$\mathcal{N}=1$ duality in the chiral limit from $\mathcal{N}=2$ duality

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We study a deformation of $\mathcal{N}=2$ supersymmetric QCD with a $U(N)$ gauge group and N_f number of quark flavors induced by the mass term μ for the adjoint matter which breaks supersymmetry down to $\mathcal{N}=1$ QCD. Recently this deformation was shown to lead to a weakly coupled dual theory only in two particular sets of vacua: the $r=N$ vacuum and the so-called zero vacua which can be found at $r<N_f-N$, where r is the number of condensed quarks. For small quark masses and intermediate values of μ, the gauge group of the dual theory is $U(N_f-N)\times U(1)^{2N-N_f}$, where the Abelian sector is heavy and can be integrated out. However, at larger values of μ, the Abelian sector enters the strong coupling regime. We show that the 't Hooft matching conditions in the chiral limit require the Seiberg neutral meson field M from this sector to become light. In the $r=N$ vacuum, M is constructed of a monopole and an antimonopole connected by confining magnetic strings, while in the zero vacua, it is built of a quark and antiquark connected by confining electric strings.

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I. INTRODUCTION

Some time ago, we started [1] a program of detailing Seiberg's duality [2,3] in $\mathcal{N}=1$ theories introducing masses for the matter fields and exploring diverse discrete vacua using additional information (see also Refs. [4,5] and additional references below) following from the Seiberg-Witten solution [6,7] of the $\mathcal{N}=2$ theory. Despite spectacular overall progress, one particular corner of the parameter space—namely, its chiral limit—has not yet been studied, as was noted in Ref. [8]. This paper is devoted to thorough studies of the chiral limit, and thus completes the program. The picture of the Seiberg duality emerging on the basis of deformations of the $\mathcal{N}=2$ theory is fully self-consistent. It provides a clear-cut understanding of the processes on both sides of duality.

Seiberg's dual of $\mathcal{N}=1$ supersymmetric QCD (SQCD) with the $SU(N)$ gauge group and N_f quark flavors is a theory with the $SU(\tilde{N})$ gauge group and the same number of dual quarks, plus a neutral meson field M_A^B. Here

$$\tilde{N}\equiv N_f-N. \tag{1.1}$$

Seiberg's duality was generalized to $\mathcal{N}=2$ supersymmetric QCD deformed by the mass term μ for the adjoint matter in the large-μ limit [9]. At large μ, the adjoint matter can be integrated out, leading to an $\mathcal{N}=1$ QCD-like theory with a quartic superpotential suppressed at large μ [9–12]. This theory has the same number of vacua as that in the original $\mathcal{N}=2$ QCD in the small-μ limit. These vacua—the so-called r vacua—are characterized by a parameter r, the number of condensed (s)quarks in the classical domain of large and generic quark mass parameters m_A ($A=1,\ldots,N_f$). Clearly, r cannot exceed N, the rank of the gauge group. In the original formulation [2,3], Seiberg's duality was suggested for the monopole vacua with $r=0$ (all other vacua become runaway vacua in the limit $\mu\to\infty$).

Chronologically, the first attempt to obtain Seiberg's duality from μ-deformed $\mathcal{N}=2$ QCD can be traced back to Ref. [10]. The dual gauge group $SU(\tilde{N})$ was identified at the root of the baryonic branch.[1] However, Seiberg's neutral mesonic fields M were not detected.

Much later, we studied a version of the theory with the $U(N)$ gauge group and $(N+1)<N_f<3/2N$. We demonstrated that the μ deformation leads to a weakly coupled dual theory only for two particular sets of the vacuum states—namely, in the $r=N$ vacuum and in the so-called zero vacua [5,13]. The latter can be found at $r<\tilde{N}$.

Both sets of vacua have vanishing gaugino condensates in the limit, in which the values of the quark masses become small. In other vacua (the so-called Λ vacua), the gaugino condensate is of the order of $\mu\Lambda^2_{\mathcal{N}=2}$, where $\Lambda_{\mathcal{N}=2}$ is the scale of $\mathcal{N}=2$ QCD. The gaugino condensate becomes large in the large-μ limit. Correspondingly, these vacua do not have a weakly coupled dual description [13].

The gauge group of the dual theory in the $r=N$ and zero vacua is

$$U(\tilde{N})\times U(1)^{N-\tilde{N}}.$$

For small quark masses and intermediate values of μ, namely

[1]It corresponds to the $r=N$ quark vacuum in the $U(N)$ version of the theory we consider in this paper.

$$m_A \ll \mu \ll \Lambda_{\mathcal{N}=2},$$

the vacuum expectation values (VEVs) of the charged scalar fields in the $U(\tilde{N})$ sector are determined by parameters

$$\xi^{\text{small}} \sim \mu m, \tag{1.2}$$

while the VEVs in the Abelian $U(1)^{N-\tilde{N}}$ sector are determined by

$$\xi^{\text{large}} \sim \mu \Lambda_{\mathcal{N}=2}. \tag{1.3}$$

Given that $m \ll \Lambda_{\mathcal{N}=2}$, the notation in Eqs. (1.2) and (1.3) is self-evident.

The dual theory is infrared free; at intermediate values of μ, both scales—ξ^{small} and ξ^{large}—are small enough to ensure weak coupling. However, the Abelian sector is much heavier and thus can be integrated out. Moreover, since $\sqrt{\xi^{\text{small}}} \ll \mu$ in this domain, the adjoint matter is also heavy and can be integrated out too. This leads to a weakly coupled low-energy dual theory with the $U(\tilde{N})$ Seiberg dual gauge group and charged light matter [5,13]. Although the correct Seiberg dual gauge group emerges in this setup, Seiberg's neutral-meson M fields are still missing. As we will see below, they will show up in the chiral limit.

To this end, we make the next step and consider larger values of μ,

$$\mu \gg \Lambda_{\mathcal{N}=2}.$$

In this domain, we pass to the chiral limit, or small quark masses, keeping the parameter ξ^{small} fixed and small enough to ensure the weak coupling in the $U(\tilde{N})$ sector. At the same time, the Abelian $U(1)^{N-\tilde{N}}$ sector enters a strong coupling regime. Then we use the 't Hooft anomaly-matching conditions [14] to show that neutral M mesons coming from this sector must become light. We find a physical interpretation of the Seiberg M mesons: in the $r = N$ vacuum, M is constructed of a monopole and antimonopole connected by confining magnetic strings, while in the zero vacua, M is constructed of a quark and antiquark connected by confining electric strings. The match of our dual description in these sets of vacua with Seiberg's dual theory becomes complete.

In the first part of the paper (Secs. II and III), we briefly summarize our previous results on r duality outside the chiral limit, emphasizing its peculiarities, such as the "instead-of-confinement" mechanism. In Sec. IV, we pass to the exploration of the chiral limit and discover that the neutral Seiberg M_A^B mesons show up in the light sector. Thus, r duality proves to be completely woven into the fabric of Seiberg's duality.

The paper is organized as follows: In Sec. II, we review duality and the instead-of-confinement mechanism in an $r = N$ vacuum in the $\mathcal{N}=2$ limit of small μ. In Sec. III, we review the dual theory at intermediate μ. Next, in Sec. IV, we consider large μ and use anomaly-matching conditions to show that monopole-antimonopole stringy mesons originating from the Abelian $U(1)^{N-\tilde{N}}$ sector of the theory should become light. We also present the dual low-energy theory in this region and discuss its mass spectrum. In Sec. V, we review the low-energy description in r vacua with $r < N_f/2$ at small μ. In Sec. VI, we consider a subset of these vacua—namely, zero vacua at intermediate and large μ—and show that stringy quark-antiquark mesonic states should become light as we increase μ. Sec. VII contains our summary and conclusions.

II. DUALITY IN THE $r = N$ VACUUM AT SMALL μ

In this section, we briefly review non-Abelian duality in the $r = N$ vacua at small μ established in Refs. [4,15]. The gauge symmetry of our basic model is

$$U(N) = SU(N) \times U(1).$$

In the absence of deformation, the model under consideration is $\mathcal{N}=2$ SQCD with N_f massive quark hypermultiplets. We assume that $N_f > N + 1$, but $N_f < \frac{3}{2}N$. The latter inequality ensures that the dual theory can be infrared free.

Our basic theory is described in detail in our previous papers (e.g., Refs. [16,17]; see also the reviews in Ref. [18]). The field content is as follows: The $\mathcal{N}=2$ vector multiplet consists of the $U(1)$ gauge field A_μ and the $SU(N)$ gauge field A_μ^a ($a = 1, \ldots, N^2 - 1$) and their Weyl fermion superpartners; plus the complex scalar fields a and a^a, and their Weyl superpartners, respectively.

As for the matter sector, the N_f quark multiplets of the $U(N)$ theory consist of the complex scalar fields q^{kA} and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners—all in the fundamental representation of the $SU(N)$ gauge group. Here $k = 1, \ldots, N$ is the color index, while A is the flavor index, $A = 1, \ldots, N_f$. We will treat q^{kA} and $\tilde{q}_{Ak}$ as rectangular matrices with N rows and N_f columns.

In addition, we introduce the mass term μ for the adjoint matter breaking $\mathcal{N}=2$ supersymmetry down to $\mathcal{N}=1$. This deformation term

$$\mathcal{W}_{\text{def}} = \mu \, \text{Tr}\Phi^2, \qquad \Phi \equiv \frac{1}{2}\mathcal{A} + T^a \mathcal{A}^a \tag{2.1}$$

does not break $\mathcal{N}=2$ supersymmetry in the small-μ limit; see Refs. [16,19,20]. At large μ, this theory obviously flows to $\mathcal{N}=1$. The fields $\mathcal{A}$ and $\mathcal{A}^a$ in Eq. (2.1) are chiral superfields, the $\mathcal{N}=2$ superpartners of the $U(1)$ and $SU(N)$ gauge bosons.

