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Gravitational collapse of generalized Vaidya spacetime

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We study the gravitational collapse of a generalized Vaidya spacetime in the context of the cosmic censorship hypothesis. We develop a general mathematical framework to study the conditions on the mass function so that future directed nonspacelike geodesics can terminate at the singularity in the past. Thus our result generalizes earlier works on gravitational collapse of the combinations of Type-I and Type-II matter fields. Our analysis shows transparently that there exist classes of generalized Vaidya mass functions for which the collapse terminates with a locally naked central singularity. We calculate the strength of these singularities to show that they are strong curvature singularities and there can be no extension of spacetime through them.

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I. INTRODUCTION

The Vaidya spacetime [1], also known as the radiating Schwarzschild spacetime, describes the geometry outside a radiating spherically symmetric star. The radiation effects are important in the later stages of gravitational collapse of a star, when a considerable amount of energy in the form of photons or neutrinos is ejected from the star. This makes the collapsing star to be surrounded by an ever expanding zone of radiation. If we treat the complete nonstatic configuration of the radiating star and the zone of radiation as an isolated system within an asymptotically flat universe, then beyond the expanding zone of radiation the spacetime may be described by the Schwarszchild solution. The Vaidya solution is of Petrov type D and possesses a normal shear-free null congruence with nonzero expansion. In terms of exploding (imploding) null coordinates the metric is given as

$$ds^2 = -\left[1 - \frac{2m(v)}{r}\right] dv^2 + 2\epsilon dv dr + r^2 d\Omega^2, \qquad (1)$$

where $\epsilon = \pm 1$ describes incoming (outgoing) radiation shells, respectively, the function $m(v)$ is the mass function and $d\Omega^2$ describes the line element on the 2-sphere.

One of the earliest counterexamples of cosmic censorship conjecture (CCC), with a reasonable matter field satisfying physically reasonable energy conditions, was found in the shell focusing singularity formed by imploding shells of radiation in the Vaidya-Papapetrou model [2,3]. In this case, radially injected radiation flows into an initially flat and empty region, and is focused into a central singularity of growing mass. It was shown that the central singularity at $(v = 0, r = 0)$ becomes a node with a definite tangent for families of nonspacelike geodesics, for a nonzero measure of parameters in the model. Hence the singularity at $(v = 0,\ r = 0)$ is naked in the sense that families of future directed nonspacelike geodesics going to future null infinity terminate at the singularity in the past. For a detailed discussion on the censorship violation in radiation collapse we refer to [4].

The generalization of the Vaidya solution, also known as the generalized Vaidya spacetime, that includes all the known solutions of Einstein's field equations with a combination of Type-I and Type-II matter fields, was given by Wang and Wu [5]. This generalization comes from the observation that the energy momentum tensor for these matter fields are linear in terms of the mass function. As a result, the linear superposition of particular solutions is also a solution to the field equations. Hence, by superposition we can explicitly construct solutions such as the monopole–de Sitter–charged Vaidya solution and the Husain solution. Generalized Vaidya spacetimes are also widely used in describing the formation of regular black holes [6], dynamical black holes [7] and black holes with closed trapped regions [8]. The generalized Vaidya model can be matched to a heat conducting interior of a radiating star as recently shown by [9]. Also, generalized Vaidya spacetime emerges naturally while solving many other astrophysical and cosmological scenarios [10,11].

The main goal of this paper is to study the gravitational collapse of generalized Vaidya spacetimes in the context of CCC. We develop a general mathematical framework to study the conditions on the mass function so that future directed nonspacelike geodesics can terminate at the singularity in the past. Thus our result generalizes the earlier works on gravitational collapse of Type-II matter fields and also shows transparently that there exist classes of the generalized Vaidya mass function (of nonzero

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measure) for which the collapse terminates with a locally naked singularity. We also calculate the strength of the naked singularities to show that they are strong curvature singularities and there is no extension of spacetime through these singularities.

The paper is organized as follows: In the next section we discuss the generalized Vaidya solution. In Sec. III we consider a combination of Type I and Type II matter fields undergoing gravitational collapse to a spacetime singularity. In Sec. IV we analyze the central singularity (at $v = 0$, $r = 0$) to find out the conditions on the mass function such that the singularity is a node with a definite tangent for nonspacelike geodesics. In Sec. V we discuss the strength of the naked central singularity. Finally in the last section we apply our result to some well known collapse models to recover the conditions for CCC violation.

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) throughout this paper, and Latin indices run from 0 to 3. The symbol ∇ represents the usual covariant derivative and ∂ corresponds to partial differentiation. We use the $(-,+,+,+)$ signature, and the Riemann tensor is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ce} - \Gamma^e{}_{bc}\Gamma^a{}_{de}, \tag{2}$$

where the $\Gamma^a{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices) defined by

$$\Gamma^a{}_{bd} = \frac{1}{2} g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \tag{3}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices

$$R_{ab} = g^{cd} R_{cadb}. \tag{4}$$

The Hilbert-Einstein action in the presence of matter is given by

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}[R - 2\Lambda - 2\mathcal{L}_m], \tag{5}$$

variation of which gives the Einstein field equations

$$G_{ab} + \Lambda g_{ab} = T_{ab}. \tag{6}$$

II. GENERALIZED VAIDYA SPACETIME

We know that the most general spherically symmetric line element for an arbitrary combination of Type-I matter fields (whose energy momentum tensor has one timelike and three spacelike eigenvectors) and Type-II matter fields (whose energy momentum tensor has double null eigenvectors) is given by [12]

$$ds^2 = -e^{2\psi(v,r)}\left[1 - \frac{2m(v,r)}{r}\right]dv^2 + 2\epsilon e^{\psi(v,r)} dv dr + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (\epsilon = \pm 1). \tag{7}$$

Here $m(v,r)$ is the mass function related to the gravitational energy within a given radius r [13]. When $\epsilon = +1$, the null coordinate v represents the Eddington advanced time, where r is decreasing toward the future along a ray $v = \text{const}$ and depicts ingoing null congruence while $\epsilon = -1$ depicts an outgoing null congruence.

