Exact solution of vacuum field equation in Finsler spacetime

Xin Li[1,2,3,*] and Zhe Chang[2,3,†]

[1]*Department of Physics, Chongqing University, Chongqing 401331, China*
[2]*Institute of High Energy Physics, Chinese Academy of Sciences, 100049 Beijing, China*
[3]*State Key Laboratory Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

(Received 26 January 2014; revised manuscript received 14 August 2014; published 29 September 2014)

We suggest that the vacuum field equation in Finsler spacetime is equivalent to the vanishing of the Ricci scalar. The Schwarzschild metric can be deduced from a solution of our field equation if the spacetime preserves spherical symmetry. Supposing that the spacetime preserves the symmetry of the "Finslerian sphere," we find a non-Riemannian exact solution of the Finslerian vacuum field equation. The solution is similar to the Schwarzschild metric. It reduces to the Schwarzschild metric as the Finslerian parameter ϵ vanishes. It is proven that the Finslerian covariant derivative of the geometrical part of the gravitational field equation is conserved. The interior solution is also given. We get solutions of the geodesic equation in such a Schwarzschild-like spacetime, and show that the geodesic equation returns to its counterpart in Newtonian gravity in the weak-field approximation. Celestial observations give a constraint on the Finslerian parameter $\epsilon < 10^{-4}$, and the recent Michelson-Morley experiment requires $\epsilon < 10^{-16}$. A counterpart of Birkhoff's theorem exists in the Finslerian vacuum. This shows that the Finslerian gravitational field with the symmetry of the "Finslerian sphere" in vacuum must be static.

DOI: 10.1103/PhysRevD.90.064049 PACS numbers: 04.50.Kd, 02.40.-k, 04.20.Cv

I. INTRODUCTION

In 1912, Einstein proposed his famous general relativity which gives the connection between Riemannian geometry and gravitation. In general relativity, the effects of gravitation are ascribed to spacetime curvature instead of a force. In four-dimensional spacetime, two solutions of the Einstein vacuum field equation are well known [1]. These are the Schwarzschild solution, which preserves spherical symmetry, and the Kerr solution, which preserves axial symmetry. The Schwarzschild solution is of vital importance for general relativity. The physics of the Schwarzschild solution is quite different from Newtonian gravity. The success of general relativity is attributed to four classical tests [2]. The predictions of these four classical tests directly come from the Schwarzschild solution. Most celestial bodies can be approximately treated as a sphere. Thus, the Schwarzschild solution is widely used in investigating astronomical phenomena. However, recent astronomical observations show that the gravitational field of galaxy clusters is offset from its baryonic matter content [3]. This implies that spherical symmetry may not be preserved on the scale of galaxy clusters.

Finsler geometry [4] is a new geometry which includes Riemannian geometry as its special case. Chern pointed out that Finsler geometry is just Riemannian geometry without the quadratic restriction. The symmetry of spacetime is described by the isometric group. The generators of the isometric group are directly connected with the Killing vectors. It is well known that the isometric group is a Lie group on a Riemannian manifold. This fact also holds on a Finslerian manifold [5]. Generally, Finsler spacetime admits less Killing vectors than Riemannian spacetime [6]. The number of independent Killing vectors of an n-dimensional non-Riemannian Finsler spacetime should be no more than $\frac{n(n-1)}{2}+1$ [7]. The causal structure of Finsler spacetime is determined by the vanishing of the Finslerian length [8] and the speed of light is modified. It has been shown that translation symmetry is preserved in flat Finsler spacetime [6]. Thus, the energy and momentum are well defined in Finsler spacetime. In flat Finsler spacetime, inertial motion preserves the Finslerian length and admits a modified dispersion relation.

Lorentz invariance (LI) is one of the foundations of the standard model of particle physics. Of course, it is very interesting to test the fate of LI in both experiments and theories. The theoretical approach of investigating LI violation is to study possible spacetime symmetries, and to erect some counterparts of special relativity. Recently, a few counterparts of special relativity have emerged. The first one is doubly special relativity (DSR) [9–11]. In DSR, Planck-scale effects are taken into account by introducing an invariant Planckian parameter. Together with the speed of light, DSR has two invariant parameters. The second counterpart of special relativity is very special relativity (VSR) [12]. Coleman and Glashow have set up a perturbative framework for investigating possible departures of local quantum field theory from LI. The symmetry group of

*lixin1981@cqu.edu.cn
†changz@ihep.ac.cn

VSR is some certain subgroups of the Poincare group, which contains the spacetime translations and proper subgroups of Lorentz transformations. Gibbons *et al.* [13] have pointed out that Glashow's VSR is Finsler geometry. Generally, the flat Finsler spacetime breaks the Lorentz symmetry. Thus, it is a possible mechanism of Lorentz violation [14].

The standard cosmological model, i.e., the ΛCDM model [15,16] has been well established. It is consistent with several precise astronomical observations coming from the Wilkinson Microwave Anisotropy Probe (WMAP) [17], the Planck satellite [18], and Supernova Cosmology Project [19]. One of the most important and basic assumptions of the ΛCDM model states that the Universe is homogeneous and isotropic on large scales. However, such a principle faces challenges [20]. The Union2 type Ia supernova data hint that the Universe has a preferred direction, $(l, b) = (309°, 18°)$ in the galactic coordinate system [21]. The Universe has a maximum expansion velocity in this direction. Astronomical observations [22] found that the dipole moment of the peculiar velocity field in the direction $(l, b) = (287° \pm 9°, 8° \pm 6°)$ at a scale of $50h^{-1}$ Mpc has a magnitude of $407 \pm 81\ \mathrm{km\,s^{-1}}$. This peculiar velocity is much larger than the value $110\ \mathrm{km\,s^{-1}}$ given by WMAP5 [23]. The recently released data of the Planck Collaboration show deviations from isotropy with a level of significance $\sim 3\sigma$ [24]. The Planck Collaboration confirms the asymmetry of the power spectra between two preferred opposite hemispheres. These facts hint that the Universe may have a preferred direction.

Many models have been proposed to resolve the asymmetric anomaly of the astronomical observations. An incomplete and succinct list includes an imperfect fluid dark energy [25], the local void scenario [26,27], a noncommutative spacetime effect [28], anisotropic curvature in cosmology [29], and the Finsler gravity scenario [30]. Instead of Minkowski spacetime, Finsler spacetime involves a preferred direction. It is a natural candidate for describing the cosmological anisotropy.

Both the ΛCDM model and the standard model require no variation of fundamental physical constants in principle, such as the fine-structure constant $\alpha = e^2/\hbar c$. Recently, the observations of quasar absorption spectra show that the fine-structure constant varies on cosmological scales [31]. Furthermore, in high-redshift regions ($z > 1.6$), it has been shown that the variation of α is well represented by an angular dipole model pointing in the direction $(l, b) = (330°, -15°)$ with a level of significance $\sim 4.2\sigma$. Mariano and Perivolaropoulos [32] have shown that the dipole of α is anomalously aligned with the corresponding dark energy dipole obtained through the Union2 sample. One direct reason for the variation of α is the variation of the speed of light. Finsler geometry, as a natural tool for investigating both the cosmological preferred direction and Lorentz violations, may also be considered as a natural framework to describe the dipole structure of the fine-structure constant. Models [33–35] based on Finsler spacetime have been developed to study the cosmological preferred direction.

The counterpart of special relativity has been established in flat Finsler spacetime; however, Finslerian gravity is still incomplete. There are various types of gravitational field equations in Finsler spacetime [36–39]. Models have been built to study the gravitational theories that are constructed in Finsler spacetime. Stavrinos *et al.* [34,40] used the method of an osculating Riemannian space to study the cosmological anisotropy in Finsler spacetime. Vacaru *et al.* studied high-dimensional gravity in Finsler spacetime [41]. Pfeifer *et al.* [42] have constructed gravitational dynamics for Finsler spacetimes in terms of an action integral on the unit tangent bundle. However, these equations (or models) are not equivalent to one another for the following reasons. It is well known that there is only a torsion-free connection—the Christoffel connection—in Riemannian geometry. However, there are many types of connections in Finsler geometry. Therefore, the covariant derivatives that depend on the connections are different. The Finslerian length element F is constructed on a tangent bundle [4]. Thus, the gravitational field equation should be constructed on the tangent bundle in principle. However, the corresponding energy-momentum tensor, which should be constructed on the tangent bundle, is rather obscure.

