# Neutrino magnetic moment, *CP* violation, and flavor oscillations in matter

Y. Pehlivan,[1,2,*] A. B. Balantekin,[3,†] and Toshitaka Kajino[2,4,‡]
[1]*Mimar Sinan Fine Arts University, Sisli, Istanbul 34380, Turkey*
[2]*National Astronomical Observatory of Japan 2-21-1 Osawa, Mitaka, Tokyo 181-8588, Japan*
[3]*Department of Physics, University of Wisconsin-Madison, Wisconsin 53706, USA*
[4]*Department of Astronomy, University of Tokyo, Tokyo 113-0033, Japan*
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We consider collective oscillations of neutrinos, which are emergent nonlinear flavor evolution phenomena instigated by neutrino-neutrino interactions in astrophysical environments with sufficiently high neutrino densities. We investigate the symmetries of the problem in the full three-flavor mixing scheme and in the exact many-body formulation by including the effects of *CP* violation and the neutrino magnetic moment. We show that, similar to the two-flavor scheme, several dynamical symmetries exist for three flavors in the single-angle approximation if the net electron background in the environment and the effects of the neutrino magnetic moment are negligible. Moreover, we show that these dynamical symmetries are present even when the *CP* symmetry is violated in neutrino oscillations. We explicitly write down the constants of motion through which these dynamical symmetries manifest themselves in terms of the generators of the SU(3) flavor transformations. We also show that the effects due to the *CP*-violating Dirac phase factor out of the many-body evolution operator and evolve independently of nonlinear flavor transformations if neutrino electromagnetic interactions are ignored. In the presence of a strong magnetic field, *CP*-violating effects can still be considered independently provided that an effective definition for the neutrino magnetic moment is used.

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## I. INTRODUCTION

Among the known species neutrinos are the second most abundant particles in the Universe after photons. Many of them were created shortly after the big bang and today form the cosmic neutrino background the presence of which can be inferred from the anisotropies of the cosmic microwave background and the cosmic large-scale structure [1–4]. Neutrinos also originate from various astrophysical sources such as core collapse supernovae [5–9] and black hole accretion disks [10–15] where they are produced in copious amounts. Given their abundance in the Universe and the prevalence of extraordinary physical conditions in their sources, it can be expected that even the tiniest anomalous electromagnetic properties or *CP*-violation features of neutrinos would have consequences in cosmology and astrophysics.

In both the early Universe and in the intense astrophysical sources mentioned above, neutrinos are believed to undergo nonlinear forms of flavor evolution which are generally termed *collective neutrino oscillations*. These oscillations follow from the self-interactions of neutrinos which become important when their number density is sufficiently high [16] and turn the flavor evolution into a many-body phenomenon [17–25]. The designation "collective" originates from the strong correlations that may develop between neutrinos [25–30].

Although the collective oscillations of neutrinos are highly nonlinear, they were shown to possess several dynamical symmetries under a set of idealized conditions such as the absence of a net electron background, a two-flavor mixing scenario, and the so-called single-angle approximation for the neutrino-neutrino interactions [31]. These symmetries are dynamical in the sense that the corresponding constants of motion depend on the interaction parameters (unlike the space-time symmetries). It was also demonstrated that a well-known collective behavior of neutrinos, namely the spectral splits, is connected to one of these symmetries [31]. One of the purposes of this paper is to show that similar symmetries also exist for the realistic three-flavor mixing case and in the presence of a *CP*-violating Dirac phase.

The second purpose of this paper is to carefully examine the interplay between the possible *CP*-violation features of neutrinos, their anomalous electromagnetic moments, and the collective flavor oscillations with a particular focus on the inherent many-body nature and the symmetries of the latter.

Our current understanding of particle physics already entails small anomalous electric and magnetic dipole moments for neutrinos because the existence of the neutrino mass calls for at least one right-handed neutrino degree of freedom which allows the neutrino to couple to a photon at the one-loop level [32,33]. In the Standard Model, minimally extended to include right-handed neutrinos, this yields

*yamac.pehlivan@msgsu.edu.tr
†baha@physics.wisc.edu
‡kajino@nao.ac.jp

 

very small values for the neutrino dipole moments which are of the order of $10^{-19}\ \mu_B$ or smaller, where $\mu_B$ denotes the Bohr magneton [34]. However, various theories beyond the Standard Model predict larger values. The current experimental upper limit on the anomalous magnetic moment of the neutrino is of the order of $10^{-11}\ \mu_B$ [35] whereas a slightly better upper limit of $10^{-12}\ \mu_B$ can be obtained from the constraints on the additional cooling mechanism for red giant stars due to plasmon decay into neutrinos [36,37]. For a recent review of the electromagnetic properties of neutrinos, see Ref. [38].

The spin-flavor precession of neutrinos in magnetic fields was studied some time ago [39]. The effects of the neutrino magnetic moment on the collective oscillations of neutrinos were recently examined in numerical simulations [40,41] and it was shown that neutrinos and antineutrinos can swap their energy spectra if they propagate through a strong magnetic field. Such a spectral swap can play an important role in the *r*-process nucleosynthesis which could take place in the hot bubble region of a core-collapse supernova by transferring energy from the relatively energetic antineutrinos of all flavors to the electron neutrinos and thereby changing the electron fraction in the environment.

On the other hand, the third neutrino mixing angle was shown to be nonzero by the recent Daya Bay [42], RENO [43] and Double Chooz [44] experiments and this opens up the possibility of *CP* violation in neutrino flavor oscillations. If the *CP* symmetry is broken by the neutrino oscillations, the value of the corresponding Dirac phase may be within the reach of the next generation of very long base-line experiments such as LBNE [45] or LBNO [46].

The effects of a possible *CP* violation in supernovae were considered by several authors [47–49] in connection with the collective oscillations. In particular, it was shown that, in the mean-field approximation, the term which contains the *CP*-violating phase factors out of the evolution operator so that *CP*-violating effects evolve independently of the nonlinear flavor transformations [50]. However, a mean-field treatment depicts an interacting many-body system only approximately in terms of independent particles moving in an average field which is collectively formed by the particles themselves in a self-consistent way. Such a treatment by definition ignores the quantum entanglements and takes into account only those states in which each particle can be described in an effective one-particle picture where the mean-field consistency conditions can be met. It is not clear whether such a formulation allows us to easily distinguish the effects that are induced by the dynamics from those that originate from a particular choice of the initial conditions or from the reduction of the Hilbert space to unentangled states.

In this paper we show that the factorization of the *CP*-violating effects from the flavor evolution during collective oscillations is more general than is implied in its original derivation. We use a formulation of the *CP* violation which is independent of the mean-field techniques and relies only on the symmetry principles. We therefore show that even in the regime where quantum entanglements due to many-body effects may be important, the *CP*-violating effects factor out of the full many-body evolution operator and evolve independently of the nonlinear flavor transformations. However, we also show that, when the neutrino magnetic moment comes into play in the presence of a strong magnetic field, the *CP* factorization procedure requires us to define an *effective* magnetic moment. This effective magnetic moment includes the *CP*-violating Dirac phase and is different for neutrinos and antineutrinos, indicating that the effects due to *CP* violation and the magnetic moment are intertwined in the neutrino flavor evolution. But, as we argue in Sec. V, the formulation introduced in this paper allows us to factor the *CP*-violating phase out of the entire Hamiltonian including the effects due to vacuum oscillations, matter refraction, self-interactions and the electromagnetic properties of the neutrinos at the expense of using an effective definition for the neutrino magnetic moment which is a small term and can be treated perturbatively to the first order in most cases.

This paper is organized as follows. In Sec. II, we introduce an operator formalism to describe flavor mixing of neutrinos. This operator formulation is somewhat different from the commonly used mixing matrix formalism, but it is better suited for the full many-body description of the problem and for an analysis of the its symmetries. In Sec. III, we express the vacuum oscillations of neutrinos together with the refracting effects including neutrino self-interactions in this formalism, and describe how the *CP*-violating Dirac phase can be factored out of the total Hamiltonian and the evolution operator (we do not consider the neutrino magnetic moment at this point). In Sec. IV, we examine the dynamical symmetries of the problem in the single-angle approximation by ignoring effects of a possible net electron background and present the corresponding many-body constants of motion. Although the formulation of the neutrino self-interactions are carried out entirely in the exact many-body picture in this paper, in Sec. IV B we briefly consider an effective one-particle approximation in the form of a mean-field formulation and show that the expectation values of the many-body constants of motion remain invariant under the mean-field evolution of the system. In Sec. V, we include the effects of the neutrino magnetic moment in the presence of a uniform magnetic field and show that the factorization of the *CP*-violating Dirac phase out of the full flavor evolution Hamiltonian can be carried out using an effective definition for the neutrino magnetic moment.

## II. FLAVOR TRANSFORMATIONS

In this paper, we use $a_{ih}$ and $b_{ih}$ to denote the annihilation operators for neutrinos and antineutrinos, respectively, in the $i$th mass eigenstate with chirality $h$.

We consider only the ultrarelativistic case for which the helicity and chirality are the same for neutrinos and opposite for antineutrinos. In other words, $a_{ih}$ annihilate neutrinos with helicity $h$ and $b_{ih}$ annihilate antineutrinos with helicity $-h$. If one does not take account of the neutrino magnetic moment, which can cause chirality to change, then it is sufficient to consider only the left-handed particles, i.e., the negative-helicity neutrinos and the positive-helicity antineutrinos. For this reason, we drop the helicity index from our notation and use

$$a_i \equiv a_{i-} \quad \text{and} \quad b_i \equiv -b_{i-} \tag{1}$$

until Sec. V where we take the neutrino magnetic moment into account.

In the literature, an isospin-type formalism is typically employed in order to describe a simplified two-neutrino mixing scenario by introducing a *neutrino doublet* $(\nu_1, \nu_2)$ and the associated *isospin operators* (see, for example, Ref. [31])

$$J^+(\vec{p}) = a_1^\dagger(\vec{p})a_2(\vec{p}), \qquad J^-(\vec{p}) = a_2^\dagger(\vec{p})a_1(\vec{p}),$$
$$J^z(\vec{p}) = \frac{1}{2}(a_1^\dagger(\vec{p})a_1(\vec{p}) - a_2^\dagger(\vec{p})a_2(\vec{p})), \tag{2}$$

where $\vec{p}$ denotes the neutrino momentum. These operators form an SU(2) algebra.

