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$F(R)$ nonlinear massive theories of gravity and their cosmological implications

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We propose a nonlinear massive gravitational theory which includes $F(R)$ modifications. This construction inherits the benefits of the de Rham-Gabadadze-Tolley model and is free of the Boulware-Deser ghost due to the existence of a Hamiltonian constraint accompanied by a nontrivial secondary one. The scalar perturbations in a cosmological background can be stabilized at the linear level for a wide class of the $F(R)$ models. The linear scalar mode arisen from the $F(R)$ sector can absorb the nonlinear longitudinal graviton, and hence, our scenario demonstrates the possibility of a gravitational Goldstone theorem. Finally, due to the combined contribution of the $F(R)$ and graviton-mass sectors, the proposed theory allows for a large class of cosmological evolutions, such as the simultaneous and unified description of inflation and late-time acceleration.

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I. INTRODUCTION

The search for a consistent theory of massive gravity has been open for decades. Its motivations arise from both theoretical considerations, namely, to understand the construction procedure of a massive spin-2 theory, as well as (lately) from observational requirements, that is, to explain the Universe's acceleration through such an infrared (IR) modification of general relativity. However, since the first, linear approach [1], the subject remains a theoretically intriguing problem.

In the instructive idea of Fierz and Pauli [1], general relativity is extended by introducing a linear mass term, and thus, the theory involves at least 5 degrees of freedom (DOF), representing a massive spin-2 field in a Poincaré invariant background. However, it turns out that the graviton's longitudinal DOF remains coupled to the trace of the energy-momentum tensor regardless of the smallness of the graviton mass. This leads to the famous van Dam-Veltman-Zakharov discontinuity [2] and thus to a severe challenge by experiments and observations. This discontinuity can be alleviated at the nonlinear level through the Vainshtein mechanism [3]; however, due to the constraints on the dynamical variables, the same nonlinearities give rise to a ghost instability, called the Boulware-Deser (BD) ghost [4]. Using the effective field theory approach, one can show that the BD ghost is related to the Goldstone boson associated with the broken general covariance [5].

The above inconsistencies puzzled physicists for years. Recently, de Rham, Gabadadze, and Tolley (dRGT) showed that the BD ghost can be removed in a suitable nonlinear massive gravitational theory [6]. In particular, due to a delicate construction of the graviton potential, the Hamiltonian constraint and the associated secondary one are restored, and thus, this IR modified theory becomes free from BD ghosts [7]. Apart from the theoretical interest, dRGT construction has the additional advantage that its application to a cosmological framework leads to late-time cosmic acceleration, where a sufficiently small value of the graviton mass mimics an effective cosmological constant [8–10]. However, as was shown in Ref. [11], cosmological perturbations of the dRGT massive gravity around background solutions exhibit instabilities.

On the other hand, after the 1960s, physicists realized that although general relativity is not renormalizable, possible high-energy corrections could improve renormalizability and thus quantization [12,13]. Although these ultraviolet (UV) corrections are expected to be of quantum origin or to arise from an underlying fundamental theory such as string theory (for example, see Refs. [14,15]), one can describe them effectively, by investigating a classical, modified, gravitational action. The simplest model of such an UV modified gravity, that can sufficiently encapsulate the basic properties of higher-order gravitational theories, is the $F(R)$ paradigm, in which the gravitational Lagrangian is extended to an arbitrary function of the Ricci scalar (see Ref. [16] for a review). The corresponding $F(R)$ cosmology is able to describe the inflationary epoch, and, in particular, the well-known Starobinsky R^2-inflation scenario [17] proves to be the best-fitted scenario with the recently released Planck data [18].

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Inspired by the above discussion, in this paper, we propose a modification of general relativity both in the UV and IR regimes that is the $F(R)$ nonlinear massive gravity. In this theory, the extra scalar DOF of the $F(R)$ sector, clearly seen through a conformal transformation, has a positive-defined kinetic term as usual, and its interaction with the massive sector can stabilize metric perturbation of scalar type at linear order (this is a novel feature, not present in Ref. [19]). In summary, the total theory is not only free of BD ghosts at the fundamental level, but it is also free of linear cosmological perturbative instabilities for the largest part of its parameter space, even in homogeneous and isotropic geometries. Finally, the increased freedom of both $F(R)$ and massive-graviton sectors can lead to a large class of interesting cosmological behaviors at early and late times, in agreement with observations.