A. The $r = N$ vacuum at large ξ

This theory has a set of r vacua, where r is the number of condensed (s)quarks in the classical domain of large generic quark masses m_A ($A=1,\ldots,N_f$, and $r\le N$). In the first part of this paper, we consider the $r=N$ vacua (for a review, see Ref. [18]). These vacua have the maximal possible number of condensed quarks, $r=N$. Moreover, the gauge group $U(N)$ is completely Higgsed in these vacua, and as a result, they support non-Abelian strings [16,21–23]. The occurrence of these strings ensures the confinement of monopoles in these vacua.

First, we will assume that μ is small, much smaller than the quark masses:

$$|\mu|\ll|m_A|,\quad A=1,\ldots,N_f. \tag{2.2}$$

In the quasiclassical region of large quark masses, scalar quarks develop VEVs triggered by the deformation parameter μ. They are given by

$$\langle q^{kA}\rangle=\langle\bar{\tilde{q}}^{kA}\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots & \sqrt{\xi_N} & 0 & \ldots & 0\end{pmatrix},$$

$$k=1,\ldots,N,\quad A=1,\ldots,N_f, \tag{2.3}$$

where we present the quark fields as matrices in the color (k) and flavor (A) indices, while the parameters ξ are given in the quasiclassical approximation by

$$\xi_P\approx 2\mu m_P,\quad P=1,\ldots,N. \tag{2.4}$$

The quark condensate [Eq. (2.3)] results in the spontaneous breaking of both the gauge and flavor symmetries. A diagonal global $SU(N)$ combining the gauge $SU(N)$ and an $SU(N)$ subgroup of the flavor $SU(N_f)$ group survives in the limit of (almost) equal quark masses. This is color-flavor locking.

Thus, the unbroken global symmetry is as follows:

$$SU(N)_{C+F}\times SU(\tilde{N})\times U(1). \tag{2.5}$$

Here $SU(N)_{C+F}$ is a global unbroken color-flavor rotation, which involves the first N flavors, while the $SU(\tilde{N})$ factor stands for the flavor rotation of the $\tilde{N}$ quarks.

The presence of the global $SU(N)_{C+F}$ group is the reason for the formation of the non-Abelian strings [16,17,21–23]. At small μ, these strings are BPS saturated [19,20], and their tensions are determined by the parameters ξ_P [17]; see Eq. (2.4):

$$T_P=2\pi|\xi_P|,\quad P=1,\ldots,N. \tag{2.6}$$

These strings confine monopoles. In fact, in the $U(N)$ theories, confined elementary monopoles are junctions of two “neighboring” Pth and $(P+1)$th strings; see Ref. [18] for a review.

Now, let us briefly discuss the perturbative excitation spectrum. Since both the $U(1)$ and $SU(N)$ gauge groups are broken by the squark condensation, all gauge bosons become massive.

To the leading order in μ, $\mathcal{N}=2$ supersymmetry is not broken. In fact, with nonvanishing ξ_P's [see Eq. (2.4)], both the quarks and the adjoint scalars combine with the gauge bosons to form long $\mathcal{N}=2$ supermultiplets [20]. In the equal-mass limit, $\xi_P\equiv\xi$, and all states come in representations of the unbroken global group [Eq. (2.5)]—namely, in the singlet and adjoint representations of $SU(N)_{C+F}$,

$$(1,1),\quad (N^2-1,1), \tag{2.7}$$

and in the bifundamental representations

$$(\bar{N},\tilde{N}),\quad (N,\bar{\tilde{N}}). \tag{2.8}$$

The representations in Eqs. (2.7) and (2.8) are marked with respect to two non-Abelian factors in Eq. (2.5). The singlet and adjoint fields are (i) the gauge bosons and (ii) the first N flavors of the squarks q^{kP} ($P=1,\ldots,N$), together with their fermion superpartners. The bifundamental fields are the quarks q^{kK} with $K=N+1,\ldots,N_f$. Quarks transform in the two-index representations of the global group [Eq. (2.5)] due to the color-flavor locking.

The above quasiclassical analysis is valid if the theory is at weak coupling. From Eq. (2.3), we see that the weak coupling condition is

$$\sqrt{\xi}\sim\sqrt{\mu m}\gg\Lambda_{\mathcal{N}=2}, \tag{2.9}$$

where we assume all quark masses to be of the same order, $m_A\sim m$. This condition means that the quark masses are large enough to compensate the smallness of μ.

B. r-dual theory

Now we will relax the condition in Eq. (2.9) and pass to the strong coupling domain at

$$|\sqrt{\xi_P}|\ll\Lambda_{\mathcal{N}=2},\qquad |m_A|\ll\Lambda_{\mathcal{N}=2}, \tag{2.10}$$

still keeping μ small.

As was shown in Refs. [4,5], in the $r=N$ vacuum, $\mathcal{N}=2$ QCD undergoes a crossover transition as the value of ξ decreases. The domain [Eq. (2.10)] can be described in terms of weakly coupled (infrared free) r-dual theory with the gauge group

$$U(\tilde{N})\times U(1)^{N-\tilde{N}} \tag{2.11}$$

and N_f flavors of light quark-like dyons.[2] Note that we call our dual theory the “r dual” because the $\mathcal{N}=2$ duality

[2]Previously, the $SU(\tilde{N})$ gauge group was identified [10] at the root of the baryonic Higgs branch in the $\mathcal{N}=2$ supersymmetric $SU(N)$ Yang-Mills theory with massless quarks and vanishing ξ parameters.

described here can be generalized to other r vacua with $r > N_f/2$. This leads to a theory with the dual gauge group $U(N_f - r) \times U(1)^{N-N_f+r}$ [24]. However, the deformation of these r-dual theories to $\mathcal{N} = 1$ theory at larger μ can be performed within the weak coupling regime only in the $r = N$ vacuum [13], which we discuss here.

The light dyons D^{lA} ($l = 1, \ldots, \tilde{N}$ and $A = 1, \ldots, N_f$) are in the fundamental representation of the gauge group $SU(\tilde{N})$ and are charged under the Abelian factors indicated in Eq. (2.11). In addition, there are $(N - \tilde{N})$ light dyons D^J ($J = \tilde{N} + 1, \ldots, N$), neutral under the $SU(\tilde{N})$ group, but charged under the $U(1)$ factors.

The color charges of all these dyons are identical to those of quarks.[3] This is the reason we call them quark-like dyons. However, these dyons are not quarks [4]. As we will review below, they belong to a different representation of the global color-flavor locked group. Most importantly, condensation of these dyons still leads to the confinement of monopoles.

The dyon condensates have the form [5,17]

$$\langle D^{lA} \rangle = \langle \bar{\tilde{D}}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \ldots & 0 & \sqrt{\xi_1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & 0 & \ldots & \sqrt{\xi_{\tilde{N}}} \end{pmatrix}, \tag{2.12}$$

$$\langle D^J \rangle = \langle \bar{\tilde{D}}^J \rangle = \sqrt{\frac{\xi_J}{2}}, \quad J = (\tilde{N} + 1), \ldots, N. \tag{2.13}$$

The important feature apparent in Eq. (2.12), as compared to the squark VEVs in the original theory [Eq. (2.3)], is a "vacuum leap" [4]. Namely, if we pick up the vacuum with nonvanishing VEVs of the first N quark flavors in the original theory at large ξ, and then reduce ξ below $\Lambda_{\mathcal{N}=2}$, the system goes through a crossover transition and ends up in the vacuum of the r-dual theory with the dual gauge group [Eq. (2.11)] and nonvanishing VEVs of the $\tilde{N}$ last dyons [plus VEVs of the $(N - \tilde{N})$ dyons that are the $SU(\tilde{N})$ singlets].

The parameters ξ_P in Eqs. (2.12) and (2.13) are determined by the quantum version of the classical expressions in Eq. (2.4) [17]. They can be expressed in terms of the roots of the Seiberg-Witten curve [6,7]. The Seiberg-Witten curve in our theory has the form [10]

$$y^2 = \prod_{P=1}^{N} (x - \phi_P)^2 - 4 \left(\frac{\Lambda_{\mathcal{N}=2}}{\sqrt{2}} \right)^{N-\tilde{N}} \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right), \tag{2.14}$$

[3]Because of monodromies [6,7,25], the quarks pick up root-like color-magnetic charges in addition to their weight-like color-electric charges at strong coupling [4].

where ϕ_P are gauge-invariant parameters on the Coulomb branch.

In the $r = N$ vacuum, the curve in Eq. (2.14) has N double roots and reduces to

$$y^2 = \prod_{P=1}^{N} (x - e_P)^2. \tag{2.15}$$

This reflects the condensation of N quarks. Quasiclassically, at large masses, e_P's are given by the mass parameters, $\sqrt{2} e_P \approx -m_P$ ($P = 1, \ldots, N$).

The dyon condensates [Eq. (2.12)] at small masses in the $r = N$ vacuum are determined by [5,17]

$$\xi_P = -2\sqrt{2}\mu e_P. \tag{2.16}$$

As long as we keep ξ_P and the masses small enough [i.e., in the domain given in Eq. (2.10)], the coupling constants of the infrared-free r-dual theory (frozen at the scale of the dyon VEVs) are small: the r-dual theory is at weak coupling.