In the case of antineutrinos, the doublet $(-\bar{\nu}_2, \bar{\nu}_1)$ is typically used instead of $(\bar{\nu}_1, \bar{\nu}_2)$ because it leads to a unified treatment of neutrinos and antineutrinos and greatly simplifies the formulation (see, for example, Refs. [51,52]). We can do so since under the SU(2) group the doublets $(-\bar{\nu}_2, \bar{\nu}_1)$ and $(\bar{\nu}_1, \bar{\nu}_2)$ transform with the same group element. Accordingly, the antineutrino isospin operators are defined as

$$\bar{J}^+(\vec{p}) = -b_2^\dagger(\vec{p})b_1(\vec{p}), \qquad \bar{J}^-(\vec{p}) = -b_1^\dagger(\vec{p})b_2(\vec{p}),$$
$$\bar{J}^z(\vec{p}) = \frac{1}{2}(b_2^\dagger(\vec{p})b_2(\vec{p}) - b_1^\dagger(\vec{p})b_1(\vec{p})). \tag{3}$$

The isospin formalism can be generalized to accommodate three-generation mixing by introducing the following neutrino and antineutrino bilinears:

$$T_{ij}(p, \vec{p}) = a_i^\dagger(\vec{p})a_j(\vec{p}),$$
$$T_{ij}(-p, \vec{p}) = -b_j^\dagger(\vec{p})b_i(\vec{p}), \tag{4}$$

for $i, j = 1, 2, 3$. Here $p = |\vec{p}|$ denotes the energy of the neutrino. Note that we use the same notation for neutrino and antineutrino bilinears except that the neutrino bilinears are labeled by the energy whereas the antineutrino bilinears are labeled by minus the energy. Such a notation allows us to consolidate the neutrino and antineutrino degrees of freedom into one simple formulation in which energy is allowed to run over both negative and positive values representing antineutrinos and neutrinos, respectively. In order to do this, we introduce the convention

$$T_{ij}(E, \vec{p}) \quad \text{where} \begin{cases} E = p & \text{for neutrinos} \\ E = -p & \text{for antineutrinos,} \end{cases} \tag{5}$$

and use the word *energy* to refer to both positive and negative values in the rest of this paper. Let us also note here that we use the word *particle* generically to refer to both neutrinos and antineutrinos.

The operators defined in Eq. (4) obey U(3) commutation relations[1]

$$[T_{ij}(E, \vec{p}), T_{kl}(E', \vec{p}')]$$
$$= \delta_{E,E'}\delta_{\vec{p},\vec{p}'}(\delta_{kj}T_{il}(E, \vec{p}) - \delta_{il}T_{kj}(E, \vec{p})). \tag{6}$$

In this equation, the factor $\delta_{\vec{p},\vec{p}'}$ reflects the fact that the particle operators corresponding to different momenta commute with one another, whereas the factor $\delta_{E,E'}$ guarantees that the neutrino and antineutrino operators commute with each other even when they have the same momentum.

It is useful to introduce the sum[2]

$$T_{ij}(E) \equiv \sum_{\substack{\vec{p} \\ (|\vec{p}|=p)}} T_{ij}(E, \vec{p}). \tag{7}$$

This sum runs over all neutrinos ($E = p$) or antineutrinos ($E = -p$) which travel in different directions but have the same energy. We would like to point out that, since the collective oscillations of neutrinos are many-body phenomena, one typically needs additional quantum numbers besides the momentum to label the individual particles. However, we do not show these quantum numbers explicitly in our formulas for ease of reading. Instead, when we

[1]We refer to this group as U(3) although it is technically the tensor product of as many U(3) algebras as the number of particles in the system. We use this offhand terminology throughout the paper for simplicity.

[2]In this paper we use sums over discrete momentum values rather than integrals over the continuum values until Sec. V where we switch back to the continuum integration (see footnote 6 below for the motivation behind this choice). Technically this requires the use of a normalization volume $V$ such that every discrete sum over momentum is multiplied by a factor of $1/V$ which yields

$$\frac{1}{V}\sum_{\vec{p}} \to \frac{1}{(2\pi)^3}\int d^3\vec{p}$$

in the continuum limit. This introduces an overall $1/V$ factor multiplying our Hamiltonian but we do not show this factor explicitly because the normalization volume becomes unimportant as soon as we take the continuum limit in the sense that the physical quantities are independent of it.

use $\vec{p}$ as in Eq. (4), for example, we view it as a collective attribute which includes all the quantum numbers needed to label an individual particle. In any case, we consider these additional quantum numbers to be also summed over in Eq. (7).

We also introduce the sum over all particles of all energies

$$T_{ij} \equiv \sum_E T_{ij}(E). \tag{8}$$

In this paper, a summation over energy such as the one in Eq. (8), always runs over both positive and negative values so that the resulting quantity incorporates both neutrinos and antineutrinos. Of course, we can always separate neutrino and antineutrino energy spectra when we need them.

For three neutrino species, the transformation from the mass basis to the flavor basis can be decomposed into three successive schemes of two-generation mixing. For this reason we first consider a transformation involving only the $i$th and $j$th mass eigenstates. Note that the change from the mass basis to the flavor basis is a global transformation in the sense that all neutrinos transform in the same way irrespective of their energies. The same is also true for the antineutrinos although neutrinos and antineutrinos transform differently in the presence of $CP$ violation. Such a transformation can be formulated in terms of the total particle bilinears defined in Eq. (8). In particular, the operators

$$T_{ij}, \qquad T_{ji}, \quad \text{and} \quad \frac{1}{2}(T_{ii} - T_{jj}), \tag{9}$$

form an SU(2) subalgebra[3] and generate the mixing between the $i$th and $j$th mass eigenstates through the operator

$$Q_{ij}(z) = e^{zT_{ij}} e^{\ln(1+|z|^2)\frac{1}{2}(T_{ii}-T_{jj})} e^{-z^* T_{ji}}. \tag{10}$$

Here $z$ is a complex variable which is related to the mixing angle $\theta$ and a possible $CP$-violating phase $\delta$ by

$$z = e^{-i\delta} \tan\theta. \tag{11}$$

The operator in Eq. (10) transforms the neutrinos as

$$\begin{aligned} Q_{ij}^\dagger a_i(\vec{p}) Q_{ij} &= \cos\theta a_i(\vec{p}) + e^{-i\delta} \sin\theta a_j(\vec{p}), \\ Q_{ij}^\dagger a_j(\vec{p}) Q_{ij} &= -e^{i\delta} \sin\theta a_i(\vec{p}) + \cos\theta a_j(\vec{p}), \end{aligned} \tag{12a}$$

and the antineutrinos as

$$\begin{aligned} Q_{ij}^\dagger b_i(\vec{p}) Q_{ij} &= \cos\theta b_i(\vec{p}) + e^{i\delta} \sin\theta b_j(\vec{p}), \\ Q_{ij}^\dagger b_j(\vec{p}) Q_{ij} &= -e^{-i\delta} \sin\theta b_i(\vec{p}) + \cos\theta b_j(\vec{p}), \end{aligned} \tag{12b}$$

[3]This algebra is SU(2) rather than U(2) because we did not include the symmetric combination $T_{ii} + T_{jj}$.

as can be easily shown by using the Baker-Champbell-Hausdorf formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \ldots \tag{13}$$

Note that, although neutrino and antineutrino bilinears appear symmetrically in the definition of the operator $Q_{ij}$ [see Eqs. (8) and (10)], the transformation of antineutrinos differs from that of neutrinos in Eq. (12) by a complex phase in the presence of $CP$ violation, i.e., when $\delta \neq 0$. This is due to the difference in the definitions of neutrino and antineutrino bilinears in Eq. (4).

Mixing between three generations of neutrinos can be decomposed into three consecutive transformations of two-flavor mixing in the form of Eq. (12). The relevant operator is

$$Q = Q_{23}(t_{\mathrm{A}}) Q_{13}(e^{-i\delta} t_{\mathrm{R}}) Q_{12}(t_\odot), \tag{14a}$$

with

$$t_\odot = \tan\theta_\odot, \qquad t_{\mathrm{R}} = \tan\theta_{\mathrm{R}}, \qquad t_{\mathrm{A}} = \tan\theta_{\mathrm{A}}, \tag{14b}$$

where $\theta_\odot$, $\theta_{\mathrm{R}}$ and $\theta_{\mathrm{A}}$ refer to solar, reactor and atmospheric mixing angles, respectively, and $\delta$ is the $CP$-violating Dirac phase. With these definitions, the flavor and mass bases are simply related by

$$a_{\alpha_i}(\vec{p}) = Q^\dagger a_i(\vec{p}) Q \quad \text{and} \quad b_{\alpha_i}(\vec{p}) = Q^\dagger b_i(\vec{p}) Q, \tag{15a}$$

where we set

$$\alpha_1 = e, \qquad \alpha_2 = \mu, \qquad \alpha_3 = \tau. \tag{15b}$$

In the literature, it is more common to express the relation between the mass and weak interaction bases with a mixing matrix rather than with an operator as in Eq. (15). In fact, considering the successive two-flavor transformations in Eq. (14) together with Eq. (12) one can write Eq. (15) in the familiar form as

$$\begin{pmatrix} a_e \\ a_\mu \\ a_\tau \end{pmatrix} = W \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \begin{pmatrix} b_e \\ b_\mu \\ b_\tau \end{pmatrix} = W^* \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \tag{16a}$$

where $W$ is a unitary matrix given by

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{16b}$$

with $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$. But the operator form of the neutrino mixing introduced in Eq. (15) is more suitable for our purpose of formulating the many-body dynamics.

The particle bilinears defined in Eq. (4) can also be transformed into the flavor basis using Eq. (15), e.g.,

$$T_{\alpha_i\alpha_j}(E,\vec{p}) \equiv Q^\dagger T_{ij}(E,\vec{p})Q = \begin{cases} a^\dagger_{\alpha_i}(\vec{p})a_{\alpha_j}(\vec{p}), & \text{for} E>0, \\ -b^\dagger_{\alpha_j}(\vec{p})b_{\alpha_i}(\vec{p}), & \text{for} E<0. \end{cases} \tag{17}$$

These operators are subject to summation conventions which are analogous to those introduced in Eqs. (7) and (8). Note that the transformation operator $Q$ has exactly the same form in both the flavor and mass bases. This can be shown as follows:

$$Q = Q^\dagger QQ = Q^\dagger Q_{23}Q_{13}Q_{12}Q = Q_{\mu\tau}Q_{e\tau}Q_{e\mu}. \tag{18}$$

Here $Q_{\alpha_i\alpha_j}(z)$ has the same form as $Q_{ij}(z)$ given in Eq. (10) except that $i$ and $j$ are replaced by $\alpha_i$ and $\alpha_j$, respectively.