The analogy between geodesic deviation equations in Finsler spacetime and Riemannian spacetime gives the vacuum field equation in Finsler gravity [43]. It is the vanishing of the Ricci scalar. The vanishing of the Ricci scalar implies that the geodesic rays are parallel to one another. The fact that the Ricci scalar is geometry invariant implies that the vacuum field equation is insensitive to the connection, which is an essential physical requirement. In this paper, we present an exact solution of the vacuum field equation in Finsler spacetime. The interior solution is also shown.

This paper is organized as follows. Section II is divided into two subsections. In Sec. II A, we briefly introduce some basic geometric objects in Finsler geometry and we show the symmetry of flat Finsler spacetime. The spherical symmetry can be represented by the metric of the Riemann sphere. In Sec. II B, we introduce the "Finslerian sphere," which is a counterpart of the Riemann sphere. Section III is divided into four subsections. In Sec. III A we introduce the vacuum field equation in Finsler spacetime by analogizing the geodesic deviation equation in Finsler spacetime and that in Riemannian spacetime; in Sec. III B we present an exact solution of the vacuum field equation in Finsler spacetime; in Sec. III C we investigate the Newtonian limit of our solution; and in Sec. III D we show the boundary conditions of the vacuum field equation. This will distinguish the Schwarzschild solution from our solution. In Sec. IV, we propose a gravitational field equation with a source. We prove that the Finslerian covariant derivative of the geometrical part of the gravitational field equation is conserved. Then, an interior solution of the gravitational field equation is shown. In Sec. V, we investigate the experimental constraint on the Schwarzschild-like spacetime that is given in Sec. III. Particle motion is shown in Sec. V A; we get three solutions of the geodesic equations. Solar System constraints on our Finslerian parameter are given in Sec. V B. In Sec. VI, we show the counterpart of Birkhoff's theorem. It states that the Finslerian gravitational field with the symmetry of the "Finslerian sphere" in vacuum must be static. Conclusions and remarks are given in Sec. VII.

II. SYMMETRY OF FINSLER SPACETIME

A. Killing vectors

Instead of defining an inner product structure over the tangent bundle, as in Riemannian geometry, Finsler geometry is based on the so-called Finsler structure F with the property $F(x,\lambda y)=\lambda F(x,y)$ for all $\lambda>0$, where $x\in M$ represents position and $y\equiv\frac{dx}{d\tau}$ represents velocity. The Finslerian metric is given as [4]

$$g_{\mu\nu}\equiv\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}\left(\frac{1}{2}F^2\right). \tag{1}$$

In physics, the Finsler structure F is not positive-definite at every point of the Finsler manifold. We focus on investigating Finsler spacetime with a Lorentz signature. A positive, zero, or negative F corresponds to time-like, null, or space-like curves, respectively. For massless particles, the stipulation is $F=0$. Two types of Finsler space should be noted. One is the Riemannian space. A Finslerian metric is said to be Riemannian if F^2 is quadratic in y. Another is Randers spacetime [44]. It is given as

$$F(x,y)\equiv\alpha(x,y)+\beta(x,y), \tag{2}$$

where

$$\alpha(x,y)\equiv\sqrt{\tilde{a}_{\mu\nu}(x)y^\mu y^\nu}, \tag{3}$$

$$\beta(x,y)\equiv\tilde{b}_\mu(x)y^\mu, \tag{4}$$

and $\tilde{a}_{ij}$ is the Riemannian metric. Throughout this paper, the indices are lowered and raised by $g_{\mu\nu}$ and its inverse matrix $g^{\mu\nu}$, and the objects with a tilde are lowered and raised by $\tilde{a}_{\mu\nu}$ and its inverse matrix $\tilde{a}^{\mu\nu}$.

To investigate the Killing vectors, we should construct the isometric transformations of the Finsler structure. It is convenient to discuss the isometric transformations under an infinitesimal coordinate transformation for x,

$$\bar{x}^\mu=x^\mu+\epsilon\tilde{V}^\mu, \tag{5}$$

together with a corresponding transformation for y,

$$\bar{y}^\mu=y^\mu+\epsilon\frac{\partial\tilde{V}^\mu}{\partial x^\nu}y^\nu, \tag{6}$$

where $|\epsilon|\ll 1$. Under the coordinate transformation (5) and (6), to first order in $|\epsilon|$, we obtain the expansion of the Finsler structure,

$$\bar{F}(\bar{x},\bar{y})=\bar{F}(x,y)+\epsilon\tilde{V}^\mu\frac{\partial F}{\partial x^\mu}+\epsilon y^\nu\frac{\partial\tilde{V}^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu}, \tag{7}$$

where $\bar{F}(\bar{x},\bar{y})$ should be equal to $F(x,y)$. Under the transformation (5) and (6), a Finsler structure is called an isometry if and only if

$$F(x,y)=\bar{F}(x,y). \tag{8}$$

From Eq. (7) we obtain the Killing equation $K_V(F)$ in Finsler space,

$$K_V(F)\equiv\tilde{V}^\mu\frac{\partial F}{\partial x^\mu}+y^\nu\frac{\partial\tilde{V}^\mu}{\partial x^\nu}\frac{\partial F}{\partial y^\mu}=0. \tag{9}$$

Plugging the length element of Randers spacetime (2) into the Killing equation (10), and noticing that the rational and irrational parts of the Killing equation are independent, we obtain that

$$\tilde{V}_{\mu|\nu}+\tilde{V}_{\nu|\mu}=0, \tag{10}$$

$$\tilde{V}^\mu\frac{\partial\tilde{b}_\nu}{\partial x^\mu}+\tilde{b}_\mu\frac{\partial\tilde{V}^\mu}{\partial x^\nu}=0, \tag{11}$$

where "|" denotes the covariant derivative with respect to the Riemannian metric α. Equations (10) and (11) are equivalent to the statements $L_V\alpha=0$ and $L_V\beta=0$, respectively, where α is the obvious Riemannian metric and β is the 1-form. Here L_V is the Lie derivative along V. It is obvious that the Killing equation (10) is the same as the Riemannian one, but the other Killing equation (11) constrains Eq. (10). Thus, the number of independent Killing vectors in Randers-Finsler spacetime (2) is less than that in Riemannian spacetime α [6].

The geodesic equation for the Finsler manifold is given as

$$\frac{d^2x^\mu}{d\tau^2}+2G^\mu=0, \tag{12}$$

where

$$G^\mu=\frac{1}{4}g^{\mu\nu}\left(\frac{\partial^2F^2}{\partial x^\lambda\partial y^\nu}y^\lambda-\frac{\partial F^2}{\partial x^\nu}\right) \tag{13}$$

are called geodesic spray coefficients. It can be proven from the geodesic equation (12) that the Finslerian structure $F(x, \frac{dx}{d\tau})$ is constant along the geodesic.

A Finsler metric is said to be locally Minkowskian if at every point there is a local coordinate system, such that $F = F(y)$ is independent of the position x [4]. It can be proven that all types of curvature tensors vanish in locally Minkowskian spacetime. Thus, a locally Minkowskian spacetime is flat Finsler spacetime. The momenta in flat Finsler spacetime are defined as $p^\mu \equiv m\frac{dx^\mu}{d\tau}$. Since the Finsler structure F does not depend on x, it is clear from the geodesic equation that $\frac{dp^\mu}{d\tau} = 0$. Therefore, the momenta p^μ are conserved in flat Finsler spacetime. The dispersion relation of flat Finsler spacetime is given as

$$\eta_{\mu\nu}(p)p^\mu p^\nu = m^2. \tag{14}$$

One should notice that $\eta_{\mu\nu}(p)$ is not a constant in flat Finsler spacetime. For example, in flat Randers spacetime, its dispersion relation is given as

$$\sqrt{\tilde{\eta}_{\mu\nu}p^\mu p^\nu} + \tilde{b}_\rho p^\rho = m, \tag{15}$$

where $\tilde{\eta}_{\mu\nu}$ is the Minkowski metric. The flat Finsler spacetime gives a modified dispersion relation that plays the role of testing Lorentz invariance, and it is taken as a boundary condition to solve the gravitational field equation in Finsler spacetime.

