus require

$$c_3^{(a)} + c_4^{(a)} = - c_2^{(a)} - c_4^{(a)} = c \quad (39)$$

with a *universal* constant c identical for all hard legs. We now move on to pose our distributional constraint (26) linking $S_{\rm G}^{(1)}$ to $S_{\rm G}^{(0)}$. One finds

$$[S_{\rm G}^{(1)}, \delta^{(D)}(P)] = \sum_a (c_1^{(a)} + c_2^{(a)}) \frac{(p_a^\mu p_a^\nu E_{\mu\nu})}{q \cdot p_a} q \cdot \partial \delta^{(D)}(P) + cP^\mu E_\mu{}^\nu \frac{\partial}{\partial P^\nu}\delta^{(D)}(P) \stackrel{!}{=} S_{\rm G}^{(0)} q \cdot \partial\delta^{(D)}(P) + \chi\delta^{(D)}(P). \quad (40)$$

One nicely sees that the first term on the rhs of the first line forms the leading Weinberg soft function for the uniform choice

$$c_1^{(a)} + c_2^{(a)} = 1. \quad (41)$$

The following term vanishes in the distributional sense by the tracelessness of $E_{\mu\nu}$. And finally we again learn that the function $\chi = 0$ implying again the identity of $\mathcal{S}^{[1]} = \tilde{\mathcal{S}}^{[1]}$ in the sense of Eq. (5). The established three equations for the four unknowns may now be solved upon expressing everything in terms of $c_4^{(a)}$:

$$c_1^{(a)} = c_4^{(a)}, \quad c_2^{(a)} = 1 - c_4^{(a)}, \quad c_3^{(a)} = - 1 - c_4^{(a)}. \quad (42)$$

One also checks that $c_3^{(a)} + c_4^{(a)} = 1$ in line with the above reasoning. Inserting this into the Ansatz (36) yields the final result

$$S_{\rm G}^{(1)} = \sum_{a=1}^n \frac{(p_a \cdot E)E_\rho q_\sigma}{p_a \cdot q}\left(p_a^\rho \frac{\partial}{\partial p_{a\sigma}} + E_a^\rho \frac{\partial}{\partial E_{a\sigma}} - \rho \leftrightarrow \sigma\right) + \sum_{a=1}^n \tilde{c}^{(a)}\left(\frac{(E \cdot p_a)(E_a \cdot q)}{p_a \cdot q} - E \cdot E_a\right) \times \left[\frac{p_a \cdot E}{p_a \cdot q} q \cdot \frac{\partial}{\partial E_a} - E \cdot \frac{\partial}{\partial E_a}\right], \quad (43)$$

where we have renamed $c_4^{(a)} = \tilde{c}^{(a)}$ and written the soft polarization $E_{\mu\nu} \to E_\mu E_\nu$ for compactness of notation.

We thus see that again the orbital part is completely determined and coincides with the results established in the literature for tree-level amplitudes.[6] The polarization-dependent parts are constrained to one numerical factor for every hard leg, just as it was the case in gauge theory.

Finally, let us also comment on the possible functional contribution $S_{\rm G\,function}^{(1)}$ in gravity. Again, our assumptions of locality constrains $S_{\rm G\,function}^{(1)}$ to be a sum of functions that are linear in $E \cdot p_a$ and arbitrary functions of $q \cdot p_a$. This, together with the dimensionality of $S_{\rm G}^{(1)}$ and the fact that $S_{\rm G\,function}^{(1)} \sim$ const. as $q \to 0$ again rules out any nonvanishing contribution.

IV. SUBSUBLEADING SOFT GRAVITON OPERATOR

The discussion for the subsubleading soft operator for graviton amplitudes is analogous to the subleading case. The starting point is an Ansatz for $S_{\rm G}^{(2)}$ of the form

$$S_{\rm G}^{(2)} = \sum_{a=1}^n E_{\mu\nu}\Omega_a^{\mu\nu\rho\sigma\gamma\lambda}\Lambda_{a,\rho\sigma}\Lambda_{a,\gamma\lambda}, \quad (44)$$

where we used $\Lambda_{a,\rho\sigma} \coloneqq p_{a,\rho}\frac{\partial}{\partial p_a^\sigma} + E_{a,\rho}\frac{\partial}{\partial E_a^\sigma}$ as in (25). Again, Ω_a must obey some constraints; specifically, it must have mass dimension zero, it must vanish linearly in the limit $q \to 0$, it must be symmetric in the exchange $\mu \leftrightarrow \nu$, and it must be symmetric in the simultaneous exchange $\rho \leftrightarrow \gamma, \sigma \leftrightarrow \lambda$. The most general Ansatz satisfying these constraints is