## III. FLAVOR EVOLUTION OF NEUTRINOS

### A. Vacuum oscillations

The propagation of neutrinos and antineutrinos in vacuum is described by the Hamiltonian

$$H_\mathrm{v} = \sum_{\vec{p}}\sum_{i=1}^{3}\sqrt{p^2+m_i^2}(T_{ii}(p,\vec{p}) - T_{ii}(-p,\vec{p})). \tag{19}$$

Here $T_{ii}(E,\vec{p})$ is a number operator in the mass basis and is clearly conserved by the vacuum Hamiltonian, i.e.,

$$[H_\mathrm{v}, T_{ii}(E,\vec{p})] = 0. \tag{20}$$

But since the neutrinos and antineutrinos are created in flavor states, the initial state is not an eigenstate of the number operators in the mass basis. As a result, although $T_{ii}(E,\vec{p})$ is a constant of motion, it is not proportional to identity and cannot be subtracted from the Hamiltonian. However the sum of the number operators over three generations has the same value in both the mass and flavor bases because of the unitarity of the transformation. In other words, the initial state is an eigenstate of the total number operator

$$\sum_{i=1}^{3} T_{ii}(E,\vec{p}) = \sum_{i=1}^{3} T_{\alpha_i\alpha_i}(E,\vec{p}). \tag{21}$$

Therefore the operator in Eq. (21) is both constant *and* proportional to identity which tells us that any multiple of it can be subtracted from the Hamiltonian. In particular, applying the ultrarelativistic approximation,

$$\sqrt{p^2+m_i^2} \cong p + \frac{m_i^2}{2p}, \tag{22}$$

and subtracting the quantity

$$\sum_E\left[\left(E + \frac{m_1^2+m_2^2+m_3^2}{6E}\right)\sum_{i=1}^{3}T_{ii}(E)\right] \tag{23}$$

allows us to express the Hamiltonian in Eq. (19) in terms of the squared-mass differences which are the relevant parameters for neutrino oscillations. This yields

$$H_\mathrm{v} = \sum_E\sum_{i=1}^{3}\frac{\Delta_i^2}{6E}T_{ii}(E), \tag{24}$$

where we defined

$$\Delta_i^2 = \sum_{j(\neq i)}\delta m_{ij}^2, \tag{25}$$

and used the summation convention introduced in Eqs. (7) and (8). As noted earlier, the sum over $E$ in Eqs. (23) and (24) runs over both neutrino ($E>0$) and antineutrino ($E<0$) degrees of freedom.

The vacuum Hamiltonian given in Eq. (24) can be expressed in the flavor basis by inverting Eq. (17), i.e.,

$$H_\mathrm{v} = \sum_E\sum_i\frac{\Delta_i^2}{6E}QT_{\alpha_i\alpha_i}(E)Q^\dagger. \tag{26}$$

Here, all the information about the mixing angles and the $CP$-violating Dirac phase is hidden in the operator $Q$. If one applies the transformation imposed by $Q$ using Eqs. (12), (17) and (18), then flavor off-diagonal terms in the form of $T_{\alpha_i\alpha_j}(p)$ appear in the Hamiltonian in Eq. (26) together with the mixing parameters.

### B. Coherent scattering in an ordinary background

Neutrinos interact very weakly with matter but their flavor transformations are nevertheless modified as they propagate in matter because their diminutive scattering amplitudes primarily superpose coherently in the forward direction. As a result, the dispersion relation is changed for each neutrino flavor depending on its interactions in a way which is very similar to the refraction of light in a medium [53–55]. Although all neutrinos undergo refraction, flavor oscillations are only sensitive to how different flavors are distinguished from one another as they propagate. In this paper, we consider an ordinary matter background, i.e., a neutral and unpolarized background composed of protons, neutrons, electrons and positrons. Such an environment singles out electron neutrinos which experience an additional charged-current interaction. Therefore the net matter refraction effect is captured by the Hamiltonian

$$H_\mathrm{m} = \sqrt{2}G_F\mathcal{N}_e T_{ee}. \tag{27}$$

Here $\mathcal{N}_e$ denotes the net number density of electrons (electrons minus positrons) in the background and $T_{ee}$ is the total number of electron neutrinos minus the total number of electron antineutrinos, i.e.,

$$T_{ee} = \sum_{\vec{p}} (a_e^\dagger(\vec{p}) a_e(\vec{p}) - b_e^\dagger(\vec{p}) b_e(\vec{p})), \tag{28}$$

as implied by Eqs. (4), (7) and (8).

### C. Self-interactions of neutrinos

For sufficiently high neutrino densities, neutrino-neutrino scatterings can contribute to flavor evolution by creating a self-refraction effect [16]. In the case of self-interactions, it is not only the forward-scattering diagrams that add up coherently but also those diagrams in which particles exchange their flavors [17]. The contribution of self-interactions to the neutrino flavor evolution can be described by the following effective Hamiltonian [19]:

$$H_{\mathrm{s}} = \frac{G_F}{\sqrt{2}V} \sum_{i,j=1}^{3} \sum_{E,\vec{p}} \sum_{E',\vec{p}'} R_{\vec{p}\vec{p}'} T_{\alpha_i\alpha_j}(E,\vec{p}) T_{\alpha_j\alpha_i}(E',\vec{p}'). \tag{29}$$

Here

$$R_{\vec{p}\vec{p}'} = 1 - \cos\theta_{\vec{p}\vec{p}'}, \tag{30}$$

where $\theta_{\vec{p}\vec{p}'}$ is the angle between the momenta of the interacting neutrinos and $V$ is the quantization volume.[4]

Self-interactions turn neutrino flavor conversion into a many-body phenomenon because the coherent superposition of flavor exchange diagrams couples the flavor evolution of each neutrino to that of the entire ensemble. This poses a formidable problem because the resulting dynamics is nonlinear and the presence of the entangled states makes the dimension of the Hilbert space astronomically large. The latter difficulty can be avoided by adopting an effective one-particle approximation which reduces the dimension of the Hilbert space by omitting entangled many-body states. Such an approach was developed in Refs. [17,18] in the form of a mean-field formalism and was widely adopted by subsequent authors. However, the nonlinearity of the original many-body problem is inherited by the resulting mean-field consistency equations and renders them very difficult to solve in general.

[4]We remarked earlier that we do not show the normalization volumes because they are *physically* not relevant, (see footnote 2). In the case of neutrino self-interactions, however, the normalization volume is important because it determines the density of neutrinos which controls the strength of the neutrino potential. Another way of saying this is that although, for example, the vacuum oscillation term has only one $1/V$ factor, the self-interaction term has two such factors, one of which tells us how many other neutrinos our *test neutrino* interacts with.

Here, we do not necessarily resort to an effective one-particle formulation but neither do we attempt to solve the many-body problem. Our purpose is to examine the full many-body system from the perspective of its symmetries in connection with $CP$ violation and dynamical invariants. However, we also study the manifestations of these symmetries under the effective one-particle approximation in Sec. IV B.

Note that each term in the self-interaction Hamiltonian given in Eq. (29) in the form of

$$\sum_{i,j=1}^{3} T_{\alpha_i\alpha_j}(E,\vec{p}) T_{\alpha_j\alpha_i}(E',\vec{p}'), \tag{31}$$

is a scalar in the flavor space, i.e., it is invariant under any global unitary transformation. This follows from the fact that they all commute with the global operators $T_{\alpha_k\alpha_l}$:

$$\left[\sum_{i,j=1}^{3} T_{\alpha_i\alpha_j}(E,\vec{p}) T_{\alpha_j\alpha_i}(E',\vec{p}'), T_{\alpha_k\alpha_l}\right] = 0. \tag{32}$$

Equation (32), together with Eqs. (10) and (18), tells us that the self-interaction Hamiltonian itself is rotationally invariant, i.e.,

$$[H_{\mathrm{s}}, Q_{\alpha_i\alpha_j}] = 0 \tag{33}$$

is satisfied for every $i, j = 1, 2, 3$. As a result, it has the same form in both the mass and flavor bases:

$$\begin{aligned} H_{\mathrm{s}} &= Q H_{\mathrm{s}} Q^\dagger \\ &= \frac{G_F}{\sqrt{2}V} \sum_{i,j=1}^{3} \sum_{E,\vec{p}} \sum_{E',\vec{p}'} R_{\vec{p}\vec{p}'} T_{ij}(E,\vec{p}) T_{ji}(E',\vec{p}'). \end{aligned} \tag{34}$$

### D. Neutrino propagation with *CP* violation

The full problem of neutrino flavor evolution in an astrophysical environment, including vacuum oscillations, matter effects and self-interactions, is represented by the sum of the Hamiltonians given in Eqs. (26), (27) and (29):

$$H = H_{\mathrm{v}} + H_{\mathrm{m}} + H_{\mathrm{s}}. \tag{35}$$

Here, the only term which explicitly involves the $CP$-violating phase is the vacuum oscillation term through the operator $Q$. The matter term $H_{\mathrm{m}}$ includes the net electron number density and it can introduce a $CP$ violation due to the matter-antimatter asymmetry of the background. But the matter Hamiltonian itself does not explicitly depend on the *intrinsic* $CP$-violating Dirac phase. A similar statement is also true for the self-interaction Hamiltonian $H_{\mathrm{s}}$, i.e., although it can introduce a $CP$ asymmetry if the initial neutrino and antineutrino backgrounds are not the same, neutrino-neutrino interactions are independent of the intrinsic *CP*-violating phase. This can be seen from the fact that $H_\mathrm{s}$ has the same form in both the mass and flavor bases as indicated by Eqs. (29) and (34).

The factorization of the *CP*-violating phase from the flavor evolution stems from the following identity which is true for any unitary operator in the form of Eq. (10):

$$Q_{e\tau}(e^{-i\delta}t_\mathrm{R}) = S_\tau^\dagger Q_{e\tau}(t_\mathrm{R})S_\tau. \tag{36a}$$

Here the operator $Q_{e\tau}(t_\mathrm{R})$ does not contain the *CP*-violating Dirac phase $\delta$ which is now incorporated into the operator

$$S_\tau = e^{-i\delta(T_{\tau\tau}+\bar{T}_{\tau\tau})}. \tag{36b}$$

Therefore, we can write the transformation operator $Q$ in Eq. (18) as

$$Q = Q_{\mu\tau}(t_\mathrm{A})S_\tau^\dagger Q_{e\tau}(t_\mathrm{R})Q_{e\mu}(t_\odot)S_\tau, \tag{37}$$

where we used Eq. (36) together with the fact that $S_\tau$ and $Q_{e\mu}$ commute with each other because they live in orthogonal flavor subspaces. However, $S_\tau$ does not commute with $Q_{\mu\tau}$ and for this reason *CP* factorization cannot be realized in the ordinary flavor basis. Instead, one has to transform into another basis in which $\mu$ and $\tau$ eigenstates are suitably mixed with one another. This is usually referred to as the *rotated flavor basis* and is defined as

$$\begin{aligned} a_{\tilde{\alpha}_i}(\vec{p}) &\equiv Q_{\mu\tau}a_{\alpha_i}(\vec{p})Q_{\mu\tau}^\dagger, \\ b_{\tilde{\alpha}_i}(\vec{p}) &\equiv Q_{\mu\tau}b_{\alpha_i}(\vec{p})Q_{\mu\tau}^\dagger. \end{aligned} \tag{38a}$$

From Eq. (12), we see that this specifically yields[5]

$$\begin{aligned} a_{\tilde{e}}(\vec{p}) &= a_e(\vec{p}), \\ a_{\tilde{\mu}}(\vec{p}) &= \cos\theta_\mathrm{A} a_\mu(\vec{p}) + \sin\theta_\mathrm{A} a_\tau(\vec{p}), \\ a_{\tilde{\tau}}(\vec{p}) &= -\sin\theta_\mathrm{A} a_\mu(\vec{p}) + \cos\theta_\mathrm{A} a_\tau(\vec{p}), \end{aligned} \tag{38b}$$

for the neutrinos and

$$\begin{aligned} b_{\tilde{e}}(\vec{p}) &= b_e(\vec{p}), \\ b_{\tilde{\mu}}(\vec{p}) &= \cos\theta_\mathrm{A} b_\mu(\vec{p}) + \sin\theta_\mathrm{A} b_\tau(\vec{p}), \\ b_{\tilde{\tau}}(\vec{p}) &= -\sin\theta_\mathrm{A} b_\mu(\vec{p}) + \cos\theta_\mathrm{A} b_\tau(\vec{p}), \end{aligned} \tag{38c}$$

for the antineutrinos. In most cases, the rotated and ordinary flavor bases are physically equivalent to each other. For example, in the case of neutrinos emanating from a supernova, $\nu_\mu$, $\nu_\tau$, $\bar{\nu}_\mu$ and $\bar{\nu}_\tau$ spectra are almost identical. These neutrinos also undergo the same neutral-current weak interactions as they propagate in the mantle. As a result, one has the same set of initial conditions and the same dispersion relation in both the rotated and the ordinary flavor bases.

