PHYSICAL REVIEW D **90**, 064008 (2014)

Torsion-bar antenna in the proper reference frame with rotation

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(Received 15 May 2014; published 4 September 2014)

The resultant response of the rotating torsion-bar antenna for gravitational waves discussed in M. Ando *et al.* [Phys. Rev. Lett. 105, 161101 (2010)] is reinvestigated from a general-relativistic point of view. To do this, the equation of motion of a free-falling particle in the proper reference frame of a rotating observer is used. As a result, the resultant response derived in the above paper is also valid even when $\omega_g \sim \Omega$, where ω_g and Ω are the angular frequencies of gravitational waves and the rotation of the antenna, respectively.

DOI: 10.1103/PhysRevD.90.064008 PACS numbers: 04.30.-w, 04.80.Nn, 95.55.Ym

The torsion-bar antenna (TOBA) is a novel type of gravitational-wave antenna for low-frequency observations. This antenna is formed by two bar-shaped test masses, arranged parallel to the x–y plane and orthogonal to each other. Each bar is supported at its center, so as to rotate around the z axis. When gravitational waves pass through this antenna, tidal forces by the gravitational waves will appear as differential angular changes in these bars. These changes are extracted as a gravitational-wave signal by using a sensitive sensor, such as a laser interferometer. A characteristic feature of this antenna is that it can expand the observation band to lower frequencies by using modulation and up-conversion of gravitational-wave signals though rotating the antenna.

A similar concept was proposed more than 40 years ago as a heterodyne detector for circular-polarized gravitational waves [1,2]. Recently, the idea for the up-conversion of low-frequency gravitational waves was reinvestigated [3] and a space-borne prototype antenna was operated [4]. In Ref. [3], the situation was considered in which the frequency of gravitational waves is much smaller than the rotation of the antenna. In this situation, the tidal force due to gravitational waves is almost stationary and the antenna rotates in this stationary tidal force field. Based on this intuitive picture, the response of the antenna was derived as

$$\ddot{\theta}_{\rm diff} = \alpha[\ddot{h}_\times \cos(2\Omega t) + \ddot{h}_+ \sin(2\Omega t)], \qquad (1)$$

where $\theta_{\rm diff}$ is the resultant output of the antenna, α is the shape factor of the antenna, $h_\times$ and h_+ are the two independent polarization components of gravitational waves propagating along the z axis, and Ω is the angular velocity of the rotation of the antenna. From this equation, it was concluded that the gravitational-wave signal is modulated by the rotation; a gravitational-wave signal with an angular frequency of ω_g is up- and down-converted to appear at $\omega_g \pm 2\Omega$ frequencies. However, due to the above intuitive picture, the resultant output (1) was valid only when $\omega_g \ll \Omega$.

The purpose of this paper is the rederivation of the resultant output (1) of the rotating TOBA from a general-relativistic point of view. Usually, the geodesic deviation equation is the basic equation to estimate the force that affects gravitational-wave detectors. However, in the case of the response of the rotating TOBA, we cannot apply the geodesic deviation equation, because the world line of the test mass in the rotating TOBA is not geodesic. Therefore, in this article, we estimate the torque, which affects the rotating TOBA test mass, through the proper reference frame for a rotating observer.

The proper reference frame for an accelerating and rotating observer was discussed in Ref. [2]. After the publication of this textbook, Ni and Zimmermann [5] derived the metric which is accurate to the second order with respect to the proper distance from the origin of coordinates. They also derived the equations of motion for freely falling particles, which is accurate to the first order. Their equation of motion contains many terms, which represent many types of effects of inertial forces as well as forces due to the Riemann curvature.

The situation discussed by Ni and Zimmermann [5] is appropriate to the rotating TOBA. Therefore, we use the proper reference frame with the rotation discussed by Ni and Zimmermann [5] to estimate the gravitational-wave torque which affects the rotating TOBA response. The metric and the equation of motion discussed in this paper is the special case of the proper reference frame discussed by Ni and Zimmermann [5]. Through this metric, we show the resultant output (1) is valid even if $\omega_g \sim \Omega$. Throughout this paper, we use natural units, in which the light velocity is unity.

