

**Positronium contribution to the electron  $g-2$** M. Fael<sup>\*</sup>

*Dipartimento di Fisica e Astronomia, Università di Padova Istituto Nazionale Fisica Nucleare,  
Sezione di Padova, I-35131 Padova, Italy  
and Physik-Institut, Universität Zürich, CH-8057, Zürich, Switzerland*

M. Passera<sup>†</sup>

*Istituto Nazionale Fisica Nucleare, Sezione di Padova, I-35131 Padova, Italy  
(Received 21 May 2014; published 12 September 2014)*

The contribution of positronium to the electron  $g-2$  ( $a_e$ ) has been computed in G. Mishima, [arXiv:1311.7109](#), and found to be of the same order of  $\alpha$  as that of five-loop perturbative QED. We confirm this result and correct a few errors in its first derivation. As recently calculated in K. Melnikov, A. Vainshtein, and M. Voloshin, [arXiv:1402.5690](#), a continuum nonperturbative contribution to  $a_e$  cancels one-half of the positronium one. We show by explicit calculation that the remaining half is already included in the five-loop perturbative result. We also show that it arises from the class I(i) of five-loop diagrams containing only one closed electron loop.

DOI: [10.1103/PhysRevD.90.056004](#)

PACS numbers: 11.10.St, 12.20.Ds, 13.40.Em, 14.60.Cd

**I. INTRODUCTION**

The leading contribution of positronium, the  $e^+e^-$  bound state, to the anomalous magnetic moment of the electron ( $a_e$ ) has been computed in Ref. [1]. The result of this calculation,

$$a_e^{\text{P}} = \frac{\alpha^5}{4\pi} \zeta(3) \left( 8 \ln 2 - \frac{11}{2} \right) = 0.89 \times 10^{-13}, \quad (1)$$

where  $\zeta(3) = 1.202\dots$  and  $\alpha$  is the fine-structure constant, is of the same order of  $\alpha$  as the perturbative QED five-loop contribution  $a_e^{(10)} = 9.16(58)(\alpha/\pi)^5$  [2]. This bound-state contribution is also comparable with the electroweak one,  $a_e^{\text{EW}} = 0.2973(52) \times 10^{-13}$  [3,4], and with the present experimental uncertainty of  $a_e$ ,  $2.8 \times 10^{-13}$  [5]. It seems reasonable to expect a reduction of this experimental error to a part in  $10^{-13}$  (or better) in ongoing efforts to improve the measurement of the electron (and positron) anomalous magnetic moment [6]. Work is also in progress to reduce the error induced by the uncertainty of  $\alpha$  in the theoretical prediction for  $a_e$  [7].

A test of the electron  $g-2$  at the level of  $10^{-13}$  (or below) is therefore a goal that may be achieved not too far in the future with ongoing experimental work. This will bring  $a_e$  to play a pivotal role in probing new physics [8]. It will also provide the opportunity to test whether the longstanding  $3 - 4\sigma$  discrepancy  $\Delta a_\mu$  in the muon  $g-2$  manifests itself in the electron one [8,9]. In fact, as shown in Ref. [8], in a large class of new-physics models, new contributions to lepton magnetic moments scale with the square of the lepton masses, so that the anomaly in  $\Delta a_\mu$  suggests a

new-physics effect in  $a_e$  of  $(0.7 \pm 0.2) \times 10^{-13}$ , a value comparable with  $a_e^{\text{P}}$ . A check of Eq. (1) is therefore clearly warranted. This is presented in Sec. II, where we confirm the result of Eq. (1) and correct a few errors in its derivation in Ref. [1].

Recently, the authors of Ref. [10] pointed out the presence of the continuum nonperturbative contribution

$$a_e(\text{vp})^{\text{cont,np}} = -\frac{|\alpha|^5}{8\pi} \zeta(3) \left( 8 \ln 2 - \frac{11}{2} \right) \quad (2)$$

arising from the region right above the  $s = 4m^2$  threshold, which corresponds to  $e^+e^-$  scattering states with the exchange of Coulomb photons. Comparing Eqs. (1) and (2) they showed that this additional  $\mathcal{O}(\alpha^5)$  nonperturbative contribution cancels one-half of that of the positronium poles. The question is therefore how to deal with the remaining half: should one add it to the perturbative five-loop QED result of Ref. [2]? The authors of Ref. [10] argued that this remaining  $a_e^{\text{P}}/2$  term is already contained in the perturbative  $\mathcal{O}(\alpha^5)$  contribution to  $a_e$  computed in Ref. [2] and, therefore, it should not be added to it. On the other hand, one of the authors of the five-loop calculation in [2] has recently claimed that positronium contributes to  $a_e$  only through diagrams of  $\mathcal{O}(\alpha^7)$  or higher [11]. Also, on more general grounds [12], Ref. [13] argued that  $a_e^{\text{P}}$  simply does not exist.