That the desired factorization of the *CP*-violating phase is achieved in the rotated flavor basis can be seen by multiplying Eq. (37) on the right with $Q_{\mu\tau}^\dagger Q_{\mu\tau}$ and using Eq. (38). The result is as follows:

$$Q = S_{\tilde{\tau}}^\dagger Q_{\tilde{e}\,\tilde{\tau}}(t_\mathrm{R})Q_{\tilde{e}\,\tilde{\mu}}(t_\odot)S_{\tilde{\tau}}Q_{\mu\tau}(t_\mathrm{A}). \tag{39}$$

Here, all operators with a tilde have the same form as they are originally defined except that $a_{\alpha_i}$ and $b_{\alpha_i}$ are replaced by $a_{\tilde{\alpha}_i}$ and $b_{\tilde{\alpha}_i}$, respectively. In Eq. (39), the part of the transformation operator $Q$ excluding the rightmost $Q_{\mu\tau}$ is now expressed in the rotated flavor basis and properly factorized so as to separate the *CP*-violating phase from the flavor evolution. The function of the rightmost $Q_{\mu\tau}$ is to transform the object on which $Q$ is acting into the rotated flavor basis where the factorization is realized. For example, using Eq. (39), we can express the vacuum Hamiltonian given in Eq. (26) as

$$\begin{aligned} H_\mathrm{v} = {}& S_{\tilde{\tau}}^\dagger Q_{\tilde{e}\,\tilde{\tau}}(t_\mathrm{R})Q_{\tilde{e}\,\tilde{\mu}}(t_\odot)S_{\tilde{\tau}} \\ & \times \sum_E \sum_i \frac{\Delta_i^2}{6E} T_{\tilde{\alpha}_i\tilde{\alpha}_i}(E) S_{\tilde{\tau}}^\dagger Q_{\tilde{e}\,\tilde{\mu}}^\dagger(t_\odot)Q_{\tilde{e}\,\tilde{\tau}}^\dagger(t_\mathrm{R})S_{\tilde{\tau}}, \end{aligned} \tag{40}$$

where we applied the definition of the rotated flavor basis from Eq. (38) in order to transform $T_{\alpha_i\alpha_i}(E)$ to $T_{\tilde{\alpha}_i\tilde{\alpha}_i}(E)$. One should also note that $S_{\tilde{\tau}}$ commutes with $T_{\tilde{\alpha}_i\tilde{\alpha}_i}(E)$ because $S_{\tilde{\tau}}$ involves only the number operators in the rotated flavor basis and $T_{\tilde{\alpha}_i\tilde{\alpha}_i}(E)$ are also number operators themselves. Applying this to Eq. (40) leads to the result

$$H_\mathrm{v} = S_{\tilde{\tau}}^\dagger H_{\tilde{\mathrm{v}}}^{(0)} S_{\tilde{\tau}}, \tag{41}$$

where $H_{\tilde{\mathrm{v}}}^{(0)}$ is the Hamiltonian which represents the vacuum oscillations in the rotated flavor basis in the absence of any *CP*-violating phase. It is given by

$$\begin{aligned} H_{\tilde{\mathrm{v}}}^{(0)} = {}& Q_{\tilde{e}\,\tilde{\tau}}(t_\mathrm{R})Q_{\tilde{e}\,\tilde{\mu}}(t_\odot) \\ & \times \sum_E \sum_i \frac{\Delta_i^2}{6E} T_{\tilde{\alpha}_i\tilde{\alpha}_i}(E) Q_{\tilde{e}\,\tilde{\mu}}^\dagger(t_\odot)Q_{\tilde{e}\,\tilde{\tau}}^\dagger(t_\mathrm{R}). \end{aligned} \tag{42}$$

The matter Hamiltonian given in Eq. (27) and the self-interaction Hamiltonian given in Eq. (29) both commute with the *CP* violation term $S_{\tilde{\tau}}$:

$$[H_\mathrm{m}, S_{\tilde{\tau}}] = 0 \quad \text{and} \quad [H_\mathrm{s}, S_{\tilde{\tau}}] = 0. \tag{43}$$

The first commutator above is trivially true because $H_\mathrm{m}$ and $S_{\tilde{\tau}}$ live in orthogonal flavor spaces and the second

[5]Although $\nu_e$ and $\bar{\nu}_e$ remain the same under this transformation which takes place in the orthogonal subspace, we introduce the notation $\nu_{\tilde{e}} = \nu_e$ and $\bar{\nu}_{\tilde{e}} = \bar{\nu}_e$ because it simplifies our formulas in subsequent sections.

commutator immediately follows from Eq. (32) with $k = l$. At a more intuitive level, the second commutator in Eq. (43) is a result of the fact that the scattering of neutrinos from one another does not change the total number of neutrinos or antineutrinos in any flavor eigenstate and that the operator $S_{\tilde{\tau}}$ includes only the total number operators for $\nu_{\tilde{\tau}}$ and $\bar{\nu}_{\tilde{\tau}}$ as can be seen from its definition in Eq. (36b). Therefore, the Hamiltonian in Eq. (35) can be written as

$$H = S_{\tilde{\tau}}^{\dagger}(H_{\tilde{\mathrm{v}}}^{(0)} + H_{\mathrm{m}} + H_{\mathrm{s}})S_{\tilde{\tau}}. \tag{44}$$

The $CP$-violating phase is now factorized in such a way that the Hamiltonian inside the parentheses in Eq. (44) has no $CP$-violating phases. In this Hamiltonian the vacuum term is expressed in the rotated flavor basis whereas the other terms are written in the ordinary flavor basis. However, both $H_{\mathrm{m}}$ and $H_{\mathrm{s}}$ do not change under the transformation from the ordinary flavor basis to rotated flavor basis because they both commute with the transformation operator $Q_{\mu\tau}$:

$$[H_{\mathrm{m}}, Q_{\mu\tau}] = 0 \quad \text{and} \quad [H_{\mathrm{s}}, Q_{\mu\tau}] = 0. \tag{45}$$

The first commutator in Eq. (45) is again trivially true since $\nu_e$ is orthogonal to the $\nu_\mu$-$\nu_\tau$ subspace and it leads to

$$H_{\mathrm{m}} = Q_{\mu\tau} H_{\mathrm{m}} Q_{\mu\tau}^{\dagger} = H_{\tilde{\mathrm{m}}} = \sqrt{2} G_F \mathcal{N}_e T_{\tilde{e}\,\tilde{e}}. \tag{46a}$$

The second commutator in Eq. (45) is a special case of Eq. (33) and it allows us to write

$$\begin{aligned} H_{\mathrm{s}} &= Q_{\mu\tau} H_{\mathrm{s}} Q_{\mu\tau}^{\dagger} = H_{\tilde{\mathrm{s}}} \\ &= \frac{G_F}{\sqrt{2}V} \sum_{i,j=1}^{3} \sum_{E,\vec{p}} \sum_{E',\vec{p}'} R_{\vec{p}\vec{p}'} T_{\tilde{\alpha}_i\tilde{\alpha}_j}(E,\vec{p}) T_{\tilde{\alpha}_j\tilde{\alpha}_i}(E',\vec{p}'). \end{aligned} \tag{46b}$$

Therefore, the total Hamiltonian given in Eq. (44) can be written as

$$H = S_{\tilde{\tau}}^{\dagger} \tilde{H}^{(0)} S_{\tilde{\tau}}, \tag{47a}$$

where $\tilde{H}^{(0)}$ is a Hamiltonian which describes the vacuum oscillations and coherent scatterings of neutrinos from the background particles as well as from each other in the rotated flavor space and includes no $CP$-violating phase. It is given by

$$\tilde{H}^{(0)} = H_{\tilde{\mathrm{v}}}^{(0)} + H_{\tilde{\mathrm{m}}} + H_{\tilde{\mathrm{s}}}. \tag{47b}$$

Equation (47a) and (47b) tell us that the collective flavor transformations of neutrinos *as a many-body system* can be described by an evolution operator

$$U(t) = S_{\tilde{\tau}}^{\dagger} \tilde{U}^{(0)}(t) S_{\tilde{\tau}}, \tag{48a}$$

where $\tilde{U}_0(t)$ is the evolution operator corresponding to the Hamiltonian $\tilde{H}^{(0)}$. In other words, it is the solution of

$$i\hbar \frac{d}{dt} \tilde{U}^{(0)}(t) = \tilde{H}^{(0)} \tilde{U}^{(0)}(t), \tag{48b}$$

with the initial condition $\tilde{U}_0(t=0) = I$.

## IV. CONSTANTS OF MOTION

Self-interactions turn the problem of neutrino flavor transformation in an astrophysical environment into a many-body phenomenon and give rise to highly nonlinear forms of flavor evolution. Still, numerical simulations reveal that some forms of collective regular behavior can emerge from the apparent complexity. *Synchronized oscillations* in which all neutrinos oscillate with a single frequency [26] and *bipolar oscillations* in which the whole ensemble can be described in terms of two frequencies [27] are the earliest discoveries of such collective behavior and both were observed in a simplified two-neutrino mixing scheme under the mean-field approximation. Another noteworthy emergent behavior is the phenomenon of spectral splits in which neutrinos or antineutrinos exchange their energy spectra at certain critical energies under the adiabatic evolution conditions [29]. These splits are observed in numerical simulations for both two- and three-flavor mixing scenarios in the mean-field case. For a review, see Ref. [25].

Such collective modes of regular behavior call attention to possible symmetries which may underline the dynamics of the system. In fact, an earlier study [31] by the present authors pointed out some parallels between self-interacting neutrinos in a two-flavor mixing scheme and the BCS model of superconductivity [56] describing the Cooper pairs of electrons in the conduction band of a metal. In particular, the role of the neutrino flavor isospin [see Eqs. (2) and (3)] in the former case is played by the pair quasispin in the latter. We used this analogy to show that certain dynamical symmetries, which were already known in the context of the BCS model [57–59], are also respected by flavor oscillations of self-interacting neutrinos in the exact many-body case if the following conditions are satisfied:

(1) The single-angle approximation is adopted,
(2) no net leptonic background is present, and
(3) the neutrino density is fixed.

