Covariantizing the interaction between dark energy and dark matter

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Coupling dark energy and dark matter through an effective fluid description is a very common procedure in cosmology; however, it always remains in comoving coordinates in the special FLRW space. We construct a consistent, general, and covariant formulation, where the interaction is a natural implication of the imperfectness of the fluids. This imperfectness makes difficult the final step towards a robust formulation of interacting fluids, namely the construction of a Lagrangian whose variation would give rise to the interacting equations. Nevertheless, we present a formal solution to this problem for a single fluid, through the introduction of an effective metric.

I. INTRODUCTION

Dark energy and dark matter are the basic constituents of the Universe [1]. Although there are theories postulating that they may correspond to a unified "dark sector" (for instance in Chaplygin-gas-like theories [2]), detailed cosmological observations, and especially the clustering properties of dark matter [3–5] in contrast with the homogeneity of dark energy, suggest with a great certainty that dark energy and dark matter are two separate sectors. Hence, one could construct scenarios in which the dark energy and dark matter sectors interact [6], since this interaction, apart from being theoretically allowed, could have an important phenomenological implication, namely alleviating the coincidence problem (i.e., why are the current dark energy and matter densities of the same order although they evolve differently).

In the existing literature the interaction is described in a very simple way, that is with the arbitrary modification of the equations of motion. In particular, one handles both dark energy and dark matter as perfect fluids in the framework of General Relativity,[1] whose total conservation is arbitrarily split into nonconserved "interacting" parts,

$$\nabla^b T_{ab}^{(\text{tot})} = \nabla^b (T_{ab}^{(\text{DM})} + T_{ab}^{(\text{DE})}) = 0 \qquad (1)$$

$$\Rightarrow$$

$$\nabla^b T_{ab}^{(\text{DM})} = Q_a \quad \text{and} \quad \nabla^b T_{ab}^{(\text{DE})} = -Q_a, \qquad (2)$$

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[1]We mention that in the framework of modified gravity one can obtain interactions of dark matter with the extra degrees of freedom of gravitational modification which play the role of an effective dark energy, through the transformation to the Einstein frame, but this is a completely different issue [7,8].

where the quantity Q_a is introduced as a phenomenological descriptor of the interaction, the form of which is assumed arbitrarily, too. Although this arbitrary splitting is mathematically correct, there is not a procedure determining how the system described in Eqs. (1) and (2) could physically arise, and especially how to derive Q_a (see, however, Refs. [9]).

In principle, any fundamental theory should be characterized by a Lagrangian whose variation gives rise to the equations of motion. If the microscopic nature of dark matter and especially of dark energy were known, one could write down a Lagrangian with all possible interaction terms, and then varying it one could obtain the complete and exact interacting equations of motion and the corresponding interaction terms, similarly to the interactions within the Standard Model. Since such a microscopic description is currently impossible, one could still hope to describe the dark energy-dark matter interaction in an effective way, writing an effective Lagrangian whose variation could give rise to (2). There has been a recent wealth of explorations of dark matter direct and indirect detection, as well as collider production through the means of the effective field theory approach (for a small sampling of the field see for example [10–17]). Similarly there have been forays into describing dark energy and modified gravity via effective theories [18,19]. Nevertheless, in the existing literature regarding the coupling of dark energy and dark matter, neither the microscopic description nor the effective field theory approach are employed, and the relations given in (2) are imposed by hand.

Therefore the important question that arises naturally is the following: Is this widespread formalism consistent? In spite of being always presented only in comoving coordinates in Friedmann-Lemaître-Robertson-Walker (FLRW) geometry, can it be given a covariant formulation? And

ideally, can we write down (effective) Lagrangians, whose variation would give rise to (2)? This is the field of interest of the present work.