$$\Omega_a^{\mu\nu\rho\sigma\gamma\lambda} = \frac{c_1^{(a)}}{(q \cdot p_a)^3} p_a^\mu p_a^\nu q^\rho q^\sigma q^\gamma q^\lambda + \frac{c_2^{(a)}}{(q \cdot p_a)}\eta^{\sigma(\mu}\eta^{\nu)\lambda}q^\rho q^\gamma + \frac{c_3^{(a)}}{(q \cdot p_a)}\eta^{\rho(\mu}\eta^{\nu)\gamma}q^\sigma q^\lambda + \frac{c_4^{(a)}}{(q \cdot p_a)}[\eta^{\rho(\mu}\eta^{\nu)\sigma}q^\gamma q^\lambda + \eta^{\gamma(\mu}\eta^{\nu)\lambda}q^\rho q^\sigma] + \frac{c_5^{(a)}}{(q \cdot p_a)}[\eta^{\rho(\mu}\eta^{\nu)\lambda}q^\gamma q^\sigma + \eta^{\gamma(\mu}\eta^{\nu)\sigma}q^\rho q^\lambda] + \frac{c_6^{(a)}}{(q \cdot p_a)^2} p_a^{(\mu}\eta^{\nu)(\rho}q^{\gamma)}q^\lambda q^\sigma + \frac{c_7^{(a)}}{(q \cdot p_a)^2} p_a^{(\mu}\eta^{\nu)(\lambda}q^{\sigma)}q^\rho q^\gamma. \quad (45)$$

[6] In fact it is in accordance with the expression for $S_{\rm G}^{(1)}$ given in Ref. [20] and differs by an overall normalization factor in the expression of Ref. [4].

Furthermore $S_{\rm G}^{(2)}$ must obey the distributional constraint Eq. (16) and the gauge invariance constraint on the soft leg. We recall these constraints here

$$[S_{\rm G}^{(2)},\delta^{(D)}(P)]=\frac{1}{2}S_{\rm G}^{(0)}\left((q\cdot\partial)^2\delta^{(D)}(P)\right)+\left(q\cdot\partial\delta^{(D)}(P)\right)S_{\rm G}^{(1)}+\chi'\delta^{(D)}(P), \tag{46}$$

$$S_{\rm G}^{(2)}[E\to\Lambda q]\cdot\mathcal{A}_n=0. \tag{47}$$

Imposing these constraints yields a total of five linear equations for the seven unknowns $c_j^{(a)}$ residing in the Ansatz (45) at each leg.[7] A tedious but straightforward computation shows that the solution is $\chi'=0$ and

$$\begin{aligned}S_{\rm G}^{(2)}&=\frac{1}{2}\sum_{a=1}^{n}\frac{1}{q\cdot p_a}E^{\lambda\sigma}q^\rho q^\gamma J_{a,\rho\sigma}J_{a,\gamma\lambda}\\&+\sum_{a=1}^{n}\left\{\frac{\tilde c_1^{(a)}}{q\cdot p_a}\left(\frac{(p_a\cdot E)(q\cdot E_a)}{q\cdot p_a}-E\cdot E_a\right)^2\left(q\cdot\frac{\partial}{\partial E_a}\right)^2\right.\\&+\tilde c_2^{(a)}\left(\frac{(p_a\cdot E)(q\cdot E_a)}{q\cdot p_a}-E\cdot E_a\right)\\&\left.\times\left(\frac{E\cdot p_a}{q\cdot p_a}q\cdot\frac{\partial}{\partial E_a}-E\cdot\frac{\partial}{\partial E_a}\right)\left(\frac{E_a\cdot q}{q\cdot p_a}q\cdot\frac{\partial}{\partial E_a}+q\cdot\frac{\partial}{\partial p_a}\right)\right\},\end{aligned} \tag{48}$$

where for convenience we expressed the soft graviton polarization as $E_{\mu\nu}=E_\mu E_\nu$ and $J_{a,\rho\sigma}\coloneqq p_{a,\rho}\frac{\partial}{\partial p_a^\sigma}+E_{a,\rho}\frac{\partial}{\partial E_a^\sigma}-(\rho\leftrightarrow\sigma)$ as before. In the notation of the Ansatz Eq. (45) we have $\tilde c_1^{(a)}=-c_4^{(a)}+c_6^{(a)}$ and $\tilde c_2^{(a)}=\tilde c^{(a)}$ the undetermined parameter of $S_{\rm G}^{(1)}$. Note that the first line of Eq. (48) accounts for the full known subsubleading soft factor in gravity as can be calculated from explicit amplitude expressions. The remaining terms are mixed orbital-helicity operators or only helicity operators, which are allowed by our constraints.