The specific combination of matter fields that makes $\psi(v,r) = 0$ gives the generalized Vaidya geometry. In this paper, as we are considering a collapse scenario, we take $\epsilon = +1$. Particularly, we consider a line element of the form

$$ds^2 = -\left(1 - \frac{2m(v,r)}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{8}$$

Using the following definitions:

$$\dot{m}(v,r) \equiv \frac{\partial m(v,r)}{\partial v}, \qquad m'(v,r) \equiv \frac{\partial m(v,r)}{\partial r}, \tag{9}$$

the nonvanishing components of the Ricci tensor can be written as

$$R^v_v = R^r_r = \frac{m''(v,r)}{r}, \tag{10a}$$

$$R^\theta_\theta = R^\phi_\phi = \frac{2m'(v,r)}{r^2}, \tag{10b}$$

while the Ricci scalar is given by

$$R = \frac{2m''(v,r)}{r} + \frac{4m'(v,r)}{r^2}. \tag{11}$$

The nonvanishing components of the Einstein tensor can be written as

$$G^v_v = G^r_r = -\frac{2m'(v,r)}{r^2}, \tag{12a}$$

$$G^r_v = \frac{2\dot{m}(v,r)}{r^2}, \tag{12b}$$

$$G^\theta_\theta = G^\phi_\phi = -\frac{m''(v,r)}{r}. \tag{12c}$$

Using the Einstein field equations, the corresponding energy momentum tensor can be written in the form [5,14]

$$T_{\mu\nu} = T^{(n)}_{\mu\nu} + T^{(m)}_{\mu\nu}, \tag{13}$$

where

$$T^{(n)}_{\mu\nu} = \mu l_\mu l_\nu, \tag{14a}$$

$$T^{(m)}_{\mu\nu} = (\rho + \varrho)(l_\mu n_\nu + l_\nu n_\mu) + \varrho g_{\mu\nu}, \tag{14b}$$

and

$$\mu = \frac{2\dot{m}(v,r)}{r^2}, \quad \rho = \frac{2m'(v,r)}{r^2}, \quad \varrho = -\frac{m''(v,r)}{r}. \tag{15}$$

In the above l_μ and n_μ are two null vectors

$$l_\mu = \delta^0_\mu, \qquad n_\mu = \frac{1}{2}\left[1 - \frac{2m(v,r)}{r}\right]\delta^0_\mu - \delta^1_\mu, \tag{16}$$

where $l_\mu l^\mu = n_\mu n^\mu = 0$ and $l_\mu n^\mu = -1$.

Equation (13) can be considered as the energy momentum tensor of the generalized Vaidya solution, where the component $T^{(n)}_{\mu\nu}$ is the matter field that moves along the null hypersurfaces $v = \text{const}$, while $T^{(m)}_{\mu\nu}$ describes the matter moving along timelike trajectories. When $\rho = \varrho = 0$, the solution reduces to the Vaidya solution with $m = m(v)$.

If the energy momentum tensor of Eq. (13) is projected to the orthonormal basis, defined by the four vectors,

$$E^\mu_{(0)} = \frac{l_\mu + n_\mu}{\sqrt{2}}, \qquad E^\mu_{(1)} = \frac{l_\mu - n_\mu}{\sqrt{2}},$$
$$E^\mu_{(2)} = \frac{1}{r}\delta^\mu_2, \qquad E^\mu_{(3)} = \frac{1}{r\sin\theta}\delta^\mu_3, \tag{17}$$

it can be shown that [5]

$$[T_{(\mu)(\nu)}] = \begin{bmatrix} \frac{\mu}{2}+\rho & \frac{\mu}{2} & 0 & 0 \\ \frac{\mu}{2} & \frac{\mu}{2}-\rho & 0 & 0 \\ 0 & 0 & \varrho & 0 \\ 0 & 0 & 0 & \varrho \end{bmatrix}. \tag{18}$$

This form of the energy momentum is a combination of Type-I and Type-II fluids as defined in [15], with the following energy conditions:

(a) *The weak and strong energy conditions*

$$\mu \geq 0, \qquad \rho \geq 0, \qquad \varrho \geq 0, \qquad (\mu \neq 0). \tag{19}$$

(b) *The dominant energy conditions*

$$\mu \geq 0, \qquad \rho \geq \varrho \geq 0, \qquad (\mu \neq 0). \tag{20}$$

These energy conditions can be satisfied by choosing the mass function $m(v,r)$ suitably. In particular, when $m = m(v)$, all the energy conditions (weak, strong, and dominant) reduce to $\mu \geq 0$, while when $m = m(r)$ we have $\mu = 0$, and the matter field degenerates to a Type-I fluid with the usual energy conditions [15].