In order to clarify this point, in Sec. III we first use the closed form for the QED vacuum polarization function near the  $s = 4m^2$  threshold of Refs. [12,14] to verify that the total (positronium poles plus continuum) nonperturbative contribution to  $a_e$  arising from the threshold region is equal to  $a_e^{\text{P}}/2$ . Then, using the analytic QED vacuum polarization at four-loop recently computed in Ref. [15], we show

<sup>\*</sup>matteo.fael@pd.infn.it  
<sup>†</sup>passera@pd.infn.it

explicitly that the perturbative five-loop calculation of  $a_e$  of Ref. [2] does indeed contain the remaining term  $a_e^{\text{P}}/2$ , in agreement with the arguments of Ref. [10]. Conclusions are drawn in Sec. IV.

## II. POSITRONIUM POLES

Let us consider QED with only electrons and photons. The vacuum polarization tensor is given by

$$\begin{aligned} i\Pi^{\mu\nu}(q) &= i\Pi(q^2)(g^{\mu\nu}q^2 - q^\mu q^\nu) \\ &= \int d^4x e^{iqx} \langle 0|T\{j^\mu(x)j^\nu(0)\}|0\rangle, \end{aligned} \quad (3)$$

where  $j^\mu(x) = -e\bar{\psi}(x)\gamma^\mu\psi(x)$  is the electromagnetic current. In perturbative calculations,  $\Pi(q^2)$  is analytic in the complex  $q^2$  plane except for cuts along the positive real axis beginning at  $q^2 = (2lm)^2$ , where  $m$  is the electron mass and  $l = 0, 1, 2, \dots$ . The  $q^2 = 0$  branch point ( $l = 0$ ) is the threshold value for production of three (or a higher odd number of) real photons, while  $l = 1$  corresponds to the threshold for the creation of a real  $e^+e^-$  pair by a virtual photon.

An electron-positron bound state will appear as an additional pole singularity in  $\Pi(q^2)$  below the  $q^2 = (2m)^2$  branch point. In fact, there is an infinite number of such poles, each corresponding to an energy state of positronium. In any of its  $n$  discrete states ( $n = 1, 2, 3, \dots$  is the principal quantum number), positronium may be regarded as an (unstable) particle with mass  $M_n = 2m - \mathcal{E}_n$ , where  $\mathcal{E}_n > 0$  is the binding energy. To leading order in  $\alpha$ ,  $\mathcal{E}_n = m\alpha^2/4n^2$  [16]. To compute the leading-order contribution of positronium to  $a_e$  we can use the approximation  $M_n \approx 2m$ . Positronium will be treated as a two-particle nonrelativistic bound state.

To determine the contribution of positronium to  $\Pi^{\mu\nu}(q)$  in the neighborhood of its poles, we write explicitly the time-ordered product appearing in Eq. (3)

$$\begin{aligned} \langle 0|T\{j^\mu(x)j^\nu(0)\}|0\rangle &= \theta(x^0)\langle 0|j^\mu(x)j^\nu(0)|0\rangle \\ &+ \theta(-x^0)\langle 0|j^\nu(0)j^\mu(x)|0\rangle \end{aligned} \quad (4)$$

and compute  $\langle 0|j^\mu(x)j^\nu(0)|0\rangle$  by inserting between the two currents the completeness relation

$$(1)_{\text{P}} = \sum_{n,\sigma} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{n,\mathbf{p}}} |n, \mathbf{p}, \sigma\rangle \langle n, \mathbf{p}, \sigma| \quad (5)$$

for the positronium one-particle states

$$|n, \mathbf{p}, \sigma\rangle = \int \frac{d^3k}{(2\pi)^3} \sqrt{\frac{2E_{n,\mathbf{p}}}{2E_+ 2E_-}} \tilde{\phi}_{n,\mathbf{p}}(\mathbf{k}) |\mathbf{k}_+, \mathbf{k}_-, \sigma\rangle. \quad (6)$$

In Eq. (5),  $\mathbf{p}$  and  $E_{n,\mathbf{p}} = \sqrt{\mathbf{p}^2 + M_n^2}$  are the three-momentum and energy of positronium, and  $\sigma$  indicates its four spin

states: three spin-1 states (triplet) and one spin-0 state (singlet). In Eq. (6), positronium states have been expressed as a linear superposition of free  $e^+$  and  $e^-$  states with three-momenta  $\mathbf{k}_\pm$ , respectively, and energies  $E_\pm = \sqrt{\mathbf{k}_\pm^2 + m^2}$ , with  $\mathbf{p} = \mathbf{k}_+ + \mathbf{k}_-$  and  $\mathbf{k} = (\mathbf{k}_+ - \mathbf{k}_-)/2$ . This superposition is weighted by the momentum-space Coulomb wave function  $\tilde{\phi}_{n,\mathbf{p}}(\mathbf{k})$ , which gives the amplitude for finding a particular value of  $\mathbf{k}$  for a positronium state  $n$  with total momentum  $\mathbf{p}$ . In the nonrelativistic bound-state approximation employed in this paper,  $|\mathbf{k}| \sim \mathcal{O}(am) \ll m$  [17,18].