Again we see that the orbital part of the subsubleading soft graviton operator $S_{\rm G}^{(2)}$ is entirely determined. The polarization-dependent parts on the other hand are now determined up to two numerical factors for every hard leg; as already stated, the coefficient $\tilde c_2^{(a)}$ equals $\tilde c^{(a)}$, where the latter are the undetermined coefficients appearing in the final form of $S_{\rm G}^{(1)}$. We thus have one additional free coefficient for each hard leg. It is also worth noticing that the additional, polarization-dependent terms are manifestly gauge invariant.

V. FOUR DIMENSIONS AND SPINOR HELICITY FORMALISM

Let us now consider the four-dimensional case, where we can use the spinor-helicity formalism[8] and obtain additional constraints from little-group scalings. Those constraints are particularly easy to access in four dimensions, because the Lorentz group factorizes into two parts acting on holomorphic and antiholomorphic spinors respectively.

A. Gauge theory

Taking a positive-helicity soft gluon for concreteness, its polarization vector can be expressed in terms of a holomorphic reference spinor μ_α as

$$E_{\alpha\dot\alpha}^{(+)}=\frac{\mu_\alpha\tilde\lambda_{q,\dot\alpha}}{\langle\mu\lambda_q\rangle}. \tag{49}$$

The Ansatz for $S_{\rm YM}^{(1)}$ in spinor-helicity variables reads

$$S_{\rm YM}^{(1)}=\sum_{a=1}^{n}E_{\alpha\dot\alpha}^{(+)}\left[\Omega_a^{\alpha\dot\alpha\beta}\frac{\partial}{\partial\lambda_a^\beta}+\bar\Omega_a^{\alpha\dot\alpha\dot\beta}\frac{\partial}{\partial\tilde\lambda_a^{\dot\beta}}\right]. \tag{50}$$

To yield the correct mass dimension for $S_{\rm YM}^{(1)}$, the coefficients $\Omega_a,\bar\Omega_a$ must be of mass dimension $-\frac{1}{2}$. In addition Ω_a carries helicity $\frac{1}{2}$ and $\bar\Omega_a$ helicity $-\frac{1}{2}$ on leg a. Moreover, terms where the open index $\dot\alpha$ comes from $\tilde\lambda_q$ do not contribute to $S_{\rm YM}^{(1)}$, as those terms vanish after contracting with the polarization tensor Eq. (49). Combining these constraints, the most general Ansatz reads

$$\Omega_a^{\alpha\dot\alpha\beta}=\frac{c_1^{(a)}}{\langle aq\rangle[aq]}\lambda_a^\alpha\lambda_a^\beta\tilde\lambda_a^{\dot\alpha}, \tag{51}$$

$$\bar\Omega_a^{\alpha\dot\alpha\dot\beta}=\frac{\bar c_1^{(a)}}{\langle aq\rangle[aq]}\lambda_a^\alpha\tilde\lambda_a^{\dot\alpha}\tilde\lambda_a^{\dot\beta}+\frac{\bar c_2^{(a)}}{\langle aq\rangle[aq]}\lambda_q^\alpha\tilde\lambda_a^{\dot\alpha}\tilde\lambda_q^{\dot\beta}+\frac{\bar c_3^{(a)}}{\langle aq\rangle}\lambda_q^\alpha\epsilon^{\dot\alpha\dot\beta}. \tag{52}$$

Gauge invariance on the soft leg implies that the operator obtained by the substitution $E_q\to q$ in $S_{\rm YM}^{(1)}$ annihilates the amplitude. The resulting operator is

$$S_{\rm YM}^{(1)}[E_q\to q]=-\sum_{a=1}^{n}\left[c_1^{(a)}\lambda_a^\beta\frac{\partial}{\partial\lambda_a^\beta}+\bar c_1^{(a)}\tilde\lambda_a^{\dot\alpha}\frac{\partial}{\partial\tilde\lambda_a^{\dot\alpha}}\right]. \tag{53}$$

In principle we can allow the above operator to be any operator annihilating the n-point tree-level gluon

[7] There is actually one additional equation that identifies the coefficients $c_4^{(a)}$ in the Ansatz with the undetermined coefficients $\frac{1}{2}\tilde c^{(a)}$ appearing in the final form of $S_{\rm G}^{(1)}$, Eq. (43).