B. Two-dimensional Finsler space with constant flag curvature

In general relativity, the Schwarzschild metric preserves spherical symmetry. In Riemannian geometry, at a fixed radial coordinate r, if the metric is of the form

$$F_{\rm RS} = \sqrt{(y^\theta)^2 + \sin\theta(y^\varphi)^2} \tag{16}$$

then we say it possesses spherical symmetry. It is clear that the Riemannian space with the metric (16) has constant curvature, which equals 1. By using the Killing equation (9), one can find that the metric (16) has three independent Killing vectors. In this paper, we want to find a Schwarzschild-like solution in Finsler spacetime. In Finsler geometry, the counterpart of spherical symmetry is the "Finslerian sphere." In order to represent the features of most celestial bodies, the "Finslerian sphere" should look like a sphere. In mathematics, it should be topologically equivalent to a sphere. Also, the "Finslerian sphere" should preserve as much symmetry as it can. The theorem proven in Ref. [7] showed that the two-dimensional Finsler space has only one independent Killing vector, as does the "Finslerian sphere." In Finsler geometry, the generalization of the Riemannian sectional curvature is flag curvature. A constant flag curvature is equivalent to a constant Ricci scalar. Thus, the counterpart of the Riemann sphere, the "Finslerian sphere," should have constant flag curvature. In the following, we present a specific example of the "Finslerian sphere."

Bao *et al.* [45] have given a complete classification of Randers-Finsler space [44] with constant flag curvature. A two-dimensional Randers-Finsler space with constant positive flag curvature $\lambda = 1$ is given as

$$F_{\rm FS} = \frac{\sqrt{(1-\epsilon^2\sin^2\theta)y^\theta y^\theta + \sin^2\theta y^\varphi y^\varphi}}{1-\epsilon^2\sin^2\theta} - \frac{\epsilon\sin^2\theta y^\varphi}{1-\epsilon^2\sin^2\theta}, \tag{17}$$

where $0 \le \epsilon < 1$. It is obvious that the metric (17) returns to the Riemann sphere when $\epsilon = 0$, and that the metric (17) is nonreversible for $\varphi \to -\varphi$. The Randers-Finsler space (17) has two geometrically distinct closed geodesics [46] if ϵ is irrational: the two geodesics located at $\theta = \frac{\pi}{2}$ with length $L_\pm = 2\pi(1\pm\epsilon)^{-1}$. This fact can be proven by plugging the metric (17) into the geodesic equation (12). Then, one can find that $\theta = \frac{\pi}{2}$ and $\varphi = u\tau + v$ (u, v are integral constants) are the solutions of the geodesic equation. The Randers-Finsler space (17) is homotopy equivalent to the two-dimensional sphere [46].

In terms of the Busemann-Hausdorff volume form, the volume of a closed Randers-Finsler surface $F = \sqrt{a_{ij}(x)y^iy^j} + b_i(x)y^i$ is given as [47]

$$\mathrm{Vol}_F = \int (1-(a^{ij}b_ib_j))^{\frac{3}{2}}\sqrt{\det(\mathrm{a_{ij}})}dx^1\wedge dx^2. \tag{18}$$

Plugging the Randers metric (17) into Eq. (18), we obtain that its volume is 4π, which is the same as the unit Riemann sphere.

III. THE EXACT SOLUTION OF THE VACUUM FIELD EQUATION

A. Vacuum field equation

In this paper, we introduce the vacuum field equation in the way first discussed by Pirani [48,49]. In Newton's theory of gravity, the equation of motion of a test particle is given as

$$\frac{d^2x^i}{d\tau^2} = -\eta^{ij}\frac{\partial\phi}{\partial x^i}, \tag{19}$$

where $\phi = \phi(x)$ is the gravitational potential and $\eta^{ij} = \mathrm{diag}(+1, +1, +1)$ is the Euclidean metric. For an infinitesimal transformation $x^i \to x^i + \epsilon\xi^i(|\epsilon| \ll 1)$, Eq. (19) becomes, to first order in ϵ,

$$\frac{d^2x^i}{d\tau^2} + \epsilon\frac{d^2\xi^i}{d\tau^2} = -\eta^{ij}\frac{\partial\phi}{\partial x^i} - \epsilon\eta^{ij}\xi^k\frac{\partial^2\phi}{\partial x^j\partial x^k}. \tag{20}$$

Combining Eqs. (19) and (20), we obtain

$$\frac{d^2\xi^i}{d\tau^2} = \eta^{ij}\xi^k\frac{\partial^2\phi}{\partial x^j\partial x^k} \equiv \xi^k H^i{}_k. \tag{21}$$

For the vacuum field equation, one has $H^i{}_i = \nabla^2\phi = 0$.

In general relativity, the geodesic deviation gives a similar equation,

$$\frac{D^2\xi^\mu}{D\tau^2} = \xi^\nu\tilde{R}^\mu{}_\nu, \tag{22}$$

where $\tilde{R}^\mu{}_\nu = \tilde{R}_\lambda{}^\mu{}_{\nu\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}$. Here, $\tilde{R}_\lambda{}^\mu{}_{\nu\rho}$ is the Riemannian curvature tensor and D denotes the covariant derivative along the curve $x^\mu(t)$. The vacuum field equation in general relativity gives $\tilde{R}_\mu{}^\lambda{}_{\lambda\nu} = 0$ [2]. This implies that the tensor $\tilde{R}^\mu{}_\nu$ is also traceless, $\tilde{R} \equiv \tilde{R}^\mu{}_\mu = 0$.

In Finsler spacetime, the geodesic deviation yields [4]

$$\frac{D^2\xi^\mu}{D\tau^2} = \xi^\nu R^\mu{}_\nu, \tag{23}$$

where $R^\mu{}_\nu = R_\lambda{}^\mu{}_{\nu\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau}$. Here, $R_\lambda{}^\mu{}_{\nu\rho}$ is the Finsler curvature tensor [4] and D denotes the covariant derivative $\frac{D\xi^\mu}{D\tau} = \frac{d\xi^\mu}{d\tau} + \xi^\nu\frac{dx^\lambda}{d\tau}\Gamma^\mu{}_{\nu\lambda}(x, \frac{dx}{d\tau})$. Since the vacuum field equations of Newton's gravity and general relativity are of similar forms, we may assume that the vacuum field equation in Finsler spacetime faces similar requirements as those in the cases of Netwonian gravity and general relativity. This implies that the tensor $R^\mu{}_\nu$ in the Finsler geodesic deviation equation should be traceless, $R^\mu{}_\mu = 0$. Since the Riemannian curvature tensor $R_\lambda{}^\mu{}_{\nu\rho}$ does not depend on $\frac{dx}{d\tau}$, the vanishing of $R^\mu{}_\mu$ is equal to the vanishing of $R_\lambda{}^\mu{}_{\nu\rho}$, which is just the vacuum field equation in general relativity.

