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Solving Bethe-Salpeter scattering state equation in Minkowski space

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We present a method to directly solve the Bethe-Salpeter equation in Minkowski space, both for bound and scattering states. It is based on a proper treatment of the singularities which appear in the kernel, propagators, and the Bethe-Salpeter amplitude itself. The off-mass-shell scattering amplitude for spinless particles interacting by a one-boson exchange is computed for the first time.

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I. INTRODUCTION

The interest in solving the Bethe-Salpeter (BS) equation [1] in its natural Minkowski-space formulation has increased in recent years [2–18]. There are several reasons for dealing with Minkowski solutions. One of them is the fact that the Wick rotation is not directly applicable for computing electromagnetic form factors due to the singularities in the complex momentum plane [6,8]. The Euclidean solutions are still used in the context of BS-Schwinger-Dyson equations for computing bound-state form factors, but this requires a huge numerical effort [19–26]. They can also be used to obtain on-shell observables like binding energies or phase shifts [27,28], whereas the computation of the off-shell BS scattering amplitude—mandatory for computing, e.g., the transition electromagnetic form factor or for solving the three- and many-body BS equations—is possible only using a full Minkowski solution.

A method for computing these solutions based on the Nakanishi representation [29] of the BS amplitude was first developed in Refs. [2,3]. A similar approach combined with the light-front projection was proposed in Refs. [4,5]. It led to a different integral equation which involved only smooth functions and was numerically easy to treat. The bound-state Minkowski amplitude and, later on [6,8], the corresponding form factors were in this way computed for the first time. A modified method to the one developed in Refs. [4,5] aimed to compute the scattering states was proposed in Ref. [9]. It has already been successfully tested for the bound states [15].

Although our approach [4,5] could also be naturally extended to the scattering states, we have developed a new method [10–13] which allows us to solve the Minkowski BS equation in a simpler and more straightforward way. It consists of a direct solution of the equation, which takes properly into account the many singularities, without making use of the Nakanishi integral representation. The aim of this paper is to present this method in detail with applications to the problem of two scalar particles interacting by a one-boson exchange kernel.

Some of the results have been presented in short publications [10–12] and reviewed in Ref. [13] without a detailed explanation of the method. Until now, the off-shell BS amplitude has been computed only for a separable kernel [30].

In Sec. II, we transform the Bethe-Salpeter equation to the form which does not contain the pole singularities. In Sec. III, we analyze the kernel singularities. Section IV is devoted to the extraction of the phase shifts from the computed Minkowski amplitude. We derive in Sec. V the system of equations which couples the Euclidean amplitude with the Minkowski one for a particular value of its arguments. The comparison between the direct solution in Minkowski space and the one found using this system of equations constitutes a strong test for our approach. The numerical results are presented in Sec. VI. They concern the half-off-shell BS amplitude, the scattering length, and the elastic and inelastic phase shifts. Section VII contains some concluding remarks. Technical details are given in Appendixes A–C.

II. TRANSFORMING THE BS EQUATION

Let us consider the scattering of two equal-mass (m) particles with initial (k_{is}) and final (k_i) four-momenta, respectively:

$$k_{1s} + k_{2s} \to k_1 + k_2.$$

The corresponding BS amplitude F is parametrized in terms of the total

$$p = k_1 + k_2 = k_{1s} + k_{2s} \qquad (1)$$

and relative momenta

$$2k = k_1 - k_2,$$
$$2k_s = k_{1s} - k_{2s}. \qquad (2)$$

The subscript s means "scattering" (on-mass-shell) momenta. For a scattering process, F obeys the inhomogeneous integral equation graphically represented in Fig. 1.