At small masses, in the region given in Eq. (2.10), the double roots of the Seiberg-Witten curve are

$$\begin{aligned} &\sqrt{2} e_I = -m_{I+N}, \\ &\sqrt{2} e_J = \Lambda_{\mathcal{N}=2} \exp\left(\frac{2\pi i}{N - \tilde{N}} J \right), \\ &I = 1, \ldots, \tilde{N}, \quad \text{and} \\ &J = (\tilde{N} + 1), \ldots, N. \end{aligned} \tag{2.17}$$

In particular, the $\tilde{N}$ first roots are determined by the masses of the last $\tilde{N}$ quarks—a reflection of the fact that the non-Abelian sector of the dual theory is infrared free and is at weak coupling in the domain of Eq. (2.10).

C. "Instead-of-confinement" mechanism

Now, we will consider the limit of almost equal quark masses. Both the gauge group and the global flavor $SU(N_f)$ group are broken in the vacuum. However, the form of the dyon VEVs in Eq. (2.12) shows that the r-dual theory is also in the color-flavor locked phase. Namely, the unbroken global group of the dual theory is

$$SU(N) \times SU(\tilde{N})_{C+F} \times U(1), \tag{2.18}$$

where this time the $SU(\tilde{N})$ global group arises from color-flavor locking.

In much the same way as in the original theory, the presence of the global $SU(\tilde{N})_{C+F}$ symmetry is the reason behind formation of the non-Abelian strings. Their tensions are still given by Eq. (2.6), where the parameters ξ_P are determined by Eq. (2.16) [5,17]. These strings still confine monopoles [4].[4]

In the equal-mass limit, the global unbroken symmetry [Eq. (2.18)] of the dual theory at small ξ coincides with the global group [Eq. (2.5)] of the original theory in the $r=N$ vacuum at large ξ. However, this global symmetry is realized in two very distinct ways in the dual pair at hand. As was already mentioned, the quarks and $U(N)$ gauge bosons of the original theory at large ξ come in the following representations of the global group [Eq. (2.5)]:

$$(1,1),(N^2-1,1),(\bar{N},\tilde{N}),\quad \text{and}\quad (N,\bar{\tilde{N}}).$$

At the same time, the dyons and $\mathrm{U}(\tilde{N})$ gauge bosons of the r-dual theory form

$$(1,1),(1,\tilde{N}^2-1),\quad (N,\bar{\tilde{N}}),\quad \text{and}\quad (\bar{N},\tilde{N}) \tag{2.19}$$

representations of Eq. (2.18). We see that the adjoint representations of the $(C+F)$ subgroup are different in two theories.

The quarks and gauge bosons which form the adjoint (N^2-1) representation of $SU(N)$ at large ξ, and the quark-like dyons and dual gauge bosons which form the adjoint $(\tilde{N}^2-1)$ representation of $SU(\tilde{N})$ at small ξ are, in fact, *distinct* states [4].

Thus, the quark-like dyons are not quarks. At large ξ, they are heavy solitonic states. However, below the crossover, at small ξ, they become light and form the fundamental "elementary" states D^{lA} of the r-dual theory. Vice versa, quarks are light at large ξ but become heavy below the crossover.

This raises the question: what exactly happens with quarks when we reduce ξ?

They are in the "instead-of-confinement" phase. The Higgs-screened quarks and gauge bosons at small ξ decay into the monopole-antimonopole pairs on the curves of marginal stability (the so-called wall crossing) [4,15]. The general rule is that the only states that exist at strong coupling inside the curves of marginal stability are those which can become massless on the Coulomb branch [6,7,25]. For the r-dual theory, these are light dyons shown in Eq. (2.12), gauge bosons of the dual gauge group, and monopoles.

At small nonvanishing values of ξ, the monopoles and antimonopoles produced in the decay process of the adjoint $(N^2-1,1)$ states cannot escape from each other and fly to opposite infinities because they are confined. Therefore, the (screened) quarks and gauge bosons evolve into stringy mesons (in the strong coupling domain of small ξ) as shown in Fig. 1—namely, monopole-antimonopole pairs connected by two strings [4,5].

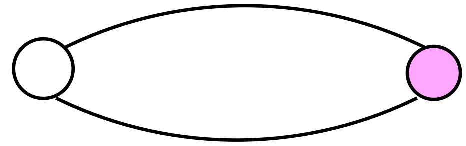

FIG. 1 (color online). Meson formed by a monopole-antimonopole pair connected by two strings. Open and closed circles denote the monopole and antimonopole, respectively.

The flavor quantum numbers of stringy monopole-antimonopole mesons were studied in Ref. [15] in the framework of an appropriate two-dimensional $CP(N-1)$ model which describes world-sheet dynamics of the non-Abelian strings [16,21–23]. In particular, confined monopoles are seen as kinks in this world-sheet theory. If two strings in Fig. 1 are "neighboring" strings P and $P+1$ $[P=1,\ldots,(N-1)]$, each meson is in the two-index representation $M_A^B(P,P+1)$ of the flavor group, where the flavor indices are $A,B=1,\ldots,N_f$. It splits into singlet, adjoint, and bifundamental representations of the global unbroken group [Eq. (2.18)]. In particular, at small ξ, the adjoint representation of $SU(N)$ contains former (screened) quarks and gauge bosons of the original theory.

Masses of these stringy mesons are determined by string tensions given by the parameters ξ_P and ξ_{P+1}; see Eqs. (2.16) and (2.17). In particular, in the r-dual theory the tensions of $\tilde{N}$ non-Abelian strings from the $U(\tilde{N})$ sector are light, of the order of $\xi^{\text{small}}\sim\mu m$, while the tensions of $(N-\tilde{N})$ "Abelian" strings from the $U(1)^{N-\tilde{N}}$ sector are much heavier, of the order of $\xi^{\text{large}}\sim\mu\Lambda_{\mathcal{N}=2}$. The majority of stringy mesons are unstable and decay into each other or into the "elementary" states [Eq. (2.19)] of the r-dual theory, the dyons and gauge bosons. For example, the mesons $M_A^B(P,P+1)$ which form representations [Eq. (2.19)] can decay into elementary states with the same quantum numbers [4,15].

III. INTERMEDIATE μ

In this section, we will discuss what happens to the r-dual theory in the $r=N$ vacuum described above once we increase μ to intermediate values, which are large enough to decouple the adjoint matter [5,24]. We also discuss the relation of our dual theory to Seiberg's dual.

A. Emergence of the $U(\tilde{N})$ gauge group

Combining Eqs. (2.12), (2.13), (2.16), and (2.17), we see that the VEVs of the non-Abelian dyons D^{lA} are determined by

[4]An explanatory remark regarding our terminology is in order. Strictly speaking, the dyons carrying root-like electric charges are confined as well. We refer to all such states collectively as "monopoles." This is to avoid confusion with the quark-like dyons which appear in Eqs. (2.12) and (2.13). The latter dyons carry weight-like electric charges. As was already mentioned, their color charges are identical to those of quarks; see Ref. [4] for further details.

$$\sqrt{\xi^{\text{small}}} \sim \sqrt{\mu m} \tag{3.1}$$

and are much smaller than the VEVs of the Abelian dyons D^J in the domain given in Eq. (2.10). The latter are of the order of

$$\sqrt{\xi^{\text{large}}} \sim \sqrt{\mu \Lambda_{\mathcal{N}=2}}. \tag{3.2}$$

This circumstance is most crucial. It allows us to increase μ and decouple the adjoint fields without violating the weak coupling condition in the dual theory [5].

Let us uplift μ to the intermediate domain:

$$|\mu| \gg |m_A|, \quad A = 1, \ldots, N_f, \quad \mu \ll \Lambda_{\mathcal{N}=2}. \tag{3.3}$$

The VEVs of the Abelian dyons [Eq. (2.13)] are large. This makes $U(1)$ gauge fields of the dual group [Eq. (2.11)] heavy. Decoupling these gauge factors, together with the adjoint matter and the Abelian dyons themselves, we obtain the low-energy theory with the

$$U(\tilde{N}) \tag{3.4}$$

gauge fields and the following set of non-Abelian dyons: D^{lA} ($l = 1, \ldots, \tilde{N}$, $A = 1, \ldots, N_f$). The superpotential for D^{lA} has the form [5]

$$\mathcal{W} = -\frac{1}{2\mu}(\tilde{D}_A D^B)(\tilde{D}_B D^A) + m_A(\tilde{D}_A D^A), \tag{3.5}$$

where the color indices are contracted inside each parenthesis. Minimization of this superpotential leads to the VEVs [Eq. (2.12)] of non-Abelian dyons determined by ξ^{small}; see Eq. (2.17).

Below the scale μ, our theory becomes dual to $\mathcal{N} = 1$ SQCD with the scale

$$\tilde{\Lambda}^{N-2\tilde{N}}_{\mathcal{N}=1} = \frac{\Lambda^{N-\tilde{N}}_{\mathcal{N}=2}}{\mu^{\tilde{N}}}. \tag{3.6}$$

In order to keep this infrared-free theory in the weak coupling regime, we impose that

$$|\sqrt{\mu m}| \ll \tilde{\Lambda}_{\mathcal{N}=1}. \tag{3.7}$$

This means that at large μ, we must keep the quark masses sufficiently small.