We have proven that the analogy of the geodesic deviation equation is valid at least in a Finsler spacetime of Berwald type [50]. We assume that this analogy still holds in a general Finsler spacetime. In Finsler geometry, there is a geometrical invariant: the Ricci scalar Ric. It is of the form [4]

$$\mathrm{Ric} \equiv R^\mu{}_\mu = \frac{1}{F^2}\left(2\frac{\partial G^\mu}{\partial x^\mu} - y^\lambda\frac{\partial^2 G^\mu}{\partial x^\lambda\partial y^\mu} + 2G^\lambda\frac{\partial^2 G^\mu}{\partial y^\lambda\partial y^\mu} - \frac{\partial G^\mu}{\partial y^\lambda}\frac{\partial G^\lambda}{\partial y^\mu}\right). \tag{24}$$

The Ricci scalar depends only on the Finsler structure F and is insensitive to the connection. For a tangent plane $\Pi \subset T_xM$ and a nonzero vector $y \in T_xM$, the flag curvature is defined as

$$K(\Pi, y) \equiv \frac{g_{\lambda\mu}R^\mu{}_\nu u^\nu u^\lambda}{F^2 g_{\rho\theta}u^\rho u^\theta - (g_{\sigma\kappa}y^\sigma u^\kappa)^2}, \tag{25}$$

where $u \in \Pi$. The flag curvature is a geometrical invariant and a generalization of the sectional curvature in Riemannian geometry. The Ricci scalar Ric is the trace of $R^\mu{}_\nu$, which is the predecessor of the flag curvature. Thus the value of the Ricci scalar Ric is invariant under the coordinate transformation.

Furthermore, the significance of the Ricci scalar Ric is very clear. It plays an important role in the geodesic deviation equation [4,51,52]. The vanishing of the Ricci scalar Ric implies that the geodesic rays are parallel to one another, which means that there is a vacuum outside the gravitational source.

Therefore, it is reasonable to believe that the gravitational vacuum field equation in Finsler geometry has its essence in $\mathrm{Ric} = 0$. Pfeifer and Wohlfarth [42] have constructed gravitational dynamics for Finsler spacetime in terms of an action integral on the unit tangent bundle. Their results show that the gravitational field equation in Finsler spacetime is given as

$$S - 6\mathrm{Ric} + 2g^{\mu\nu}(\nabla_\mu S_\nu + S_\mu S_\nu + \partial_{y^\mu}\nabla S_\nu) = -4\pi GT. \tag{26}$$

The S_μ terms can be written as $S_\mu = y^\nu P_\nu{}^\lambda{}_{\lambda\mu}/F$, where $P_\nu{}^\lambda{}_{\lambda\mu}$ are the coefficients of the cross basis $dx\wedge\frac{\delta y}{F}$ [4]. Accordingly, the energy-momentum tensor can also be divided into two parts in terms of the basis of $dx\wedge dx$ and $dx\wedge\frac{\delta y}{F}$, respectively. Thus, the S_μ terms contribute to the energy-momentum tensor that belongs to the basis $dx\wedge\frac{\delta y}{F}$. The vacuum field equation constructed by Pfeifer and Wohlfarth implies that each coefficient of a different basis should vanish. Thus, the stipulation $\mathrm{Ric} = 0$ here is compatible with Pfeifer and Wohlfarth's results for the gravitational field equation.

B. Vacuum solution

Here, we propose an ansatz that the Finsler structure is of the form

$$F^2 = B(r)y^ty^t - A(r)y^ry^r - r^2\bar{F}^2(\theta, \varphi, y^\theta, y^\varphi). \tag{27}$$

Then, the Finsler metric can be derived as

$$g_{\mu\nu} = \mathrm{diag}(B, -A, -r^2\bar{g}_{ij}), \tag{28}$$

$$g^{\mu\nu} = \mathrm{diag}(B^{-1}, -A^{-1}, -r^{-2}\bar{g}^{ij}), \tag{29}$$

where $\bar{g}_{ij}$ and its reverse are the metrics derived from $\bar{F}$ and the indices i, j run over the angular coordinates θ, φ.

Plugging the Finsler structure (27) into Eq. (13), we obtain that

$$G^t = \frac{B'}{2B} y^t y^r, \tag{30}$$

$$G^r = \frac{A'}{4A} y^r y^r + \frac{B'}{4A} y^t y^t - \frac{r}{2A} \bar{F}^2, \tag{31}$$

$$G^\theta = \frac{1}{r} y^\theta y^r + \bar{G}^\theta, \tag{32}$$

$$G^\varphi = \frac{1}{r} y^\varphi y^r + \bar{G}^\varphi, \tag{33}$$

where a prime denotes a derivative with respect to r, and $\bar{G}$ are the geodesic spray coefficients derived from $\bar{F}$. Plugging the geodesic coefficients (30)–(33) into the formula for the Ricci scalar [Eq. (24)], we obtain that

$$\begin{aligned} F^2\mathrm{Ric} = & \left[\frac{B''}{2A} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rA}\right] y^t y^t \\ & + \left[-\frac{B''}{2B} + \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rA}\right] y^r y^r \\ & + \left[\bar{R}ic - \frac{1}{A} + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] \bar{F}^2, \end{aligned} \tag{34}$$

where $\bar{R}ic$ denotes the Ricci scalar of the Finsler structure $\bar{F}$. Since $\bar{F}$ is independent of y^t and y^r, the vanishing of Ricci scalar implies that the terms in each square bracket of Eq. (34) should vanish as well. These equations are given as

$$0 = \frac{B''}{2A} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rA}, \tag{35}$$

$$0 = -\frac{B''}{2B} + \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rA}, \tag{36}$$

$$0 = \bar{R}ic - \frac{1}{A} + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right). \tag{37}$$

Noticing that $\bar{R}ic$ is independent of r, and thus Eq. (37) is satisfied if and only if $\bar{R}ic$ is constant. This means that the two-dimensional Finsler space $\bar{F}$ is a constant-flag-curvature space. The flag curvature is a generalization of the sectional curvature in Riemannian geometry. Here, we label the constant flag curvature as λ. Therefore, $\bar{R}ic = \lambda$. Equations (35)–(37) are similar to the Schwarzschild vacuum field equation in general relativity. The solutions of Eqs. (35)–(37) are given as

$$B = a\lambda + \frac{b}{r}, \tag{38}$$

$$A = \left(\lambda + \frac{b}{ra}\right)^{-1}, \tag{39}$$

where a and b are integral constants.

C. The Newtonian limit

In the above subsection we obtained the vacuum field solution in Finsler spacetime. The integral constants of Eqs. (38) and (39) should be determined by specific boundary conditions, which are given by physical requirements. Here we require that the solutions should return to Newtonian gravity in the weak-field approximation [2]. In order to compare with Newtonian gravity, we only consider the radial motion of particles. Plugging the solutions (38) and (39) into the geodesic coefficients (30) and (31), and noticing that the velocity of a particle $\frac{dr}{dt}$ is small, we obtain the geodesic equations

$$\frac{d^2 t}{d\tau^2} = 0, \tag{40}$$

$$\frac{d^2 r}{d\tau^2} - \frac{b\lambda}{2r^2}\left(\frac{dt}{d\tau}\right)^2 = 0. \tag{41}$$

Combining the geodesic equations (40) and (41), we obtain that

$$\frac{d^2 r}{dt^2} = \frac{b\lambda}{2r^2}. \tag{42}$$

Comparing Eq. (42) with Newtonian gravity, we conclude that

$$b\lambda = -2GM, \tag{43}$$

where M denotes the total mass of the gravitational source.

D. Boundary conditions

The solution of the vacuum field equation $Ric = 0$ gives a specific form of the functions $B(r)$ and $A(r)$, and requires the two-dimensional subspace $\bar{F}$ to be a constant-curvature space. Two integral constants a and b and the specific form of the subspace $\bar{F}$ need to be determined by boundary conditions. These boundary conditions are given by physical requirements. In Sec. III C, we obtained $b = -2GM/\lambda$ using the Newtonian limit. In the following section, we will show that the interior solution is consistent with the exterior solution at the boundary of the gravitational source if $a = 1/\lambda$. The value of the constant curvature λ can be set to 1 by a redefinition of the curve parameter τ. Now one boundary condition is left to determine the specific form of the subspace $\bar{F}$.