II. TWO FLUID INTERACTION: THE STANDARD PROCEDURE

In the discussions of coupled dark energy and dark matter in cosmology, one considers two coupled fluids in a FLRW space. Let us restrict, for simplicity, to a spatially flat FLRW geometry (which is anyway the one encountered in the literature) described by the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2), \qquad (3)$$

where $a(t)$ is the scale factor. The two fluids commonly considered in the literature are assumed to have energy densities $\rho_{1,2}$ and pressures $P_{1,2}$ depending only on time, in order to respect spatial homogeneity and isotropy, and apart from the interaction they mimic perfect fluid behavior. They are usually assumed to satisfy the equations of motion,

$$\dot{\rho}_1 + 3H(P_1 + \rho_1) = Q, \qquad (4)$$

$$\dot{\rho}_2 + 3H(P_2 + \rho_2) = -Q. \qquad (5)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and an overdot denotes differentiation with respect to the comoving time t. The quantity Q quantifies the interaction and its forms are considered completely arbitrarily, with the obvious requirement to depend only on time due to homogeneity and isotropy. The usual choices of Q encountered in the literature make this quantity proportional to $\rho_{1,2}$ or to the Hubble parameter H, and their powers [20], and one can additionally use observations in order to constrain their forms [21].

The two equations (4) and (5) are concocted so that, by adding them together, a "total fluid" of energy density

$$\rho_{\text{tot}} = \rho_1 + \rho_2 \qquad (6)$$

and pressure

$$P_{\text{tot}} = P_1 + P_2 \qquad (7)$$

satisfies the conservation equation

$$\dot{\rho}_{\text{tot}} + 3H(P_{\text{tot}} + \rho_{\text{tot}}) = 0. \qquad (8)$$

Actually, as we discussed in the Introduction, the aforementioned arbitrary splitting exactly arises from this conservation of the total fluid. The "total" fluid has effective equation of state parameter

$$w_{\text{tot}}(t) \equiv \frac{P_{\text{tot}}}{\rho_{\text{tot}}} = \frac{P_1 + P_2}{\rho_1 + \rho_2} = \frac{w_1\rho_1 + w_2\rho_2}{\rho_1 + \rho_2}, \qquad (9)$$

namely it is an average of the equation of state parameters of the individual fluids w_i weighted by their energy fractions (density parameters) ρ_i/ρ_{tot}. Although the individual w_1 and w_2 may both be constant, the resulting w_{tot} is not, except for the trivial cases $w_1 = w_2$ (in which case there is a single fluid with density 2ρ and pressure $2P$) or constant ρ_1 and ρ_2. Based on this formulation, a noninsignificant amount of literature (e.g., [20–24]) has appeared.

III. TWO FLUID INTERACTION: A CONSISTENT COVARIANT PICTURE

Equations (4) and (5) can be obtained in a consistently covariant picture if the two fluids are described by the stress-energy tensors

$$T_{ab}^{(1)} = (P_1 + \rho_1)u_a u_b + P_1 g_{ab} + q_a u_b + q_b u_a, \qquad (10)$$

$$T_{ab}^{(2)} = (P_2 + \rho_2)u_a u_b + P_2 g_{ab} - q_a u_b - q_b u_a, \qquad (11)$$

where u^a is the common 4-velocity of the two fluids, a timelike unit vector pointing in the time direction. The two fluids are not tilted with respect to each other, that is, they have the same 4-velocity u^a and they "see" the same three-space orthogonal to u^a with three-metric $h_{ab} = g_{ab} + u_a u_b$ ($h^a{}_b$ is the projection operator on this three-space). q^c is a current energy density, a timelike vector which describes the transfer of energy between the two fluids. Due to spatial isotropy, q^c cannot have any spatial component and must point in the time direction,

$$q^c = \alpha(t)u^c, \qquad (12)$$

where α is a function of time which must be non-negative for q^c to be future-oriented.

We mention here that the two fluids are imperfect fluids, but not in the usual sense [25]. Usually, the term $q_a u_b + q_b u_a$ in an imperfect fluid is associated with a purely spatial energy current density (that is, one satisfying $q^c u_c = 0$ [25]), but this is not the case here: the flux density of energy must be parallel to u^c in order not to violate spatial isotropy. Because of this, and contrary to the standard textbook imperfect fluid, the traces of $T_{ab}^{(i)}$ are not the same as those of a perfect fluid, namely