[8] See, e.g., Ref. [26] for a textbook treatment.

amplitude. Since tree-level gluon amplitudes in four dimensions are invariant under conformal transformations, we could in principle allow $c_1^{(a)} = \bar{c}_1^{(a)} = c$ for some fixed constant c for all hard legs a, so that the above operator is the dilation operator $\mathfrak{D}$ [29].[9]

The distributional constraint Eq. (14) then reads

$$\sum_{a=1}^{n}\left[2c\frac{\langle\mu a\rangle}{\langle aq\rangle\langle\mu q\rangle}\lambda_a^\alpha\tilde{\lambda}_a^{\dot\alpha} + (\bar{c}_2^{(a)} + \bar{c}_3^{(a)})\frac{1}{\langle aq\rangle}\lambda_a^\alpha\tilde{\lambda}_q^{\dot\alpha}\right]\frac{\partial}{\partial P^{\alpha\dot\alpha}}\delta^4(P) \stackrel{!}{=} \frac{\langle n1\rangle}{\langle nq\rangle\langle q1\rangle}\left(\lambda_q^\alpha\tilde{\lambda}_q^{\dot\alpha}\frac{\partial}{\partial P^{\alpha\dot\alpha}}\delta^4(P)\right) + \chi\delta^4(P),$$

where we have inserted the spinor-helicity form of $S_{\mathrm{YM}}^{(0)}$ on the rhs. Since the first term in the lhs cannot conspire to build a Lorentz generator, we see that upon using Schouten's identity the solution to this equation is[10]

$$k = 0, \qquad \chi = 0, \qquad \bar{c}_2^{(a)} + \bar{c}_3^{(a)} = \begin{cases} 1 & \text{for } a = 1, n \\ 0 & \text{otherwise.} \end{cases} \tag{54}$$

This leads to the known form for the four-dimensional subleading soft factor for gluon amplitudes,

$$S_{\mathrm{YM}}^{(1)} = \frac{\tilde{\lambda}_q^{\dot\alpha}}{\langle q1\rangle}\frac{\partial}{\partial\tilde{\lambda}_1^{\dot\alpha}} - \frac{\tilde{\lambda}_q^{\dot\alpha}}{\langle qn\rangle}\frac{\partial}{\partial\tilde{\lambda}_n^{\dot\alpha}}. \tag{55}$$

B. Gravity

For what concerns graviton amplitudes, we consider a positive-helicity soft graviton; the polarization vector is expressed in terms of two reference spinors λ_x and λ_y as

$$E_{\alpha\dot\alpha\beta\dot\beta}^{(+)} \coloneqq \frac{1}{\langle xq\rangle\langle yq\rangle}(\lambda_{x,\alpha}\lambda_{y,\beta} + \lambda_{y,\alpha}\lambda_{x,\beta})\tilde{\lambda}_{s,\dot\alpha}\tilde{\lambda}_{s,\dot\beta}. \tag{56}$$

We now consider the usual local first-order Ansatz for $S_{\mathrm{G}}^{(1)}$:

$$S_{\mathrm{G}}^{(1)} = \sum_{a=1}^{n} E_{\alpha\dot\alpha\beta\dot\beta}^{(+)}\left[\Omega_a^{\alpha\dot\alpha\beta\dot\beta\gamma}\frac{\partial}{\partial\lambda_a^\gamma} + \bar{\Omega}_a^{\alpha\dot\alpha\beta\dot\beta\dot\gamma}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot\gamma}}\right]. \tag{57}$$