In general relativity, the Schwarzschild metric returns to the Minkowski metric if $r \to \infty$. This means that the spacetime is Minkowski in the absence of gravity. The Finsler spacetime (27) in the absence of gravity is another physical boundary condition. If $r \to \infty$ or $M = 0$, the Finsler spacetime (27) reduces to Minkowski spacetime, and our solution for the vacuum field equation is simply the Schwarzschild solution. This fact implies that Finsler geometry is a generation of Riemannian geometry, and Finslerian gravity involves the physical contents of general relativity. If $r \to \infty$ or $M = 0$, the Finsler spacetime (27) violates Lorentz symmetry, and we get a Finslerian solution of the vacuum field equation. For example, $\bar{F}$ is of the form $F_{\rm FS}$ [Eq. (17)].

IV. INTERIOR SOLUTION

It is well known that Schwarzschild spacetime has an interior solution, which can be used to deduce the famous Oppenheimer-Volkoff equation. The interior behavior of Finsler spacetime [Eq. (27)] is worth investigating. However, as we mentioned in the Introduction, there are obstructions to constructing a gravitational field equation in Finsler spacetime. In this section, we will show that there is a self-consistent gravitational field equation in Finsler spacetime.

The notion of a Ricci tensor in Finsler geometry was first introduced by Akbar-Zadeh [53],

$$Ric_{\mu\nu} = \frac{\partial^2(\frac{1}{2}F^2 Ric)}{\partial y^\mu \partial y^\nu}, \tag{44}$$

and the scalar curvature in Finsler geometry is given as $S = g^{\mu\nu} Ric_{\mu\nu}$. Here, we define the modified Einstein tensor in Finsler spacetime

$$G_{\mu\nu} \equiv Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} S. \tag{45}$$

Plugging the equation for the Ricci scalar (34) into Eq. (45), and noticing that $\bar{F}$ is a two-dimensional Finsler spacetime with constant flag curvature λ, we obtain

$$G^t_t = \frac{A'}{rA^2} - \frac{1}{r^2 A} + \frac{\lambda}{r^2}, \tag{46}$$

$$G^r_r = -\frac{B'}{rAB} - \frac{1}{r^2 A} + \frac{\lambda}{r^2}, \tag{47}$$

$$G^\theta_\theta = G^\varphi_\varphi = -\frac{B''}{2AB} - \frac{B'}{2rAB} + \frac{A'}{2rA^2} + \frac{B'}{4AB}\left(\frac{A'}{A} + \frac{B'}{B}\right). \tag{48}$$

Next, we investigate the covariant conserved properties of the tensor G^μ_ν. The covariant derivative of G^μ_ν in Finsler spacetime is given as [4]

$$G^\mu_{\nu|\mu} = \frac{\delta}{\delta x^\mu} G^\mu_\nu + \Gamma^\mu_{\mu\rho} G^\rho_\nu - \Gamma^\rho_{\mu\nu} G^\mu_\rho, \tag{49}$$

where

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} - \frac{\partial G^\rho}{\partial y^\mu}\frac{\partial}{\partial y^\rho}, \tag{50}$$

and $\Gamma^\mu_{\mu\rho}$ is the Chern connection. Here, we have used "|" to denote the covariant derivative. The forms of the covariant derivative (49) and "δ" derivative (50) are well defined such that they transform as tensors under a coordinate change in Finsler spacetime [4]. The Chern connection can be expressed in terms of the geodesic spray coefficients G^μ and the Cartan connection $A_{\lambda\mu\nu} \equiv \frac{F}{4}\frac{\partial}{\partial y^\lambda}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial y^\nu}(F^2)$,

$$\Gamma^\rho_{\mu\nu} = \frac{\partial^2 G^\rho}{\partial y^\mu \partial y^\nu} - A^\rho_{\mu\nu|\kappa}\frac{y^\kappa}{F}. \tag{51}$$

By noticing that the modified Einstein tensor G^μ_ν only depends on r and does not have any y dependence, and that the Cartan tensor $A^\rho_{\mu\nu} = A^i_{jk}$ (indices i, j, k run over θ, φ), one can easily get that $G^\mu_{t|\mu} = G^\mu_{\theta|\mu} = G^\mu_{\varphi|\mu} = 0$. The proof of $G^\mu_{r|\mu} = 0$ is somewhat subtle. By making use of Eqs. (30), (32), and (33), we find from Eq. (51) and $A^\rho_{\mu\nu} = A^i_{jk}$ that

$$\Gamma^t_{rt} = \frac{B'}{2B}, \qquad \Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = \frac{1}{r}. \tag{52}$$

Then, after a tedious calculation, one can check that the equation $G^\mu_{r|\mu} = 0$ is indeed satisfied. Following the spirit of general relativity, we propose that the gravitational field equation in the given Finsler spacetime [Eq. (27)] should be of the form

$$G^\mu_\nu = 8\pi_F G T^\mu_\nu, \tag{53}$$

where T^μ_ν is the energy-momentum tensor. The volume of Finsler space [47] is generally different than that of Riemannian geometry. We have used $4\pi_F$ to denote the volume of $\bar{F}$ in Eq. (53). The boundary condition gives a specific form of the subspace $\bar{F}$. The volume of the surface of the subspace $\bar{F}$ can identify the value of π_F. For example, if we take $\bar{F}$ to be of the form $F_{\rm FS}$ [Eq. (17)], then $\pi_F = \pi$ according to the discussion in Sec. II B. The proposed form of the field equation (53) given here is not inconsistent with properties of the ansatz (27). The gravitational field equation (53) is valid for a specific Finsler spacetime (27). A general field equation that is valid for an arbitrary Finsler spacetime was proposed by Pfeifer and Wohlfarth [42], and Vacaru [39]. However, these field equations do not satisfy the requirement of covariant conservation, and they do not force the energy-momentum tensor to be constructed on the tangent bundle.

For simplicity, we set the energy-momentum tensor to be of the form

$$T^\mu_\nu = {\rm diag}(\rho(r), -p(r), -p(r), -p(r)), \tag{54}$$

where $\rho(r)$ and $p(r)$ are the energy density and pressure of the gravitational source, respectively. Then, by making use of Eqs. (46)–(48), we reduce the gravitational field equation to three independent equations,

$$\frac{2p'}{\rho+p}=-\frac{B'}{B},\tag{55}$$

$$\frac{A'}{rA^2}-\frac{1}{r^2A}+\frac{\lambda}{r^2}=8\pi_F G\rho,\tag{56}$$

$$\frac{B'}{rAB}+\frac{1}{r^2A}-\frac{\lambda}{r^2}=8\pi_F Gp.\tag{57}$$

The solution of Eq. (56) is given as

$$A^{-1}=\lambda-\frac{2Gm(r)}{r},\tag{58}$$

where $m(r)\equiv\int_0^r 4\pi_F x^2\rho(x)dx$. By making use of Eq. (58), and plugging Eq. (57) into Eq. (55), we obtain that

$$-r^2p'=(\rho+p)(4\pi_F Gpr^3+Gm)\left(\lambda-\frac{2Gm}{r}\right)^{-1}.\tag{59}$$

Equation (59) reduces to the famous Oppenheimer-Volkoff equation if the Finsler spacetime $\bar{F}$ reduces to the two-dimensional Riemann sphere. By combining the modified Oppenheimer-Volkoff equation (59) with the equation of state, one can obtain the interior structure of the gravitational source.