III. COLLAPSING MODEL

In this section, we examine the gravitational collapse of imploding radiation and matter described by the generalized Vaidya spacetime. In this situation, a thick shell of radiation and Type-I matter collapses at the center of symmetry [4].

If K^μ is the tangent to nonspacelike geodesics with $K^\mu = \frac{dx^\mu}{dk}$, where k is the affine parameter, then $K^\mu_{;\nu}K^\nu = 0$ and

$$g_{\mu\nu}K^\mu K^\nu = \beta, \tag{21}$$

where β is a constant that characterizes different classes of geodesics. $\beta = 0$ describes null geodesics, while $\beta < 0$ applies to timelike geodesics. The equations for dK^v/dk and dK^r/dk are calculated from the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\alpha} - \frac{d}{dk}\left(\frac{\partial L}{\partial \dot{x^\alpha}}\right) = 0, \tag{22}$$

with the Lagrangian

$$L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu, \tag{23}$$

where the dot is a derivative with respect to the affine parameter k. These equations are given by

$$\frac{dK^v}{dk} + \left(\frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}\right)(K^v)^2 - \frac{\ell^2}{r^3} = 0, \tag{24a}$$

$$\frac{dK^r}{dk} + \frac{\dot{m}(v,r)}{r}(K^v)^2 - \frac{\ell^2}{r^3}\left(1 - \frac{2m(v,r)}{r}\right) - \beta\left(\frac{m(v,r)}{r^2} - \frac{m'(v,r)}{r}\right) = 0. \tag{24b}$$

The components K^θ and K^ϕ of the tangent vector are given by [4]

$$K^\theta = \frac{\ell\cos\varphi}{r^2\sin^2\theta}, \tag{25a}$$

$$K^\phi = \frac{\ell\sin\varphi\cos\phi}{r^2}, \tag{25b}$$

where ℓ is the impact parameter and φ is the isotropy parameter defined by the relation $\sin\phi\tan\varphi = \cot\theta$.

If we follow [3] and write K^v as

$$K^v = \frac{P}{r}, \tag{26}$$

where $P = P(v,r)$ is an arbitrary function, then $g_{\mu\nu}K^\mu K^\nu = \beta$ gives

$$K^r = \frac{P}{2r}\left[1 - \frac{2m(v,r)}{r}\right] - \frac{\ell^2}{2rP} + \frac{\beta r}{2P}. \tag{27}$$

From Eq. (26), we have

$$\frac{dK^v}{dk} = \frac{d}{dk}\left(\frac{P}{r}\right) = \frac{1}{r}\frac{dP}{dk} - \frac{P}{r^2}\frac{dr}{dk}. \tag{28}$$

Thus

$$\frac{dP}{dk} = \frac{1}{r}\left(r^2\frac{dK^v}{dk} + P\frac{dr}{dk}\right). \tag{29}$$

Substituting Eqs. (24a) and (27) into Eq. (29) gives the differential equation satisfied by the function P,

$$\frac{dP}{dk} = \frac{P^2}{2r^2}\left(1 - \frac{4m(v,r)}{r} + 2m'(v,r)\right) + \frac{\ell^2}{2r^2} + \frac{\beta}{2}. \tag{30}$$

The function $P(v,r)$ can be found if the mass function $m(v,r)$ and the initial conditions are defined (see, for example, [3]).

IV. CONDITIONS FOR LOCALLY NAKED SINGULARITY

In this section we examine, given the generalized Vaidya mass function, how the final fate of collapse is determined in terms of either a black hole or a naked singularity. If there are families of future directed nonspacelike trajectories reaching faraway observers in spacetime, which terminate in the past at the singularity, then we have a naked singularity forming as the collapse final state. Otherwise when no such families exist and an event horizon forms sufficiently early to cover the singularity, we have a black hole. The equation for the radial null geodesics ($\ell = 0, \beta = 0$) for the line element (8) can easily be found, using Eqs. (26) and (27), which is given by

$$\frac{dv}{dr} = \frac{2r}{r - 2m(v,r)}. \tag{31}$$

The above differential equation has a singularity at $r = 0$, $v = 0$. The nature of this singularity can be analyzed by the usual techniques of the theory of ordinary differential equations [16,17]. Whereas the procedures used below are standard, we shall describe the case treated here in some detail so as to give the exact picture of the nature of the central singularity at $r = 0$, $v = 0$.

A. Structure of the central singularity

We can generally write Eq. (31) in the form

$$\frac{dv}{dr} = \frac{M(v,r)}{N(v,r)}, \tag{32}$$

with the singular point at $r = v = 0$, where both the functions $M(v,r)$ and $N(v,r)$ vanish. Hence we should carefully analyze the existence and uniqueness of the solution of the above differential equation in the vicinity of this singularity. At this point it is useful to introduce a new independent variable t with differential dt such that

$$\frac{dv}{M(v,r)} = \frac{dr}{N(v,r)} = dt, \tag{33}$$

so that the differential equation (32) can be replaced by a system

$$\frac{dv(t)}{dt} = M(v,r),$$
$$\frac{dr(t)}{dt} = N(v,r). \tag{34}$$

We would like to emphasize here that all the solutions of Eq. (32) are solutions of the system (34), and hence we study the behavior of this system of equations near the singular point $r = v = 0$ in the (r, v) plane. We can easily see that the singular point of (32) is a fixed point of the system (34). To find the necessary and sufficient conditions for the existence of the solutions of this system in the vicinity of the fixed point $r = v = 0$, let us write (34) as a differential equation of the vector $\boldsymbol{x}(t) = [v(t), r(t)]^T$ on $\mathbb{R}^2$ as

$$\frac{d\boldsymbol{x}(t)}{dt} = \boldsymbol{f}(\boldsymbol{x}(t)). \tag{35}$$

Now to show the existence and uniqueness of the solution with respect to the initial conditions arbitrarily near the fixed point of the above system (since the initial conditions on the fixed point will imply the system stays on the fixed point) we give the following definitions.