Its analytic expression in Minkowski space reads

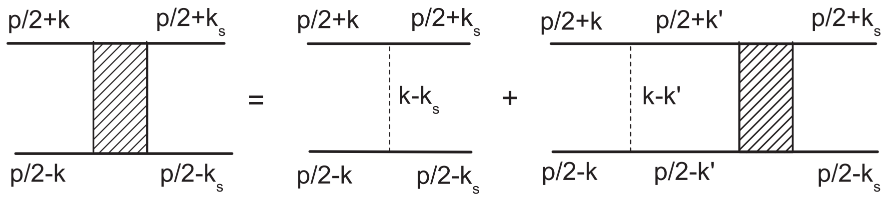

FIG. 1. Bethe-Salpeter equation for a scattering state.

$$F(k,k_s;p) = K(k,k_s;p) - i\int \frac{d^4k'}{(2\pi)^4} \frac{K(k,k';p)F(k',k_s;p)}{[(\frac{p}{2}+k')^2 - m^2 + i\epsilon][(\frac{p}{2}-k')^2 - m^2 + i\epsilon]}. \tag{3}$$

We will consider all along in this article the case of spinless particles interacting by the one-boson exchange kernel K:

$$K(k,k';p) = -\frac{16\pi m^2\alpha}{(k-k')^2 - \mu^2 + i\epsilon}, \tag{4}$$

where $\alpha = g^2/(16\pi m^2)$ is the dimensionless coupling constant. In the nonrelativistic limit, this kernel leads to the Yukawa potential $V(r) = -\alpha \exp(-\mu r)/r$. We denote by $M^2 = p^2$ the squared total invariant mass of the system.

The amplitude F depends on the three four-momenta k, k_s, and p. In the center-of-mass frame, defined by $\vec{p} = 0$, one has $k_s = (0, \vec{k}_s)$ and $p_0 = M = 2\varepsilon_{k_s} = 2\sqrt{m^2 + k_s^2}$. For a given incident momentum $\vec{k}_s$, F depends on the three scalar variables k_0, $|\vec{k}|$, and $z = \cos(\vec{k}, \vec{k}_s)$. It will be hereafter denoted by $F(k_0, k, z)$, setting abusively $k = |\vec{k}|$ and $k_s = |\vec{k}_s|$. The modulus of the incident momentum k_s plays the role of a parameter (like the bound-state mass M in the bound-state equation), and therefore, it will not be included in the arguments of the amplitude. However, in contrast to the bound-state case, one has $M > 2m$ and F depends also on the extra variable $z = \cos\theta$, where θ is the scattering angle.

Notice that the solution thus obtained is the half-off-mass-shell amplitude. It is a particular case of the so-called full off-shell amplitude $F(k_0, k, z; k_{0s}, k_s; M)$. The latter, in addition to the variables k_0, k, and z, depends also on the off-shell independent variables k_{0s} and k_s, now with $k_{0s} \neq 0$ and $k_{0s} \neq \varepsilon_{k_s}$, and the total mass M, which is an independent parameter neither equal to $2\varepsilon_{k_s}$ nor related to k_{0s}. By "off-shell amplitude," we will hereafter mean the half-off-shell amplitude. The method we have developed can also be applied to the full off-shell amplitude, although the dependence of the latter on two extra variables k_{0s} and k_s requires much more extensive numerical calculations and will not be considered here.

The difficulty in computing the off-shell amplitude $F(k_0, k, z)$ in the entire domain of its arguments is due to the singular character of the inhomogeneous term K as well as of each of the factors in the integrand of Eq. (3). In particular, the singular character of the amplitude F itself makes it hardly representable in terms of smooth functions. These singularities are integrable in the mathematical sense, due to $i\epsilon$ in the denominators of propagators, but their integration is a quite delicate task and requires the use of appropriate analytical as well as numerical methods.

To avoid these problems, Eq. (3) was first solved on shell [27] by rotating the integration contour $k_0 \to ik_4$ and taking into account the contributions of the crossed singularities. These singularities are absent in the bound-state case but exist for the scattering states. A similar method will be developed in Sec. V as a test of our approach.