There are as many such dynamical symmetries as the number of energy modes under consideration and they manifest themselves as a set of *constants of motion*, i.e., quantities which depend nontrivially on the initial flavor content of the ensemble and do not change as neutrinos propagate and undergo flavor evolution.

It was also shown in Ref. [31] that under the adoption of an effective one-particle approximation (i.e., in the mean-field picture), these dynamical symmetries are no longer exact but the expectation values of the corresponding constants of motion continue to remain invariant. These mean-field invariants are closely related to the *N-mode* coherence modes considered in Ref. [30] which are also known as *degenerate solutions* in the context of the BCS model [60,61].

It should be noted that, although these dynamical symmetries are exact only under the assumptions listed above, they can still be relevant when the system is away from these idealized conditions. For example, it was demonstrated in Ref. [31] that the two-flavor spectral split phenomenon, which emerges as neutrinos adiabatically evolve from a high-density region into the vacuum can be analytically understood in terms of one of the dynamical symmetries although condition 3 is violated in this case. In this scheme, the split frequency corresponds to the chemical potential in the BCS model of superconductivity.

These observations clearly call for a thorough analysis of collective neutrino oscillation modes in connection with the dynamical symmetries which will be the subject of a future study. In this paper, we restrict ourselves solely to a study of the symmetries themselves. In particular we show that the dynamical symmetries and the associated constants of motion, which were originally found in the two-flavor mixing scheme using an analogy to the BCS model, can be generalized to the full three-flavor mixing case. We also show that these dynamical symmetries continue to be exact even when the *CP* symmetry is broken by neutrino oscillations.

### A. In the exact many-body picture

In light of the above comments, we ignore any net electron background in this section, adopt the single-angle approximation for neutrino self-interactions and assume that neutrinos occupy a fixed volume. The single-angle approximation assumes that all neutrinos experience the same flavor transformation regardless of their direction of travel, which amounts to replacing the angular factor $R_{\vec{p}\,\vec{q}}$ introduced in Eq. (30) with a suitable representative value $R$. In this case, the Hamiltonian describing the flavor evolution of neutrinos reduces to

$$H = \sum_E \sum_{i=1}^{3} \frac{\Delta_i^2}{6E} T_{ii}(E) + \frac{\mu}{2} \sum_{i,j=1}^{3} T_{ij} T_{ji}, \tag{49}$$

where $\mu$ is given by

$$\mu = R\frac{\sqrt{2}G_F}{V}. \tag{50}$$

Here we used Eqs. (24) and (34) in order to express the Hamiltonian in the mass basis where it takes a simpler form and we employed the summation conventions introduced in Eqs. (7) and (8).

Using the U(3) commutators given in Eq. (6), one can easily show that the operators[6]

$$h_E = \sum_{i=1}^{3} \frac{\Delta_i^2}{3} T_{ii}(E) + \mu \sum_{i,j=1}^{3} \sum_{\substack{E' \\ (E' \neq E)}} \frac{T_{ij}(E) T_{ji}(E')}{\frac{1}{2E} - \frac{1}{2E'}} \tag{51}$$

are constants of motion of the Hamiltonian given in Eq. (49) because they commute with the Hamiltonian and with each other, i.e., for every $E$ and $E'$

$$[H, h_E] = 0 \quad \text{and} \quad [h_E, h_{E'}] = 0 \tag{52}$$

are satisfied. Note that in Eqs. (51) and (52), the energies $E$ and $E'$ can take both positive and negative values. This tells us that for every *physical energy* mode $p$ in the system, there are two constants of motion given by $h_p$ and $h_{-p}$ corresponding to neutrino and antineutrino degrees of freedom, respectively. The Hamiltonian itself, which is given in Eq. (49), can be written as a sum of these invariants, i.e.,

$$H = \sum_E \frac{1}{2E} h_E, \tag{53}$$

up to some terms which are proportional to identity.

We would like to note that in the limit of $\mu \to 0$, self-interactions of neutrinos disappear and the Hamiltonian given in Eq. (49) reduces to the vacuum propagation Hamiltonian only. In this limit, the invariants presented in Eq. (51) reduce to number operators for mass eigenstates and we recover Eq. (20). However away from the $\mu \to 0$ limit, the invariants given in Eq. (51) are nontrivial and cannot be reduced to a combination of number operators. We also would like to note that both the Hamiltonian given in Eq. (49) and the invariants given in Eq. (51) reduce to their two-flavor counterparts presented in Ref. [31] if one restricts the sums over three mass eigenstates to include only two of them (see the Appendix).

One can express the constants of motion in the flavor basis using the inverse of Eq. (17) as

$$h_E = Q \sum_{i=1}^{3} \frac{\Delta_i^2}{3} T_{\alpha_i \alpha_i}(E) Q^\dagger + \mu \sum_{i,j=1}^{3} \sum_{E'(\neq E)} \frac{T_{\alpha_i \alpha_j}(E) T_{\alpha_j \alpha_i}(E')}{\frac{1}{2E} - \frac{1}{2E'}}, \tag{54}$$

[6]In the continuum limit, the sum over $E'$ is replaced by an integral which has a singularity at $E' = E$. But the integral does not diverge as will be seen in Sec. IV B below. In presenting the constants of motion in this paper, we choose to use a sum over discrete values of energy-momentum because in practice one usually carries out the calculation over a discretized spectrum and we wanted to emphasize that in the discrete case, the $E' = E$ term should be removed from the sum in Eq. (51).

where we also used Eq. (32) which tells us that the quadratic part of the constants of motion will have the same form in both the flavor and mass bases.

It is important to note that Eq. (52) is valid even in the presence of the $CP$-violating phase. In other words, the many-body dynamical symmetries of the system are not broken when the neutrino oscillations are not $CP$ invariant. However, the $CP$-violating phase can be factored out in a way similar to Eq. (47), i.e.,

$$h_E = S_{\tilde{\tau}}^{\dagger} \tilde{h}_E^{(0)} S_{\tilde{\tau}}, \tag{55}$$

where $\tilde{h}_E^{(0)}$ are the constants of motion of the Hamiltonian

$$\tilde{H}^{(0)} = H_{\tilde{\mathrm{v}}}^{(0)} + H_{\tilde{\mathrm{s}}}, \tag{56}$$

which represents the vacuum oscillations and self-interactions of neutrinos and antineutrinos in the rotated flavor basis in the absence of any $CP$-violating phase. They are given by

$$\begin{aligned} h_E^{(0)} &= Q_{\tilde{e}\tilde{\tau}}(t_{\mathrm{R}}) Q_{\tilde{e}\tilde{\mu}}(t_{\odot}) \sum_{i=1}^{3} \frac{\Delta_i^2}{3} T_{\tilde{\alpha}_i\tilde{\alpha}_i}(E) Q_{\tilde{e}\tilde{\mu}}^{\dagger}(t_{\odot}) Q_{\tilde{e}\tilde{\tau}}^{\dagger}(t_{\mathrm{R}}) \\ &\quad + \mu \sum_{i,j=1}^{3} \sum_{E'(\neq E)} \frac{T_{\tilde{\alpha}_i\tilde{\alpha}_j}(E) T_{\tilde{\alpha}_j\tilde{\alpha}_i}(E')}{\frac{1}{2E} - \frac{1}{2E'}}. \end{aligned} \tag{57}$$

In order to show that Eq. (55) is true, one should substitute the factored form of the operator $Q$ given in Eq. (39) into Eq. (54) and use the definition of the rotated flavor basis given in Eq. (32). Note that the quadratic parts of the constants of motion have the same form in the rotated flavor basis as implied by Eq. (32).

### B. Effective one-particle approximation

The Hilbert space of a self-interacting neutrino ensemble grows exponentially with the number of particles so that even with the symmetries described in this paper, the diagonalization of the full many-body Hamiltonian is a formidable task. For this reason one usually resorts to an *effective one-particle* approximation which replaces the system of mutually interacting neutrinos with a system of free particles moving in an average (mean) field. This approach can be formulated with the *operator product linearization* in which the quadratic term representing mutual interactions of particles is approximated by

$$\mathcal{O}_1\mathcal{O}_2 \sim \mathcal{O}_1\langle\mathcal{O}_2\rangle + \langle\mathcal{O}_1\rangle\mathcal{O}_2 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle. \tag{58a}$$

Here the expectation values are calculated with respect to a state $|\Psi\rangle$ which represents the whole system and it is assumed that this state satisfies the condition

$$\langle\mathcal{O}_1\mathcal{O}_2\rangle = \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle, \tag{58b}$$

so that the expectation values of both sides of Eq. (58a) agree with each other. Usually, the condition in Eq. (58b) can only be satisfied by a restricted class of states in the Hilbert space. In Ref. [24] two of us showed that SU(2) or SU(3) coherent states can be used for this purpose in the case of two or three flavors, respectively.

The application of the operator product linearization to the neutrino Hamiltonian given in Eq. (49) yields

$$H_{\mathrm{MF}} = \sum_E \sum_{i=1}^{3} \frac{\Delta_i^2}{6E} T_{ii}(E) + \frac{\mu}{2} \sum_{i,j=1}^{3} S_{ij} T_{ji}, \tag{59}$$

where we define

$$S_{ij}(E,\vec{p}) = 2\langle T_{ij}(E,\vec{p})\rangle, \tag{60}$$

and adopt the same summation conventions for $S_{ij}(E,\vec{p})$ as in Eqs. (7) and (8). The factor of 2 in Eq. (60) is introduced to account for the fact that when we linearize a quadratic term as in Eq. (58a), two linear terms appear on the right-hand side.