The interior solution (58) should be consistent with the exterior solution (39) at the boundary of the gravitational source. Therefore, we get

$$a\lambda=1.\tag{60}$$

At last, combining the boundary condition (60) with the requirement of the Newtonian limit [Eq. (43)], we get the exterior solutions $B(r)$ and $A(r)$ as

$$B(r)=1-\frac{2GM}{\lambda r},\tag{61}$$

$$A(r)=\left(\lambda-\frac{2GM}{r}\right)^{-1}.\tag{62}$$

V. EXPERIMENTAL CONSTRAINTS ON FINSLERIAN GRAVITY

A. The motion of particles

Plugging the equations for the geodesic spray coefficients [Eqs. (30)–(33)] into the geodesic equation (12), we obtain the geodesic equation of Finsler spacetime [Eq. (27)],

$$0=\frac{d^2t}{d\tau^2}+\frac{B'}{B}\frac{dr}{d\tau}\frac{dt}{d\tau},\tag{63}$$

$$0=\frac{d^2r}{d\tau^2}+\frac{B'}{2A}\left(\frac{dt}{d\tau}\right)^2+\frac{A'}{2A}\left(\frac{dr}{d\tau}\right)^2-\frac{r}{A}\bar{F}^2\left(\frac{d\theta}{d\tau},\frac{d\varphi}{d\tau}\right),\tag{64}$$

$$0=\frac{d^2\theta}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}+2\bar{G}^\theta,\tag{65}$$

$$0=\frac{d^2\varphi}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\varphi}{d\tau}+2\bar{G}^\varphi.\tag{66}$$

The solution of Eq. (63) is

$$B\frac{dt}{d\tau}=1,\tag{67}$$

where we have set the integral constant to be 1 by the normalization of τ. Noticing that y^μ is equal to $\frac{dx^\mu}{d\tau}$ along the geodesic, and by making use of Eqs. (65) and (66), we find that

$$\begin{aligned}\frac{d\bar{F}}{d\tau}&=\frac{\partial\bar{F}}{\partial x^i}\frac{dx^i}{d\tau}+\frac{\partial\bar{F}}{\partial y^i}\frac{dy^i}{d\tau}\\&=y^i\left(\frac{\partial\bar{F}}{\partial x^i}-\frac{2\bar{G}_i}{\bar{F}}\right)-2y^r\bar{F}=-\bar{F}\frac{d\ln r^2}{d\tau},\end{aligned}\tag{68}$$

where $\bar{G}_i=\bar{g}_{ij}\bar{G}^j$ (i,j run over θ,φ), and we have used the fact that $\bar{F}$ is a homogenous function of y of degree 1 to derive the third equation of Eq. (68). The solution of Eq. (68) is given as

$$r^2\bar{F}=J,\tag{69}$$

where J is an integral constant. By making use of Eqs. (67) and (69), we find from the geodesic equation (64) that

$$\frac{d}{d\tau}\left(A\left(\frac{dr}{d\tau}\right)^2+\frac{J^2}{r^2}-\frac{1}{B}\right)=0.\tag{70}$$

The solution of Eq. (70) is given as

$$A\left(\frac{dr}{d\tau}\right)^2+\frac{J^2}{r^2}-\frac{1}{B}=A\left(\frac{dr}{d\tau}\right)^2+r^2\bar{F}^2-B\left(\frac{dt}{d\tau}\right)^2=-F^2,\tag{71}$$

where we have used Eqs. (67) and (69) to derive the second equation of Eq. (71). Equation (71) means that F is constant along the geodesic.

Now, we have three solutions [Eqs. (67), (69), and (71)] of the geodesic equations; the fourth one depends on the explicit form of the two-dimensional Finsler space $\bar{F}$. However, we can still find some information about particle motion from the obtained solutions. Consider a particle move along the radial direction: by combining Eq. (67) with Eq. (69), and by making use of the exterior solutions (61) and (62) of $B(r)$ and $A(r)$, we obtain that

$$\frac{dr}{dt} = \sqrt{\lambda^{-1} - F^2\left(1 - \frac{2GM}{\lambda r}\right)}\left(1 - \frac{2GM}{\lambda r}\right). \tag{72}$$

It is obvious from Eq. (72) that $\frac{dr}{dt} \to 0$ while $r \to 2GM/\lambda$. The modified Schwarzschild radius in Finsler spacetime is

$$r_s = \frac{2GM}{\lambda}. \tag{73}$$

B. Classical tests

The predictions of general relativity have been proven by four classical tests [2]. If our spacetime is Finslerian, then it is necessary to test the validity of Finslerian gravity. In this subsection, our discussion is based on the following Finsler structure:

$$F^2 = \left(1 - \frac{2GM}{r}\right)y^t y^t - \left(1 - \frac{2GM}{r}\right)^{-1} y^r y^r - r^2 F_{\mathrm{FS}}^2, \tag{74}$$

where the "Finslerian sphere" F_{FS} is of the form of Eq. (17). The difference between the Finslerian solution (74) and the Schwarzschild solution lies on the "Finslerian sphere" F_{FS}. Two of the four classical tests—namely, radar echo delay and gravitational redshift—are related to the radial motion of a particle. One can also find from Eq. (72) that $\frac{dr}{dt}$ is the same as that in Schwarzschild spacetime when $\lambda = 1$.

It is expected that the motion of a particle in a bounded or unbounded orbit in Finsler spacetime is different from that in Schwarzschild spacetime. Since the "Finslerian sphere" is nonreversible for $\varphi \to -\varphi$, it implies that particle motion along a given direction (φ) is different from its counterpart ($-\varphi$) in orbital motion. It is convenient to consider the orbit of a particle confined to the equatorial plane $\theta = \pi/2$. Then, by making use of the metric of the "Finslerian sphere" [Eq. (17)], Eq. (69) simplifies as

$$r^2 \frac{d\varphi}{d\tau} = J_\pm, \tag{75}$$

where $J_\pm \equiv (1 \pm \epsilon)J$, J_+ corresponds to a given direction and J_- corresponds to its counterpart. Plugging Eq. (75) into the solution of the geodesic equation (71), we get the equation for orbital motion,

$$(1 \pm \epsilon)^2 \left(\frac{dr}{d\varphi}\right)^2 = \frac{r^4}{J^2} - \left(1 - \frac{2GM}{r}\right)\left(\frac{F^2 r^4}{J^2} + r^2\right). \tag{76}$$

One should notice that F is constant. By reparametrizing the curve parameter τ, one can set F equal to 1 and 0 for massive and massless particles, respectively. The solution of the orbital equation (76) gives the deflection angle and precession of the orbit per revolution for unbounded and bounded orbits, respectively. Note that the orbital equation is the same as that in Schwarzschild spacetime if we transform $(1 \pm \epsilon)\varphi$ into φ; thus—recalling the results in Schwarzschild spacetime—in Finsler spacetime [Eq. (74)] we obtain the deflection angle for the gravitational deflection of light,

$$\delta\alpha = (1 \pm \epsilon)\frac{4GM}{\xi}, \tag{77}$$

where ξ is the distance of closest approach, and the precession of the orbit per revolution is given as

$$\delta\varphi = (1 \pm \epsilon)6\pi\frac{GM}{L}, \tag{78}$$

where L is the semilatus rectum of the orbit.

The observations of very-long-baseline radio interferometry [54,55] give constraints on the gravitational deflection of light in the Solar System. Its results yield a constraint on the Finslerian parameter ϵ. It is given as $\epsilon \sim 2 \times 10^{-4}$. The observations of the perihelion shift of Mercury [55] also give a constraint on the Finslerian parameter ϵ. Its result yields $\epsilon < 3 \times 10^{-3}$. The recent Michelson-Morley experiment carried out by Müller *et al.* [56] gives a precise limit on Lorentz invariance violation. Their experiment showed that the change of resonance frequencies of the optical resonators is of the magnitude $|\frac{\delta\omega}{\omega}| \sim 10^{-16}$. This implies that the Finslerian parameter ϵ should be less than 10^{-16}.