$$T^{(i)} = -\rho_i + 3P_i \mp 2\alpha. \qquad (13)$$

Note that the "total" stress-energy tensor

$$T_{ab}^{(\text{tot})} = T_{ab}^{(1)} + T_{ab}^{(2)} \qquad (14)$$

is covariantly conserved

$$\nabla^b T_{ab}^{(\text{tot})} = 0, \qquad (15)$$

and the "total" energy density and pressure associated with it are $\rho_{\text{tot}} = \rho_1 + \rho_2$ and $P_{\text{tot}} = P_1 + P_2$. On the other

hand, the covariant divergence of the i-th fluid ($i = 1, 2$) stress-energy tensor $T_{ab}^{(i)}$ is

$$\nabla^b T_{ab}^{(i)} = u_a u^b \nabla_b P_i + u_a u_b \nabla^b (\rho_i \pm 2\alpha) + \nabla_a P_i$$
$$+ (P_i + \rho_i \pm 2\alpha) u^b \nabla_b u_a$$
$$+ (P_i + \rho_i \pm 2\alpha) u_a \nabla^b u_b, \qquad (16)$$

where the upper sign corresponds to fluid 1 and the lower one to fluid 2. Projection along the time direction u^a gives

$$u^a \nabla^b T_{ab}^{(i)} = (\dot{\rho}_i \pm 2\dot{\alpha}) + 3H(P_i + \rho_i \pm 2\alpha), \qquad (17)$$

where $\dot{\rho}_i \equiv u^a \nabla_a \rho_i$, etc. By imposing that $u^a \nabla^b T_{ab}^{(i)} = 0$, the two fluids are conserved separately, but their perfect-fluid components $(P_i + \rho_i) u_a u_b + P_i g_{ab}$ are not, having nonzero covariant divergences which satisfy

$$u^a \nabla^b [(P_i + \rho_i) u_a u_b + P_i g_{ab}] = \pm 2(\dot{\alpha} + \alpha \nabla_b u^b). \quad (18)$$

In an FLRW background this equation becomes

$$\dot{\rho}_i + 3H(P_i + \rho_i) = \mp 2(\dot{\alpha} + 3H\alpha). \qquad (19)$$

Hence, one can clearly see that imposing the right hand side of this equation to be equal to $\pm Q$, Eqs. (4) and (5) are reproduced. In this case α and Q satisfy the relation

$$\dot{\alpha} + 3H\alpha + \frac{Q(t)}{2} = 0. \qquad (20)$$

This equation can be rewritten as

$$\frac{1}{a^3} \frac{d}{dt} (\alpha a^3) + \frac{Q(t)}{2} = 0, \qquad (21)$$

which integrates to

$$\alpha(t) = -\frac{1}{2a^3(t)} \int dt a^3(t) Q(t). \qquad (22)$$

Note that in the case $\alpha = 0$ the two fluids become perfect and noninteracting, that is, $Q = 0$.

A possible physical interpretation is the following. Fluid 1, described by $T_{ab}^{(1)}$, is not a perfect fluid and its effective energy density is not ρ_1 but

$$T_{ab}^{(1)} u^a u^b = \rho_1 + 2\alpha, \qquad (23)$$

while its effective pressure is still

$$\frac{1}{3} T_{ab}^{(1)} h^{ab} = P_1. \qquad (24)$$

Fluid 2, instead, has effective energy density and pressure

$$T_{ab}^{(2)} u^a u^b = \rho_2 - 2\alpha, \qquad (25)$$

$$\frac{1}{3} T_{ab}^{(2)} h^{ab} = P_2, \qquad (26)$$

respectively. In this picture, it would be incorrect to think of these two fluids as perfect fluids. The terms $\pm(q_a u_b + q_b u_a)$ in $T_{ab}^{(i)}$ describe an energy transfer which happens simultaneously at all points of space, without transfer of three-dimensional momentum, and spoil the perfect fluid nature of these fluids. The amount of energy lost by fluid 1 per unit time and per unit volume is instantaneously gained by fluid 2, and vice versa. This picture provides the underlying explanation for the splitting (4)–(5) in the standard approach, where an energy transfer occurring simultaneously at all points of space is introduced by hand.