Our result for the positronium contribution to  $\langle 0|j^\mu(x)j^\nu(0)|0\rangle$  is

$$\begin{aligned} \langle 0|j^\mu(x)j^\nu(0)|0\rangle_{\text{P}} &= -16\pi\alpha \sum_n \frac{|\phi_{n,0}(0)|^2}{M_n} \\ &\times \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ipx}}{2E_{n,\mathbf{p}}} (M_n^2 g^{\mu\nu} - p^\mu p^\nu), \end{aligned} \quad (7)$$

where  $p^\mu = (E_{n,\mathbf{p}}, \mathbf{p})$  and  $\phi_{n,0}(0)$  is the position-space wave function at the origin in the rest frame of positronium. Our result in Eq. (7) differs from that in Eq. (6) of Ref. [1]. Ours has an additional factor

$$\xi_{n,\mathbf{p}} = -\frac{M_n}{E_{n,\mathbf{p}}}. \quad (8)$$

Apart from the sign difference, this factor  $\xi_{n,\mathbf{p}}$  renders our expression in Eq. (7) Lorentz invariant (we note that, in the  $|\mathbf{k}| \ll m$  limit, the ratio  $\phi_{n,0}(0)/\sqrt{M_n}$  is a Lorentz scalar under boosts with momentum  $\mathbf{p}$ ). On the contrary, the result for the positronium contribution to  $\langle 0|j^\mu(x)j^\nu(0)|0\rangle$  of Ref. [1] is not Lorentz invariant.

Contrary to Ref. [1], Eq. (7) has been obtained summing over all spin states of positronium. However, the spin-0 state (singlet) does not contribute because, in the nonrelativistic bound-state approximation employed, the expression for  $\langle 0|j^\mu(x)j^\nu(0)|0\rangle_{\text{P}}$  has no angular dependence. The  $e^+e^-$  bound state is therefore in an  $s$  wave with zero orbital angular momentum, and angular momentum conservation requires that the total spin of the bound state is equal to 1 (triplet).

The leading contribution of positronium to  $\Pi(q^2)$  can be immediately obtained from Eqs. (4) and (7) using the integral representation  $\theta(t) = -i \int (d\omega/2\pi) e^{i\omega t}/(\omega - i\epsilon)$  for the step function ( $\epsilon > 0$ ). The result is

$$\Pi_{\text{P}}(q^2) = -16\pi\alpha \sum_n \frac{|\phi_{n,0}(0)|^2}{M_n} \frac{1}{q^2 - M_n^2 + i\epsilon}. \quad (9)$$

Once again, our Eq. (9) differs from Eq. (8) in Ref. [1] by a factor  $\xi_{n,\mathbf{q}}$ , which renders our result for  $\Pi_{\text{P}}(q^2)$  Lorentz invariant, while that in [1] is not. Also, the nonrelativistic limit  $E_{n,\mathbf{q}} \rightarrow M_n$  taken in [1] to compute the contribution of  $\Pi_{\text{P}}(q^2)$  to  $a_e$  (which should not be confused with the

nonrelativistic bound-state approximation  $|\mathbf{k}| \ll m$  is not tenable. The sign of the residues of the poles in our Eq. (9),

$$Z_n = -16\pi\alpha \frac{|\phi_{n,0}(0)|^2}{M_n} < 0, \quad (10)$$

is in agreement with the sign of the spectral density of the Källén-Lehmann representation for  $\langle 0|T\{j^\mu(x)j^\nu(0)\}|0\rangle$  [19]. The leading contribution of positronium to the imaginary part of  $\Pi(q^2)$  is given by

$$\text{Im}\Pi_P(q^2) = -\pi \sum_n Z_n \delta(q^2 - M_n^2). \quad (11)$$

This result differs from that reported in Ref. [14], ours being twice theirs, while it agrees with that of Ref. [20] obtained via the nonrelativistic Coulomb Green's function [12,21–23] (see also Eq. (22) below).