Let us briefly summarize the mass spectrum of our $U(\tilde{N})$ r-dual theory at intermediate μ [5]. The lightest states are $4N\tilde{N}$ bifundamental dyons (we count real bosonic degrees of freedom). Their masses are of the order of quark mass differences $(m_A - m_B)$. Half of the dyons, namely $2\tilde{N}^2$, from singlet and adjoint representations of $SU(\tilde{N})$, are also light with masses of the order of $m \sim m_A$. Another $\tilde{N}^2$ dyonic states become scalar superpartners for the massive gauge bosons of the $U(\tilde{N})$ gauge group (altogether $4\tilde{N}^2$ states). These are much heavier, with masses of the order of $\tilde{g}\sqrt{\xi^{\text{small}}}$, where $\tilde{g}$ is the gauge coupling constant of the r-dual theory. On top of that we have stringy monopole-antimonopole mesons (see Fig. 1) $M^B_A(P, P+1)$, where $P = 1, \ldots, (\tilde{N} - 1)$, while $A, B = 1, \ldots, N_f$. Their masses are of the order of $\sqrt{\xi^{\text{small}}}$; they are determined by tensions of light non-Abelian strings.

Note that in the intermediate domain of μ [Eq. (3.3)], we assume that $\mu \ll \Lambda_{\mathcal{N}=2}$. This condition ensures that the heavy Abelian $U(1)^{(N-\tilde{N})}$ sector is at weak coupling too and really heavy. At weak coupling, the masses of the states in this sector can be determined in the quasiclassical approximation. They are of the order of $g_{U(1)}\sqrt{\xi^{\text{large}}}$ for "elementary" states, where $g_{U(1)}$ are couplings in the $U(1)$ factors, and are of the order of $\sqrt{\xi^{\text{large}}}$ for stringy mesons $M^B_A(P, P+1)$ with $P = \tilde{N}, \ldots, (N-1)$.

If we relax the condition $\mu \ll \Lambda_{\mathcal{N}=2}$, this sector enters a strong coupling regime, and certain states could in principle become light and couple to our low-energy $U(\tilde{N})$ theory. We will see in the next section that this is exactly what happens at larger values of μ and is, in fact, required by the 't Hooft anomaly matching [14].

B. Connection to Seiberg's duality

The gauge group of our r-dual theory is $U(\tilde{N})$, the same as the gauge group of Seiberg's dual theory [2,3]. This suggests that there should be a close relation between two duals. For intermediate values of μ, this relation was found in Refs. [13,26].

Originally Seiberg's duality was formulated for $\mathcal{N} = 1$ SQCD, which in our setup corresponds to the limit $\mu \to \infty$. Therefore, in the original formulation, Seiberg's duality referred to the monopole vacua with $r = 0$. Other vacua, with $r \neq 0$, have condensates of r quark flavors $\langle \tilde{q}q \rangle_A \sim \mu m_A$, and therefore disappear in the limit $\mu \to \infty$: they become runaway vacua. However, as was already mentioned in Sec. I, Seiberg's duality can be generalized to the μ-deformed $\mathcal{N} = 2$ QCD [9,12]. At large μ, μ-deformed $\mathcal{N} = 2$ QCD flows to $\mathcal{N} = 1$ QCD with an additional quartic quark superpotential. This theory has all r vacua which were present in the original $\mathcal{N} = 2$ QCD in the small-μ limit. The generalized Seiberg dual theory for the μ-deformed $U(N)$ $\mathcal{N} = 2$ SQCD at large but finite μ has the gauge group $U(\tilde{N})$, N_f flavors of Seiberg's "dual quarks" h^{lA} ($l = 1, \ldots, \tilde{N}$ and $A = 1, \ldots, N_f$), and the superpotential

$$\mathcal{W}_S = -\frac{\kappa^2}{2\mu}\,\mathrm{Tr}(M^2) + \kappa m_A M^A_A + \tilde{h}_{Al} h^{lB} M^A_B, \tag{3.8}$$

where M^B_A is Seiberg's neutral mesonic field, defined as

$$(\tilde{q}_A q^B) = \kappa M_A^B. \tag{3.9}$$

Here, κ is a parameter of the dimension of mass needed to formulate Seiberg's duality [2,3]. The last two terms in Eq. (3.8) were originally suggested by Seiberg, while the first term is a generalization to finite μ which originates from the quartic quark potential [9,12].

Now let us assume the fields M_A^B to be heavy and integrate them out. This implies that κ is large. Integrating out the M fields in Eq. (3.8), we get

$$\mathcal{W}_S^{\rm LE} = \frac{\mu}{2\kappa^2}(\tilde{h}_A h^B)(\tilde{h}_B h^A) + \frac{\mu}{\kappa} m_A (\tilde{h}_A h^A). \tag{3.10}$$

The change of variables

$$D^{lA} = \sqrt{-\frac{\mu}{\kappa}} h^{lA}, \quad l = 1, \ldots, \tilde{N}, \quad A = 1, \ldots, N_f \tag{3.11}$$

brings this superpotential to the form

$$\mathcal{W}_S^{\rm LE} = \frac{1}{2\mu}(\tilde{D}_A D^B)(\tilde{D}_B D^A) - m_A (\tilde{D}_A D^A). \tag{3.12}$$

We see that (up to a sign) this superpotential coincides with the superpotential of our r-dual theory [Eq. (3.5)]. As was already mentioned, the dual gauge groups also coincide for Seiberg's and r-dual theories in the $r = N$ vacuum. Note that the kinetic terms are not known in Seiberg's dual theory; thus, normalization of the h fields is not fixed.

We see that the r-dual and Seiberg's dual theories match. However, it seems that this match is not complete. The mesonic field M_A^B is supposed to be light in the Seiberg duality.

It seems there is no apparent candidate for a light neutral field with these flavor quantum numbers in the r-dual theory. Moreover, the match outlined above assumes that the M field is heavy and can be integrated out.

In principle, there are candidates for the Seiberg M field with correct flavor quantum numbers in the r-dual theory. These are the monopole-antimonopole stringy mesons $M_A^B(P, P+1)$ from the Abelian sector with $P = \tilde{N}, \ldots, (N-1)$. They could produce the Seiberg M field.

But at intermediate μ (3.3), the $U(1)^{(N-\tilde{N})}$ Abelian sector is at weak coupling. This ensures that the masses of the Abelian $M_A^B(P, P+1)$ mesons can be determined quasiclassically. As was discussed in Sec. III A, they are of the order of $\sqrt{\xi^{\rm large}}$ and cannot possibly become light. We will come back to this issue in Sec. IV B.

The resolution of this puzzle is that Seiberg's duality refers to much larger values of μ than those given by the upper bound in Eq. (3.3). In fact, the generalized Seiberg duality assumes that

$$\mu \gg \Lambda_{\mathcal{N}=1}, \tag{3.13}$$

where $\Lambda_{\mathcal{N}=1}$ is the scale of the original $\mathcal{N}=1$ QCD,

$$\Lambda_{\mathcal{N}=1}^{2N-\tilde{N}} = \mu^N \Lambda_{\mathcal{N}=2}^{N-\tilde{N}}. \tag{3.14}$$

The domain given in Eq. (3.13) is above the intermediate-μ domain considered in this section.

This leads us to the conclusion that at intermediate μ, we have a perfect match between the r-dual and Seiberg's dual theories. In this domain, the Seiberg M meson is heavy and should be integrated out, implying the superpotential in Eq. (3.12), which agrees with the superpotential in Eq. (3.5) obtained in the r-dual theory.

This match, together with the identification in Eq. (3.11), reveals the physical nature of Seiberg's "dual quarks." They are not monopoles, as naive duality suggests. Instead, they are quark-like dyons appearing in the r-dual theory below the crossover. Their condensation leads to the confinement of monopoles and the instead-of-confinement phase [24] for quarks and gauge bosons of the original theory.

IV. LARGE μ

Now we turn to the large-μ domain. Increasing μ, we simultaneously reduce m while keeping $\xi^{\rm small}$ sufficiently small; see Eq. (3.7). Namely, we assume

$$\xi^{\rm small} \sim \mu m \ll \tilde{\Lambda}_{\mathcal{N}=1}, \quad \mu \gg \Lambda_{\mathcal{N}=1}. \tag{4.1}$$

This ensures that our low-energy $U(\tilde{N})$ r-dual theory is at weak coupling. However, the Abelian $U(1)^{(N-\tilde{N})}$ sector ultimately enters the strong coupling regime. As was already mentioned, we lose analytic control over this sector and, in particular, certain states can become light and couple to our low-energy $U(\tilde{N})$ theory. Below, we will show that this indeed happens, as required by the 't Hooft anomaly matching.

The anomaly matching was previously analyzed in Ref. [2] as a basis for the very formulation of the Seiberg duality. In particular, the anomaly matching requires us to have a light neutral meson M field in the dual theory. Without Seiberg's M meson, the anomalies do not match. A novelty of our discussion in this section is that we have a symmetry breaking in the r-dual theory at the scale $\sqrt{\xi^{\rm small}}$ and have to match anomalies at energies above and below this scale. This leads to a rather restrictive bound for the M-meson mass. Also, since we μ-deform our r-dual theory and start from a well understood $\mathcal{N}=2$ limit, we can reveal a physical interpretation for the M meson.

A. Anomaly matching

The limit [Eq. (4.1)] ensures that the quark masses are rather small. They are the smallest parameters of the theory. Thus, we are in the chiral limit. Above the scale m, the global group of our theory before the symmetry breaking includes independent left and right chiral rotations, namely

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_R, \tag{4.2}$$

where $U(1)_R$ is the nonanomalous (with respect to the non-Abelian gauge bosons) R symmetry [2].[5] Note that here we use the fact that μ is large and the adjoint matter is decoupled. For example, in the $\mathcal{N}=2$ limit in which the adjoint matter is present, the chiral group in Eq. (4.2) is broken by the Yukawa couplings to the adjoint matter even at small values of the quark masses.