Alternatively, one could describe our situation as follows: when $\alpha > 0$, the correction $2(\dot{\alpha} + 3H\alpha)$ to the perfect fluid part of fluid 1 can be visualized as a dust with zero pressure and energy density 2α which supplies energy to fluid 1, while taking it from fluid 2 through an immediate transfer. From the point of view of fluid 2, one can think of a perfect fluid from which a dust with negative energy density -2α removes energy to transfer it to fluid 1. Clearly, this second dust would violate the weak energy condition, but this is not a significant problem since a similar case occurs in the standard imperfect fluid, where a purely spatial heat flux density q^c describes a spacelike, instantaneous transfer of energy which violates the energy conditions and is clearly unphysical, but is still useful as a toy model for a consistent relativistic theory without all its complications. In this sense, the model described by Eqs. (4) and (5) may indeed be acceptable as a phenomenological toy model.

Finally, note that the usual quantity $Q(t)$ introduced in the literature is related to $\alpha(t)$ through (21) as

$$Q(t) = -\frac{2(\alpha a^3)^{\cdot}}{a^3}. \qquad (27)$$

The physical meaning becomes apparent if we consider a region of three-dimensional space with unit comoving volume and physical volume a^3. Then $-2\alpha a^3$ is just the energy transferred between the two fluids in this volume, $-(2\alpha a^3)^{\cdot}$ is the rate at which this energy transfer occurs, and $Q(t)$ is the rate at which this energy is transferred per unit volume.

IV. A LAGRANGIAN DESCRIPTION

Having constructed a consistent covariant description of the two-fluid interaction, the question that arises naturally is whether these equations can arise from a Lagrangian. The disadvantage is that the two fluids are not perfect and thus, as it is well known, there is not a robust Lagrangian formulation for imperfect fluids. Additionally, there is not

even a consensus on how one should proceed in order to approach it. If such a Lagrangian density is found, it could be possible to give a Lagrangian and covariant description of two interacting fluids.

Given the high degree of symmetry of FLRW geometry, for inspiration one can proceed along the lines of a less-known treatment of the classical dissipative oscillator [26], in which the oscillating position $x(t)$ of a point particle is ruled by the usual equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0, \tag{28}$$

where γ and ω_0 are positive constants. The change of variable $x(t) = e^{-\gamma t}q(t)$ transforms the equation of motion (28) into the new equation,

$$\ddot{q} + \omega^2 q = 0, \tag{29}$$

where $\omega \equiv \sqrt{\omega_0^2 - \gamma^2}$, which is dissipationless [26]. The change of variable $x(t) \to q(t)$ is a canonical transformation which makes the system Lagrangian, with Lagrangian function

$$L(q, \dot{q}) = \frac{\dot{q}^2}{2} - \frac{\omega^2 q^2}{2}, \tag{30}$$

which does not depend explicitly on time. The momentum conjugated to q is

$$p \equiv \frac{dL}{d\dot{q}} = e^{\gamma t}(\dot{x} + \gamma x), \tag{31}$$

and the associated Hamiltonian is

$$\mathcal{H} = p\dot{q} - L = \frac{p^2}{2} + \frac{\omega^2 q^2}{2}. \tag{32}$$

Much has been written on this way of removing dissipation from the oscillator, the physical interpretation of this procedure and of the new Hamiltonian variables (q, p), and on possible quantizations of the dissipative oscillator [26].

Let us now be inspired by the above treatment, and try to follow the general idea of removing dissipation from the physical system by changing the variables which describe the motion. In particular, since we desire to remove the combination $q_a u_b + q_b u_a$ from the stress-energy tensor (10), we should redefine the spacetime variables themselves, that is redefine the metric. Since $q^c = \alpha u^c$, we can transform the metric $g_{ab} \to \bar{g}_{ab}$ according to the Kerr-Schild transformation [27,28],

$$\bar{g}_{ab} = g_{ab} + 2\lambda\alpha u_a u_b, \tag{33}$$

where λ is a constant with the dimensions of an inverse density, thus making the product $\lambda\alpha$ and the metric $\bar{g}_{ab}$ dimensionless (the condition $\lambda \geq 0$ guarantees that the metric $\bar{g}_{ab}$ has the same signature as g_{ab}). The inverse metric straightforwardly reads

$$\bar{g}^{ab} = g^{ab} + \frac{2\lambda\alpha}{2\lambda\alpha - 1} u^a u^b \tag{34}$$

and is defined for $\alpha \neq \frac{1}{2\lambda}$. We restrict ourselves to this case: in the pathological situation $\alpha = \frac{1}{2\lambda}$ the metric $\bar{g}_{ab}$ degenerates into the three-dimensional metric $h_{ab} \equiv g_{ab} + u_a u_b$ with Euclidean signature, which cannot describe the full spacetime metric.