The contribution to  $a_e$  of the diagram in Fig. 1, containing the vacuum polarization insertion in the internal photon line of the one-loop electron vertex diagram, can be computed using a (subtracted) dispersion relation for the vacuum polarization. The result can be cast in the form [19,24,25]

$$a_e(\text{vp}) = \frac{\alpha}{\pi^2} \int_0^\infty \frac{ds}{s} \text{Im}\Pi(s + i\epsilon) K(s), \quad (12)$$

where

$$K(s) = \int_0^1 dx \frac{x^2(1-x)}{x^2 + (1-x)(s/m^2)} \quad (13)$$

is a positive function. The  $i\epsilon$  prescription indicates that, in correspondence of a cut, the function  $\text{Im}\Pi(s)$  must be evaluated right above it, at  $s + i\epsilon$ . Equation (12) differs from Eq. (13) of Ref. [1] by an overall minus sign. This sign can be checked, for example, inserting in Eq. (12) the imaginary part of the second-order (one-loop) contribution to  $\Pi(q^2)$

$$\text{Im}\Pi^{(2)}(s + i\epsilon) = \theta(s - 4m^2) \frac{\alpha}{3} \sqrt{1 - \frac{4m^2}{s}} \left(1 + \frac{2m^2}{s}\right). \quad (14)$$

One obtains  $a_e^{(4)}(\text{vp}) = (119/36 - \pi^2/3)(\alpha/\pi)^2$ , the well-known positive result for the two-loop QED contribution to  $a_e$  originated by the one-loop  $e^+e^-$  contribution to the photon self-energy (see e.g. [25,26]). Similarly, including hadronic effects, the leading-order hadronic contribution to

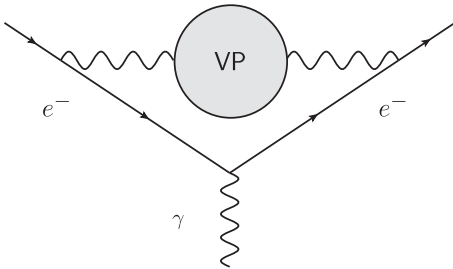


FIG. 1. Leading vacuum polarization contribution to  $a_e$ .

$a_e$  can be obtained via the dispersive integral in Eq. (12) with  $\text{Im}\Pi_h(s) = s\sigma_h(s)/4\pi\alpha$ , where  $\sigma_h(s)$  is the total cross section for  $e^+e^-$  annihilation into any hadronic state (with vacuum polarization and initial-state QED corrections subtracted off), leading to  $a_e^{\text{HLO}} = 18.66(11) \times 10^{-13}$  [25,27], once again a positive contribution.

The leading contribution of positronium to  $a_e$ , depicted in Fig. 2, can be immediately derived inserting Eq. (11) into the integral in Eq. (12). Using the explicit expression for the position-space wave function  $\phi_{n,0}(0)$  at the origin in the rest frame of positronium [16]

$$|\phi_{n,0}(0)|^2 = \frac{(m\alpha)^3}{8\pi n^3}, \quad (15)$$

the approximation  $M_n \approx 2m$  [thus neglecting terms of  $\mathcal{O}(m\alpha^2)$ ], and  $K(4m^2) = 8\ln 2 - 11/2$ , we obtain Eq. (1). We note that the Riemann zeta function  $\zeta(3) = \sum_{n=1}^\infty 1/n^3$  is due to the sum over the residues of the poles. Equation (1) can equivalently be computed by direct integration of the Feynman diagram in Fig. 1 with the subtracted vacuum polarization function

$$\Pi_P(q^2) - \Pi_P(0) = \sum_n \frac{Z_n}{M_n^2} \frac{q^2}{q^2 - M_n^2 + i\epsilon} \quad (16)$$

without employing its dispersion representation.

Our result for  $a_e^P$  agrees with that of Ref. [1]. In fact, the sign error in the calculation of  $\langle 0|j^\mu(x)j^\nu(0)|0\rangle$  in [1] is compensated by the incorrect sign in Eq. (13) of that reference. Also, as we discussed earlier, the erroneous additional factor  $E_{n,q}/M_n$  present in Eq. (8) of Ref. [1] was set to one taking the incorrect limit  $E_{n,q} \rightarrow M_n$ . In spite of these shortcomings, Ref. [1] provides the correct contribution of positronium to the  $g$ -2 of the electron and was the first, to our knowledge, to compute it.

### III. THRESHOLD CONTRIBUTION

In this section we study the nonperturbative contribution to  $a_e(\text{vp})$  arising from the region near the electron-positron threshold, both below and above  $q^2 = 4m^2$ , and discuss its relation with perturbative QED results.