The general prescription of the anomaly matching is as follows: the anomalies of all unbroken global currents must be the same at all energies well above m (below m, chiral symmetries are broken). In particular, we calculate the anomalies in the ultraviolet (UV) domain in terms of quarks and gauge bosons of the original theory and match them with the anomalies calculated in the infrared (IR) domain in terms of the relevant degrees of freedom of the dual theory. The UV energy should be large enough to ensure the original theory is at weak coupling, $E_{UV} \gg \Lambda_{\mathcal{N}=1}$. Note that μ should be even larger, $\mu \gg E_{UV}$, so that the adjoint matter really decouples and we do have chiral symmetry. This explains why we do not check the anomaly matching at intermediate values of μ (see Sec. III).

Under the symmetry in Eq. (4.2), the squark fields transform as [2,3]

$$q\colon \left(N_f, 1, \frac{\tilde{N}}{N_f}\right), \qquad \tilde{q}\colon \left(1, \bar{N}_f, \frac{\tilde{N}}{N_f}\right). \tag{4.3}$$

In particular, the R charges of the squarks under $U(1)_R$ are determined by the number of flavors N_f and the rank of the gauge group N. Note that the fermions (quarks) have R charges $R-1$, where R is the charge of the boson component of a given multiplet, while gauginos have the unit R charge.

Quark-like dyons of the r-dual theory transform as

$$D\colon \left(\bar{N}_f, 1, \frac{N}{N_f}\right), \qquad \tilde{D}\colon \left(1, N_f, \frac{N}{N_f}\right), \tag{4.4}$$

where the R charges of dyons are determined by N_f and $\tilde{N}$, the rank of the dual gauge group. Also, in much the same way as in Ref. [2], we assume that D is in the antifundamental representation of $SU(N_f)_L$. We μ-deform our r-dual theory starting from the $\mathcal{N}=2$ limit in which the chiral symmetries are broken. Hence, no memory remains as to which of the two $SU(N_f)$ factors in Eq. (4.2) was left handed or right handed in the original quark theory. It is possible that the dyon appears in the fundamental representation of $SU(N_f)_R$ at large μ. Then the transformations in Eq. (4.4) ensue (upon redefinition of D and $\tilde{D}$).

The anomaly matching in the IR domain $E_{IR} \gg \sqrt{\xi^{\rm small}}$ closely follows the calculation in Ref. [2], and we skip it here. The main result is that without the M meson, the anomalies do not match. Including the M meson, we see that it has quantum numbers of $\tilde{q}_A q^B$ and transforms as [2]

$$M\colon \left(N_f, \bar{N}_f, \frac{2\tilde{N}}{N_f}\right). \tag{4.5}$$

Thus, the anomaly matching requires the presence of the M meson.

So far, we have considered the anomaly matching at energies $E_{IR} \gg \sqrt{\xi^{\rm small}}$, which ensures that the M meson cannot be heavier than $\sqrt{\xi^{\rm small}}$. Below, we will show that in fact the upper bound on the M-meson mass is much more restrictive.

To this end, let us consider energies $E_{IR} \ll \sqrt{\xi^{\rm small}}$ still well above the scale of the chiral symmetry breaking. At these energies, the unbroken global group is

$$SU(N) \times SU(\tilde{N}) \times U(1)_V \times U(1)_{R'}, \tag{4.6}$$

where the first three factors are vector-like symmetries [Eq. (2.18)], while the additional R symmetry appears in the chiral limit.

Let us check that we have an unbroken R symmetry. Consider first dyons of the r-dual theory. We can combine the $U(1)_R$ transformation with the axial subgroup of the non-Abelian factors in Eq. (4.2) to make the R' charges of the last $\tilde{N}$ dyons vanish. In this way, we arrive at

$$\begin{aligned} R'_D &= \frac{N}{N_f} + \left(\frac{\tilde{N}}{N_f}, \ldots, \frac{\tilde{N}}{N_f}, -\frac{N}{N_f}, \ldots, -\frac{N}{N_f}\right) \\ &= (1, \ldots, 1, 0, \ldots, 0), \end{aligned} \tag{4.7}$$

where we divide the charges of N_f dyons into $N+\tilde{N}$ entries shown in the brackets. This $U(1)_{R'}$ symmetry is unbroken by the dyon VEVs; see Eq. (2.12).

This leads to the following transformation law of dyons under the unbroken symmetry in Eq. (4.6):

$$\begin{aligned} &D^P\colon \left(\bar{N}, 1, \frac{N_f}{2N}, 1\right), \qquad \tilde{D}_P\colon \left(N, 1, -\frac{N_f}{2N}, 1\right), \\ &D^K\colon (1, \bar{\tilde{N}}, 0, 0), \qquad \tilde{D}_K\colon (1, \tilde{N}, 0, 0), \end{aligned} \tag{4.8}$$

where $P = 1, \ldots, N$ and $K = (N+1), \ldots, N_f$. Here we also combine the vector flavor $SU(N_f)$ transformation with the $U(1)$ gauge transformation to get vanishing charges under $U(1)_V$ of the last $\tilde{N}$ dyons.

[5] The gauge group of our original theory is $U(N)$; thus, it includes Abelian $U(1)$ gauge fields. $U(1)_R$ symmetry is anomalous with respect to $U(1)$ gauge fields. Still, we have the freedom to make the $U(1)$ gauge coupling small so that the $U(1)_R$ current is approximately conserved.

Now, let us find the quark R charges. We will see below that the diagonal entries of the $N\times N$ upper-left block of the meson matrix M_A^B also develops VEVs in the vacuum of the dual theory. Since the M mesons are defined as quark-antquark pairs of the original theory, this means that the $U(1)_{R'}$ symmetry is unbroken if the first N quarks have vanishing R' charges. We define

$$R'_q=\frac{\tilde N}{N_f}+\left(-\frac{\tilde N}{N_f},\ldots,-\frac{\tilde N}{N_f},\frac{N}{N_f},\ldots,\frac{N}{N_f}\right)$$
$$=(0,\ldots,0,1,\ldots,1). \tag{4.9}$$

Thus, the quarks transform under the unbroken symmetry [Eq. (4.6)] as follows:

$$q^P:\ (N,1,0,0),\qquad \tilde q_P:\ (\bar N,1,0,0),$$
$$q^K:\ \left(1,\tilde N,\frac{N_f}{2\tilde N},1\right),\qquad \tilde q_K:\ \left(1,\bar{\tilde N},-\frac{N_f}{2\tilde N},1\right). \tag{4.10}$$

Here we again combine the vector flavor $SU(N_f)$ transformation with the $U(1)$ gauge transformation to get vanishing charges of the first N quarks under $U(1)_V$. The transformation properties of the M field ensue from Eq. (4.10):

$$M^P_{P'}:\ (N\bar N,1,0,0),\qquad M^P_K:\ (N,\bar{\tilde N},0,1),$$
$$M^K_P:\ (\bar N,\tilde N,0,1),\qquad M^K_{K'}:\ (1,\tilde N\bar{\tilde N},0,2), \tag{4.11}$$

where $P,P'=1,\ldots,N$ and $K,K'=(N+1),\ldots,N_f$.

The list of anomalies to be checked is

$$U(1)_{R'}\times SU(N)^2:\ -\frac{\delta^{mn}}{2}N|_{UV}=-\frac{\delta^{mn}}{2}N|_{IR},$$
$$U(1)_{R'}\times SU(\tilde N)^2:\ 0|_{UV}=\frac{\delta^{ps}}{2}(-\tilde N+\tilde N)|_{IR},$$
$$U(1)_{R'}\times U(1)_V^2:\ 0|_{UV}=0|_{IR},$$
$$U(1)_{R'}:\ -2N^2+N^2|_{UV}=-N^2=-\tilde N^2-N^2+\tilde N^2|_{IR},$$
$$U(1)^3_{R'}:\ -2N^2+N^2|_{UV}=-N^2=-\tilde N^2-N^2+\tilde N^2|_{IR}, \tag{4.12}$$

where n,m and p,s are the adjoint indices in $SU(N)$ and $SU(\tilde N)$, respectively. Here, the UV contributions are calculated in terms of the fermion quarks and gauginos, while the IR contributions come from the fermion components of (screened) dyons and M fields. For example, in the second line, the IR anomaly is saturated by D^K and $M^K_{K'}$. In the fourth line, the UV contribution comes from the quarks q^P, $\tilde q_P$ and gauginos. The IR contribution comes from the light dyons (a half of D^K and $\tilde D_K$ states; see Sec. III A), $M^P_{P'}$ and $M^K_{K'}$, respectively.

Needless to say, all anomalies match. The contribution of the M meson is *essential*. Since E_{IR} can lie in the window $m\ll E_{IR}\ll\sqrt{\xi^{\rm small}}$, we find the upper bound for the M-meson mass,

$$m_M\lesssim m. \tag{4.13}$$

We see that the M meson is rather light; its mass is determined by the small-scale m of the chiral symmetry breaking. Thus, the M mesons play the role of π mesons in our theory.

B. Interpretation of the Seiberg M mesons

As was already discussed, the candidates for the Seiberg M mesons in the r-dual theory are stringy mesons $M_A^B(P,P+1)$ $[P=\tilde N,\ldots,(N-1)]$ from the Abelian $U(1)^{(N-\tilde N)}$ sector. This sector is at strong coupling at large μ; therefore, certain states from this sector can become light. Perturbative states from this sector (quark-like dyons and Abelian gauge fields) are singlets with respect to the global group [Eq. (4.6)] and cannot play the M-meson role. Note that stringy mesons $M_A^B(P,P+1)$ [where $P=1,\ldots,(\tilde N-1)$] from the $U(\tilde N)$ low-energy theory also cannot play the M-meson role. First, they are represented in the $U(\tilde N)$ low-energy theory by themselves as nonperturbative solitonic states and cannot be added to this theory as new "fundamental" or "elementary" fields. Second, they are too heavy, with mass of the order of $\sqrt{\xi^{\rm small}}$ determined by the tensions of the non-Abelian strings, which can be calculated at weak coupling.