Definition 1: The function $\boldsymbol{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at $\boldsymbol{x} = \boldsymbol{x}_0$, if the partial derivatives of the functions M and N with respect to r and v exist at that point. The derivative of the function, $\boldsymbol{Df}$, is given by the 2×2 Jacobian matrix

$$\begin{bmatrix} M_{,v} & M_{,r} \\ N_{,v} & N_{,r} \end{bmatrix}.$$

Definition 2: Suppose U is an open subset of $\mathbb{R}^2$, then $\boldsymbol{f}\colon U \to \mathbb{R}^2$ is of class C^1 iff the partial derivatives $M_{,v}$, $M_{,r}$, $N_{,v}$, $N_{,r}$ exist and are continuous on U.

Henceforth we will consider the function $\boldsymbol{f}$ to be of class C^1 throughout the spacetime. Let us now show that there exists a unique solution to the system (35) subject to the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$, where $\boldsymbol{x}_0$ is arbitrarily near the fixed point of the equation. Let us define an operator T in the following way.

Definition 3: Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an operator acting on all continuous and differentiable vectors $\boldsymbol{y}(t)$ on $\mathbb{R}^2$ that takes them to the image $T\boldsymbol{y}(t)$ defined as

$$T\boldsymbol{y}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{y}(s))ds.$$

We now prove an important property of this operator T, subject to the function $\boldsymbol{f}$ being class C^1.

Lemma 1: Let $U \ni \boldsymbol{x}_0$ be an open subset of $\mathbb{R}^2$, $\boldsymbol{f}\colon U \to \mathbb{R}^2$ is of class C^1, and $\boldsymbol{y}(t)$, $\boldsymbol{z}(t)$ are continuous and differentiable vectors on U. Then there always exists an ϵ-neighborhood $B_\epsilon(\boldsymbol{x}_0)$ of $\boldsymbol{x}_0$ in which $|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| \le \kappa|\boldsymbol{y}(t) - \boldsymbol{z}(t)|$ where $0 \le \kappa \le 1$. In other words T is a contraction mapping on $B_\epsilon(\boldsymbol{x}_0)$.

Proof 1: Let $K_0 = \max_{|\boldsymbol{x}-\boldsymbol{x}_0|\le\epsilon} \|\boldsymbol{Df}(\boldsymbol{x})\|$. Then we have

$$|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| = \left| \int_{t_0}^{t} (\boldsymbol{f}(\boldsymbol{y}(s)) - \boldsymbol{f}(\boldsymbol{z}(s)))ds \right|. \tag{36}$$

The above equation can be written as

$$|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| = \left| \int_{t_0}^{t} \left(\int_{\boldsymbol{z}(s)}^{\boldsymbol{y}(s)} \boldsymbol{Df}(\boldsymbol{r})d\boldsymbol{r} \right) ds \right|, \tag{37}$$

and therefore we get the inequality

$$|T\boldsymbol{y}(t) - T\boldsymbol{z}(t)| \le K_0|(t - t_0)||\boldsymbol{y}(t) - \boldsymbol{z}(t)|. \tag{38}$$

Hence there always exists an open interval $(t_0 - h, t_0 + h)$ (that corresponds to a neighborhood around $\boldsymbol{x}_0$) where $K_0|(t - t_0)| \le 1$ and T is a contraction mapping. □

Having established the existence of a contraction mapping in a neighborhood of the point $\boldsymbol{x}_0$ and recalling that $\mathbb{R}^2$ is a complete metric space, we now use the following theorem to establish the existence and uniqueness of the solution of the system (35) subject to the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$.

Theorem 1: If $T\colon \mathbb{X} \to \mathbb{X}$ is a contraction mapping on a complete metric space $\mathbb{X}$, then there is exactly one solution of the equation $T\boldsymbol{x} = \boldsymbol{x}$.

The above theorem establishes a unique solution of the system (35) with the initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ in an ϵ-neighborhood of the point $\boldsymbol{x}_0$ which is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\boldsymbol{x}(s))ds. \tag{39}$$

The assumption that $\boldsymbol{f}\colon U \to \mathbb{R}^2$ is of class C^1 assures the solution to be continuous and differentiable in this neighborhood. Let us now find the nature of the fixed point $r = v = 0$ of the system (35). As the partial derivatives of the functions M and N exist and are continuous in the neighborhood of the fixed point, we can linearize the system near the fixed point, and hence the general behavior of this system near the singular point is similar to the characteristic equations [16]

$$\frac{dv}{dt} = Av + Br,$$
$$\frac{dr}{dt} = Cv + Dr, \tag{40}$$

where $A = \dot{M}(0,0)$, $B = M'(0,0)$, $C = \dot{N}(0,0)$, $N'(0,0) = D$, with the dot denoting partial differentiation with respect to the variable v while the dash denotes partial differentiation with respect to the coordinate r and $AD - BC \neq 0$. By using a linear substitution of the type