Here, we consider the situation where the TOBA is rotating around the Z axis, where the center of the TOBA is setting $X = Y = Z = 0$. Consider the world line $P_0(\tau)$ of the center of the TOBA with four-velocity $u^a(\tau)$ and four-rotation ω^a in gravitational waves with Riemann

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tensor $R_{abc}{}^{d}$. The orthogonal tetrad $e^{a}_{(\alpha)}$, which is carried by observers at the center of the TOBA, transports according to [2]

$$u^{b}\nabla_{b}e^{a}_{(\alpha)} = -\Omega^{ab}e_{b(\alpha)}, \tag{2}$$

where

$$\Omega^{ab} := a^{a}u^{b} - a^{b}u^{a} + \epsilon^{abcd}u_{c}\omega_{d}, \tag{3}$$

$$a^{a}(\tau) := u^{b}\nabla_{b}u^{a}. \tag{4}$$

Even when the TOBA is rotating, the world line $P_0(\tau)$ of the center of the TOBA is still a geodesic with the four-velocity u^a, if the laboratory can be regarded as a local inertial system. Furthermore, we concentrate only on the case where the angular velocity of rotation is constant. Therefore, in this paper, we may treat the case where

$$a^{a}(\tau) = 0, \qquad u^{b}\nabla_{b}\omega^{a}(\tau) = 0 \tag{5}$$

along $P_{(0)}(\tau)$.

At any event $P_0(\tau)$, we consider geodesics $P(\tau; n^a; s)$ orthogonal to $u^a(\tau)$, where n^a is the unit vector tangent to a particular geodesic at $P_0(\tau)$, and $n^a u_a = 0$. At each event $P_0(\tau)$, a proper distance s along any geodesic $P(\tau; n^a; s)$ with tangent vector n^a is assigned by the local coordinates

$$X^{\hat{0}} := \tau, \tag{6}$$

$$X^{\hat{j}} := s n^{a} e^{(\hat{j})}_{a} = s\alpha^{\hat{j}}, \tag{7}$$

where $\alpha^{\hat{j}}$ is the spatial direction cosine and $\alpha^{\hat{j}} = X^{\hat{j}}/s$ with $s^2 = (X^{\hat{i}})^2 = (X^{\hat{1}})^2 + (X^{\hat{2}})^2 + (X^{\hat{3}})^2$, and $X^{\hat{0}} = \tau = T$, $X^{\hat{1}} = X$, $X^{\hat{2}} = Y$, and $X^{\hat{3}} = Z$. This means $n^a = \alpha^{\hat{j}} e^{a}_{(\hat{j})}$. This coordinate system is well defined for events near the world line $P_0(\tau)$ if the "light cylinder" has not been reached ($s \ll 1/|\Omega|$), or if the curvature has not yet caused geodesics to cross ($s \ll 1/|R_{abc}{}^{d}|^{1/2}$), and the Riemann tensor has not yet changed much from its value on the world line $P_0(\tau)$ ($s \ll |R_{abc}{}^{d}|/|\partial_a R_{abc}{}^{e}|$). In the case of gravitational waves, the last condition corresponds to the fact that s should be much smaller than the wavelength of gravitational waves.

Using the above coordinate system $\{X^{\hat{\mu}}\}$, Ni and Zimmermann [5] derived the second-order expansion of the metric near $P_0(\tau)$ as

$$\begin{aligned} ds^2 = &-(dX^{\hat{0}})^2[1 + (\omega^{\hat{l}}X^{\hat{l}})^2 - (\omega)^2 X^{\hat{l}}X^{\hat{l}} + R_{\hat{0}\hat{l}\hat{0}\hat{m}}X^{\hat{l}}X^{\hat{m}}] \\ &+ 2dX^{\hat{0}}dX^{\hat{i}}\left(\epsilon_{\hat{i}\hat{j}\hat{k}}\omega^{\hat{j}}X^{\hat{k}} - \frac{2}{3}R_{\hat{0}\hat{l}\hat{i}\hat{m}}X^{\hat{l}}X^{\hat{m}}\right) \\ &+ dX^{\hat{i}}dX^{\hat{j}}\left(\delta_{\hat{i}\hat{j}} - \frac{1}{3}R_{\hat{i}\hat{l}\hat{j}\hat{m}}X^{\hat{l}}X^{\hat{m}}\right) \\ &+ O(dX^{\hat{\mu}}dX^{\hat{\nu}}X^{\hat{l}}X^{\hat{m}}X^{\hat{k}}), \end{aligned} \tag{8}$$

where $\omega^{\hat{l}}$ and $R_{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}}$ are evaluated on the world line $P(\tau)$ at time $X^{\hat{0}} = \tau$.

To calculate the coordinate acceleration of a freely falling body, we use the geodesic equation in the form

$$\frac{d^2X^{\hat{i}}}{(dX^{\hat{0}})^2} + \left(\Gamma^{\hat{i}}{}_{\hat{\mu}\hat{\nu}} - \Gamma^{\hat{0}}{}_{\hat{\mu}\hat{\nu}}\frac{dX^{\hat{i}}}{dX^{\hat{0}}}\right)\frac{dX^{\hat{\mu}}}{dX^{\hat{0}}}\frac{dX^{\hat{\nu}}}{dX^{\hat{0}}} = 0 \tag{9}$$

and substitute into it the first-order expansion of the Γ's. Defining $W^{i} := dX^{\hat{i}}/dX^{\hat{0}}$, the velocity measured by the accelerated rotating observer, the resulting coordinate acceleration is