II. THE SETUP

Imposing both the UV [$F(R)$ sector] and IR (graviton-mass sector) modifications, the total action becomes

$$S = \frac{M_p^2}{2}\int d^4x\sqrt{|g|}[F(R) + 2m_g^2\mathcal{U}_M], \tag{1}$$

where M_p is the Planck mass, g is the physical metric, and m_g is the graviton mass. As usual in dRGT construction, to build the dimensionless graviton potential $\mathcal{U}_M$, one needs to define the matrix $\mathbb{K} \equiv \mathbb{I} - \mathbb{X}$, where $\mathbb{X} \equiv \sqrt{g^{-1}f}$ involves a nondynamical (fiducial) metric f.[1] Then, the regular anti-symmetrization in 4D space-time yields the following polynomials

$$\mathcal{U}_2 = \mathbb{K}^{\mu}_{[\mu}\mathbb{K}^{\nu}_{\nu]}, \qquad \mathcal{U}_3 = \mathbb{K}^{\mu}_{[\mu}\mathbb{K}^{\nu}_{\nu}\mathbb{K}^{\sigma}_{\sigma]}, \qquad \mathcal{U}_4 = \mathbb{K}^{\mu}_{[\mu}\mathbb{K}^{\nu}_{\nu}\mathbb{K}^{\sigma}_{\sigma}\mathbb{K}^{\rho}_{\rho]}, \tag{2}$$

and the graviton potential is given by $\mathcal{U}_M = \mathcal{U}_2 + \alpha_3\mathcal{U}_3 + \alpha_4\mathcal{U}_4$, containing two dimensionless parameters (α_3, α_4).

The UV sector inherits the remarkable properties of the $F(R)$ term. In particular, by performing the conformal transformation $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ with $\Omega = \exp\left[\frac{\varphi}{\sqrt{6}M_p}\right]$, the $F(R)$ part can be reformulated as the standard general relativity minimally coupled to a canonical scalar field φ, with effective potential

$$U(\varphi) = M_p^2(RF_{,R} - F)/2F_{,R}^2, \tag{3}$$

where $F_{,R} \equiv \partial F/\partial R$. Additionally, the conformal transformation acts on the IR sector, too, with the graviton potentials transforming as

$$\tilde{\mathcal{U}}_M = \sum_{i=0}^{4}\Omega^{i-4}\beta_i\mathcal{E}_i, \tag{4}$$

where $\beta_i = (-1)^i[(4-i)(3-i)/2 + (4-i)\alpha_3 + \alpha_4]$. In this expression, based on the transformed matrix $\tilde{\mathbb{X}} \equiv \sqrt{\tilde{g}^{-1}f}$, we have introduced the elementary symmetric polynomial $\mathcal{E}_i$ as

$$\mathcal{E}_0 = 1, \qquad \mathcal{E}_1 = \tilde{\mathbb{X}}^{\mu}_{\mu}, \qquad \mathcal{E}_2 = \tilde{\mathbb{X}}^{\mu}_{[\mu}\tilde{\mathbb{X}}^{\nu}_{\nu]},$$
$$\mathcal{E}_3 = \tilde{\mathbb{X}}^{\mu}_{[\mu}\tilde{\mathbb{X}}^{\nu}_{\nu}\tilde{\mathbb{X}}^{\sigma}_{\sigma]}, \qquad \mathcal{E}_4 = \tilde{\mathbb{X}}^{\mu}_{[\mu}\tilde{\mathbb{X}}^{\nu}_{\nu}\tilde{\mathbb{X}}^{\sigma}_{\sigma}\tilde{\mathbb{X}}^{\rho}_{\rho]}. \tag{5}$$

Then, the resulting Lagrangian in the Einstein frame can be written as

$$\mathcal{L} = \sqrt{|\tilde{g}|}\left[\frac{M_p^2}{2}(\tilde{R} + 2m_g^2\tilde{\mathcal{U}}_M) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi)\right]. \tag{6}$$

At first sight, one might feel that our construction has a relation with the quasidilaton massive gravity [20] and the mass-varying massive gravity [21,22]. However, these scenarios are radically different, straightaway from the starting point of the model building, and moreover, they obey completely different symmetries. In particular, in the quasidilaton massive gravity, the coefficient in front of the kinetic term of the scalar field is a free parameter, while in our model, it results in being unity, and this feature has a crucial effect on the perturbational analysis, reducing the number of degrees of freedom, as we will see. Additionally, while in mass-varying massive gravity the separate gravitational terms acquire a common overall factor, in the present construction, they result obtaining different scalar-field dependencies, which make the two models radically different.