The off-shell amplitude $F(k_0, k, z)$ can be obtained by directly solving the corresponding three-dimensional equation derived from (3) after integrating over the azimuthal variable. However, we prefer to present in what follows its partial wave solution. This procedure, apart from the much smaller numerical cost, has the advantage of smoothing the kernel singularities, in particular, in the inhomogeneous term. The partial wave amplitude $F_L(k_0, k)$ is defined as [31]

$$F(k_0, k, z) = 16\pi \sum_{L=0}^{\infty} (2L+1) F_L(k_0, k) P_L(z), \tag{5}$$

where $P_L(z)$ is the Legendre polynomial and

$$F_L(k_0, k) = \frac{1}{32\pi}\int_{-1}^{1} dz P_L(z) F(k_0, k, z). \tag{6}$$

By inserting (5) into (3), we obtain a set of uncoupled two-dimensional equations for the partial amplitudes F_L:

$$F_L(k_0,k) = F_L^B(k_0,k) - i\int_0^\infty k'^2 dk' \int_{-\infty}^{\infty} dk'_0 \frac{W_L(k_0,k,k'_0,k')F_L(k'_0,k')}{(k'_0 - a_- + i\epsilon)(k'_0 + a_- - i\epsilon)(k'_0 - a_+ + i\epsilon)(k'_0 + a_+ - i\epsilon)}, \tag{7}$$

with

$$W_L(k_0,k,k_0',k') = \frac{1}{(2\pi)^3}\int_{-1}^{1} dz P_L(z) K(k,k';p),$$

and the inhomogeneous (Born) term F_L^B is given in terms of W_L by

$$F_L^B(k_0,k) = \frac{\pi^2}{4} W_L(k_0,k,k_0'=0,k'=k_s). \quad (8)$$

The denominator of (3) has been factorized by writing (in the c.m. frame)

$$\left(\frac{p}{2}+k'\right)^2 - m^2 + i\epsilon = (\varepsilon_{k_s}+k_0')^2 - (\varepsilon_{k'}-i\epsilon)^2,$$
$$\left(\frac{p}{2}-k'\right)^2 - m^2 + i\epsilon = (\varepsilon_{k_s}-k_0')^2 - (\varepsilon_{k'}-i\epsilon)^2$$

and making the replacement $-\varepsilon_{k'}^2 + i\epsilon \to -(\varepsilon_{k'}-i\epsilon)^2$ valid since it does not change the sign of the imaginary contribution. This leads to the expression displayed in (7) where the four propagator poles are made explicit. They are symmetric with respect to the origin in the complex k_0' plane and are given by

$$\begin{aligned} k_0'^{(1)} &= \varepsilon_{k_s} + \varepsilon_{k'} - i\epsilon = +a_+ - i\epsilon, \\ k_0'^{(2)} &= \varepsilon_{k_s} - \varepsilon_{k'} + i\epsilon = -a_- + i\epsilon, \\ k_0'^{(3)} &= -\varepsilon_{k_s} + \varepsilon_{k'} - i\epsilon = +a_- - i\epsilon, \\ k_0'^{(4)} &= -\varepsilon_{k_s} - \varepsilon_{k'} + i\epsilon = -a_+ + i\epsilon, \end{aligned} \quad (9)$$

with

$$a_\pm = \varepsilon_{k'} \pm \varepsilon_{k_s}. \quad (10)$$

Notice that $a_+ > 0$, while for the scattering process, a_- versus k' changes sign and $a_-(k'=k_s) = 0$.

We will be hereafter restricted to the Bethe-Salpeter solutions for the S wave. The corresponding kernel W_0 is given by

$$W_0(k_0,k,k_0',k') \equiv \frac{1}{kk'} w_0(\eta) = -\frac{\alpha m^2}{\pi k k'}\left\{\frac{1}{\pi}\log\left|\frac{(\eta+1)}{(\eta-1)}\right| - iI(\eta)\right\}, \quad (11)$$