Thus, we propose that the Seiberg M_A^B meson is one of a multitude of the monopole-antimonopole stringy mesons $M_A^B(P,P+1)$ [where $P=\tilde N,\ldots,(N-1)$] from the Abelian $U(1)^{(N-\tilde N)}$ sector. At large μ, this meson should become light, with mass of the order of m. It should be incorporated in the $U(\tilde N)$ low-energy theory as a new "fundamental" or "elementary" field. Note that other states from the Abelian sector are still heavy and decouple.

C. Effective action

Since our $U(\tilde N)$ r-dual theory is at weak coupling, we can write down its effective action. In particular, since this theory is a μ deformation of a particular $\mathcal{N}=2$ r-dual theory, the quark-like dyons D^{lA} have canonically normalized kinetic terms. Using the procedure described in Sec. III B in the opposite direction, we "integrate the M-meson in" the superpotential [Eq. (3.5)]. In this way, we arrive at

$$\mathcal{W}=\frac{\kappa^2}{2\mu}\,{\rm Tr}(M^2)-\kappa m_A M_A^A+\frac{\kappa}{\mu}\tilde D_{Al}D^{lB}M_B^A. \tag{4.14}$$

We suggest that Eq. (4.14) is a correct continuation of the superpotential [Eq. (3.5)] of the r-dual theory to large μ.

Then the effective action of the r-dual theory at large μ takes the form

$$S=\int d^4x\Big\{\frac{1}{4\tilde{g}^2}(F^a_{\mu\nu})^2+\frac{1}{4\tilde{g}^2_{U(1)}}(F_{\mu\nu})^2+|\nabla_\mu D^A|^2+|\nabla_\mu\bar{\tilde{D}}^A|^2 +\frac{2}{\gamma}\mathrm{Tr}|\partial_\mu M|^2+\frac{\tilde{g}^2}{2}(\bar{D}_A T^a D^A-\tilde{D}_A T^a\bar{\tilde{D}}^A)^2 +\frac{\tilde{g}^2_{U(1)}}{8}(\bar{D}_A D^A-\tilde{D}_A\bar{\tilde{D}}^A)^2+\frac{\kappa^2}{\mu^2}\mathrm{Tr}|DM|^2+\frac{\kappa^2}{\mu^2}\mathrm{Tr}|\bar{\tilde{D}}M|^2 +\frac{\gamma\kappa^2}{2\mu^2}|\tilde{D}_A D^B-\mu m_A\delta^B_A+\kappa M^B_A|^2\Big\}, \tag{4.15}$$

where the covariant derivative is defined as

$$\nabla_\mu=\partial_\mu-\frac{i}{2}A_\mu-iT^aA^a_\mu, \tag{4.16}$$

and we introduce gauge potentials for $SU(\tilde{N})$ and $U(1)$ gauge groups, while $\tilde{g}$ and $\tilde{g}_{U(1)}$ are associated dual gauge couplings. We also introduce the coupling constant γ for the M field.

We assume that κ is a function of μ and m with the following behavior:

$$\kappa\sim\begin{cases}\mu^{\frac{3}{4}}\Lambda^{\frac{1}{4}}_{\mathcal{N}=2}, & \mu\ll\Lambda_{\mathcal{N}=2},\\ \sqrt{\mu m}, & \mu\gg\Lambda_{\mathcal{N}=2}.\end{cases} \tag{4.17}$$

This dependence ensures that the M meson is heavy, with mass of the order of $\sqrt{\xi^{\text{large}}}$, at intermediate μ, and it becomes light, with mass of the order of m, at large μ.

Minimization of the potential in Eq. (4.15) gives VEVs [Eq. (2.12)] for dyons [see also Eqs. (2.16) and (2.17)], while the M-field VEVs are

$$\mathrm{diag}\langle M^B_A\rangle=\frac{\mu}{\kappa}(m_1,\ldots,m_N,0,\ldots,0). \tag{4.18}$$

These VEVs ensure chiral symmetry breaking [Eq. (4.6)] in the (almost) equal-mass limit.

Now, let us briefly discuss the mass spectrum of r-dual theory [Eq. (4.15)]. Much in the same way as at intermediate μ, the lightest states are $4N\tilde{N}$ bifundamental dyons with masses of the order of the quark mass differences (m_A-m_B). Half $(2\tilde{N}^2)$ of the dyons from the singlet and adjoint representations of $SU(\tilde{N})$ have masses of the order of m. Moreover, the M mesons are also light, with masses of the order of m.

Other $\tilde{N}^2$ dyonic states, together with the gauge bosons of the $U(\tilde{N})$ gauge group, are much heavier, with masses of the order of $\tilde{g}\sqrt{\xi^{\text{small}}}$. In addition, we have stringy monopole-antimonopole mesons $M^B_A(P,P+1)$, where $P=1,\ldots,(\tilde{N}-1)$, with masses of the order of $\sqrt{\xi^{\text{small}}}$.

However, now at large μ, all these stringy monopole-antimonopole mesons can decay into light Seiberg M mesons.

V. VACUA WITH $r<N_f/2$ AT SMALL μ

Now consider r vacua with $r<N$, in which the first r quarks develop nonvanishing VEVs in the large-mass limit. In the classically unbroken $U(N-r)$ pure gauge sector, the gauge symmetry gets broken through the Seiberg-Witten mechanism [6]: first down to $U(1)^{N-r}$ by the condensation of the adjoint fields, and then almost completely by the condensation of $(N-r-1)$ monopoles. A single $U(1)$ gauge factor survives, though, because the monopoles are charged only with respect to the Cartan generators of the $SU(N-r)$ group.

The presence of this unbroken $U(1)$ factor in all $r<N$ vacua makes them different from the $r=N$ vacuum: in the latter, there are no long-range forces.

The low-energy theory in the given r vacuum has the gauge group

$$U(r)\times U(1)^{N-r} \tag{5.1}$$

if the quark masses are almost equal. Moreover, N_f quarks are charged under the $U(r)$ factor, while $(N-r-1)$ monopoles are charged under the $U(1)$ factors. If $0<r<(N-1)$, then the r vacua are hybrid vacua in which both quarks and monopoles are condensed. Note that the quarks and monopoles are charged with respect to orthogonal subgroups of $U(N)$ and therefore are mutually local (i.e., they can be described by a local Lagrangian). The low-energy theory is infrared free, and it is at weak coupling as long as VEVs of quarks and monopoles are small. The quark VEVs are given by

$$\langle q^{kA}\rangle=\langle\bar{\tilde{q}}^{kA}\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0\end{pmatrix}, \quad k=1,\ldots,r,\quad A=1,\ldots,N_f, \tag{5.2}$$

where in the quasiclassical domain of large quark masses the r parameters $\xi_{1,\ldots,r}$ are

$$\xi_P\approx 2\mu m_P,\quad P=1,\ldots,r. \tag{5.3}$$

These parameters can be made small in the limit of large m_A if μ is sufficiently small.

In quantum theory, the ξ_P parameters are determined by the roots of the Seiberg-Witten curve [Eq. (2.14)]; see Refs. [24,27]. The Seiberg-Witten curve in the $r<N$ vacuum has $N-1$ double roots which are associated with r condensed quarks and $(N-r-1)$ condensed monopoles.

Namely, the Seiberg-Witten curve factorizes [28]

$$y^2 = \prod_{P=1}^{r}(x-e_P)^2 \prod_{K=r+1}^{N-1}(x-e_K)^2(x-e_N^+)(x-e_N^-). \tag{5.4}$$

The first r-quark double roots are associated with the mass parameters in the large-mass limit, $\sqrt{2}e_P \approx -m_P$, where $P=1,\ldots,r$. The other $(N-r-1)$ double roots associated with the light monopoles are much smaller and are determined by $\Lambda_{\mathcal{N}=2}$. The last two unpaired roots are also much smaller. For the single-trace deformation superpotential [Eq. (2.1)], their sum vanishes [28]:

$$e_N^+ + e_N^- = 0. \tag{5.5}$$

The root e_N^+ determines the value of the gaugino condensate [29]:

$$e_N^2 = \frac{2S}{\mu}, \qquad S = \frac{1}{32\pi^2}\langle \mathrm{Tr} W_\alpha W^\alpha \rangle. \tag{5.6}$$

The superfield W_α includes the gauge field strength tensor.

In terms of the roots of the Seiberg-Witten curve, the quark VEVs are given by the formula [24,27]

$$\xi_P = -2\sqrt{2}\mu\sqrt{(e_P-e_N^+)(e_P-e_N^-)},$$
$$P = 1,\ldots,(N-1). \tag{5.7}$$

In fact, this formula is universal: it determines both the VEVs of r quarks and $(N-r-1)$ monopoles [27]. Namely, the index P runs over $P=1,\ldots,(N-1)$ in Eq. (5.7) with quark and monopole VEVs given by Eq. (5.2) and

$$\langle m_{P(P+1)}\rangle = \langle \bar{\tilde{m}}_{P(P+1)}\rangle = \sqrt{\frac{\xi_P}{2}},$$
$$P = (r+1),\ldots,(N-1), \tag{5.8}$$

respectively. Here $m_{PP'}$ denotes the monopole with the charge given by the root $\alpha_{PP'} = w_P - w_{P'}$ of the $SU(N)$ algebra with the weights w_P ($P<P'$).