For the spatially flat FLRW spacetime, the line element would become

$$d\bar{s}^2 = -[1 - 2\lambda\alpha(t)]dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{35}$$

It can then be transformed back to the form $d\bar{s}^2 = -d\bar{t}^2 + a^2(t)d\vec{x}^2$ by the redefinition of the time coordinate

$$\bar{t}(t) = \int dt \sqrt{1 - 2\lambda\alpha(t)}. \tag{36}$$

The stress-energy tensor becomes

$$T_{ab} = (P + \rho)u_a u_b + Pg_{ab} + q_a u_b + q_b u_a \tag{37}$$

$$= (P - 2\lambda\alpha P + \rho + 2\alpha)u_a u_b + P\bar{g}_{ab}. \tag{38}$$

This expression formally describes the stress-energy tensor of a perfect fluid with energy density

$$\bar{\rho} = \rho + 2\alpha - 2\lambda\alpha P \tag{39}$$

and pressure $\bar{P} = P$ in the spacetime metric $\bar{g}_{ab}$. Therefore, this stress energy tensor will be covariantly conserved according to the covariant derivative $\bar{\nabla}_c$ *of the metric $\bar{g}_{ab}$* [29], namely,

$$\bar{\nabla}^b T_{ab} = 0. \tag{40}$$

Thus, since in the metric $\bar{g}_{ab}$ the fluid is perfect, and having in mind the well-known result that the Lagrangian density of a perfect fluid is just $\sqrt{-g}P$ [30–32], we can write down the Lagrangian density associated with the stress-energy tensor T_{ab} as

$$\mathcal{L} = \sqrt{-\bar{g}}P, \tag{41}$$

where $\bar{g}$ is the determinant of $\bar{g}_{ab}$. In summary, the "dissipative" term $q_a u_b + q_b u_a$ has indeed been eliminated from the stress-energy tensor and a Lagrangian description has been found for this fluid, but at the price of introducing a fictitious metric that depends on that particular fluid.

The metric $\bar{g}_{ab}$, in which the imperfect fluid becomes perfect, is not universal: if two different fluids are considered simultaneously, there will be two different metrics $\bar{g}_{ab}^{(1)}$ and $\bar{g}_{ab}^{(2)}$ and one cannot give a consistent description of the two fluids in the same "effective spacetime." In the case of

the two fluids (4) and (5), the metrics $\bar{g}_{ab}^{(1)}$ and $\bar{g}_{ab}^{(2)}$ given by Eq. (33) with α and $-\alpha$, respectively, are different.

The situation is similar to that occurring in the classical mechanics of point particles, in which one can eliminate the (conservative) forces acting on a particle, by introducing a fictitious space such that the particle follows geodesics of an effective metric in this fictitious space[2]—the Jacobi form of the least action principle [35]. More generally, one can remove forces acting on a particle, or self-interaction terms in the equation for a field, by introducing a fictitious metric[3] in a fictitious space [34,36–38]. However, there is a different effective space for each particle or field considered, and one cannot consider two (or more) particles or fields simultaneously in this kind of approach, but only self-interactions (this statement is true also for test fluids, see Appendix A). Nevertheless, the formal result of this section may still be useful for a single fluid.

V. SCALAR FIELD FLUIDS

In this section we desire to go one step further, and investigate the case where the fluid is the effective description of a scalar field (see [39] and references therein). Scalars are the simplest fundamental physical fields, and since there is no shortage of scalar fields in high energy theories, a scalar field is often used in the cosmology of the early and late universe. In principle, a scalar field can be coupled to a fluid or to another field. Thus, in this section we briefly discuss a covariant description of this possible coupling.