III. HAMILTONIAN CONSTRAINT ANALYSIS

To examine the BD ghost issue, one must perform the Hamiltonian constraint analysis [7]. For simplicity, we work within the Einstein frame and expand the metrics using the famous Arnowitt-Deser-Misner formalism:

$$\tilde{g}_{\mu\nu}dx^\mu dx^\nu = -(N_g^0)^2dt^2 + \gamma_{ij}(dx^i + N_g^i dt)(dx^j + N_g^j dt),$$
$$f_{\sigma\rho}dx^\sigma dx^\rho = -(N_f^0)^2dt^2 + \omega_{ab}(dx^a + N_f^a dt)(dx^b + N_f^b dt). \tag{7}$$

The lapse N_g^0 and shift $\vec{N}_g$ (the three N_g^i's expressed as vectors) of the physical metric, as well as the corresponding ones for the fiducial metric N_f^0 and $\vec{N}_f$, respectively, are all nondynamical. In massive gravity, γ_{ij} allows for at most six propagating modes, one of them being the origin of the BD ghost. A potentially healthy theory must maintain a single constraint on $\bar{\gamma}$ (from now on, a bar denotes the matrix form) and the conjugate momenta, along with an associated

[1] We use lower case f's to denote the fiducial metric and upper case F's to denote the function $F(R)$.

secondary constraint, which will lead to the elimination of the ghost DOF. In the following, we briefly show the existence of these constraints in $F(R)$ massive gravity; one can find additional details in Ref. [23].

In order to introduce the Lagrange multiplier explicitly, we define a new shift $\vec{\eta}$ through $\vec{N}_g - \vec{N}_f = (N_f^0 \mathbb{I} + N_g^0 \mathbb{D})\vec{\eta}$, where $\mathbb{I}$ is the 3×3 unit matrix and the 3×3 matrix $\mathbb{D}$ satisfies $\lambda \mathbb{D}\mathbb{D} = [\bar{\gamma}^{-1} - (\mathbb{D}\vec{\eta})(\mathbb{D}\vec{\eta})^{\mathrm{T}}]\bar{\omega}$ with $\lambda = 1 - \vec{\eta}^{\mathrm{T}}\bar{\omega}\,\vec{\eta}$. The conjugate momenta are defined as $\pi \equiv \frac{\delta S_{\mathrm{G}}}{\delta \dot{\varphi}}$ and $\bar{\Pi} \equiv \frac{\delta S_{\mathrm{G}}}{\delta \dot{\bar{\gamma}}}$. Then, we can derive the Hamiltonian as

$$\mathrm{H} = \int d^3x [\mathcal{H} - N_g^0 \mathcal{C}(\varphi, \pi, \bar{\gamma}, \bar{\Pi}, \vec{\eta})], \tag{8}$$

where

$$\mathcal{H} = -(N_f^0 \vec{\eta} + \vec{N}_f)\vec{\mathcal{R}} - N_f^0 \sqrt{|\bar{\gamma}|}\mathfrak{F}_{\mathrm{H}} m_g^2 M_p^2, \tag{9}$$

$$\mathcal{C} = \mathcal{R} + \vec{\mathcal{R}}^{\mathrm{T}}\mathbb{D}\vec{\eta} + \sqrt{|\bar{\gamma}|}\mathfrak{F}_{\mathrm{C}} m_g^2 M_p^2. \tag{10}$$