Let us start considering the vacuum polarization function close to  $q^2 \approx 4m^2$  given by [12,14,20]

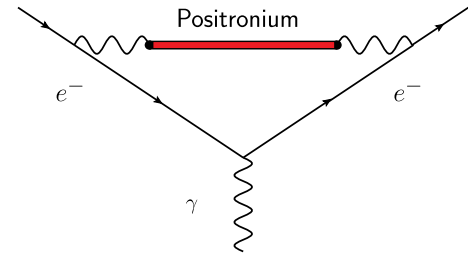


FIG. 2 (color online). Leading contribution of positronium to  $a_e$ .

$$\Pi_{\text{thr}}(q^2) = \Pi_{\text{thr}}^{(2)}(q^2) + \Pi_{\text{thr}}^{(4)}(q^2) + A(\beta), \quad (17)$$

where

$$\Pi_{\text{thr}}^{(2)}(q^2) = \alpha \left( \frac{8}{9\pi} + \frac{i}{2}\beta \right), \quad (18)$$

$$\Pi_{\text{thr}}^{(4)}(q^2) = \alpha^2 \left[ \frac{1}{4\pi^2} \left( 3 - \frac{21}{2}\zeta(3) \right) + \frac{11}{32} - \frac{3}{4} \ln 2 - \frac{1}{2} \ln(-i\beta) \right], \quad (19)$$

$$A(\beta) = -\frac{\alpha^2}{2} \left[ \gamma + \psi \left( 1 - \frac{i\alpha}{2\beta} \right) \right], \quad (20)$$

$\gamma = 0.577\dots$  is Euler's constant,  $\psi(z) = d \ln \Gamma(z)/dz$  is the digamma function, and  $\beta = \sqrt{1 - 4m^2/q^2}$  (for  $q^2 > 4m^2$ ,  $\beta$  corresponds to the velocity of the electron and the positron in their c.m. frame). The functions  $\Pi_{\text{thr}}^{(2)}(q^2)$  and  $\Pi_{\text{thr}}^{(4)}(q^2)$  are the leading terms of the one- and two-loop functions  $\Pi^{(2)}(q^2)$  and  $\Pi^{(4)}(q^2)$ , respectively, in the non-relativistic limit  $\beta \rightarrow 0$ . For example, Eq. (14) shows that the leading term of  $\text{Im}\Pi^{(2)}(q^2)$  in the limit  $\beta \rightarrow 0$  is  $\alpha\beta/2$ , in agreement with Eq. (18). The function  $A(\beta)$ , obtained via the nonrelativistic Coulomb Green's function, resums the nonrelativistic vacuum polarization diagrams with the exchange of two or more photons between the electron-positron pair, therefore corresponding to the sum of the leading contributions for  $\beta \rightarrow 0$  of all vacuum polarization diagrams with three or more loops [28]. For  $|\beta| \lesssim \alpha$ ,  $A(\beta)$  is of  $\mathcal{O}(\alpha^2)$ , whereas for  $|\beta| \gg \alpha$  it contains terms of  $\mathcal{O}(\alpha^3)$  and higher, as it can be immediately seen expanding it for  $|\beta| > \alpha/2$ ,

$$A(\beta) = \frac{\alpha^2}{2} \sum_{k=1}^{\infty} \zeta(k+1) \left( \frac{i\alpha}{2\beta} \right)^k. \quad (21)$$

Remarkably, the function  $A(\beta)$  catches the threshold effects both above and below  $q^2 = 4m^2$ . In fact, as the digamma function  $\psi(z)$  has simple poles at  $z = 0, -1, -2, \dots$ ,  $A(\beta)$  has poles at  $\beta = i\alpha/2n$  which, to leading order in  $\alpha$ , correspond to  $q^2 = M_n^2$ , the energy states of positronium. Developing the Laurent expansion of Eq. (20) about the positronium poles and selecting the imaginary part of  $\Pi_{\text{thr}}(q^2)$  for all values of  $q^2$ , one obtains [20]

$$\begin{aligned} \text{Im}\Pi_{\text{thr}}(q^2) &= 16\pi^2 \alpha \sum_n \frac{|\phi_{n,0}(0)|^2}{M_n} \delta(q^2 - M_n^2) \\ &+ \theta(\beta) \frac{\pi\alpha^2/2}{1 - e^{-\pi\alpha/\beta}}. \end{aligned} \quad (22)$$

The first line of Eq. (22) agrees with the contribution of the positronium poles to  $\text{Im}\Pi(q^2)$  in Eq. (11). The second line, which provides the continuum contribution, is the Sommerfeld factor.