Again Ω_a and $\bar{\Omega}_a$ must obey some constraints. The mass dimensions are $[\Omega_a] = [\bar{\Omega}_a] = \frac{1}{2}$, the helicity of the soft leg should be zero for both Ω_a and $\bar{\Omega}_a$, and the helicities for leg a are $\frac{1}{2}$ $(-\frac{1}{2})$ for Ω_a $(\bar{\Omega}_a)$. Moreover, the open indices $\dot\alpha, \dot\beta$ cannot come from $\tilde{\lambda}_q$, and both Ω_a and $\bar{\Omega}_a$ must be symmetric in the pairs (α, β) and $(\dot\alpha, \dot\beta)$ These constraints imply that the possible forms of Ω_a and $\bar{\Omega}_a$ are

$$\Omega_a^{\alpha\dot\alpha\beta\dot\beta\gamma} = \frac{c_1^{(a)}}{\langle aq\rangle[aq]}\tilde{\lambda}_a^{\dot\alpha}\tilde{\lambda}_a^{\dot\beta}\lambda_a^\alpha\lambda_a^\beta\lambda_a^\gamma, \tag{58}$$

$$\bar{\Omega}_a^{\alpha\dot\alpha\beta\dot\beta\dot\gamma} = \frac{\bar{c}_1^{(a)}}{\langle aq\rangle[aq]}\tilde{\lambda}_a^{\dot\alpha}\tilde{\lambda}_a^{\dot\beta}\lambda_a^\alpha\lambda_a^\beta\tilde{\lambda}_a^{\dot\gamma} + \frac{\bar{c}_2^{(a)}}{\langle aq\rangle[aq]}\tilde{\lambda}_a^{\dot\alpha}\tilde{\lambda}_a^{\dot\beta}\lambda_a^{(\alpha}\lambda_q^{\beta)}\tilde{\lambda}_q^{\dot\gamma} + \frac{\bar{c}_3^{(a)}}{\langle aq\rangle}\epsilon^{\dot\gamma(\dot\alpha}\tilde{\lambda}_a^{\dot\beta)}\lambda_a^{(\alpha}\lambda_q^{\beta)}. \tag{59}$$

An infinitesimal gauge transformation amounts to the shift

$$\lambda_x \to \lambda_x + \eta\lambda_q, \qquad \lambda_y \to \lambda_y + \eta'\lambda_q, \tag{60}$$

for some (infinitesimal) η, η'. Gauge invariance then implies that

$$c_1^{(a)} = \bar{c}_1^{(a)} = 0, \qquad \bar{c}_3^{(a)} - \bar{c}_2^{(a)} = c, \quad \forall\ a \tag{61}$$

for some universal constant c. $S_{\mathrm{G}}^{(1)}$ then reads[11]

$$S_{\mathrm{G}}^{(1)} = c\sum_{a=1}^{n}\frac{[qa]}{\langle aq\rangle}\frac{1}{\langle xq\rangle\langle yq\rangle}(\langle ax\rangle\langle qy\rangle + \langle ay\rangle\langle qx\rangle)\tilde{\lambda}_q^{\dot\gamma}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot\gamma}}. \tag{62}$$

We can now impose the distributional constraint again, Eq. (14). This reads (after using Schouten's identity)

$$\left(\frac{\partial}{\partial P^{\gamma\dot\gamma}}\delta^4(P)\right)\left[\sum_{a=1}^{n}2k\frac{\langle xa\rangle\langle ya\rangle}{\langle xq\rangle\langle yq\rangle}\frac{[qa]}{\langle aq\rangle}\lambda_q^\gamma\tilde{\lambda}_q^{\dot\gamma} - \frac{k}{\langle xq\rangle\langle yq\rangle}\tilde{\lambda}_q^{\dot\alpha}\tilde{\lambda}_q^{\dot\gamma}(\lambda_x^\alpha\lambda_y^\gamma + \lambda_y^\alpha\lambda_x^\gamma)\sum_{a=1}^{n}\lambda_{a,\alpha}\tilde{\lambda}_{a,\dot\alpha}\right] \stackrel{!}{=} \left(\lambda_q^\gamma\tilde{\lambda}_q^{\dot\gamma}\frac{\partial}{\partial P^{\gamma\dot\gamma}}\delta^4(P)\right)\sum_{a=1}^{n}\frac{\langle xa\rangle\langle ya\rangle}{\langle xq\rangle\langle yq\rangle}\frac{[aq]}{\langle aq\rangle} + \chi\delta^4(P), \tag{63}$$

where we wrote explicitly the form of $S_{\mathrm{G}}^{(0)}$ in spinor-helicity variables. Notice that the second term in the first line is zero in the distributional sense. We may therefore conclude that the solution is

$$\chi = 0, \qquad c = -\frac{1}{2}. \tag{64}$$

This fixes the form of $S_{\mathrm{G}}^{(1)}$ in Eq. (62) to be

[9] Strictly speaking, what appears here is $\mathfrak{D} - 2$; however, the constant piece could be in principle restored by adding a constant term $\sum_a E_{\alpha\dot\alpha}\frac{\lambda_a^\alpha\tilde{\lambda}_a^{\dot\alpha}}{\langle aq\rangle[aq]}$ to $S_{\mathrm{YM}}^{(1)}$. We will, however, see that, although allowed by gauge invariance, these terms are ruled out by the distributional constraint.