Here, we have introduced the coefficients

$$\mathcal{R} = \frac{\sqrt{|\bar{\gamma}|}}{2}[M_p^2 R_3 - \varphi_{,i}\varphi^{,i} - U(\varphi)] + \frac{1}{\sqrt{|\bar{\gamma}|}}\left[\frac{(\mathrm{Tr}\bar{\Pi})^2}{M_p^2} - \frac{2\bar{\Pi}^2}{M_p^2} - \frac{\pi^2}{2}\right], \tag{11}$$

and

$$\mathcal{R}_i = 2\gamma_{ij}\Pi^{jk}_{;k} - \pi\varphi_{,i}, \tag{12}$$

while the coefficients appearing in the mass terms are

$$\mathfrak{F}_{\mathrm{H}} = \frac{\beta_1 \lambda^{\frac{1}{2}}}{\Omega^3} + \frac{\beta_2}{\Omega^2}[\lambda \mathrm{Tr}\mathbb{D} + \vec{\eta}^{\mathrm{T}}\bar{\omega}\mathbb{D}\vec{\eta}] + \frac{\beta_3}{\Omega}\left[2\lambda^{\frac{1}{2}}\mathbb{D}^{[l}_{l}\eta^{i]}\omega_{ij}\mathbb{D}^j_k\eta^k + \lambda^{\frac{3}{2}}\mathbb{D}^{[i}_{i}\mathbb{D}^{j]}_{j}\right] + \frac{\beta_4 |\bar{\omega}|^{\frac{1}{2}}}{|\bar{\gamma}|^{\frac{1}{2}}}, \tag{13}$$

$$\mathfrak{F}_{\mathrm{C}} = \frac{\beta_0}{\Omega^4} + \frac{\beta_1 \lambda^{\frac{1}{2}}\mathrm{Tr}\mathbb{D}}{\Omega^3} + \frac{\beta_2 \lambda \mathbb{D}^{[i}_{i}\mathbb{D}^{j]}_{j}}{\Omega^2} + \frac{\beta_3 \lambda^{\frac{3}{2}}}{\Omega}\mathbb{D}^{[i}_{i}\mathbb{D}^{j}_{j}\mathbb{D}^{k]}_{k}. \tag{14}$$

Varying the Hamiltonian with respect to the new shift $\vec{\eta}$ does not yield any constraints, but due to this new shift vector, the variation with the lapse function N_g^0 does give a constraint which reads

$$\mathcal{C}(\varphi, \pi, \bar{\gamma}, \bar{\Pi}, \vec{\eta}) = 0. \tag{15}$$

In order for this constraint to hold at all times, we must also demand that

$$\mathcal{C}^{(2)} = \{\mathcal{C}, \mathrm{H}\} = 0, \tag{16}$$

where the Poisson brackets of two quantities are defined as

$$\{\mathcal{O}_1(x), \mathcal{O}_2(y)\} = \sum_i \int d^3z \left[\frac{\delta \mathcal{O}_1(x)}{\delta q_i}\frac{\delta \mathcal{O}_2(y)}{\delta p^i} - \frac{\delta \mathcal{O}_1(x)}{\delta p^i}\frac{\delta \mathcal{O}_2(y)}{\delta q_i}\right], \tag{17}$$

with the q_i being the canonical variables (γ_{ij}, φ) and p^i their conjugate momenta (Π^{ij}, π).

Equation (16) must generate a needed second constraint on $\varphi, \bar{\gamma}$, and their conjugate momenta; therefore, it must not vanish identically and/or must not be an equation that determines the lapse function N_g^0. The latter condition will not hold if $\{\mathcal{C}(x), \mathcal{C}(y)\}$ does not vanish as it appears in $\mathcal{C}^{(2)}$ with N_g^0 as its coefficient. Fortunately, one can show that $\{\mathcal{C}(x), \mathcal{C}(y)\}$=0 identically [7,23]. The remaining term $\mathcal{C}^{(2)} = \{\mathcal{C}, \int d^3x \mathcal{H}(x)\}$ becomes

$$\begin{aligned}\mathcal{C}^{(2)} &= \mathcal{C}\nabla_i(N_f\eta^i + N_f^i) + m_g^2 M_p^2(\gamma_{mn}\Pi^k_k - 2\Pi_{mn})\mathfrak{F}_{\mathrm{H}}^{mn} \\ &\quad + m_g^2 M_p^2 N_f \mathbb{D}^i_k \eta^k\left(2\sqrt{\gamma}(\nabla_m F^{mn}_{\mathrm{H}})\gamma_{ni} - \frac{\partial \mathfrak{F}_{\mathrm{H}}}{\partial \varphi}\partial_i\varphi\right) \\ &\quad + \nabla_i(N_f\eta^i + N_f^i)(\mathcal{R}_j\mathbb{D}^j_k\eta^k - m_g^2 M_p^2\sqrt{\gamma}\gamma_{jk}B^{kj}) \\ &\quad - m_g^2 M_p^2(N_f\eta^i + N_f^i)\sqrt{\gamma}\frac{\partial \mathfrak{F}_{\mathrm{C}}}{\partial \varphi}\partial_i\varphi,\end{aligned} \tag{18}$$