With  $\Pi_{\text{thr}}(q^2)$  at our disposal, we will now follow an argument similar to one in [10] to verify that the total (positronium poles plus continuum) nonperturbative contribution to the electron  $g-2$  arising from the threshold

region is equal to  $a_e^{\text{P}}/2$ . Starting from  $a_e(\text{vp})$  in Eq. (12), this contribution is given by

$$a_e^{\text{thr}}(\text{vp}) = \frac{\alpha}{\pi^2} \int_{M_1^2}^{q_0^2} \frac{ds}{4m^2} \text{Im} \left\{ A(\beta) - \frac{i\pi^2 \alpha^3}{24\beta} \right\} K(4m^2), \quad (23)$$

where  $M_1 = 2m - \mathcal{E}_1$  is the energy of the positronium ground state and  $q_0^2 > 4m^2$  corresponds to  $\beta_0 = \beta(q_0^2)$  with  $\pi\alpha \ll \beta_0 \ll 1$ . With these integration limits,  $a_e^{\text{thr}}(\text{vp})$  catches the contribution of the entire threshold region. The expression in braces in Eq. (23) is  $\Pi_{\text{thr}}(q^2)$  subtracted of the  $\mathcal{O}(\alpha)$ ,  $\mathcal{O}(\alpha^2)$ , and  $\mathcal{O}(\alpha^3)$  terms of its perturbative expansion [see Eqs. (17) and (21)]; this subtracted quantity selects the nonperturbative contribution of the threshold region, which arises at  $\mathcal{O}(\alpha^4)$ . Equation (23) can be split into its poles and continuum parts, and, using Eq. (22), can be written in the form (note that  $\beta$  is imaginary at the poles)

$$\begin{aligned} a_e^{\text{thr}}(\text{vp}) &= \frac{\alpha}{\pi^2} K(4m^2) \left\{ \int_{M_1^2}^{4m^2} \frac{ds}{4m^2} \text{Im}A(\beta) \right. \\ &\left. + \int_0^{\beta_0} 2\beta d\beta \left[ \frac{\pi\alpha^2/2}{1 - e^{-\pi\alpha/\beta}} - \frac{\alpha\beta}{2} - \frac{\pi\alpha^2}{4} - \frac{\pi^2\alpha^3}{24\beta} \right] \right\}. \end{aligned} \quad (24)$$

The function  $A(\beta)$  has branch points at  $q^2 = 0$  and  $4m^2$  and, as discussed above, simple poles at  $q^2 = M_n^2$ . Employing a dispersion relation for the real part of  $A(\beta)$ , Eq. (24) can be expressed in terms of  $\text{Re}A(\beta)$  at  $|q^2| \rightarrow \infty$ , i.e.  $\beta \rightarrow 1$ . To leading order in  $\alpha$  we obtain

$$a_e^{\text{thr}}(\text{vp}) = -\frac{\alpha}{\pi} K(4m^2) \text{Re}A(1). \quad (25)$$

This very simple formula can be immediately evaluated using Eq. (21) at leading order. The result is

$$a_e^{\text{thr}}(\text{vp}) = \frac{\alpha^5}{8\pi} \zeta(3) K(4m^2) = \frac{a_e^{\text{P}}}{2}. \quad (26)$$

This consistency check agrees with Eqs. (21) and (25) of Ref. [10], and confirms that the total contribution of the threshold region to  $a_e(\text{vp})$  is equal to the sum of the poles' contribution in Eq. (1) and the continuum one in Eq. (2).

We will now show that the above derived threshold contribution  $a_e^{\text{thr}}(\text{vp}) = a_e^{\text{P}}/2$  is already included in the usual perturbative QED calculations of Refs. [2,15]. To this end, we use the explicit expressions for  $\Pi^{(8)}(q^2)$ , the QED vacuum polarization function at four loops recently computed in Ref. [15]. The authors provide expansions for the low-energy, high-energy, and threshold regions. In particular, in the threshold region  $\Pi^{(8)}(q^2)$  can be written as

$$\Pi^{(8)}(q^2) = \sum_{k=-2}^{\infty} \Pi_k^{(8)}(q^2) \beta^k. \quad (27)$$

The five-loop QED contribution to  $a_e$  arising from the insertion of the eight-order (four-loop) vacuum polarization in the photon line of the second-order vertex diagram has been computed via the formula [29,30]



$$a_e^{(10)}(\text{vp}) = -\frac{\alpha}{\pi} \int_0^1 dx (1-x) \Pi^{(8)}\left(-\frac{m^2 x^2}{1-x}\right). \quad (28)$$

If we select the first term in the expansion of  $\Pi^{(8)}(q^2)$  in powers of  $\beta$  given by Eq. (27), and replace in it  $1/\beta^2 = x^2/[x^2 + 4(1-x)]$ , we obtain

$$a_e^{(10)}(\text{vp}) = -\frac{\alpha}{\pi} \int_0^1 (1-x) \left[ \frac{x^2 \Pi_{-2}^{(8)}}{x^2 + 4(1-x)} \right] dx + \dots \quad (29)$$