Note that the quadratic interaction term that we linearize involves SU(3) generators for which Eq. (58b) is only satisfied by SU(3) coherent states [24]. These coherent states involve no quantum entanglement, i.e., they are in the form of a product of the one-particle states:

$$\begin{aligned} |\Psi\rangle \equiv\; & |\psi(\vec{p}_1)\rangle \otimes |\psi(\vec{p}_2)\rangle \otimes \ldots \otimes |\psi(\vec{p}_N)\rangle \\ & \otimes |\bar{\psi}(\vec{p}_1)\rangle \otimes |\bar{\psi}(\vec{p}_2)\rangle \otimes \ldots \otimes |\bar{\psi}(\vec{p}_N)\rangle. \end{aligned} \tag{61}$$

Here the states $|\psi(\vec{p}_k)\rangle$ and $|\bar{\psi}(\vec{p}_k)\rangle$ represent a single neutrino and antineutrino, respectively. They are not necessarily flavor states but can be a superposition of different flavor or mass eigenstates. These single-particle states are computed as a function of time by solving a set of *mean-field consistency equations* which guarantee that the mean field evolves in line with the evolution of the individual particles in the system because all particles contribute to the mean field. In order to find these equations, one should first note that the Heisenberg equation of motion for the operator $T_{ij}(E,\vec{p})$ is given by

$$\begin{aligned} -i\frac{d}{dt} T_{ij}(E,\vec{p}) &= [H_{\mathrm{MF}}, T_{ij}(E,\vec{p})] \\ &= \frac{\delta m_{ij}^2}{2E} T_{ij}(E,\vec{p}) \\ &\quad + \frac{\mu}{2} \sum_{k=1}^{3} (S_{ik} T_{kj}(E,\vec{p}) - S_{kj} T_{ik}(E,\vec{p})). \end{aligned} \tag{62}$$

Taking the expectation value of both sides of Eq. (62) gives

$$-i\frac{d}{dt}S_{ij}(E,\vec{p}) = \frac{\delta m_{ij}^2}{2E}S_{ij}(E,\vec{p}) + \frac{\mu}{2}\sum_{k=1}^{3}(S_{ik}S_{kj}(E,\vec{p}) - S_{kj}S_{ik}(E,\vec{p})), \quad (63)$$

which are the mean-field consistency equations to be solved to determine $S_{ij}(E,\vec{p})$ and hence the state in Eq. (61).

In the mean-field approximation, the many-body invariants considered above are no longer exactly conserved. This is not surprising because when the state of a particle undergoes a small change as a consequence of its interaction with another particle, the conservation principle requires the latter to undergo exactly the opposite change. This requirement obviously cannot be satisfied in a mean-field-type approximation [62]. However, the expectation values of the many-body invariants considered in the previous subsection still remain constant under the mean-field dynamics. In other words, the quantities

$$I_E \equiv 2\langle h_E\rangle = \sum_{i=1}^{3}\frac{\Delta_i^2}{3}S_{ii}(E) + \frac{\mu}{2}\sum_{i,j=1}^{3}\sum_{\substack{E' \\ (E'\neq E)}}\frac{S_{ij}(E)S_{ji}(E')}{\frac{1}{2E}-\frac{1}{2E'}} \quad (64)$$

obey

$$\frac{d}{dt}\langle I_E\rangle = 0, \quad (65)$$

for every $E$. One can easily confirm Eq. (65) by taking the derivative of Eq. (64) and using the mean-field equations given in Eq. (63).

It is instructive to calculate the values of the constants of motion in the mean-field approximation for neutrinos which are emitted during the cooling phase of a proto-neutron star after a core-collapse supernova explosion. Initially $S_{\alpha_i\alpha_j}(E)$ are nonzero only for $i=j$ because all neutrinos are emitted in flavor states and there is no mixing near the neutron star surface. Neutrinos reach a thermal equilibrium before they are released from the proto-neutron star so that the diagonal elements are given by

$$S_{\alpha_i\alpha_i}(p) = \frac{2L}{2\pi R^2F_3(0)}\frac{1}{T_{\alpha_i}^4}\frac{p^2}{1+e^{p/T_{\alpha_i}}},$$
$$S_{\alpha_i\alpha_i}(-p) = \frac{-2L}{2\pi R^2F_3(0)}\frac{1}{T_{\bar{\alpha}_i}^4}\frac{p^2}{1+e^{p/T_{\bar{\alpha}_i}}}. \quad (66)$$

Here $L$ denotes the neutrino luminosity and $R$ denotes the radius of the neutrino sphere. We assume that both quantities are the same for all neutrino and antineutrino flavors. The Fermi integral $F_3(0)$ corresponding to zero chemical potential is equal to $7\pi^2/120$. In Eq. (66), the temperature of the $\nu_{\alpha_i}$ and $\bar{\nu}_{\alpha_i}$ are, respectively, shown by $T_{\alpha_i}$ and $T_{\bar{\alpha}_i}$. Model-independent arguments tell us that these temperatures obey the hierarchy

$$T_{\nu_e} < T_{\bar{\nu}_e} < T_{\nu_x} = T_{\bar{\nu}_x}, \quad (67)$$

where $x=\mu,\tau$.

Note that near the proto-neutron star, the neutrino luminosity is very large ($L=10^{51}$ ergs/s for the cooling period of the proto-neutron star). In this case the quadratic terms in the conserved quantities given in Eq. (64) are at least 9 orders of magnitude larger than the linear terms so that the linear terms can be safely ignored. As for the quadratic terms in Eq. (64), they have the same form in both the mass and flavor bases as emphasized above [see Eq. (54) and the text that follows it]. Therefore, the values of the conserved quantities can be obtained by using Eq. (66) as follows:

$$I_p = I\sum_{i=1}^{3}\frac{1}{T_{\alpha_i}^4}\frac{p^2}{1+e^{p/T_{\alpha_i}}}\int dq\left(\frac{\frac{1}{T_{\alpha_i}^4}\frac{q^2}{1+e^{q/T_{\alpha_i}}}}{\frac{1}{2p}-\frac{1}{2q}} - \frac{\frac{1}{T_{\bar{\alpha}_i}^4}\frac{q^2}{1+e^{q/T_{\bar{\alpha}_i}}}}{\frac{1}{2p}+\frac{1}{2q}}\right),$$
$$I_{-p} = I\sum_{i=1}^{3}\frac{1}{T_{\bar{\alpha}_i}^4}\frac{p^2}{1+e^{p/T_{\bar{\alpha}_i}}}\int dq\left(\frac{\frac{1}{T_{\bar{\alpha}_i}^4}\frac{q^2}{1+e^{q/T_{\bar{\alpha}_i}}}}{\frac{1}{2p}-\frac{1}{2q}} - \frac{\frac{1}{T_{\alpha_i}^4}\frac{q^2q}{1+e^{q/T_{\alpha_i}}}}{\frac{1}{2p}+\frac{1}{2q}}\right). \quad (68)$$

In writing Eq. (68), we take the continuum limit

$$\frac{1}{V}\sum_{\vec{q}} \to \frac{1}{(2\pi)^3}\int d^3\vec{q} \quad (69)$$

and define a common proportionality constant

$$I = \frac{RG_F}{\sqrt{2}}\left(\frac{L}{2\pi^2R^2F_3(0)}\right)^2. \quad (70)$$

We also take into account that the neutrinos are all going away from the proto-neutron star so that the angular part of $d\vec{q}$ integrates to $2\pi$ rather than $4\pi$.

The values of the invariants calculated from Eq. (68) for initial neutrino distributions with a representative set of neutrino temperatures [63–65] $T_{\nu_e}=3.0$ MeV, $T_{\bar{\nu}_e}=4.0$ MeV and $T_{\nu_x}=T_{\bar{\nu}_x}=6.0$ MeV are shown in Fig. 1. Note that the values of the invariants depend on the *CP*-violating Dirac phase only through the linear term of Eq. (64) which we ignored in the case of a core-collapse supernova [see the discussion above Eq. (68)].

## V. MAGNETIC MOMENT

In those astrophysical sources where neutrinos are produced abundantly, it is also typical to find strong magnetic fields so that even tiny electromagnetic properties of neutrinos may be consequential. As mentioned in the Introduction, in the current paradigm of particle physics

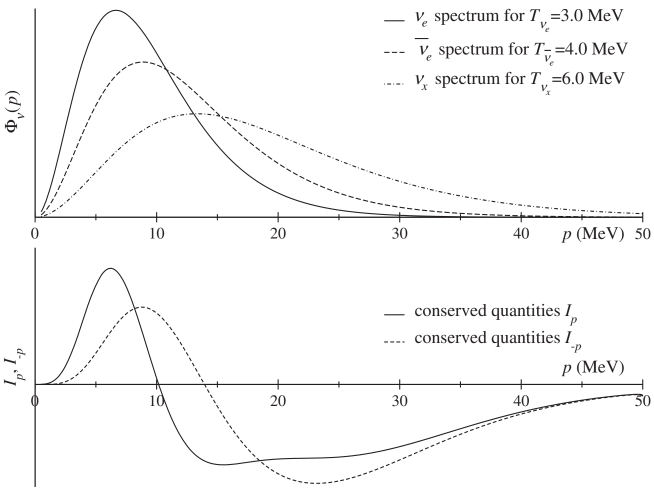


FIG. 1. Energy spectrum of the neutrinos emanating from the surface of a proto-neutron star and the corresponding invariants. We adopted a representative set of neutrino temperatures given by $T_{\nu_e} = 3.0$ MeV, $T_{\bar{\nu}_e} = 4.0$ MeV and $T_{\nu_x} = T_{\bar{\nu}_x} = 6.0$ MeV and calculated the values of the invariants from Eq. (68).

neutrinos have tiny amounts of anomalous magnetic moments due to charged-particle loops. However, various theories beyond the Standard Model predict much larger values (see Ref. [38] and references therein). In this section, we consider the effect of the neutrino dipole moments on the flavor evolution of neutrinos as they propagate in a magnetic field. In particular we show that the above-mentioned factorization of the $CP$-violating Dirac phase is not valid under these circumstances, i.e., there is an interplay between the $CP$-violating and electromagnetic effects in neutrino flavor transformation.

### A. Dirac neutrinos

The interaction of fermions with a classical electromagnetic field through the anomalous electric and magnetic dipole moments is described by the Pauli Lagrangian [see Eq. (91) of Ref. [66] or Section 2-2-3 of Ref. [67]]. In the case of neutrinos, those interactions can cause transitions between different types so that the dipole moments should be represented by matrices, i.e., the Pauli Lagrangian is given by

$$\mathcal{L}_\mu = \sum_{i,j=1}^{3} \bar{\Psi}_i \frac{1}{2} \mu_{ij} \sigma^{\mu\nu} F_{\mu\nu} \Psi_j. \tag{71}$$

Here, we use the greek indices $\mu, \nu = 0, 1, 2, 3$ to denote space-time components and the latin indices $i, j = 1, 2, 3$ to denote the neutrino mass basis. Summation convention is adopted for space-time indices but not for the neutrino mass or flavor indices. Note that although $\mu_{ij}$ in Eq. (71) contains contributions from both the electric and magnetic dipole moments of neutrinos, we follow the convention and refer to it simply as the magnetic moment. In fact, since the neutrino is ultrarelativistic, it *sees* an electric field in its rest frame and interacts with it through its electric dipole moment even when there is only a magnetic field present in the environment (see, for example, Appendix E of Ref. [38] for a detailed account). Note that the Hermiticity of the Lagrangian in Eq. (71) requires that the magnetic moment is a Hermitian matrix, i.e.,

$$\mu_{ij} = \mu_{ji}^*. \tag{72}$$

The dipole moments are defined in the mass basis as indicated in Eq. (71) but since neutrinos are produced and detected in flavor states, physically relevant quantities are effective dipole moments which depend on the mixing parameters and the energy of the neutrino as well as the distance it travels from the source.