with $F^{mn}_{\mathrm{H}} = \frac{1}{\sqrt{\gamma}}\frac{\partial(\sqrt{\gamma}\mathfrak{F}_{\mathrm{H}})}{\partial \gamma_{mn}}$ and

$$\begin{aligned}B^{ki} &= \gamma^{km}\left[\frac{\beta_1}{\lambda^{1/2}}\omega_{ma}(\mathbb{D}^{-1})^a_j + \beta_2(\omega_{ma}(\mathbb{D}^{-1})^a_j\mathbb{D}^b_b - \omega_{mj}) \right.\\ &\quad + \beta_3\lambda^{1/2}[\omega_{ma}(\mathbb{D}^{-1})^a_j - \omega_{mj}(\mathbb{D}^{-1})^a_a] \\ &\quad \left. + \frac{\beta_3\lambda^{1/2}}{2}\omega_{ma}(\mathbb{D}^{-1})^a_j(\mathbb{D}^a_a\mathbb{D}^b_b - \mathbb{D}^a_b\mathbb{D}^b_a)\right]\gamma^{ji}.\end{aligned} \tag{19}$$

From this expression, we see that $\mathcal{C}^{(2)}$ contains no mention of N_g^0 and is not proportional to the original constraint $\mathcal{C}$; hence, it does not vanish identically when $\mathcal{C} = 0$. Therefore, imposing

$$\left\{\mathcal{C}, \int d^3x \mathcal{H}(x)\right\} = 0 \tag{20}$$

gives a second nontrivial constraint. This result is not too surprising, as the effect of considering $F(R)$ gravity amounts to (in the Einstein frame) multiplying each of the graviton's mass terms by a power of the conformal factor Ω. Therefore, the structure of Eq. (2), which is responsible for the elimination of the BD ghost, is still preserved, as seen in Eq. (5).

Now, since $\mathcal{C}^{(2)} = 0$ must remain valid at all times, one must make sure that the equation $\{\mathcal{C}^{(2)}, H\} = 0$ does not lead to additional constraints but instead give an equation determining N_g^0. This is the case only if $\{\mathcal{C}^{(2)}, \int d^3x\mathcal{H}(x)\} \neq 0$ and $\{\mathcal{C}^{(2)}, \mathcal{C}\} \neq 0$. These two conditions are satisfied by the Fierz-Pauli constraints for which $\mathcal{C}^{(2)}$ and $\mathcal{C}$ reduce to lowest order in the γ_{ij}, π_{ij} fields and with $\varphi = 0$. Therefore, as argued in Ref. [7], considering the terms of higher order in the fields cannot change $\{\mathcal{C}^{(2)}, \int d^3x\mathcal{H}(x)\}$ and $\{\mathcal{C}^{(2)}, \mathcal{C}\}$ in a way that makes them vanish identically. Additionally, since $\varphi = 0$ belongs to the constraining surface, adding this new DOF will not change the fact that no tertiary constraint exists. Hence, we see that the effect of considering an $F(R)$ modification does not lead to a resurgence of the BD ghost.

IV. COSMOLOGY

When applied in cosmological frameworks, the scenario of $F(R)$ massive gravity exhibits a large class of phenomenological behaviors due to the combination of the $F(R)$ and graviton-mass sectors. Let us start with a Minkowski fiducial metric $f_{\sigma\rho} = \eta_{\sigma\rho}$. The model allows only for an open Friedmann-Robertson-Walker universe, and thus, we consider the physical metric in the Jordan frame as

$$ds^2 = -N^2dt^2 + a^2(t)\gamma^K_{ij}dx^idx^j, \tag{21}$$

with $\gamma^K_{ij}dx^idx^j = \delta_{ij}dx^idx^j - \frac{a_0^2(\delta_{ij}x^idx^j)^2}{1-a_0^2\delta_{ij}x^ix^j}$, and $a_0 = \sqrt{|K|}$ is associated with the spatial curvature. The Stückelberg scalars are $\varphi^0 = b(t)\sqrt{1 + a_0^2\delta_{ij}x^ix^j}$, $\varphi^i = a_0b(t)x^i$. Then, the polynomials defined in (2) take the forms of

$$\begin{aligned}
\mathcal{U}_2 &= 3a(a - a_0b)(2Na - b'a - Na_0b),\\
\mathcal{U}_3 &= (a - a_0b)^2(4Na - 3ab' - Na_0b),\\
\mathcal{U}_4 &= (a - a_0b)^3(N - b'),
\end{aligned}\tag{22}$$

where primes denote derivatives with respect to t. Finally, for simplicity, we assume that the gravitational sector couples minimally to the regular matter component.