$$\begin{aligned} \frac{d^2X^{\hat{i}}}{(dX^{\hat{0}})^2} = &-R_{\hat{0}\hat{j}\hat{0}}{}^{\hat{i}}X^{\hat{j}} \\ &- (\vec{\omega}\times(\vec{\omega}\times\vec{X}))^{i} - 2(\vec{\omega}\times\vec{W})^{i} \\ &- 2R_{\hat{0}\hat{k}\hat{j}}{}^{\hat{i}}X^{\hat{k}}W^{\hat{j}} + 2R_{\hat{0}\hat{l}\hat{j}}{}^{\hat{0}}X^{\hat{l}}W^{\hat{j}}W^{\hat{i}} \\ &+ \frac{2}{3}R_{\hat{m}\hat{k}\hat{j}}{}^{\hat{i}}X^{\hat{m}}W^{\hat{j}}W^{\hat{k}} - \frac{2}{3}\bar{R}_{\hat{l}\hat{k}\hat{j}}{}^{\hat{0}}X^{\hat{l}}W^{\hat{j}}W^{\hat{k}}W^{\hat{i}} \\ &+ O((X^{\hat{i}})^2). \end{aligned} \tag{10}$$

The first line in Eq. (10) is the tidal force due to the curvature near $P_0(\tau)$. The first term in the second line of Eq. (10) is the centripetal force and the second term is the Coriolis force. These terms should be neglected in the case of the configuration of the TOBA as discussed below.

Here, we consider the configuration of the rotating TOBA. The TOBA essentially measures the relative rotation of the two bars which is induced by the tidal force of gravitational waves. To measure the tidal force due to gravitational waves, the centers of the bars are fixed at the point $X = Y = 0$, but are free for rotational modes. The test mass is aligned within the X–Y plane and the rotation axis of the TOBA is chosen so that $\omega^a = \Omega e^{a}_{(Z)}$. The shape of the test mass is symmetric, which means the density distribution ρ is symmetric in all three axes.

In this configuration, the direction of the centripetal force $-(\vec{\omega}\times(\vec{\omega}\times\vec{X}))^{i}$, which is the first term in the second line of Eq. (10), is the direction along the bar. This force cancels out when we sum this force along the bar. Similarly, the velocity W^i of the bar is restricted to only the rotational motion around the $e^{a}_{(Z)}$ axis. In this case, the direction of the Coriolis force $-2(\vec{\omega}\times\vec{W})^{i}$, which is the second term in the second line of Eq. (10), is also the direction along the bar. This force also cancels out when we sum this force along the bar. Therefore, we may neglect the second line in Eq. (10). Furthermore, we regard that the velocity W^i is induced by gravitational waves, i.e., $W^i = O(h)$. As far as we concentrate only on the linear effect of gravitational waves, we may neglect the third and fourth lines in Eq. (10).

In the case where the mass distributes as the mass density ρ, the force F_i induced by gravitational waves on a volume element dV of the test mass of the TOBA is given by

$$F_{\hat{i}}dV = -\rho R_{\hat{0}\hat{j}\hat{0}\hat{i}}X^{\hat{j}}dV. \tag{11}$$

This force is also derived from the potential

$$U := -\int dV \int dX^{\hat{i}}F_{\hat{i}}, \\ = \frac{1}{2}R_{\hat{0}\hat{j}\hat{0}\hat{i}}\int dV\rho X^{\hat{i}}X^{\hat{j}}. \tag{12}$$

The torque F_{gw} induced by the gravitational wave is given by

$$F_{gw} = -\frac{\partial U}{\partial\theta} =: -\frac{1}{2}R_{\hat{0}\hat{j}\hat{0}\hat{i}}q^{\hat{i}\hat{j}}, \tag{13}$$

where $q^{\hat{i}\hat{j}}$ is the dynamic quadruple moment tensor [6]. For bar rotation, $q^{XX} = -q^{YY} = -\int\rho(2XY)dV$ and $q^{XY} = q^{YX} = \int\rho(X^2 - Y^2)dV$.

To evaluate curvature components in Eq. (13), we consider the gravitational-wave solution with a flat spacetime background $g_{ab} = \eta_{ab} + h_{ab}$, where h_{ab} is transverse traceless, i.e., $\eta^{ab}h_{ab} = 0 = \eta^{ad}\partial_d h_{ab}$. In an inertia frame, the background metric is given by $\eta_{ab} = -(dt)_a(dt)_b + (dx)_a(dx)_b + (dy)_a(dy)_b + (dz)_a(dz)_b$, and we assume the gravitational wave propagates along the z axis,

$$h_{ab} = h_+(t+z)((dx)_a(dx)_b - (dy)_a(dy)_b) \\ + 2h_\times(t+z)(dx)_{(a}(dy)_{b)}. \tag{14}$$