Condensation of r quarks leads to formation of non-Abelian magnetic strings that confine monopoles from the $SU(r)$ sector [strings are non-Abelian in the (almost) equal quark mass limit]. Tensions of the magnetic strings are determined by Eq. (2.6) with $P=1,\ldots,r$. In a similar way, condensation of $(N-r-1)$ monopoles leads to the formation of the Abelian electric strings which confine quarks from $U(1)^{N-r}$. Their tensions are also given by Eq. (2.6) with $P=(r+1),\ldots,(N-1)$; for more details on the confinement of monopoles and quarks in the hybrid vacua, see Ref. [27].

Now, let us consider the limit of small quark masses. As was already mentioned, in the r vacua with $r>N_f/2$, there is a crossover to the r-dual theory with the dual gauge group $U(N_f-r)\times U(1)^{N-N_f+r}$ [24]. The $r=N$ vacuum considered in the previous sections provides us with the simplest example of this behavior.

Now, let us focus on r vacua with smaller r. If $r<N_f/2$, the low-energy theory essentially remains the same as at large m_A—namely, infrared-free $U(r)\times U(1)^{N-r}$ gauge theory with N_f flavors of light states charged under a non-Abelian gauge factor and $(N-r-1)$ singlet monopoles charged under $U(1)^{N-r}$ [13,30]. Although the color charges of light non-Abelian states are identical to those of quarks,[6] they are not quarks. In much the same way as in the $r=N$ vacuum, we call these states quark-like dyons D^{lA}, $l=1,\ldots,r$, $A=1,\ldots,N_f$. We will see in Sec. VI B that they have chiral R charges different from those of quarks.[7] At large masses, these dyons are heavy monopole-antimonopole stringy states while below crossover; at small masses, they become light fundamental (or elementary) states of the $U(r)\times U(1)^{N-r}$ gauge theory.

The quark-like dyons from the $U(r)$ sector and the monopoles from the orthogonal $U(1)^{N-r}$ sector develop VEVs determined by Eq. (5.7). In particular, dyons develop VEVs

$$\langle D^{lA}\rangle = \langle \bar{\tilde{D}}^{lA}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_r} & 0 & \ldots & 0 \end{pmatrix},$$
$$l=1,\ldots,r, \quad A=1,\ldots,N_f. \tag{5.9}$$

The theory is at weak coupling, provided the ξ_P parameters are small.

What happens to quarks of the original theory? In much the same way as in the $r=N$ vacuum, the screened q^{kA} quarks (with $k=1,\ldots,r$) of the $U(r)$ gauge sector decay into monopole-antimonopole pairs and evolve into stringy mesons as shown in Fig. 1. These quarks are in the instead-of-confinement phase.

We would like to stress, however, that there is a peculiar distinction of this picture with the one in the $r=N$ vacuum. In the limit of small and almost equal masses, the dyon condensation breaks the global $SU(N_f)$ group down to

$$SU(r)_{C+F}\times SU(N_f-r)\times U(1)_V. \tag{5.10}$$

[6]As we reduce m, the quarks pick up root-like color-magnetic charges, in addition to their weight-like color-electric charges due to monodromies; see Ref. [30].

[7]In Ref. [13], the chiral limit was not considered. It was concluded that these states are identical to quarks. Here we correct this interpretation.

In particular, color-flavor locking takes place in the $SU(r)$ factor. In contrast to the case of the $r = N$ vacuum, both dyons and monopole-antimonopole stringy mesons, which originate from screened quarks of the large-m theory are in the same representations of this group. Namely, they form singlet and adjoint representaions of $SU(r)_{C+F}$, as well as bifundamental representations,

$$(1,1), \quad (r^2-1,1), \qquad (\bar{r}, N_f - r), \qquad (r, \bar{N}_f - \bar{r}), \tag{5.11}$$

where we mark representations with respect to two non-Abelian factors in Eq. (5.10). The $U(1)_R$ symmetry which distinguishes screened dyons and monopole-antimonopole mesons (former screened quarks) is broken. Therefore, monopole-antimonopole stringy mesons are unstable and decay into dyons, which are lighter.

There are also other quarks q^{kA} charged with respect to the Abelian $U(1)^{N-r}$ gauge group with $l = (r+1), \ldots, N$ in the original theory. These are still confined by Abelian strings formed as a result of the monopole condensation in the small-m limit.

VI. ZERO VACUA

In this section, we consider zero vacua at intermediate and large μ [13]. These vacua form a subset of r vacua with small r, $r < \tilde{N}$.

A. Intermediate μ

In the small-mass limit, r double roots of the Seiberg-Witten curve associated with light dyons are still determined by quark masses:

$$\sqrt{2} e_P = -m_P, \quad P = 1, \ldots, r. \tag{6.1}$$

The above expression is valid in r vacua with $r < N_f/2$. Other roots are much larger, of the order of $\Lambda_{\mathcal{N}=2}$. However, in contrast to the $r = N$ vacuum (see Sec. III A), this does not allow us to increase μ, keeping the $U(r)$ theory at weak coupling. The point is that dyons' VEVs which are supposed to be small to ensure weak coupling (in the IR-free theory) are not determined entirely by e_P in the $r < N$ vacua. They are given by parameters ξ_P that depend also on the gaugino condensate, which determines the values of the unpaired roots in Eq. (5.7). In the majority of the r vacua, the gaugino condensate is of the order of $S \sim \mu \Lambda^2_{\mathcal{N}=2}$. We refer to these vacua as the Λ vacua. In the Λ vacua, all parameters ξ are of the order of $\xi \sim \mu \Lambda_{\mathcal{N}=2}$, and we cannot increase μ without destroying the weak coupling condition [13].

However, there are two exceptions. One is the $r = N$ vacuum, in which the gaugino condensate vanishes, and $\tilde{N}$ parameters ξ are determined by the quark masses; see Eqs. (2.16) and (2.17) [5]. We considered this vacuum in the previous sections. Another exception is the subset of the $r < \tilde{N}$ vacua, which we call the zero vacua [13]. In the zero vacua, the gaugino condensate is extremely small [12,13]:

$$S \approx \mu \frac{m^{\frac{N_f - 2r}{\tilde{N}-r}}}{\Lambda_{\mathcal{N}=2}^{\frac{N-\tilde{N}}{\tilde{N}-r}}} e^{\frac{2\pi k}{\tilde{N}-r} i} \ll \mu m^2, \quad k = 1, \ldots, (\tilde{N} - r), \tag{6.2}$$

in the limit of small equal quark masses. This behavior can be obtained from the exact Cachazo-Seiberg-Witten solution for the chiral ring of the theory [29]; see also Ref. [13].

Thus, in the zero vacua we can neglect contributions of the unpaired roots as compared to the quark masses in Eq. (5.7). It turns out that ξ's are given by [13]

$$\xi_P \approx -2\mu \left(m_1, \ldots, m_r, 0, \ldots, 0, \Lambda_{\mathcal{N}=2}, \ldots, \Lambda_{\mathcal{N}=2} e^{\frac{2\pi i}{N-\tilde{N}}(N-\tilde{N}-1)} \right), \tag{6.3}$$

where $(\tilde{N} - r)$ entries are of the order of $\sqrt{S/\mu}$ and are taken to be zero in the quasiclassical approximation, while the last entries are large. They determine the VEVs of $(N - \tilde{N})$ monopoles.

Now we can increase μ to intermediate values:

$$|\mu| \gg |m_A|, \quad A = 1, \ldots, N_f, \quad \mu \ll \Lambda_{\mathcal{N}=2}. \tag{6.4}$$

The monopole $U(1)^{(\tilde{N}-r)}$ sector associated with almost vanishing entries in Eq. (6.3) enters the strong regime. It is shown in Ref. [13] that it goes through a crossover at $\mu \sim e_N \sim \sqrt{S/\mu}$, and the domain of intermediate μ can be described in terms of the weak coupling μ dual theory with the gauge group

$$U(\tilde{N}) \times U(1)^{N-\tilde{N}}, \tag{6.5}$$

N_f flavors of quark-like dyons charged with respect to the $U(\tilde{N})$ gauge group, and $(N - \tilde{N})$ singlet monopoles charged with respect to the $U(1)^{N-\tilde{N}}$ Abelian sector. The restoration of the $U(\tilde{N})$ gauge group occurs because $(\tilde{N} - r)$ Coulomb branch parameters ϕ_P of the Seiberg-Witten curve almost vanish, being determined by the small value of the gaugino condensate [13].

Qualitatively, the enhancement of the $U(r)$ gauge group to $U(\tilde{N})$ can be understood as follows: As we reduce m, the expectation values of monopoles in the $U(\tilde{N} - r)$ sector tend to zero; see Eq. (6.3). Confinement of quarks in this sector becomes weaker and eventually disappears. However, confined quark-antiquark pairs cannot just move apart, because they have "wrong" chiral charges; see the next subsection. They decay into a pair of quark-like dyons

$$q+\tilde{q}\to\bar{D}+\bar{\tilde{D}}+\lambda+\lambda \tag{6.6}$$

via emission of two gauginos.

These dyons and gauge fields of the $U(\tilde{N}-r)$ sector become unconfined and enter the non-Abelian Coulomb phase. Moreover, dyons of the $U(\tilde{N}-r)$ sector combine with dyons of the $U(r)$ sector to form light non-Abelian matter of the enhanced $U(\tilde{N})$ gauge group.