$$\xi = \alpha v + \omega r,$$
$$\eta = \gamma v + \delta r, \tag{41}$$

where $\alpha\delta - \omega\gamma \neq 0$, and the equation

$$\frac{d\eta}{d\xi} = \frac{\chi_2 \eta}{\chi_1 \xi}, \tag{42}$$

the system (40) can be reduced into the form

$$\frac{d\xi}{dt} = \chi_1 \xi,$$
$$\frac{d\eta}{dt} = \chi_2 \eta. \tag{43}$$

Using Eqs. (40), (41) and (43), it can be found that

$$\alpha(Av + Br) + \omega(Cv + Dr) = \chi_1(\alpha v + \omega r),$$
$$\gamma(Av + Br) + \delta(Cv + Dr) = \chi_2(\gamma v + \delta r).$$

By equating the coefficients of v and r in the above equations, we obtain

$$(A - \chi_1)\alpha + C\omega = 0$$
$$B\alpha + (D - \chi_1)\omega = 0 \tag{44}$$

and

$$(A - \chi_2)\gamma + C\delta = 0$$
$$B\gamma + (D - \chi_2)\delta = 0. \tag{45}$$

The above equations in α, ω and γ, δ may be satisfied by the values of α, ω, γ, δ not all zero if the determinant of the coefficients is zero. That is,

$$\begin{vmatrix} A - \chi & C \\ B & D - \chi \end{vmatrix} = 0, \tag{46}$$

or

$$\chi^2 - (A + D)\chi + AD - BC = 0. \tag{47}$$

This is the characteristic equation with roots (eigenvalues) χ_1 and χ_2 given by

$$\chi = \frac{1}{2}((A+D) \pm \sqrt{(A-D)^2 + 4BC}). \quad (48)$$

The singularity of Eq. (40) is classified as a node if $(A-D)^2 + 4BC \geq 0$ and $BC > 0$. Otherwise, it may be a center or focus.

Now, for Eq. (31) we have $M(v,r) = 2r$, $N(v,r) = r - 2m(v,r)$. If at the central singularity, $v = 0$, $r = 0$, we define the following limits:

$$m_0 = \lim_{v\to 0, r\to 0} m(v,r), \quad (49a)$$

$$\dot{m}_0 = \lim_{v\to 0, r\to 0} \frac{\partial}{\partial v} m(v,r), \quad (49b)$$

$$m_0' = \lim_{v\to 0, r\to 0} \frac{\partial}{\partial r} m(v,r), \quad (49c)$$

then the null geodesic equation can be linearized near the central singularity as

$$\frac{dv}{dr} = \frac{2r}{(1-2m_0')r - 2\dot{m}_0 v}. \quad (50)$$

Clearly, this equation has a singularity ar $v = 0$, $r = 0$. We can determine the nature of this singularity by observing the value of the discriminant of the characteristic equation. Using Eq. (48), the roots of the characteristic equation are given by

$$\chi = \frac{1}{2}((1-2m_0') \pm \sqrt{(1-2m_0')^2 - 16\dot{m}_0}). \quad (51)$$

For the singular point at $r = 0$, $v = 0$ to be a node, it is required that

$$(1-2m_0')^2 - 16\dot{m}_0 \geq 0 \quad \text{and} \quad \dot{m}_0 > 0. \quad (52)$$

Thus, if the mass function $m(v,r)$ is chosen such that the condition in Eq. (52) is satisfied, then the singularity at the origin $(v = 0, r = 0)$ will be a node and outgoing non-spacelike geodesics can come out of the singularity with a definite value of the tangent.

B. Existence of outgoing nonspacelike geodesics

Let us now return to the physical problem of the collapsing generalized Vaidya spacetime and let us choose the mass function that has the following properties:

(1) The mass function $m(v,r)$ obeys all the physically reasonable energy conditions throughout the spacetime.
(2) The partial derivatives of the mass function exist and are continuous on the entire spacetime.
(3) The limits of the partial derivatives of the mass function $m(v,r)$ at the central singularity obey the conditions: $(1-2m_0')^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$.

The choice of the mass function with the above properties would ensure the existence and uniqueness of the solutions of the null geodesic equation in the vicinity of the central singularity and will also make the central singularity a node of C^1 solutions with definite tangents.

To find the condition for the existence of outgoing radial nonspacelike geodesics from the nodal singularity, we consider the tangent of these curves at the singularity. Suppose X denotes the tangent to the radial null geodesic. If the limiting value of X at the singular point is positive and finite, then we can see that outgoing future directed null geodesics do terminate in the past at the central singularity. The existence of these radial null geodesics characterizes the nature (a naked singularity or a black hole) of the collapsing solutions. In order to determine the nature of the limiting value of X at $r = 0$, $v = 0$ we define

$$X_0 = \lim_{v\to 0, r\to 0} X = \lim_{v\to 0, r\to 0} \frac{v}{r}. \quad (53)$$

Using Eq. (50) and l'Hôpital's rule (for the C^1 null geodesics) we get

$$X_0 = \lim_{v\to 0, r\to 0} \frac{v}{r} = \frac{dv}{dr} = \frac{2}{(1-2m_0') - 2\dot{m}_0 \left(\frac{v}{r}\right)}, \quad (54)$$

which simplifies to

$$X_0 = \frac{2}{(1-2m_0') - 2\dot{m}_0 X_0}. \quad (55)$$

Solving for X_0 gives

$$X_0 = b_\pm = \frac{(1-2m_0') \pm \sqrt{(1-2m_0')^2 - 16\dot{m}_0}}{4\dot{m}_0}. \quad (56)$$

If we can get one or more positive real roots by solving Eq. (55), then the singularity may be locally naked if the null geodesic lies outside the trapped region. In the next subsection we will calculate the dynamics of the trapped region to find the conditions for the existence of such geodesics.