In Eq. (71), $F^{\mu\nu}$ denotes the electromagnetic field tensor and $\sigma^{\mu\nu}$ is given by

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{73}$$

We adopt the Euclidean metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ in which case the interaction term takes the form

$$\mu_{ij} \frac{1}{2} \sigma^{\mu\nu} F_{\mu\nu} = \mu_{ij} (i\vec{\alpha} \cdot \vec{E} + \vec{\Sigma} \cdot \vec{B}), \tag{74}$$

where

$$\alpha^k = \gamma^0 \gamma^k \quad \text{and} \quad \Sigma^k = \gamma^0 \gamma^k \gamma^5. \tag{75}$$

In both the Dirac and the chiral representations of the $\gamma$ matrices, $\Sigma^k$ defined in Eq. (75) is equal to

$$\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}, \tag{76}$$

where $\sigma^k$ are ordinary Pauli matrices. But to be specific, throughout the paper we use the chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}. \tag{77}$$

One can write down the Hamiltonian density for neutrinos propagating in an external magnetic field by using Eqs. (71) and (74) as

$$\mathcal{H}_\mu = \sum_{i,j=1}^{3} \bar{\Psi}_i \mu_{ij} \vec{\Sigma} \cdot \vec{B} \Psi_j, \tag{78}$$

where we set $\vec{E} = 0$. In order to obtain the corresponding many-body Hamiltonian, we integrate the Hamiltonian density over the space coordinates,

$$H_\mu = \int d^3\vec{r} \sum_{i,j=1}^{3} \bar{\psi}_i \mu_{ij} \vec{\Sigma}\cdot\vec{B}\psi_j, \tag{79}$$

and use the expansion of the field operator in terms of the plane waves with definite helicity given by

$$\psi_i(t,\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{h=\pm} (a_{ih}(\vec{p})u_h(\vec{p})e^{-i(pt-\vec{p}\cdot\vec{r})} + b^\dagger_{ih}(\vec{p})v_h(\vec{p})e^{i(pt-\vec{p}\cdot\vec{r})}). \tag{80}$$

Note that, in this section, we use an integration over the continuous values of the momentum rather than a sum over discrete values and we no longer use the convention introduced in Eq. (1). In Eq. (80), $u_h(\vec{p})$ and $v_{-h}(\vec{p})$ are plane-wave solutions for particles and antiparticles, respectively, with helicity $h$. In the ultrarelativistic limit, they are given by

$$\begin{aligned} -u_+(\vec{p}) = v_-(\vec{p}) &= \begin{pmatrix} \chi^{(+)} \\ 0 \end{pmatrix}, \\ u_-(\vec{p}) = -v_+(\vec{p}) &= \begin{pmatrix} 0 \\ \chi^{(-)} \end{pmatrix}, \end{aligned} \tag{81}$$

where $\chi^{(h)}$ are the helicity eigenstates which are given as follows:

$$\chi^{(+)} = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \chi^{(-)} = \begin{pmatrix} -e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}. \tag{82}$$

In Eq. (82), $\theta$ and $\phi$ denote the polar and azimuthal angles of the momentum $\vec{p}$, respectively. Note that according to Eqs. (80) and (81), the operators $a_{ih}(p)$ annihilate neutrinos in the $i$th mass eigenstate with helicity $h$ whereas the operators $b^\dagger_{ih}(p)$ create antineutrinos in the $i$th mass eigenstate with helicity $-h$.

Substituting the expansion of the field operator given in Eq. (80) into the interaction Hamiltonian in Eq. (79), assuming that $\vec{B}$ is a uniform field, and keeping only those terms which are relevant to the propagation of neutrinos in the limit where $E \gg m$ yields the following result for the Hamiltonian:

$$\begin{aligned} H_\mu = \int d^3\vec{p} \sum_{h,h'=\pm} \mu_{ij}\{ & a^\dagger_{ih}(\vec{p})[\bar{u}_h(\vec{p})\vec{\Sigma}\cdot\vec{B}u_{h'}(\vec{p})]a_{jh'}(\vec{p}) \\ & + b_{ih}(\vec{p})[\bar{v}_h(\vec{p})\vec{\Sigma}\cdot\vec{B}v_{h'}(\vec{p})]b^\dagger_{jh'}(\vec{p})\}. \end{aligned} \tag{83}$$

The expressions which appear in square brackets in Eq. (83) can be easily calculated using Eqs. (76) and (81). The result is given by

$$\begin{aligned} \bar{u}_h(\vec{p})\vec{\Sigma}u_{h'}(\vec{p}) &= (\hat{n}_\theta + ih\hat{n}_\phi)\delta_{h',-h}, \\ \bar{v}_h(\vec{p})\vec{\Sigma}v_{h'}(\vec{p}) &= -(\hat{n}_\theta + ih\hat{n}_\phi)\delta_{h',-h}. \end{aligned} \tag{84}$$

Here $\hat{n}_\theta$ and $\hat{n}_\phi$ are two unit vectors which are orthogonal to the direction of motion of the neutrino. In other words, $\hat{p} = \vec{p}/|\vec{p}|$, $\hat{n}_\theta$ and $\hat{n}_\phi$ form an orthonormal basis for the spherical coordinates in the momentum space. In terms of the Cartesian unit vectors they are given by

$$\begin{aligned} \hat{p} &= \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}, \\ \hat{n}_\theta &= \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}, \\ \hat{n}_\phi &= \sin\phi\hat{x} - \cos\phi\hat{y}. \end{aligned} \tag{85}$$

A substitution of the results in Eq. (84) into Eq. (83) yields the following result:

$$\begin{aligned} H_\mu = \int d^3\vec{p} \sum_{i,j=1}^{3} \mu_{ij}B_\perp (a^\dagger_{i+}(\vec{p})a_{j-}(\vec{p}) + b^\dagger_{j+}(\vec{p})b_{i-}(\vec{p})) \\ + \mathrm{H.c.} \end{aligned} \tag{86}$$

Here, $B_\perp = (\hat{n}_\theta + i\hat{n}_\phi)\cdot\vec{B}$ denotes the component of the magnetic field which is perpendicular to the direction of neutrino propagation. One can always rotate the plane perpendicular to the direction of neutrino propagation to make $\hat{n}_\phi\cdot\vec{B} = 0$ so that $B_\perp$ can be assumed to be real.

In order to express the flavor evolution of neutrinos in the presence of a strong magnetic field, one should write the Hamiltonian given in Eq. (86) in the flavor basis. The transformation of left-handed neutrinos from the mass basis to the flavor basis was discussed in Sec. II. However, the right-handed Dirac neutrinos do not take part in weak interactions so the choice of the *flavor* basis for them is completely arbitrary. For our purposes, this choice is of no practical consequence and we simply leave the right-handed neutrinos in the mass basis in our formulas.

The Hamiltonian in Eq. (86) can be expressed in the flavor basis by using the inverse of Eq. (15):

$$\begin{aligned} H_\mu = Q\int d^3\vec{p} \sum_{i,j=1}^{3} \mu_{ij}B_\perp (a^\dagger_{i+}(\vec{p})a_{\alpha_j-}(\vec{p}) \\ + b^\dagger_{j+}(\vec{p})b_{\alpha_i-}(\vec{p}))Q^\dagger + \mathrm{H.c.} \end{aligned} \tag{87}$$

The form of the transformation operator given in Eq. (39) is once again useful in examining the dependence of this Hamiltonian on the $CP$-violating Dirac phase. As was the case in Sec. III D, the rightmost $Q_{\mu\tau}$ in Eq. (39) transforms the left-handed neutrino degrees of freedom into the rotated flavor basis leading to

$$H_\mu = S^\dagger_{\tilde{\tau}} Q_{\tilde{e}\,\tilde{\tau}}(t_R) Q_{\tilde{e}\,\tilde{\mu}}(t_\odot) S_{\tilde{\tau}} \int d^3\vec{p} \sum_{i,j=1}^{3} \mu_{ij} B_\perp (a^\dagger_{i+}(\vec{p}) a_{\tilde{\alpha}_j-}(\vec{p}) + b^\dagger_{j+}(\vec{p}) b_{\tilde{\alpha}_i-}(\vec{p})) S^\dagger_{\tilde{\tau}} Q^\dagger_{\tilde{e}\,\tilde{\mu}}(t_\odot) Q^\dagger_{\tilde{e}\,\tilde{\tau}}(t_R) S_{\tilde{\tau}} + \mathrm{H.c.} \tag{88}$$

Unlike the case in the vacuum oscillations, the operator $S_{\tilde{\tau}}$ which contains the $CP$-violating phase does not commute with the terms in parentheses so, strictly speaking, we cannot disentangle the $CP$-violating effects from those of the magnetic moment. However, one can show that

$$S_{\tilde{\tau}} a_{\tilde{\alpha}_i-} S^\dagger_{\tilde{\tau}} = \sum_{j=1}^{3} S_{ij} a_{\tilde{\alpha}_j-} \quad \text{and} \quad S_{\tilde{\tau}} b_{\tilde{\alpha}_i-} S^\dagger_{\tilde{\tau}} = \sum_{j=1}^{3} S^*_{ij} b_{\tilde{\alpha}_j-} \tag{89}$$

are satisfied where $S_{ij}$ is given by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{pmatrix}. \tag{90}$$

As a result, one can define an *effective* magnetic moment $\mu^{\mathrm{eff}}$ as

$$\mu^{\mathrm{eff}} = \mu S = \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} e^{i\delta} \\ \mu^*_{12} & \mu_{22} & \mu_{23} e^{i\delta} \\ \mu^*_{13} & \mu^*_{23} & \mu_{33} e^{i\delta} \end{pmatrix} \tag{91}$$

and write the Hamiltonian in Eq. (88) as

$$H_\mu = S^\dagger_{\tilde{\tau}} \left( Q^{(0)}_{\tilde{e}\,\tilde{\tau}} Q_{\tilde{e}\,\tilde{\mu}} \int d^3\vec{p} B_\perp \sum_{i,j=1}^{3} (\mu^{\mathrm{eff}}_{ij} a^\dagger_{i+}(\vec{p}) a_{\tilde{\alpha}_j-}(\vec{p}) + \mu^{\mathrm{eff}\,*}_{ij} b^\dagger_{j+}(\vec{p}) b_{\tilde{\alpha}_i-}(\vec{p})) Q^\dagger_{\tilde{e}\,\tilde{\mu}} Q^{(0)\dagger}_{\tilde{e}\,\tilde{\tau}} + \mathrm{H.c.} \right) S_{\tilde{\tau}}. \tag{92}$$

This tells us that the Hamiltonian describing neutrinos in a strong magnetic field can be factorized as

$$H_\mu = S^\dagger_{\tilde{\tau}} \tilde{H}_{\mu^{\mathrm{eff}}} S_{\tilde{\tau}}, \tag{93}$$

where $S_{\tilde{\tau}}$ contains the $CP$-violating Dirac phase and is given by Eq. (36b). The Hamiltonian $\tilde{H}_{\mu^{\mathrm{eff}}}$ is given by the expression in parentheses in Eq. (92). It describes neutrinos with an effective magnetic moment in the rotated flavor basis (as indicated by the tilde) and does not contain the $CP$-violating Dirac phase *explicitly*. However, the effective magnetic moment defined in Eq. (91) is not a unitary matrix. The appearance of $\mu^{\mathrm{eff}}$ for neutrinos and $\mu^{\mathrm{eff}\,*}$ for antineutrinos in Eq. (92) reflects the $CP$ violation. This clearly shows that the effects of $CP$ violation and the magnetic moment are intertwined and cannot be separated.