[10] Notice that in $S_{\mathrm{YM}}^{(1)}$ only the combination $\bar{c}_2^{(a)} + \bar{c}_3^{(a)}$ appears once $\bar{\Omega}_a$ is contracted with the polarization $E^{(+)}$, and therefore we can consider this sum as a single coefficient.

[11] Again, only the combination $\bar{c}_3^{(a)} - \bar{c}_2^{(a)}$ appears in $S_{\mathrm{G}}^{(1)}$ once the contraction with $E^{(+)}$ is performed.

$$S_{\mathrm{G}}^{(1)} = \frac{1}{2}\sum_{a=1}^{n}\frac{[aq]}{\langle aq\rangle}\left(\frac{\langle ax\rangle}{\langle qx\rangle}+\frac{\langle ay\rangle}{\langle qy\rangle}\right)\tilde{\lambda}_q^{\dot{\gamma}}\frac{\partial}{\partial\tilde{\lambda}_a^{\dot{\gamma}}}. \tag{65}$$

In fact, we have also checked that the subsubleading soft factor $S_{\mathrm{G}}^{(2)}$ in four dimensions is completely fixed by gauge invariance and the distributional constraint.

VI. DISCUSSION

In this note we analyzed constraints arising for the novel subleading soft gluon and graviton theorems in general dimensions. Next to the obvious demands of Poincaré and gauge invariance we pointed out a slightly less obvious distributional constraint arising from the unbalanced arguments of the total momentum conserving Dirac delta functions on both sides of the soft theorems. The distributional constraint *requires* the subleading soft operators to be differential operators of degree 1 (subleading) or 2 (subsubleading) in the hard momenta and relates them to the leading Weinberg soft pole function.

In the D-dimensional case we started from an Ansatz compatible with dimensional analysis and soft momentum scaling. We demonstrated that the entity of those constraints determines the subleading soft gluon and graviton differential operators as well as the subsubleading soft graviton differential operator up to a single numerical constant for every leg. The undetermined constant is related to derivatives with respect to polarizations. Arbitrary functions commuting with the delta distributions could be added to these operators and are generally unconstrained. However, taking scalings and mass dimension constraints into account assuming locality, there is nothing which can be written down at tree level.

Specializing to the four-dimensional case and employing the spinor-helicity formalism, the same line of arguments was shown to *entirely* fix the subleading differential operators. This can be traced back to the factorization of the Lorentz group in four dimensions. Upon fixing a unitary gauge, however, there might be similar arguments from little-group scalings in other dimensions.

The operators so determined match the forms established in the literature at tree level. Given that our arguments are very general the question arises whether they apply to loop amplitudes as well; certainly, Poincaré invariance, gauge invariance as well as the distributional constraint Eq. (5) continue to hold.

However, in the loop scenario we have to consider at least four novel circumstances, which are not reflected in our Ansätze for the subleading soft operators in Eqs. (30), (37) and (45). First, the loop corrections may contribute to the unconstrained functional parts of $S^{(1)}$ and $S^{(2)}$, as is in fact the case in the one-loop corrections reported in Refs. [19,20]. Interestingly, the operator $S_{\mathrm{G}}^{(2)}$ may also receive first-order differential corrections which should be related to the functional corrections to $S_{\mathrm{G}}^{(1)}$ by the distributional constraint Eq. (16). Second, we construct our Ansätze employing dimensional analysis to constrain the possible terms. The dimensionality of the couplings, however, allows for dimensionless quantities such as $\log(\frac{-\mu^2}{q\cdot p_a})$ or $\frac{q\cdot p_a}{-\mu^2}$, which so far have not been accounted for in our Ansätze. These terms arise in the IR-divergent one-loop corrections to the soft operators reported in Ref. [19]. In fact, this also introduces contributions of the form $(\log\epsilon)$ in the soft momentum expansion. Third, the loop corrections may not respect our central assumption of locality. Fourth, for gauge theory the leading soft factor $S_{\mathrm{YM}}^{(0)}$ receives loop corrections, which feed into the subleading constraint equations.