[we note that the expansion in Eq. (27) is not well defined in the integration region of Eq. (29), where  $\beta \geq 1$ , and it is only employed to isolate the term of  $\mathcal{O}(1/\beta^2)$ ]. The coefficient  $\Pi_{-2}^{(8)}$  is constant and given by the explicit calculation of Ref. [15],

$$\Pi_{-2}^{(8)} = -n_e \frac{\alpha^4}{8} \zeta(3), \quad (30)$$

where the label  $n_e$  (to be set to one) indicates that this term arises from four-loop diagrams with only one closed electron loop. Inserting (30) into (29) we obtain

$$a_e^{(10)}(\text{vp}) = n_e \frac{\alpha^5}{8\pi} \zeta(3) K(4m^2) + \dots = \frac{a_e^{\text{P}}}{2} + \dots, \quad (31)$$

which shows that the contribution  $a_e^{\text{P}}/2$  is naturally included in the perturbative five-loop calculation. Equation (31) also shows that this contribution arises from the five-loop set I(i) of Ref. [30] which contains eighth-order vacuum polarization diagrams with only one closed electron loop. This is at variance with the claim of Ref. [11] that the leading-order contribution of positronium to  $a_e(\text{vp})$  occurs through diagrams of  $\mathcal{O}(\alpha^7)$  obtained from the five-loop set I(j) by adding the exchange of at least one additional photon in each of the two light-by-light scattering loops.

Finally, from Eq. (27) we note that  $\Pi_{\text{thr}}^{(8)}(q^2)$ , the leading term of the four-loop function  $\Pi^{(8)}(q^2)$  in the limit  $\beta \rightarrow 0$ , is equal to  $\Pi_{-2}^{(8)}/\beta^2$ . From Eq. (30) we see that this explicit result is in agreement with the  $\mathcal{O}(\alpha^4)$  term of the expansion of  $A(\beta)$  in Eq. (21). To leading order in  $\alpha$  we can therefore express Eq. (25) in the form

$$a_e^{\text{thr}}(\text{vp}) = -\frac{\alpha}{\pi} K(4m^2) \Pi_{\text{thr}}^{(8)}(|q^2| \rightarrow \infty). \quad (32)$$

This result shows that the contribution of the threshold region can be mapped into one at  $|q^2| \rightarrow \infty$  where, far from the positronium bound states, perturbation theory converges well. This observation, presented in Ref. [10] (where it was introduced via the nonrelativistic Coulomb Green's function in the spacelike limit  $q^2 \rightarrow -\infty$ ) led the authors to argue that the term  $a_e^{\text{P}}/2$  can be obtained through conventional perturbation theory, where loop diagrams are calculated performing a Wick rotation with subsequent

integration over spacelike momenta. Our Eq. (31) shows this point explicitly.

#### IV. CONCLUSIONS

In this paper we reexamined the contribution  $a_e^{\text{P}}$  of positronium to the electron  $g$ -2 computed in Ref. [1]. We confirmed the result of this reference and corrected a few errors in its derivation.

As shown recently in Ref. [10], the integral representation for  $a_e(\text{vp})$  also receives a continuum nonperturbative contribution from the integration region right above the electron-positron threshold. This additional nonperturbative contribution was shown in [10] to cancel one-half of that of positronium. In order to verify this partial cancellation, we introduced the closed-form QED vacuum polarization function near threshold of Refs. [12,14] and calculated the contribution to  $a_e(\text{vp})$  arising from its integration in the region below and above threshold. Our result confirms that the total contribution to  $a_e(\text{vp})$  of the region near threshold is equal to  $a_e^{\text{P}}/2$ .

We therefore addressed the question whether this remaining term  $a_e^{\text{P}}/2$  should be added to the perturbative five-loop QED result of Ref. [2]. The authors of Ref. [10] argued that this term is already included in the perturbative  $\mathcal{O}(\alpha^5)$  contribution to  $a_e(\text{vp})$  computed in Ref. [2] and, therefore, should not be added to it. On the other hand, one of the authors of Ref. [2] recently claimed that positronium contributes to  $a_e(\text{vp})$  only through a class of diagrams of  $\mathcal{O}(\alpha^7)$  [11]. Using the analytic four-loop vacuum polarization function of Ref. [15], we showed explicitly that the perturbative five-loop calculation of  $a_e(\text{vp})$  of Ref. [2] indeed includes the remaining term  $a_e^{\text{P}}/2$ . We also showed that this contribution arises from the class I(i) of five-loop diagrams containing only one closed electron loop, thus refuting the claim of Ref. [11].

In conclusion, we showed by explicit calculation that there is no additional contribution of QED bound states to  $a_e$  beyond perturbation theory.