VI. COUNTERPART OF BIRKHOFF'S THEOREM

In general relativity, Birkhoff's theorem guarantees that the solution of the vacuum field equation with spherical symmetry must be static. This means that its exterior solution must be the Schwarzschild metric regardless of the evolution of the gravitational source. It is necessary to investigate such an issue in Finsler spacetime. In Riemannian geometry, the spherical symmetry can be represented by a Riemann sphere. Its metric is of the form of Eq. (16). The counterpart of the Riemann sphere in Finsler geometry is the "Finslerian sphere" (17). In the above discussion, we gave a static solution of the Finslerian vacuum field equation with the symmetry of the "Finslerian sphere." Now, we turn to investigate the time-dependent Finsler spacetime with the symmetry of the "Finslerian sphere." Its Finsler structure is given as

$$F^2 = B(r,t)y^t y^t - A(r,t)y^r y^r - r^2 F_{\mathrm{FS}}^2. \tag{79}$$

Plugging the Finsler structure (79) into Eq. (13), we obtain that

$$G^t = \frac{B'}{2B} y^t y^r + \frac{\dot{B}}{4B} y^t y^t + \frac{\dot{A}}{4B} y^r y^r, \tag{80}$$

$$G^r = \frac{A'}{4A} y^r y^r + \frac{B'}{4A} y^t y^t - \frac{r}{2A} F_{\rm FS}^2 + \frac{\dot{A}}{2A} y^t y^r, \tag{81}$$

$$G^\theta = \frac{1}{r} y^\theta y^r + G_{\rm FS}^\theta, \tag{82}$$

$$G^\varphi = \frac{1}{r} y^\varphi y^r + G_{\rm FS}^\varphi, \tag{83}$$

where a dot denotes a derivative with respect to t, and $\bar{G}$ are the geodesic spray coefficients derived from $F_{\rm FS}$. Plugging the geodesic coefficients (80)–(83) into the formula for the Ricci scalar [Eq. (24)], we obtain

$$F^2 Ric = \left[\frac{B''}{2A} - \frac{B'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{B'}{rA} - \frac{\ddot{A}}{2A} + \frac{\dot{A}^2}{4A^2} + \frac{\dot{A}\dot{B}}{4AB}\right] y^t y^t + \left[-\frac{B''}{2B} + \frac{B'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) + \frac{A'}{rA} + \frac{\ddot{A}}{2B} - \frac{\dot{A}\dot{B}}{4B^2} - \frac{\dot{A}^2}{4AB}\right] y^r y^r + \frac{2\dot{A}}{rA} y^t y^r + \left[1 - \frac{1}{A} + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right)\right] F_{\rm FS}^2. \tag{84}$$

The vacuum field equation $Ric = 0$ means that $\frac{2\dot{A}}{rA} = 0$. This tells us that A is time independent. This fact shows that all time derivatives drop out of Eq. (84), and it becomes identical with that in the static case. Following the discussion for the static case, we obtain that

$$B = f(t)\left(1 - \frac{2GM}{r}\right), \tag{85}$$

$$A = \left(1 - \frac{2GM}{r}\right)^{-1}. \tag{86}$$

The function $f(t)$ can be made equal to 1 by defining a new time coordinate,

$$t' = \int^t \sqrt{f(t)} dt. \tag{87}$$

Now, the Finsler structure (79) is entirely time independent and is identical with the static solution (74). Thus, the counterpart of Birkhoff's theorem exists in Finslerian gravity. Unlike the requirement of spherical symmetry in general relativity, this shows that the Finslerian gravitational field with the symmetry of the "Finslerian sphere" in vacuum must be static, and its metric is of the form of Eq. (74).

VII. CONCLUSIONS AND REMARKS

In view of the geodesic deviation equation, the vacuum field equation $Ric = 0$ in Finsler spacetime implies that the geodesic rays are parallel to one another. The geometry-invariant nature of the Ricci scalar implies that the vacuum field equation is insensitive to the connection, which is an essential physical requirement. Starting from the ansatz (27), we have found an exact solution of the vacuum field equation [Eqs. (38) and (39)].

A general gravitational field equation in Finsler spacetime is still to be completed. However, we have found that the proposed form of the field equation (53) given here is not inconsistent with the properties of the ansatz (27). We have also proven that the Finslerian covariant derivative of the geometrical part of the gravitational field equation is conserved. It is obvious that the gravitational field equation (53) returns to the vacuum field equation when the energy-momentum tensor vanishes. We have found an interior solution of the gravitational field equation (53). The interior solution (58) is consistent with the exterior solution (39) at the boundary of the gravitational source, and we required that the exterior solution should return to Newtonian gravity. The two boundary conditions constrain the exterior solution to be of the same form as Eqs. (61) and (62). One should notice that the Schwarzschild metric is also a solution of $Ric = 0$. There is a boundary condition which distinguishes the Finslerian solution from the Schwarzschild solution, namely, the violation of Lorentz symmetry when the Finsler spacetime (27) has no gravitational source, $M = 0$. For example, we make the subspace $\bar{F}$ to be a "Finslerian sphere" $F_{\rm FS}$; then, the exterior metric of the vacuum field solution is given as

$$F^2 = \left(1 - \frac{2GM}{r}\right) y^t y^t - \left(1 - \frac{2GM}{r}\right)^{-1} y^r y^r - r^2 \left(\frac{\sqrt{(1 - \epsilon^2 \sin^2\theta) y^\theta y^\theta + \sin^2\theta y^\varphi y^\varphi} - \epsilon \sin^2\theta y^\varphi}{1 - \epsilon^2 \sin^2\theta}\right)^2. \tag{88}$$

The metric (88) is none other than the Schwarzschild metric except for the change from the Riemann sphere to the "Finslerian sphere" (17). We have presented three solutions [Eqs. (67), (69), and (71)] of the geodesic equations of the metric (88). The fourth one depends on the geodesic equation of the "Finslerian sphere" (17). The geometrical properties of the "Finslerian sphere" (17) are as follows: it is nonreversible for $\varphi \to -\varphi$; it has two closed geodesics located at $\theta = \frac{\pi}{2}$ with length $L_\pm = 2\pi(1 \pm \epsilon)^{-1}$; its volume is the surface volume of the unit "Finslerian sphere," equal to 4π; and it only has one independent Killing vector, $V^{\varphi}=$ constant.

We have investigated the motion of particles in Finsler spacetime. The solution of the geodesic equations are shown. Taking Finsler spacetime (74) as a example, we have shown the experimental constraints on the Finslerian parameter ϵ. In the Solar System, celestial observations require $\epsilon < 10^{-4}$, and the recent Michelson-Morley experiment requires $\epsilon < 10^{-16}$. It is expected that Finslerian spacetime may have an unavoidable effect on cosmological scales. In general relativity, as a special case of gravitational lensing, an Einstein ring has a symmetric structure. However, if the spacetime is Finslerian, one may observe an Einstein ring with an asymmetric structure.

The counterpart of Birkhoff's theorem exists in Finslerian vacuum. This shows that the Finslerian gravitational field with the symmetry of the "Finslerian sphere" in vacuum must be static, and its metric is of the form of Eq. (74).

The Schwarzschild spacetime will return to Minkowski spacetime if there is no gravitational source. As for the Finslerian vacuum spacetime (88), if there is no gravitational source—namely, $M=0$—the metric reduces to

$$F^2 = y^t y^t - y^r y^r - r^2\left(\frac{\sqrt{(1-\epsilon^2\sin^2\theta)y^\theta y^\theta + \sin^2\theta y^\varphi y^\varphi} - \epsilon\sin^2\theta y^\varphi}{1-\epsilon^2\sin^2\theta}\right)^2. \tag{89}$$

According to Eq. (34), the Ricci scalar or Ricci tensor of the metric (89) is equal to 0. This fact holds even for a three-dimensional subspace of the metric (89). However, the metric (89) or its spatial part is not a flat Finsler spacetime. The Finsler spacetime is a flat one [4] if and only if $Ric=0$ and the geodesic spray coefficients G^μ are quadratic in y. This fact is quite different than in the case of Riemannian geometry. It is well known that three-dimensional Riemannian space is flat, while its Ricci tensor equal to 0. Nevertheless, even the spacial part of the metric (89) is not a flat Finsler space.