After incorporating the issues pointed out in the last paragraph, the distributional constraint might be of use in the future in order to constrain possible loop corrections to soft theorems. While our work constrains the possible forms of the subleading soft gluon and graviton operators, it would be desirable to have a deeper understanding toward the origin of the soft theorems.

ACKNOWLEDGMENTS

We thank G. Bossard, T. McLoughlin, B. Schwab and especially C. Vergu for important discussions. J. P. thanks the Pauli Center for Theoretical Studies Zürich and the Institute for Theoretical Physics at the ETH Zürich for hospitality and support in the framework of a visiting professorship. The work of M. dL. and M. R. is partially supported by Grant No. 200021-137616 from the Swiss National Science Foundation.

[1] F. Low, Phys. Rev. **110**, 974 (1958).
[2] S. Weinberg, Phys. Rev. **135**, B1049 (1964).
[3] S. Weinberg, Phys. Rev. **140**, B516 (1965).
[4] F. Cachazo and A. Strominger, arXiv:1404.4091.
[5] R. Britto, F. Cachazo, B. Feng, and E. Witten, Phys. Rev. Lett. **94**, 181602 (2005) [arXiv:hep-th/0501052].
[6] E. Casali, J. High Energy Phys. 08 (2014) 077 [arXiv:1404.5551].
[7] T. Burnett and N. M. Kroll, Phys. Rev. Lett. **20**, 86 (1968); V. Del Duca, Nucl. Phys. **B345**, 369 (1990).
[8] E. Laenen, L. Magnea, G. Stavenga, and C. D. White, J. High Energy Phys. 01 (2011) 141 [arXiv:1010.1860];

E. Laenen, G. Stavenga, and C. D. White, J. High Energy Phys. 03 (2009) 054 [arXiv:0811.2067].
[9] D. J. Gross and R. Jackiw, Phys. Rev. **166**, 1287 (1968).
[10] C. D. White, J. High Energy Phys. 05 (2011) 060 [arXiv:1103.2981].
[11] A. Strominger, arXiv:1312.2229.
[12] T. He, V. Lysov, P. Mitra, and A. Strominger, arXiv:1401.7026.
[13] H. Bondi, M. van der Burg, and A. Metzner, Proc. R. Soc. A **269**, 21 (1962); R. Sachs, Proc. R. Soc. A **270**, 103 (1962).
[14] G. Barnich and C. Troessaert, J. High Energy Phys. 05 (2010) 062 [arXiv:1001.1541].
[15] D. Kapec, V. Lysov, S. Pasterski, and A. Strominger, J. High Energy Phys. 08 (2014) 058 [arXiv:1406.3312].
[16] T. Adamo, E. Casali, and D. Skinner, arXiv:1405.5122.
[17] Y. Geyer, A. E. Lipstein, and L. Mason, arXiv:1406.1462.
[18] B. U. W. Schwab and A. Volovich, arXiv:1404.7749.
[19] Z. Bern, S. Davies, and J. Nohle, arXiv:1405.1015.
[20] S. He, Y.-t. Huang, and C. Wen, arXiv:1405.1410.
[21] A. J. Larkoski, arXiv:1405.2346.
[22] F. Cachazo and E. Y. Yuan, arXiv:1405.3413.
[23] N. Afkhami-Jeddi, arXiv:1405.3533.
[24] B. U. W. Schwab, J. High Energy Phys. 08 (2014) 062 [arXiv:1406.4172]; M. Bianchi, S. He, Y.-t. Huang, and C. Wen, arXiv:1406.5155.
[25] Z. Bern, V. Del Duca, and C. R. Schmidt, Phys. Lett. B **445**, 168 (1998) [arXiv:hep-ph/9810409].
[26] J. M. Henn and J. C. Plefka, Lect. Notes Phys. **883**, 1 (2014); H. Elvang and Y. -t. Huang, arXiv:1308.1697.
[27] Z. Bern, L. J. Dixon, M. Perelstein, and J. Rozowsky, Nucl. Phys. **B546**, 423 (1999) [arXiv:hep-th/9811140].
[28] S. Weinberg, *The Quantum Theory of Fields: Foundations* (Cambridge University Press, Cambridge, England, 1995), Vol. 1.
[29] E. Witten, Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].