#### ACKNOWLEDGMENTS

We would like to thank G. Dall'Agata, G. D'Ambrosio, F. Feruglio, T. Gehrmann, M. Hayakawa, H. Leutwyler, W. J. Marciano, P. Paradisi, and E. Remiddi for very useful discussions. We are also very grateful to G. P. Lepage, G. Mishima, M. Steinhauser, and A. Vainshtein for valuable correspondence. The work of M. F. is supported in part by the European Program LHCPHenoNet (PITN-GA-2010-264564). M. P. also thanks the Department of Physics and Astronomy of the University of Padova for its support. His work was supported in part by the PRIN 2010-11 of the Italian MIUR and by the European Program INVISIBLES (PITN-GA-2011-289442).

- [1] G. Mishima, [arXiv:1311.7109](#).
- [2] T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, *Phys. Rev. Lett.* **109**, 111807 (2012).
- [3] P. J. Mohr, B. N. Taylor, and D. B. Newell, *Rev. Mod. Phys.* **84**, 1527 (2012).
- [4] A. Czarnecki, B. Krause, and W. J. Marciano, *Phys. Rev. Lett.* **76**, 3267 (1996); *Phys. Rev. D* **52**, R2619 (1995).
- [5] D. Hanneke, S. Fogwell, and G. Gabrielse, *Phys. Rev. Lett.* **100**, 120801 (2008).
- [6] G. Gabrielse, in *Lepton Dipole Moments*, Advanced Series on Directions in High Energy Physics, edited by B. L. Roberts and W. J. Marciano (World Scientific, Singapore, 2010), Vol. 20, pp. 157–218; G. Gabrielse (private communication).
- [7] R. Bouchendira, P. Cladé, S. Guellati-Khélifa, F. Nez, and F. Biraben, *Phys. Rev. Lett.* **106**, 080801 (2011); F. Nez (private communication).
- [8] G. F. Giudice, P. Paradisi, and M. Passera, *J. High Energy Phys.* **11** (2012) 113.
- [9] F. Terranova and G. M. Tino, *Phys. Rev. A* **89**, 052118 (2014).
- [10] K. Melnikov, A. Vainshtein, and M. Voloshin, *Phys. Rev. D* **90**, 017301 (2014).
- [11] M. Hayakawa, [arXiv:1403.0416](#).
- [12] M. A. Braun, *Zh. Eksp. Teor. Fiz.* **54**, 1220 (1968) [*Sov. Phys. JETP* **27**, 652 (1968)].
- [13] M. I. Eides, [arXiv:1402.5860](#) [*Phys. Rev. D* (to be published)].
- [14] R. Barbieri, P. Christillin, and E. Remiddi, *Phys. Rev. A* **8**, 2266 (1973). We point out a typo in Eq. (23) of this reference: the right-hand side should be multiplied by 2.
- [15] P. A. Baikov, A. Maier, and P. Marquard, *Nucl. Phys.* **B877**, 647 (2013).
- [16] V. B. Berestetskii, E. M. Lifshitz, and L. P. Pitaevskii, *Quantum Electrodynamics*, 2nd ed. (Pergamon Press, New York, 1982).
- [17] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, MA, 1995).
- [18] C. Itzykson and J. Zuber, *Quantum Field Theory* (Dover, New York, 2006).
- [19] F. Jegerlehner, The Anomalous Magnetic Moment of the Muon, Springer Tracts in Modern Physics (Springer-Verlag, Berlin, 2008), Vol. **226**.
- [20] A. H. Hoang, *Phys. Rev. D* **57**, 1615 (1998).
- [21] L. Hostler and R. H. Pratt, *Phys. Rev. Lett.* **10**, 469 (1963).
- [22] J. Schwinger, *J. Math. Phys. (N.Y.)* **5**, 1606 (1964).
- [23] W. E. Caswell and G. P. Lepage, *Phys. Rev. A* **18**, 810 (1978).
- [24] C. Bouchiat and L. Michel, *J. Phys. Radium* **22**, 121 (1961).
- [25] F. Jegerlehner and A. Nyffeler, *Phys. Rep.* **477**, 1 (2009).
- [26] M. Passera, *J. Phys. G* **31**, R75 (2005).
- [27] D. Nomura and T. Teubner, *Nucl. Phys.* **B867**, 236 (2013).
- [28] V. N. Baier and V. S. Fadin, *Zh. Eksp. Teor. Fiz.* **57**, 225 (1969) [*Sov. Phys. JETP* **30**, 127 (1970)].
- [29] B. E. Lautrup, A. Peterman, and E. de Rafael, *Phys. Rep.* **3**, 193 (1972).
- [30] T. Aoyama, M. Hayakawa, T. Kinoshita, M. Nio, and N. Watanabe, *Phys. Rev. D* **78**, 053005 (2008); T. Aoyama, M. Hayakawa, T. Kinoshita, and M. Nio, *Phys. Rev. D* **83**, 053003 (2011).