C. Apparent horizon

The occurrence of a naked singularity or black hole is usually decided by casual behavior of the trapped surfaces developing in the spacetime during the collapse evolution. The apparent horizon is the boundary of the trapped surface

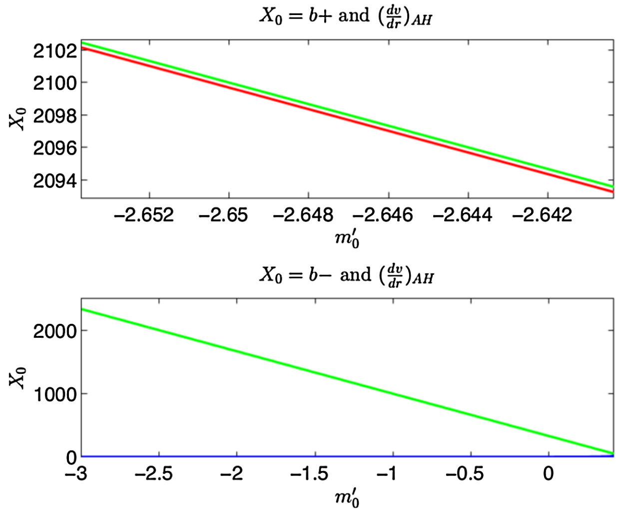

FIG. 1 (color online). Variation of X_0 (red and blue/ dark) lines and $(\frac{dv}{dr})_{AH}$ (green/lightgrey) line with m_0' at fixed value $\dot{m} = 0.0015$.

region in the spacetime. For the generalized Vaidya spacetime the equation of the apparent horizon is given as

$$\frac{2m(v,r)}{r} = 1. \tag{57}$$

Thus, the slope of the apparent horizon can be calculated in the following way: we know

$$\frac{2dm(v,r)}{dr} = 1, \tag{58a}$$

$$2\left(\frac{\partial m}{\partial v}\right)\left(\frac{dv}{dr}\right)_{AH} + \frac{2\partial m}{\partial r} = 1, \tag{58b}$$

which finally gives the slope of the apparent horizon at the central singularity ($v \to 0, r \to 0$) as

$$\left(\frac{dv}{dr}\right)_{AH} = \frac{1 - 2m_0'}{2\dot{m}_0}. \tag{59}$$

Thus now we have the sufficient conditions for the existence of a locally naked central singularity for a collapsing generalized Vaidya spacetime, which we state in the following proposition.

Proposition 1. Consider a collapsing generalized Vaidya spacetime from a regular epoch, with a mass function $m(v,r)$ that obeys all the physically reasonable energy conditions and is differentiable in the entire spacetime. If the following conditions are satisfied:

(1) The limits of the partial derivatives of the mass function $m(v,r)$ at the central singularity obey the conditions: $(1 - 2m_0')^2 - 16\dot{m}_0 \geq 0$ and $\dot{m}_0 > 0$,
(2) There exist one or more positive real roots X_0 of Eq. (56),
(3) At least one of the positive real roots is less than $(\frac{dv}{dr})_{AH}$ at the central singularity,

then the central singularity is locally naked with outgoing C^1 radial null geodesics escaping to the future.

Figure 1 shows the values of X_0 and $(\frac{dv}{dr})_{AH}$ when m_0' is varied in the interval $-3.0 \leq m_0' \leq 0.42$ for a fixed value of $\dot{m}_0$. It can be observed from the figure that the value of $X_0 = b_\pm$ is always below the value of $(\frac{dv}{dr})_{AH}$, and thus there exist open sets of parameter values for which the singularity is locally naked.

V. STRENGTH OF SINGULARITY

To compute the strength of singularity according to Tipler [18], which is the measure of its destructive capacity in the sense that whether extension of spacetime is possible through them or not [19], we consider the null geodesics parametrized by the affine parameter k and terminating at the shell focusing singularity $r = v = k = 0$. Following Clarke and Krolack [20], a singularity would be strong if the condition

$$\lim_{k\to 0} k^2 \psi = \lim_{k\to 0} k^2 R_{\mu\nu} K^\mu K^\nu > 0, \tag{60}$$

as defined by Tipler [18] (which is the sufficient condition for the singularity to be Tipler strong) where $R_{\mu\nu}$ is the Ricci tensor, is satisfied. We find the scalar $\psi = R_{\mu\nu} K^\mu K^\nu$ using Eqs. (10), (26) and (27) as

$$\psi = (2\dot{m}_0)\left(\frac{P}{r^2}\right)^2, \tag{61}$$

and therefore,

$$k^2 \psi = (2\dot{m}_0)\left(\frac{Pk}{r^2}\right)^2. \tag{62}$$

Using Eqs. (26) and (27) and l'Hôpital's rule, we can evaluate the limit along nonspacelike geodesics as $k \to 0$. This limit is found to be

$$\lim_{k\to 0} k^2 \psi = (2\dot{m}_0) \lim_{k\to 0} \left(\frac{Pk}{r^2}\right)^2. \tag{63}$$