Variation of the action with respect to b, N, and a gives, respectively, the constraint and the two Friedmann equations, namely,

$$(\dot{a} - a_0)Y_1 = 0, \tag{23}$$

$$3M_p^2F_{,R}\left(H^2 - \frac{a_0^2}{a^2}\right) = \rho_m + \rho_{\rm IR} + \rho_{\rm UV}, \tag{24}$$

$$M_p^2F_{,R}\left(-2\dot{H} - 3H^2 + \frac{a_0^2}{a^2}\right) = p_m + p_{\rm IR} + p_{\rm UV}, \tag{25}$$

with $\dot{a} = \frac{a'}{N}$ and $H = \frac{\dot{a}}{a}$. In the above expressions, we have defined the IR (massive-gravity) effective contribution

$$\begin{aligned}
\rho_{\rm IR} &= m_g^2M_p^2(\mathcal{B} - 1)(Y_1 + Y_2),\\
p_{\rm IR} &= -m_g^2M_p^2(\mathcal{B} - 1)Y_2 - m_g^2M_p^2(\dot{b} - 1)Y_1,
\end{aligned}\tag{26}$$

as well as the UV [$F(R)$ sector] effective contribution

$$\rho_{\rm UV} = M_p^2\left[\frac{RF_{,R} - F}{2} - 3H\dot{R}F_{,RR}\right], \tag{27}$$

$$p_{\rm UV} = M_p^2\left[\dot{R}^2F_{,RRR} + 2H\dot{R}F_{,RR} + \ddot{R}F_{,RR} + \frac{F - RF_{,R}}{2}\right], \tag{28}$$

where the polynomials $Y_{1,2}$ are given by $Y_1 = (3-2\mathcal{B}) + \alpha_3(3-\mathcal{B})(1-\mathcal{B}) + \alpha_4(1-\mathcal{B})^2$ and $Y_2 = (3-\mathcal{B}) + \alpha_3(1-\mathcal{B})$, with $\mathcal{B} = \frac{a_0b}{a}$.

Similar to all massive-gravity scenarios, Eq. (23) constrains the dynamics significantly. As in self-accelerating backgrounds of dRGT [9], the nontrivial solutions correspond to the case of $Y_1 = 0$ and yield

$$\mathcal{B}_\pm = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\alpha_3 + \alpha_4}. \tag{29}$$

This relation can be always fulfilled by choosing $b(t) \propto a(t)$, and therefore, it yields $\rho_{\rm IR} = -p_{\rm IR}$ to be constant, as it is expected similarly to standard nonlinear massive gravity [24]. However, the crucial issue is that in the present model, the remaining $F(R)$ sector can be taken at will, leading to a large class of cosmologies. Among them, an interesting class is when the $F(R)$ sector is important at early times and thus responsible for inflation, while the massive graviton is dominant at late times and can drive the Universe's acceleration as observed today.

In order to provide a representative example, we consider the well-known Starobinsky model with $F(R) = R + \frac{\xi}{M_p^2}R^2$ in numerical estimates. In the left panel of Fig. 1, we present the early-time inflationary solutions for three parameter choices, while in the right panel, we depict the late-time self-accelerating solutions. In this particular choice, the Ricci scalar becomes very small at late times, and thus, the $F(R)$'s contribution is dramatically suppressed by the Planck scale. Therefore, only the massive-gravity part contributes to the late-time acceleration. However, note that in the general case, the total effective dark energy is constituted of both the massive gravity as well as the $F(R)$-modification sectors; that is, $\rho_{\rm DE} \equiv \rho_{\rm IR} + \rho_{\rm UV}$. Therefore, our model is expected to be very interesting phenomenologically [23].