Since we consider the rotating TOBA with the rotating axis z, this rotational frame $\{T, X, Y, Z\}$ is given by

$$T = t, \tag{15}$$

$$X = x\cos\Omega t - y\sin\Omega t, \tag{16}$$

$$Y = x\sin\Omega t + y\cos\Omega t, \tag{17}$$

$$Z = z. \tag{18}$$

From this coordinate transformation, the flat metric η_{ab} and TT-gauge gravitational wave h_{ab} are given by [7]

$$\eta_{ab} = -(1 - \Omega^2(Y^2 + X^2))(dT)_a(dT)_b \\ + 2\Omega(dT)_{(a}(YdX - XdY)_{b)} \\ + (dX)_a(dX)_b + (dY)_a(dY)_b + (dZ)_a(dZ)_b \tag{19}$$

and

$$h_{ab} = \Omega^2(h_+(\cos(2\Omega T)(Y^2 - X^2) - 2\sin(2\Omega T)XY) + h_\times(\sin(2\Omega T)(X^2 - Y^2) - 2\cos(2\Omega T))XY))(dT)_a(dT)_b \\ + 2\Omega(h_+(\cos(2\Omega T)Y - \sin(2\Omega T)X) - h_\times(\sin(2\Omega T)Y + \cos(2\Omega T))X))(dX)_{(a}(dT)_{b)} \\ + 2\Omega(h_+(\cos(2\Omega T)X + \sin(2\Omega T)Y) + h_\times(\cos(2\Omega T))Y - \sin(2\Omega T)X))(dY)_{(a}(dT)_{b)} \\ + (h_+\cos(2\Omega T) - h_\times\sin(2\Omega T))((dX)_a(dX)_b - (dY)_a(dY)_b) \\ + 2(h_+\sin(2\Omega T) + h_\times\cos(2\Omega T)))(dX)_{(a}(dY)_{b).} \tag{20}$$

We note that the metric (19) is consistent with Eq. (8).

In this rotational coordinate system, the components of the Riemann curvature that are necessary for the evaluation of Eq. (13) are summarized as

$$R_{\mathrm{TXTX}} = \frac{1}{2}(\sin(2\Omega T)\ddot{h}_\times - \cos(2\Omega T)\ddot{h}_+) \\ = -R_{\mathrm{TYTY}}, \tag{21}$$

$$R_{\mathrm{TXTY}} = -\frac{1}{2}(\cos(2\Omega T)\ddot{h}_\times + \sin(2\Omega T)\ddot{h}_+). \tag{22}$$

Since $q^{XX} = q^{YY} = 0$ and $q^{XY} = : I\alpha$ in the case of the thin bar aligned along the X axis, the torque that affects this thin bar is given by

$$F_{gw(xbar)} = -\frac{I\alpha}{2}(\cos(2\Omega T)\ddot{h}_\times + \sin(2\Omega T)\ddot{h}_+), \tag{23}$$

where I is the inertia moment of the bar and α is the shape factor. On the other hand, in the case of the thin bar aligned along the Y axis, we have $q^{XX} = q^{YY} = 0$ and $q^{XY} =: -I\alpha$, and the torque affects this thin bar is given by

$$F_{gw(ybar)} = +\frac{I\alpha}{2}(\cos(2\Omega T)\ddot{h}_\times + \sin(2\Omega T)\ddot{h}_+). \tag{24}$$

In an approximation where the test-mass bar freely rotates around the Z axis, the equation of motion for the resultant output of the antenna θ_{diff} is given by

$$I\ddot{\theta}_{\mathrm{diff}} = F_{gw(xbar)} - F_{gw(ybar)} \\ = I\alpha(\cos(2\Omega T)\ddot{h}_\times + \sin(2\Omega T)\ddot{h}_+). \tag{25}$$

This is completely identical to the result (1) derived in Ref. [3]. In that paper, Eq. (25) was derived in the case $\omega_{gw} \ll \Omega$, as mentioned above. However, the derivation in this paper shows that the limitation $\omega_{gw} \ll \Omega$ is not necessary and that Eq. (25) is valid even in the case $\omega_{gw} \simeq \Omega$. Of course, the proper reference frame is valid only near the rotation axis, and Eq. (25) is valid only when the size s of the antenna satisfies $\omega_{gw}s, \Omega s \ll 1$. If this limitation become serious, we have to evaluate the next-order expression discussed by Li and Ni [8].

K. N. deeply thanks Professor Masa-Katsu Fujimoto for his valuable discussions and encouragement. This work was supported by JSPS KAKENHI Grant No. 24244031.

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[7] This coordinate transformation is called the first-kind gauge transformation and has nothing to do with the second-kind gauge transformation. The second-kind gauge degree of freedom is completely fixed through the TT gauge.

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