If we assume that $P \neq 0, \infty$, then by using l'Hôpital's rule we have

$$\lim_{k\to 0} \left(\frac{Pk}{r^2}\right) = \lim_{k\to 0} \left(\frac{Pdk}{2rdr}\right). \tag{64}$$

From Eq. (26), $\frac{P}{r} = \frac{dv}{dk}$. Therefore

$$\lim_{k\to 0} \left(\frac{Pk}{r^2}\right) = \frac{1}{2}\frac{dv}{dk}\frac{dk}{dr} = \frac{1}{2}\frac{dv}{dr} = \frac{1}{2} X_0. \tag{65}$$

TABLE I. Equations of tangents X_0 to the singularity curve and values of $\lim_{k\to 0} k^2\psi$ for some special subclasses of generalized Vaidya spacetime.

Spacetime	Equation for tangent to the singularity curve X_0	$\lim_{k\to 0} k^2\psi$
Vaidya	$X_0 = \frac{2}{1-\lambda X_0}$ or $X_0 = \frac{1\pm\sqrt{1-8\lambda}}{2\lambda}, 0 < \lambda \le \frac{1}{8}$	$\frac{1}{4}\lambda X_0^2$
Charged Vaidya	$\mu^2 X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0$	$\frac{1}{2} X_0^2(\lambda - \mu^2 X_0)$
Charged Vaidya–de Sitter	$\mu^2 X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0$	$\frac{1}{2} X_0^2(\lambda - \mu^2 X_0)$
Husain solution	$2\mu^2(1 - \frac{2k}{2k-1}) X_0^{2k+1} + \lambda X_0^2 - 2X_0 + 4 = 0$	$\frac{1}{4} X_0^2(\lambda - \frac{4k\mu^2}{2k-1} X_0^{2k-1})$

Thus, we finally get

$$\lim_{k\to 0} k^2\psi = \frac{1}{4} X_0^2 (2\dot{m}_0). \tag{66}$$

We observe that the strength of the central singularity depends only on the limit of the derivative of mass function with respect to v and the limiting value X_0.

With the suitable choice of the mass function (see Table I for some special cases), it can be shown that

$$\lim_{k\to 0} k^2\psi = \frac{1}{4} X_0^2 (2\dot{m}_0) > 0. \tag{67}$$

If this condition is satisfied for some real and positive root X_0, then we conclude that the observed naked singularity is strong. It is interesting to note that when the energy conditions are satisfied, and if a naked singularity is developed as an end state of the collapse, then that naked singularity is always strong.

VI. SOME SPECIAL SUBCLASSES OF GENERALIZED VAIDYA

Using Eq. (55) we calculate the equations of tangents to the null geodesics at the central singularity for some special subclasses of the generalized Vaidya spacetime with the specific mass function, $m(v, r)$. In all these mass functions, we can see that it is possible to obtain at least one or more real and positive values of X_0.

(i) *The self-similar Vaidya spacetime*

In this case we consider the situation of a radial influx of null fluid in an initially empty region of Minkowski spacetime [4,21]. The first shell arrives at $r = 0$ at time $v = 0$ and the final shell at $v = T$. A central singularity of the collapsing mass is developed at $r = 0$. For $v < 0$ we have $m(v, r) = 0$ and for $v > T$ we have $m(v, r) = M_0$ where M_0 is the constant Schwarzschild mass. For the weak energy conditions to be satisfied, it is required that $\dot{m}(v, r)$ be a non-negative. We define the mass function as

$$m(v, r) = m(v), \tag{68}$$

where

$$m(v) = \begin{cases} 0, & v < 0, \\ \frac{1}{2}\lambda v, & 0 \le v \le T, \\ M_0, & v > T. \end{cases} \tag{69}$$

The mass function is a non-negative increasing function of v for imploding radiation. For $0 \le v \le T$, the solution is the self-similar Vaidya spacetime. For this choice of mass function, using Eq. (55), we get

$$X_0 = \frac{2}{1 - \lambda X_0} \quad \text{or} \quad X_0 = \frac{1 \pm \sqrt{1 - 8\lambda}}{2\lambda}. \tag{70}$$

This is similar to the solution obtained by Joshi [4]. This equation gives positive values of X_0 for all values of λ in the range $0 < \lambda \le \frac{1}{8}$. It can also be observed that $\lim_{k\to 0} k^2\psi = \frac{1}{4}\lambda X_0^2 > 0$ for all positive values of X_0; hence the singularity is strong.

(ii) *The charged Vaidya spacetime*

This subclass of the generalized Vaidya spacetime has been studied in great detail in [22–24]. We consider here the form of the mass function in [5,21]

$$m(v, r) = f(v) - \frac{e^2(v)}{2r}, \tag{71a}$$

where $f(v)$ and $e(v)$ are arbitrary functions representing the mass and electric charge, respectively (limited only by the energy conditions), at the advanced time v. Particularly, we define these functions as [25]

$$f(v) = \begin{cases} 0, & v < 0, \\ \lambda v (\lambda > 0) & 0 \le v \le T, \\ f_0 (> 0), & v > T, \end{cases} \tag{71b}$$

and

$$e^2(v) = \begin{cases} 0, & v < 0, \\ \mu^2 v^2 (\mu^2 > 0), & 0 \le v \le T, \\ e_0^2 (> 0), & v > T. \end{cases} \tag{71c}$$

For this choice of mass function, using Eq. (55) we obtain

$$\mu^2 X_0^3 - 2\lambda X_0^2 + X_0 - 2 = 0. \tag{72}$$

This equation is a polynomial of degree three with the negative last term and positive first coefficient. By the theory of polynomial functions, every equation of this nature must have at least one root which is positive. The existence of these roots signifies that the singularity is naked. In particular, when $\mu^2 = 0.001$, $\lambda = 0.01$, then one of the roots of Eq. (72) is 2.077 and $\lim_{k\to 0} k^2\psi = \frac{1}{2}X_0^2(\lambda - \mu^2 X_0) = 0.0171 > 0$. Therefore the condition for a strong naked singularity is satisfied.