V. PERTURBATION ANALYSIS

The scenario at hand is free of the BD ghost, and its cosmological applications allow for a large class of

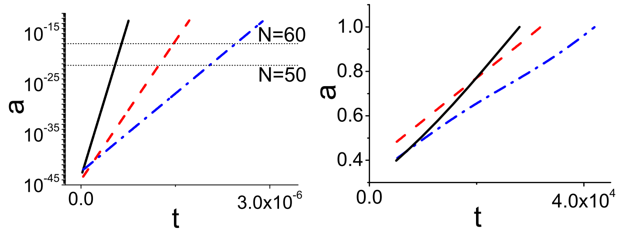

FIG. 1 (color online). The left panel presents three inflationary solutions corresponding to (a) $m_g = 10^{-50}$, $\alpha_3 = 2$, $\alpha_4 = -1$, $a_0 = 5 \times 10^{-41}$, and $\xi = 10^{10}$ (solid black line), (b) $m_g = 10^{-50}$, $\alpha_3 = 10$, $\alpha_4 = 10$, $a_0 = 10^{-41}$, and $\xi = 10^9$ (dashed red line), and (c) $m_g = 10^{-50}$, $\alpha_3 = 1$, $\alpha_4 = 1$, $a_0 = 10^{-40}$, and $\xi = 10^{10}$ (dash-dotted blue line). The two horizontal lines mark the $N = 50$ and $N = 60$ e-folding regimes. All parameters are in Planck units. The right panel depicts three late-time accelerating solutions corresponding to (a) $m_g = 3$, $\alpha_3 = 3$, $\alpha_4 = -5$, $a_0 = 0.05$, and $\xi = 0.5$ (solid black line), (b) $m_g = 1.5$, $\alpha_3 = 1$, $\alpha_4 = -2$, $a_0 = 0.01$, and $\xi = 0.5$ (dashed red line), and (c) $m_g = 3$, $\alpha_3 = 10$, $\alpha_4 = 1$, $a_0 = 0.05$, and $\xi = 0.5$ (dash-dotted blue line). All parameters are in units where the present Hubble parameter is $H_0 = 1$, and we have imposed $\Omega_{m0} \approx 0.31$, $\Omega_{\mathrm{DE}0} \approx 0.69$, and $\Omega_{k0} \approx 0.01$ at the present scale factor $a_0 = 1$.

behaviors. However, the last and necessary step is to examine whether such cosmological applications remain free of instabilities at the perturbative level, which is exactly the weak and disastrous point of standard nonlinear massive gravity pointed out in Ref. [11] (see also Refs. [25,26]). In the rest of the paper, we briefly show that the scalar perturbations in our model can be stable at the linear perturbative level under certain parameter choices.

For simplicity, we work in the Einstein frame, using the Lagrangian (6), and then consider perturbations around a homogeneous and isotropic background. The scalar perturbations of our variables involve the metric part

$$\delta g_{00} = -2\tilde{N}^2\phi, \qquad \delta g_{0i} = \tilde{N}\,\tilde{a}\,\partial_i B,$$
$$\delta g_{ij} = \tilde{a}^2\left[2\gamma^K_{ij}\psi + \left(\nabla_i\nabla_j - \frac{1}{3}\gamma^K_{ij}\nabla_k\nabla^k\right)\right]E, \quad (30)$$

and the field fluctuation $\delta\varphi$. Using the Hamiltonian and momentum constraints, as well as the background equations of motion, we can integrate out the nondynamical modes, namely, ϕ, B, and E. Therefore, the would-be BD ghost is eliminated in our model. Furthermore, since the scalar DOF of the graviton is nondynamical at the linear level on the self-accelerating solution, one can introduce the usual Bardeen potential ψ_B and define a generalized Mukhanov-Sasaki variable

$$Q \equiv \delta\varphi_B + \frac{\dot{\varphi}\psi_B}{H}. \quad (31)$$

This allows us to obtain the perturbation equation of our single propagating scalar DOF in the Fourier space as

$$\ddot{Q}_k + 3H\dot{Q}_k + \left[\frac{k^2}{a^2} + U_{,\varphi\varphi} - \frac{1}{M_p^2 a^3}\left(\frac{a^3}{H}\dot{\varphi}^2\right)^{\cdot}\right]Q_k$$
$$= \frac{2m_g^2\tilde{Y}_Q}{3\Omega^4}Q_k - \frac{2K}{a^2H^2}\left(\ddot{\varphi} - \frac{\dot{H}\dot{\varphi}}{H}\right)\psi_B, \quad (32)$$

where $\tilde{Y}_Q \equiv 4(1-\tilde{\mathcal{B}})\tilde{Y}_2$ is defined in the Einstein frame. Note that Q is the only dynamical perturbation variable since ψ_B can be determined by it as well.