A. A fluid and a scalar field

We begin by considering two coupled fluids in an FLRW universe, the first being an ordinary fluid with energy density ρ_1 and pressure P_1, and the second fluid arising from a canonical scalar field ϕ minimally coupled to the curvature (which, when decoupled from the dust fluid, is equivalent to an effective perfect fluid). We would like to offer a theoretical justification of the interaction form

$$\dot{\rho}_1 + 3H(P_1 + \rho_1) = Q, \qquad (42)$$

$$\dot{\rho}_\phi + 3H(P_\phi + \rho_\phi) = -Q. \qquad (43)$$

The effective energy density and pressure of a fluid arising from a scalar field in an FLRW space are given by the well-known formulas,

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi), \qquad (44)$$

$$P_\phi = \frac{\dot{\phi}^2}{2} - V(\phi). \qquad (45)$$

Finally, adding Eqs. (42) and (43) one obtains a conservation equation for the "total perfect fluid" characterized by energy density $\rho_{\text{tot}} = \rho_1 + \rho_\phi$ and pressure $P_{\text{tot}} = P_1 + P_\phi$.

In order to provide a theoretically justified form of the fluid-field interaction term, we are inspired by the large amount of research devoted in the 1980's literature on inflation reheating. In particular, one should find an interaction term as a phenomenological way to describe the decay of the inflaton due to its coupling to other particles, a term that would excite the production of this particle in order to end inflation after the number of e-folds of expansion needed to solve the horizon and flatness problems [40]. Later on, the scenarios for ending inflation took a more definite shape in the various works on reheating and preheating. Thus, inspired by the inflaton phenomenological interaction we consider

$$Q = \Gamma\dot{\phi}^2, \qquad (46)$$

with Γ a positive constant. Then, using Eqs. (44) and (45), the equation of motion (43) for the scalar field becomes

$$\dot{\phi}\left(\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + \frac{dV}{d\phi}\right) = 0 \qquad (47)$$

and, unless ϕ is a constant ϕ_0 (in which case the scalar field fluid reduces to a pure cosmological constant $\Lambda = V(\phi_0)$ and decouples from the first fluid), we have a Klein-Gordon equation with a potential and an extra source of "friction" with strength described by Γ and proportional to the "speed" $\dot{\phi}$ of the scalar, namely,

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + \frac{dV}{d\phi} = 0. \qquad (48)$$

Correspondingly, the perfect fluid part of fluid 1 enjoys a source $\Gamma\dot{\phi}$ in the right-hand side of Eq. (42),

$$\dot{\rho}_1 + 3H(P_1 + \rho_1) = \Gamma\dot{\phi}. \qquad (49)$$

The quantity α introduced in the previous section is

$$\alpha(t) = -\frac{\Gamma}{2a^3}\int dt a^3 \dot{\phi}^2, \qquad (50)$$

and it involves only the kinetic energy $\dot{\phi}^2/2$ of the field ϕ. The decay of the field ϕ into the fluid is due to its kinetic energy and stops if ϕ becomes static. Thus, we can apply the procedure of the previous section, with the above α quantifying the imperfectness, obtaining a covariant formulation of the fluid-field interaction.

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[2]If there are a finite number of centers of attraction or repulsion for a particle, its motion under these forces can be reduced to a geodesic flow, as mentioned in [33] and proved in [34].

[3]The fictitious metric is obtained by means of a conformal transformation in [36–38].

B. Two scalar field fluids

Now let the first fluid be also a scalar field ψ with self-interaction potential $U(\psi)$. In this case $\rho_1 = \frac{\dot{\psi}^2}{2} + U(\psi)$ and $P_1 = \frac{\dot{\psi}^2}{2} - U(\psi)$ and the equation of motion for ψ becomes

$$\ddot{\psi} + 3H\dot{\psi} - \Gamma\frac{\dot{\phi}^2}{\dot{\psi}} + \frac{dU}{d\psi} = 0 \qquad (51)$$

(we assume that $\dot{\psi} \neq 0$ and $\Gamma > 0$). Thus, when $|\dot{\psi}|$ is large (that is, a "fast-moving" ψ) and increasing, there is a comparatively small extra term $-\Gamma\frac{\dot{\phi}^2}{\dot{\psi}}$ which enhances the motion of ψ and could perhaps be interpreted as a sort of "antifriction" for this field, a force which depends on the velocities of both ψ and ϕ. However, when ψ is decreasing, this term turns into friction opposing the motion of ψ. Thus, one can also apply the formulation of the previous section, with α given by (50) quantifying the imperfectness.