(iii) *The charged Vaidya–deSitter spacetime*

The charged Vaidya–de Sitter solution is a generalized Vaidya solution of a charged null fluid in an expanding de Sitter background [25]. We define the mass function as

$$m(v,r) = m(v) - \frac{e^2(v)}{2r} + \frac{\Lambda r^3}{6}, \tag{73}$$

where $f(v)$ and $e(v)$ are arbitrary functions representing the mass and electric charge, respectively, and $\Lambda \neq 0$ is the cosmological constant. For the weak energy condition to be satisfied, it is required that $r\dot{m}(v) - e(v)\dot{e}(v)$ to be non-negative [5,25]. We specifically define the functions similar to that of charged Vaidya, and the algebraic equation that governs the behavior of the tangent vectors near the central singularity comes out to be the same.

(iv) *The Husain solution*

This is a solution of the Einstein field equations for the null fluid with the equation of state $\varrho = k\rho$ where $\rho = \frac{g(v)}{4\pi r^{2k+2}}$, $k \neq \frac{1}{2}$ [5,14]. This solution is a subclass to the generalized Vaidya solutions with the mass function given by

$$m(v,r) = \begin{cases} q(v) - \frac{g(v)}{(2k-1)r^{2k-1}}, & k \neq \frac{1}{2}, \\ q(v) + g(v)\ln r, & k = \frac{1}{2}, \end{cases} \tag{74a}$$

where $q(v)$ and $g(v)$ are arbitrary functions which are restricted only by the energy conditions. For the dominant energy conditions to be satisfied, it is required that $g(v) \geq 0$ and either $\dot{g}(v) > 0$ for $k < \frac{1}{2}$ or $\dot{g}(v) < 0$ for $k > \frac{1}{2}$. The weak or strong energy conditions are satisfied when $\rho \geq 0, \varrho \geq 0$. We consider the case when $k \neq \frac{1}{2}$ and define the mass function as

$$q(v) = \begin{cases} 0, & v < 0, \\ \frac{1}{2}\lambda v(\lambda > 0), & 0 \leq v \leq T, \\ q_0(> 0), & v > T, \end{cases} \tag{74b}$$

and

$$g(v) = \begin{cases} 0, & v < 0, \\ \mu^2 v^{2k}, & 0 \leq v \leq T, \\ g_0(> 0), & v > T. \end{cases} \tag{74c}$$

For this mass function using Eq. (55), we get

$$2\mu^2\left(1 - \frac{2k}{2k-1}\right)X_0^{2k+1} + \lambda X_0^2 - 2X_0 + 4 = 0. \tag{75}$$

This equation can be solved to get some positive roots X_0 for some particular values of μ^2, k and λ. In particular, when $\mu^2 = 0.001$, $k = \lambda = 0.01$, then one of the roots of Eq. (75) is 2.00408 and $\lim_{k\to 0} k^2\psi = \frac{1}{4}X_0^2(\lambda - \frac{4k\mu^2}{2k-1}X_0^{2k-1}) = 0.506 > 0$. This shows that the singularity is naked and strong.

Table I gives a summary of the equations of tangent to the singularity curve X_0 and the value of $\lim_{k\to 0} k^2\psi$ for chosen mass functions in some subclasses of the generalized Vaidya spacetime.

VII. CONCLUDING REMARKS

In this paper we developed a general mathematical formalism to study the gravitational collapse of the generalized Vaidya spacetime in the context of the cosmic censorship conjecture. We studied the structure of the central singularity to show that it can be a node with outgoing radial null geodesics emerging from the singular point with a definite value of the tangent, depending on the nature of the generalized Vaidya mass function and the parameters in the problem. The key points that emerged transparently from this analysis are as follows:

(i) It is quite clear that given any realistic mass function, there always exists an open set in the parameter space for which the central singularity is naked and CCC is violated. A similar result is well known for pure Type-I matter fields [26,27]. Hence we can conclude that the occurrence of naked singularity is quite a "stable" phenomenon even when the nature of the matter field changes by combining a radiation-like field along with a collapsing perfect fluid.

(ii) It is also evident that for an open set in the parameter space, these naked central singularities are strong and they cannot be regularized anyway by extension of spacetime through them. This has far reaching consequences as their presence will no longer make the global spacetime future asymptotically simple, and the proofs of black hole dynamics and thermodynamics have to be reformulated.

(iii) Finally the generalized Vaidya spacetime is a more realistic spacetime than pure dustlike matter or perfect fluid, during the later stages of gravitational collapse of a massive star. A collapsing star should always radiate, and hence there should be a combination of lightlike matter along with a perfect fluid. Therefore a violation of censorship in these models should have novel astrophysical signatures which are yet to be properly deciphered.

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