In Riemannian spacetime without torsion, at any fixed point, one can erect a local coordinate system such that the metric is Minkowskian. One necessary condition of this statement is that the Riemannian metric is quadric. However, this necessary condition does not hold in a general Finsler metric. Therefore, Finsler spacetime is not locally isometric to Minkowski spacetime. One result of this is that the speed of light is not locally isotropic. The propagation of light obeys $F=0$. One can find from the local metric that the radial speed of light is equal to 1, and the nonradial speed of light satisfies

$$c_\theta^2 + (c_\varphi - \epsilon\sin\theta)^2 = 1, \tag{90}$$

where $c_\theta \equiv \frac{d\theta}{dt}$ and $c_\varphi \equiv \frac{d\varphi}{dt}\sin\theta$.

The Schwarzschild radius forms an event horizon in Schwarzschild spacetime. The Schwarzschild solution can be maximally extended by Kruskal extension. The coordinate transformation between the Schwarzschild metric and the Kruskal metric is only related to r and t. Therefore, one can also get a maximally extended Finslerian vacuum solution [Eq. (88)] by Kruskal extension.

ACKNOWLEDGMENTS

We would like to thank Prof. C.-G. Huang for useful discussions. Projects No. 11375203 and No. 11305181 are supported by the NSFC.

[1] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
[2] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, New York, 1972).
[3] H. Y. Shan *et al.*, Mon. Not. R. Astron. Soc. **406**, 1134 (2010).
[4] D. Bao, S. S. Chern, and Z. Shen, *An Introduction to Riemann–Finsler Geometry* (Springer, New York, 2000).
[5] S. Deng and Z. Hou, Pac. J. Math. **207**, 149 (2002).
[6] X. Li and Z. Chang, Differential Geometry and its Applications **30**, 737 (2012).
[7] H. C. Wang, J. Lond. Math. Soc. **s1-22**, 5 (1947).
[8] C. Pfeifer and M. N. R. Wohlfarth, Phys. Rev. D **84**, 044039 (2011).
[9] G. Amelino-Camelia, Phys. Lett. B **510**, 255 (2001).
[10] G. Amelino-Camelia, Int. J. Mod. Phys. D **11**, 35 (2002).
[11] G. Amelino-Camelia, Nature (London) **418**, 34 (2002).
[12] S. R. Coleman and S. L. Glashow, Phys. Lett. B **405**, 249 (1997); Phys. Rev. D **59**, 116008 (1999); A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. **97**, 021601 (2006).
[13] G. W. Gibbons, J. Gomis, and C. N. Pope, Phys. Rev. D **76**, 081701 (2007).
[14] V. A. Kostelecky, Phys. Lett. B **701**, 137 (2011); V. A. Kostelecky, N. Russell, and R. Tsoc, Phys. Lett. B **716**, 470 (2012).
[15] V. Sahni, Classical Quantum Gravity **19**, 3435 (2002).
[16] T. Padmanabhan, Phys. Rep. **380**, 235 (2003).
[17] E. Komatsu *et al.* (WMAP Collaboration), Astrophys. J. Suppl. Ser. **192**, 18 (2011).

[18] Planck Collaboration, arXiv:1303.5062 [Astron. Astrophys. (to be published)].
[19] N. Suzuki *et al.*, Astrophys. J. **746**, 85 (2012).
[20] L. Perivolaropoulos, arXiv:1104.0539.
[21] I. Antoniou and L. Perivolaropoulos, J. Cosmol. Astropart. Phys. 12 (2010) 012.
[22] R. Watkins, H. A. Feldman, and M. J. Hudson, Mon. Not. R. Astron. Soc. **392**, 743 (2009).
[23] G. Hinshaw *et al.*, Astrophys. J. Suppl. Ser. **180**, 225 (2009).
[24] Planck Collaboration, arXiv:1303.5083 [Astron. Astrophys. (to be published)].
[25] T. Koivisto and D. F. Mota, Phys. Rev. D **73**, 083502 (2006).
[26] S. Alexander, T. Biswas, A. Notari, and D. Vaid, J. Cosmol. Astropart. Phys. 09 (2009) 025.
[27] J. Garcia-Bellido and T. Haugboelle, J. Cosmol. Astropart. Phys. 04 (2008) 003.
[28] E. Akofor, A. P. Balachandran, S. G. Jo, A. Joseph, and B. A. Qureshi., J. High Energy Phys. 05 (2008) 092.
[29] T. S. Koivisto, D. F. Mota, M. Quartin, and T. G. Zlosnik, Phys. Rev. D **83**, 023509 (2011).
[30] Z. Chang, M.-H. Li, X. Li, and S. Wang, Eur. Phys. J. C **73**, 2459 (2013).
[31] J. K. Webb, J. A. King, M. T. Murphy, V. V. Flambaum, R. F. Carswell, and M. B. Bainbridge, Phys. Rev. Lett. **107**, 191101 (2011); J. A. King, J. K. Webb, M. T. Murphy, V. V. Flambaum, R. F. Carswell, M. B. Bainbridge, M. R. Wilczynska, and F. Elliott Koch, Mon. Not. R. Astron. Soc. **422**, 3370 (2012).
[32] A. Mariano and L. Perivolaropoulos, Phys. Rev. D, **86**, 083517 (2012).
[33] Z. Chang, M.-H. Li, and S. Wang, Phys. Lett. B **723**, 257 (2013).
[34] Z. Chang and S. Wang, Eur. Phys. J. C **73**, 2516 (2013).
[35] Z. Chang, X. Li, and S. Wang, arXiv:1307.4542.
[36] X. Li and Z. Chang, Chin. Phys. C **34**, 28 (2010).
[37] R. Miron and M. Anastasiei, *The Geometry of Lagrange Spaces: Theory and Applications* (Kluwer, Alphen aan den Rijn, Netherlands, 1994).
[38] S. F. Rutz, Gen. Relativ. Gravit. **25**, 1139 (1993).
[39] S. Vacaru, P. Stavrinos, E. Gaburov, and D. Gonta, arXiv:gr-qc/0508023.
[40] A. P. Kouretsis, M. Stathakopoulos, and P. C. Stavrinos, Phys. Rev. D **79**, 104011 (2009); **82**, 064035 (2010).
[41] S. Vacaru, Nucl. Phys. **B494**, 590 (1997); S. Vacaru, D. Singleton, V. A. Botan, and D. A. Dotenco, Phys. Lett. B **519**, 249 (2001).
[42] C. Pfeifer and M. N. R. Wohlfarth, Phys. Rev. D **85**, 064009 (2012).
[43] X. Li, M.-H. Li, H.-N. Lin, and Z. Chang, Mon. Not. R. Astron. Soc. **428**, 2939 (2013).
[44] G. Randers, Phys. Rev. **59**, 195 (1941).
[45] D. Bao, C. Robles, and Z. Shen, J. Diff. Geom. **66**, 377 (2004).
[46] H. B. Rademacher, Math. Ann. **328**, 373 (2004).
[47] Z. Shen, *Lectures on Finsler Geometry* (World Scientific, Singapore, 2001).
[48] A. Trautman, F. A. E. Pirani, and H. Bondi, *Lectures on General Relativity: Brandeis Summer Institute in Theoretical Physics*, Vol. 1 (Prentice-Hall, Englewood Cliffs, NJ, 1965).
[49] S. F. Rutz, Comput. Phys. Commun. **115**, 300 (1998).
[50] Z. Chang and X. Li, Phys. Lett. B **668**, 453 (2008).
[51] X. Li and Z. Chang, Commun. Theor. Phys. **60**, 535 (2013).
[52] X. Li and Z. Chang, Chin. Phys. C **37**, 123103 (2013).
[53] H. Akbar-Zadeh, Acad. Roy. Belg. Bull. Cl. Sci. **74**, 281 (1988).
[54] I. H. Stairs *et al.*, Astrophys. J. **632**, 1060 (2005).
[55] C. M. Will, Living Rev. Relativity **9**, 3 (2005).
[56] H. Müller, P. Stanwix, M. Tobar, E. Ivanov, P. Wolf, S. Herrmann, A. Senger, E. Kovalchuk, and A. Peters, Phys. Rev. Lett. **99**, 050401 (2007).