Note also that VEVs of r dyons are given by $\xi^{\rm small}$, while VEVs of $(N-\tilde{N})$ monopoles are much larger and given by $\xi^{\rm large}$; see Eq. (6.3). Therefore, the monopole sector is heavy and can be integrated out together with the adjoint matter. In much the same way as in the $r=N$ vacuum, this leaves us with the low-energy theory with Seiberg's dual gauge group

$$U(\tilde{N}) \tag{6.7}$$

and N_f flavors of dyons with the superpotential [13]

$$\mathcal{W}_{\text{zero vac}}=-\frac{1}{2\mu}(\tilde{D}_AD^B)(\tilde{D}_BD^A)+m_A(\tilde{D}_AD^A). \tag{6.8}$$

This is the same superpotential as in the $r=N$ vacuum; see Eq. (3.5).

Note, that the dyons in this setup have $\tilde{N}$ colors; however, only r of them condense, $r<\tilde{N}$. Thus, our low-energy infrared-free $U(\tilde{N})$ theory is in the mixed Coulomb-Higgs phase with regards to dyons. In particular, the $U(\tilde{N}-r)$ subgroup of $U(\tilde{N})$ remains unbroken, and $(\tilde{N}-r)$ massless gauge bosons are present. The gauge bosons of the $U(r)$ subgroup and their dyon $\mathcal{N}=1$ superpartners have masses of the order of $\tilde{g}\sqrt{\xi^{\rm small}}$. Other dyons charged with respect to $U(\tilde{N})$ have masses of the order of m.

Quarks of the original theory charged with respect to $U(1)^{N-\tilde{N}}$ are confined by electric strings formed due to the condensation of monopoles in the heavy $U(1)^{N-\tilde{N}}$ Abelian sector. In much the similar way as in the $r=N$ vacuum, these stringy mesons are the candidates for Seiberg's M mesons. At intermediate values of μ, the $U(1)^{N-\tilde{N}}$ Abelian sector is at weak coupling, and these mesons are heavy, with masses of the order of $\sqrt{\xi^{\rm large}}\sim\sqrt{\mu\Lambda_{\mathcal{N}=2}}$.

We can compare our low-energy $U(\tilde{N})$ μ-dual theory to Seiberg's dual. In much the same way as in the $r=N$ vacuum, we find a perfect match [13]. Namely, if we integrate out M fields in Seiberg's dual superpotential [Eq. (3.8)] (they are heavy at intermediate values of μ) and make identification [Eq. (3.11)] similar to that in the $r=N$ vacuum, we arrive at the superpotential, which coincides (up to a sign) with our superpotential in Eq. (6.8).

The identification [Eq. (3.11)] reveals the physical nature of the Seiberg "dual quarks." In much the same way as in the $r=N$ vacuum, they are not monopoles. Instead, they are quark-like dyons, which have color charges identical to those of quarks but different global charges. Condensation of r dyons leads to the confinement of monopoles and the "instead-of-confinement" phase for quarks in the $U(r)$ sector.

B. Large μ

Now we assume that μ is large while $\sqrt{\xi^{\rm small}}$ is small enough to ensure the weak coupling regime in the low-energy $U(\tilde{N})$ μ-dual theory; see Eq. (4.1). By the same token as in the $r=N$ vacuum, we can use the anomaly matching to show that Seiberg's M mesons should become light at large μ.

If the IR energy scale is large, $E_{IR}\gg\sqrt{\xi^{\rm small}}$, the global group is given by Eq. (4.2), and in this case the anomaly matching was carried out in Ref. [2]. Namely, the transformation properties of quarks of the original theory and M mesons are given by Eqs. (4.3) and (4.5). Let us consider the dyon charges. The R charge is determined by the anomaly cancellation requirement with respect to non-Abelian gauge bosons [2]. It is determined by the number of flavors and the rank of the gauge group. Say, for quarks of the original theory it is given by $\tilde{N}/N_f$; see Eq. (4.3). The rank of the gauge group in the μ-dual theory is different, however. It equals $\tilde{N}$. Thus, the R charges of the D^{lA} dyons are given by

$$R_D=\frac{N}{N_f}. \tag{6.9}$$

This tells us that the quarks and dyons are in fact different states, as was mentioned above. We arrive at our μ-dual theory starting from the $\mathcal{N}=2$ limit by virtue of the μ deformation. Moving along in this way, we break the $U(1)_R$ symmetry. Thus, we were unable to observe the above distinction. The dyons appeared just as quarks with a truncated number of colors. Now, studying the chiral limit, we see that in fact they are different states.

As was already explained, the weakly confined quark-antiquark pairs decay into unconfined dyon pairs via a wall-crossing-like process

$$q+\tilde{q}\to\bar{D}+\bar{\tilde{D}}+\lambda+\lambda \tag{6.10}$$

upon increasing μ; see Eq. (6.6). It is easy to see that this decay respects the R-charge conservation, where we use the fact that the gaugino R charge is unity. Equation (6.10) shows that the dyon transformation laws are

$$D:\ \left(\bar{N}_f,1,\frac{N}{N_f}\right),\qquad \tilde{D}:\ \left(1,N_f,\frac{N}{N_f}\right). \tag{6.11}$$

In particular, the D^{lA} dyon transforms in the $\bar{N}_f$ representation of the $SU(N_f)_L$ rather than[8] in the representation N_f.

[8]This important circumstance was noted by Chernyak [8].

We see that the dyon transformation properties are the same in both the zero and the $r = N$ vacua [see Eq. (4.4)], and that they coincide with those for the Seiberg dual quarks [2]. Thus, the anomaly matching at the IR energy scale

$$E_{IR} \gg \sqrt{\xi^{\rm small}}$$

follows the calculation presented in Ref. [2]. The concluding result is that the light neutral M_A^B field is needed to match the anomalies.

If $E_{IR} \ll \sqrt{\xi^{\rm small}}$, the unbroken global group is

$$SU(r) \times SU(N_f - r) \times U(1)_V. \qquad (6.12)$$

In particular, it is easy to see that the chiral $U(1)_R$ symmetry is broken in the zero vacua in contradistinction with the $r = N$ case. In fact, we cannot arrange combinations similar to those in Eqs. (4.7) and (4.9) to ensure that the R' charges of the r components of quarks and $(N_f - r)$ components of M mesons (which develop VEVs) vanish. The required axial rotation from the non-Abelian subgroups in Eq. (4.2) does not respect the Yukawa interaction $(\tilde{D}_A D^B) M_B^A$. Therefore, we cannot match anomalies at energies below $\sqrt{\xi^{\rm small}}$.

Thus, in the zero vacua, the anomaly matching gives a less restrictive upper bound on the M-meson mass as compared to the $r = N$ vacuum, namely $m_M \lesssim \sqrt{\mu m}$. Still, we can obtain a more restrictive estimate for the M-meson mass using the Goldstone theorem. The number of broken generators in the breaking of Eq. (4.2) down to Eq. (6.12) is

$$r^2 + (N_f - r)^2 + 4r(N_f - r). \qquad (6.13)$$

While r^2 and $4r(N_f - r)$ broken generators can be accounted for by light dyons in the $r\bar{r}$ and bifundamental representations, respectively, the extra $(N_f - r)^2$ light states are missing. These can be accounted for by the light M meson. As a result, we conclude that the M-meson mass should be lighter, namely

$$m_M \sim m, \qquad (6.14)$$

as is the case in the $r = N$ vacuum.

The physical interpretation of Seiberg's M mesons in the zero vacua is as follows: As was already mentioned, the candidates for the M mesons can be found among mesonic states from the heavy Abelian $U(1)^{N-\tilde{N}}$ sector—quark-antiquark pairs connected by confining strings. The majority of these mesons are similar to those shown in Fig. 1, in which the monopoles should be replaced by quarks. However, a peculiar feature of all $r < N$ vacua is that there are only $(N - 1)$ strings; one of the strings is missing. Therefore, some of these mesons are formed by quarks and antiquarks connected by only one string, while the other one is missing; see Refs. [24,27] for more details.

Now, similarly to the situation in the $r = N$ vacuum, we suggest that one of these quark-antiquark stringy mesons becomes light at large μ when the $U(1)^{N-\tilde{N}}$ sector enters the strong coupling regime. This M meson should be integrated in the $U(\tilde{N})$ μ-dual theory as a "fundamental" (elementary) field. Other fields of the Abelian $U(1)^{N-\tilde{N}}$ sector are heavy and can be integrated out. The superpotential and action of the low-energy $U(\tilde{N})$ μ-dual theory are given in Eqs. (4.14) and (4.15).

VII. SUMMARY AND CONCLUSIONS

To summarize, at large μ and small $\xi^{\rm small}$, μ-deformed SQCD in the $r = N$ vacuum is described by the weakly coupled infrared-free r-dual $U(\tilde{N})$ theory [Eq. (4.15)] with N_f light quark-like dyon flavors. Condensation of the light dyons D^{lA} in this theory triggers formation of the non-Abelian strings and confinement of monopoles. The quarks and gauge bosons of the original $\mathcal{N} = 1$ SQCD are in the "instead-of-confinement" phase: they evolve into the monopole-antimonopole stringy mesons shown in Fig. 1. There is also Seiberg's neutral-meson M field which is a monopole-antimonopole stringy meson from the heavy Abelian sector. It becomes anomalously light and plays the role of a "pion" at large μ.

In the zero r-vacua, we have the weak coupling description in terms of the infrared-free μ-dual $U(\tilde{N})$ theory [Eq. (4.15)] with N_f flavors of quark-like dyons. Only r dyons condense ($r < \tilde{N}$), leading to the confinement of monopoles in the $U(r)$ sector. The $U(\tilde{N} - r)$ sector is in the non-Abelian Coulomb phase for dyons. Seiberg's M meson is a quark-antiquark stringy state which comes from the heavy Abelian sector. It becomes light at large μ.

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