VI. DISCUSSION

The increasing amount of literature on mutually coupled dark energy and dark matter, and of a scalar field explicitly coupled other forms of matter in cosmology [20,22–24], raises the problem of finding a covariant description of the widely used formulation of energy exchange between two fluids. In the present work we have constructed such a covariant formulation, where the interaction is a natural implication of the imperfectness of the fluids.

This imperfectness makes difficult the final step towards a robust formulation of interacting fluids, namely the construction of a Lagrangian, whose variation would give rise to the interacting equations, since we need to face the issue of finding Lagrangian descriptions of dissipative systems, which is notoriously difficult. We have presented a formal solution to this problem for a single fluid, entailing the introduction of an effective metric which depends on this particular fluid. However, its applicability beyond one fluid is limited, since each fluid sees a different effective metric.

In summary, we have constructed a covariant description for an otherwise *ad hoc*, coordinate-dependent formalism widely used in cosmology, introducing imperfectness. Whether imperfectness is a necessary (apart from sufficient) condition for interaction is still an open question; however, this seems reasonable from the microscopic point of view since in general one cannot easily imagine an effective sector to be simultaneously "perfect" and "interacting." If this is the case, then it will be very hard, if not impossible, to construct a Lagrangian formulation in the usual way, for the dark energy–dark matter interaction. And vice versa, if the microscopic nature of dark matter and dark energy is some day understood, their possible interacting terms in the fundamental Lagrangian will probably give rise to a different effective interacting behavior than the one used in the current literature.

APPENDIX

Consider a *single test fluid* which is not isolated but interacts with another system in FLRW geometry according to Eq. (4). The effective equation of state parameter of this fluid is defined by $w \equiv P/\rho$, and Eq. (4) takes the form

$$\dot{\rho} + 3(w + 1)H\rho = Q(t). \qquad (A1)$$

We search for a solution of this equation in the form

$$\rho(t) = \frac{C(t)}{a^{3(w+1)}(t)}. \qquad (A2)$$

Inserting this ansatz in Eq. (A1) we acquire

$$\dot{C} = Q(t)a^{3(w+1)}(t), \qquad (A3)$$

which is immediately integrated to yield

$$\rho(t) = \frac{C(t)}{a^{3(w+1)}(t)} = \frac{C_0 + \int_0^t dt' Q(t')a^{3(w+1)}(t')}{a^{3(w+1)}}, \qquad (A4)$$

where C_0 is an integration constant.

However, note that if two fluids with equation of state parameters w_1 and w_2 interact according to Eqs. (4) and (5), the solution (A4) does not apply because then, adding these equations term to term, one would obtain

$$\dot{\rho}_1 + \dot{\rho}_2 + 3(w_{\text{tot}} + 1)H\rho_{\text{tot}} = 0, \qquad (A5)$$

and the test fluid solutions would be

$$\rho_1 = \frac{C_1 + \int_0^t dt' Q(t')a^{3(w_1+1)}(t')}{a^{3(w_1+1)}}, \qquad (A6)$$

$$\rho_2 = \frac{C_2 + \int_0^t dt' Q(t')a^{3(w_2+1)}(t')}{a^{3(w_2+1)}}. \qquad (A7)$$

In this case $\rho_{\text{tot}} = \rho_1 + \rho_2$ has a complicated form that does not correspond to the "total fluid" being a perfect fluid (unless $w_1 = w_2$, which is the trivial case of a fluid interacting with itself, therefore, of a single fluid). A total perfect fluid should instead have $\rho_{\text{tot}} = \rho_1 + \rho_2$ scaling with one well-defined power of a equal to $-3(w_{\text{tot}} + 1)$.

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