But aside from proving this point, the formulation developed in this section can also be practical. For example, one can consider the neutrino propagation in the presence of a matter background and self-interactions as well as a magnetic field by using the Hamiltonian [see Eq. (47)]

$$H = S^\dagger_{\tilde{\tau}} (H^{(0)}_{\tilde{\mathrm{v}}} + H_{\tilde{\mathrm{m}}} + H_{\tilde{\mathrm{s}}} + \tilde{H}_{\mu^{\mathrm{eff}}}) S_{\tilde{\tau}}. \tag{94}$$

The term in the parentheses in Eq. (94) includes $CP$ violation only implicitly through $\mu^{\mathrm{eff}}$ which, in most cases, can be simply studied to the first order in perturbation theory. In such a calculation, the $CP$-violating phase will appear only linearly and create a minimal complication. The full effect of the $CP$-violating phase can later be included using Eq. (48).

### B. Majorana neutrinos

If the neutrinos are of Majorana type, then the part of the Lagrangian in Eq. (71) involving the symmetric component of $\mu_{ij}$ vanishes automatically once the Majorana condition $\psi^c_i = \psi_i$ is imposed. Therefore, the magnetic moment can be taken as an antisymmetric matrix for Majorana neutrinos:

$$\mu_{ij} = \mu^*_{ji} \quad \text{and} \quad \mu_{ij} = -\mu_{ji}. \tag{95}$$

This tells us that, for the Majorana neutrinos, the diagonal magnetic moments vanish and the nondiagonal ones are purely imaginary. Also note that, once we impose the Majorana condition, the Lagrangian in Eq. (71) should be divided by 2 in order to avoid double counting of neutrino and antineutrino degrees of freedom. In our notation introduced in Eqs. (80), (81) and (82), the Majorana condition amounts to

$$b_{i+}(\vec{p}) = a_{i-}(\vec{p}) \quad \text{and} \quad b_{i-}(\vec{p}) = -a_{i+}(\vec{p}). \tag{96}$$

Another important point is the fact that, although neutrinos and antineutrinos are identical [as implied by Eq. (96)], it is conventional to call Majorana neutrinos with positive helicity *antineutrinos* because, as far as the production and detection of neutrinos are concerned, the difference between positive-helicity Majorana neutrinos and positive-helicity Dirac antineutrinos is suppressed by the neutrino mass/energy. Therefore, for Majorana neutrinos, we adopt the notation

$$a_{i-}(\vec{p}) = a_i(\vec{p}) \quad \text{and} \quad a_{i+}(\vec{p}) = b_i(\vec{p}). \tag{97}$$

Substituting Eqs. (95), (96) and (97) into Eq. (86) and dividing it by 2 yields the corresponding Hamiltonian for Majorana neutrinos:

$$H_\mu = \int d^3\vec{p} \sum_{i,j=1}^{3} \mu_{ij} B_\perp b_i^\dagger(\vec{p}) a_j(\vec{p}) + \text{H.c.} \tag{98}$$

Unlike the case for Dirac neutrinos, the transformation of Majorana *antineutrinos* from the mass basis to the flavor basis is fixed by Eq. (15). Together with Eq. (39), this leads to

$$H_\mu = S_{\tilde{\tau}}^\dagger Q_{\tilde{e}\tilde{\tau}}(t_R) Q_{\tilde{e}\tilde{\mu}}(t_\odot) S_{\tilde{\tau}} \left( \int d^3\vec{p} \sum_{i,j=1}^{3} \mu_{ij} B_\perp b_{\tilde{\alpha}_i}^\dagger(\vec{p}) a_{\tilde{\alpha}_j}(\vec{p}) + \text{H.c.} \right) S_{\tilde{\tau}}^\dagger Q_{\tilde{e}\tilde{\mu}}^\dagger(t_\odot) Q_{\tilde{e}\tilde{\tau}}^\dagger(t_R) S_{\tilde{\tau}}. \tag{99}$$

Using Eqs. (89) and (90) which are still valid in the Majorana case, one obtains

$$H_\mu = S_{\tilde{\tau}}^\dagger Q_{\tilde{e}\tilde{\tau}}(t_R) Q_{\tilde{e}\tilde{\mu}}(t_\odot) \left( \int d^3\vec{p} \sum_{i,j=1}^{3} \mu_{ij}^{\text{eff}} B_\perp b_{\tilde{\alpha}_i}^\dagger(\vec{p}) a_{\tilde{\alpha}_j}(\vec{p}) + \text{H.c.} \right) Q_{\tilde{e}\tilde{\mu}}^\dagger(t_\odot) Q_{\tilde{e}\tilde{\tau}}^\dagger(t_R) S_{\tilde{\tau}}, \tag{100}$$

where $\mu^{\text{eff}}$ is defined as follows:

$$\mu^{\text{eff}} = S\mu S = \begin{pmatrix} 0 & \mu_{12} & \mu_{13}e^{i\delta} \\ -\mu_{12} & 0 & \mu_{23}e^{i\delta} \\ -\mu_{13}e^{i\delta} & -\mu_{23}e^{i\delta} & 0 \end{pmatrix}. \tag{101}$$

As is the case for Dirac neutrinos, the effective magnetic moment is not a Hermitian matrix but it is still antisymmetric. We see that Eq. (94) and the comments following that equation are also valid for Majorana neutrinos provided that the effective magnetic moment is now given by Eq. (101).

## VI. SUMMARY AND CONCLUSIONS

In this paper, we considered the flavor evolution of neutrinos which are subject to refractive effects due to both self-interactions and a matter background. We attempted a comprehensive study of the problem by taking into account its full many-body nature in the three-flavor mixing scenario with the effects of possible *CP* violation and the anomalous magnetic moment included. Since our perspective was exclusively based on the symmetries of the problem, important environmental details were left out of our analysis, such as a specific core-collapse supernova model for matter and magnetic field profiles.

We showed that, in its exact many-body formulation, the system exhibits several dynamical symmetries in such a way that one has a constant of motion for each allowed neutrino and antineutrino energy mode. We expressed these constants of motion in terms of the generators of the SU(3) flavor transformations. In the case of the effective one-particle approximation, we showed that the expectation values of these constants of motion remain invariant under the mean-field dynamics. The dynamical symmetries considered in this paper are valid under a set of *ideal* conditions, i.e., when the single-angle approximation is adopted, the *net* electron background is negligible, and the volume occupied by the neutrinos is fixed ($\mu =$ constant). We also showed that these dynamical symmetries are not broken even when *CP* symmetry is violated in neutrino oscillations.

Even away from the ideal conditions mentioned above, the constants of motion presented in this paper can still be useful by providing a convenient set of variables to work with because one can always decompose the Hamiltonian into an *ideal* and a *nonideal* part as

$$H = H_{\text{ideal}} + H_{\text{nonideal}}, \tag{102}$$

such that, although the *constants of motion* will now evolve in time, their evolution will only be due to the nonideal part, i.e.,

$$-i\frac{d}{dt} h_E = [H_{\text{nonideal}}, h_E] \tag{103}$$

since they commute with the ideal part of the Hamiltonian.

In this paper, we also showed that the $CP$ violation effects factor out of the Hamiltonian and the evolution operator not only in the effective one-particle picture adopted by the mean-field type approximations, but also in the full many-body picture. This conclusion is exact as long as the neutrino magnetic moment is not considered but even when one includes the neutrino dipole moments in the analysis, $CP$ violation can still be studied independently as long as an effective magnetic moment is defined which includes the Dirac $CP$-violating phase in an implicit way. Clearly, the effects due to $CP$ violation and the magnetic moment are intertwined in an inseparable way even in this formulation because the definition of the effective magnetic moment is different for neutrinos and antineutrinos. However, such a formulation is still useful because it allows us to include the $CP$-violating effects in a seamless and methodical way in analytical and numerical calculations. On the practical side, even when the neutrino magnetic moment is not ignored, this formulation locks the $CP$-violating phase only into the magnetic moment which is very small and can be conveniently studied only to the first order in a perturbation approach.

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## APPENDIX: REDUCTION TO THE TWO-FLAVOR SCHEME

The constants of motion given in Eq. (51) for three mixing flavors reduce to those that were presented earlier in Ref. [31] in the context of a two-flavor mixing scheme. In order to show this, we first consider the two neutrino *isospin* operators

$$J^{+}(p,\vec{p}) = T_{12}(p,\vec{p}), \qquad J^{-}(p,\vec{p}) = T_{21}(p,\vec{p}),$$
$$J^{0}(p,\vec{p}) = \frac{T_{11}(p,\vec{p}) - T_{22}(p,\vec{p})}{2}, \tag{A1}$$

which are similar to Eqs. (2) and (3) except that the negative-energy formulation for antineutrinos is now incorporated. We adopt the same summation convention for these isospin operators as in Eqs. (7) and (8). Note that we choose to work with the first two mass eigenstates but this choice is completely arbitrary. It is easy to show that

$$\sum_{i,j=1}^{2} T_{ij}(E)T_{ji}(E') = 2\vec{J}(E)\cdot\vec{J}(E') + \frac{1}{2}N_{12}(E)N_{12}(E'), \tag{A2}$$

where $N_{12}(E) = T_{11}(E) + T_{22}(E)$ is the total number of neutrinos ($E > 0$) or antineutrinos ($E < 0$) in the first two mass eigenstates with energy $E$.

Next, we consider the Hamiltonian given in Eq. (49) but restrict the range of the sums over the mass eigenstates that appear in this Hamiltonian to the first two mass eigenstates only. Note that there is no need to set $m_3 = 0$, i.e., the result is independent of the value of $m_3$. Then, using the definitions given in Eq. (A1) together with Eq. (A2) leads to

$$H_{\text{two flavors}} = \sum_{E} \frac{\delta m_{12}^2}{2E} J^0(E) + \mu\vec{J}\cdot\vec{J}. \tag{A3}$$

In deriving Eq. (A3), we discarded some terms which are proportional to $N_{12}(E)$ because it commutes with the rest of the Hamiltonian and is proportional to identity.

The constants of motion given in Eq. (51) can similarly be reduced to the two-flavor mixing scheme in a similar way. Restricting the sums over the mass eigenstates to the first two mass eigenstates only, using Eqs. (A1) and (A2), and dropping the terms proportional to $N_{12}(E)$ leads to

$$h_E = \delta m_{12}^2 J^0(E) + 2\mu \sum_{E'(\neq E)} \frac{\vec{J}(E)\cdot\vec{J}(E')}{\frac{1}{2E} - \frac{1}{2E'}}. \tag{A4}$$

Dividing Eq. (A4) by $\delta m_{12}^2$ gives the same many-body invariants which were presented in Ref. [31].

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