From the above analysis, one can clearly see the qualitative difference of the present construction, comparing to other extended nonlinear massive-gravity models, such as the quasidilaton massive gravity [20] and the mass-varying massive gravity [21]. In particular, these extensions involve 2 extra scalar DOF, as it can be verified by counting the number of nonzero eigenvalues of the matrix for the kinetic part of the perturbation action [27]. Applying the method of Ref. [27] in the present scenario, by setting the coefficient in front of the scalar-field kinetic term to unity, we find that there exists only one nonzero eigenvalue, and this implies only a single DOF. A detailed analysis of this issue can be found in the accompanying paper [23].

The lhs of the perturbation equation (32) is exactly the same as the usual one in general relativity (GR) plus a scalar field, but the rhs involves a mass term due to the graviton potential. Its positivity depends on the coefficient $\tilde{Y}_Q$ and directly determines whether the model suffers from a tachyonic instability or not. Obviously, a healthy model of $F(R)$ massive gravity requires $\tilde{Y}_Q < 0$, which provides the corresponding allowed regime of the parameter space. Additionally, the last term of (32) appears due to the spatial curvature. Since this term would easily dilute out along the cosmic expansion, it is harmless to the model when applied into cosmology. Therefore, we conclude that there exists enough parameter space for scalar perturbations to be stable throughout the cosmological evolutions of phenomenological interest.

VI. CONCLUSIONS

The study of massive gravity may be important in understanding the observed acceleration of the present cosmic expansion, which is one of the greatest mysteries in modern physics. In this regard, the question of establishing a theoretically healthy and observationally viable model of nonlinear massive gravity has attracted the interest of the literature.

The theory of $F(R)$ nonlinear massive gravity, as a possible GR modification both at the IR and UV regimes, has significant advantages both at the theoretical as well as at the cosmological levels. First, due to the usual dRGT-like graviton potential, it inherits its benefits and is free of BD ghosts. Furthermore, due to the freedom of the $F(R)$ sector combined with the graviton mass, it allows for a large class of cosmological evolutions. For instance, a simple R^2 form is able to drive both early universe inflation and late-time acceleration, determining the whole cosmic evolution in a unified way.

We would like to end by highlighting the advantage of our model that the perturbations around a cosmological background can be stabilized due to the $F(R)$ term, which introduces a scalar DOF at the linear level, and hence, it constrains the scalar metric perturbations to be as in GR. Usually, the nonlinear inclusion of the gravitational mass gives rise to a scalar DOF, that is, the longitudinal graviton. Although the inclusion of the $F(R)$ sector at first introduces another scalar mode, this mode nicely "eats" the nonlinear mode due to the graviton mass, and moreover, it imposes the constraint on the stability. This mechanism is very similar to the process of the spontaneous symmetry breaking of particle physics governed by the Goldstone theorem. In this respect, the possible instabilities that could appear at a higher nonlinear regime do not appear unless the perturbation theory itself breaks down. Additionally, we mention that once the perturbations evolve into the nonlinear regime, higher curvature terms would become important, accompanied by the high energy scale, and thus completely change the dynamics of the theory. The above features may reveal that the UV and IR behaviors of gravitation may not be independent.

The possibility of a gravitational Goldstone theorem deserves further investigation. In particular, since the $F(R)$ sector can be reformulated as a scalar field minimally coupled to the Ricci scalar with an effective potential, and since for a wide class of $F(R)$ forms the effective potential approaches an extremely flat plateau in the UV regime, then from the viewpoint of symmetry, the corresponding effective potential indicates an approximate shift symmetry along the scalar field. When the scalar field evolves into the IR regime, it is stabilized at the vacuum, and therefore, the shift symmetry can be spontaneously broken. One may make an analogue with the scalar field and the dilaton. Thus, one at first expects the second propagating scalar mode to appear in the IR regime; however, it was eaten by the dilaton field through the process of the spontaneous shift symmetry breaking. This interesting property is perhaps an indication for the aforementioned possibility of a gravitational Goldstone theorem.

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