with

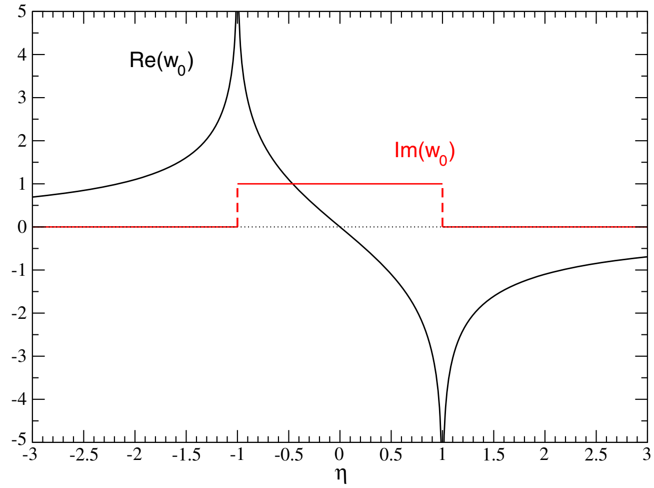

FIG. 2 (color online). S-wave reduced kernel w_0 [Eq. (11)] as a function of the variable η defined in (12).

$$I(\eta) = \begin{cases} 1 & \text{if } |\eta| \le 1 \\ 0 & \text{if } |\eta| > 1, \end{cases}$$
$$\eta = \frac{(k_0-k_0')^2 - k^2 - k'^2 - \mu^2}{2kk'}. \quad (12)$$

The reduced kernel $w_0(\eta)$ has singularities both in its real and imaginary parts. Its real part is an odd function of η with logarithmic singularities at $\eta = \pm 1$, and its imaginary part is an even function of η with discontinuities at the same points. It is represented in Fig. 2 as a function of the variable η.

The solution of (7) faces three kind of problems, all related to the unavoidable singularities when working in the Minkowski metric, and must be properly treated before a numerical solution can be tried.

First are the four propagator poles in the right-hand side of (7) which are explicitly given by (9).

Second are the logarithmic singularities of the kernel W_0, which make difficult its numerical integration both in the k_0' and k' variables.

Third are the singularities of the inhomogeneous term F_0^B. They are related to the previous ones, i.e., W_0, but generate a different type of problem: they imply the singular character of the amplitude F_0 we are interested in and thus a difficulty in being represented in terms of smooth functions when solving numerically Eq. (7).

In the following subsections, we will examine separately each of these points and detail our approach to circumvent the related difficulties.

A. Removing the pole singularities

Let us first represent the pole contributions in the integrand of (7) in the usual form:

$$\frac{1}{k'_0 - a_\pm + i\epsilon} = \mathrm{PV}\left(\frac{1}{k'_0 - a_\pm}\right) - i\pi\delta(k'_0 - a_\pm),$$
$$\frac{1}{k'_0 + a_\pm - i\epsilon} = \mathrm{PV}\left(\frac{1}{k'_0 + a_\pm}\right) + i\pi\delta(k'_0 + a_\pm),$$

where PV denotes the principal value. The integrand in the right-hand side of Eq. (7) takes the form

$$\begin{aligned}&\frac{f(k_0,k,k'_0,k')}{(k'_0 - a_+ + i\epsilon)(k'_0 + a_+ - i\epsilon)(k'_0 + a_- - i\epsilon)(k'_0 - a_- + i\epsilon)}\\ &= \left[-i\pi\delta(k'_0 - a_+) + \mathrm{PV}\frac{1}{k'_0 - a_+}\right]\left[i\pi\delta(k'_0 + a_+) + \mathrm{PV}\frac{1}{k'_0 + a_+}\right]\\ &\times\left[-i\pi\delta(k'_0 - a_-) + \mathrm{PV}\frac{1}{k'_0 - a_-}\right]\left[i\pi\delta(k'_0 + a_-) + \mathrm{PV}\frac{1}{k'_0 + a_-}\right] f(k_0,k,k'_0,k'),\end{aligned} \tag{13}$$

with the notation

$$f(k_0,k,k'_0,k') \equiv W_0(k_0,k,k'_0,k')F_0(k'_0,k'). \tag{14}$$

By expanding this expression, we obtain the integral term on the right-hand side of (7) as a sum of terms containing, respectively, products of the four-, three-, two-, one-, and zero-delta functions to be integrated over k'_0 and k' variables:

$$I = I_4 + I_3 + I_2 + I_1 + I_0.$$

The terms I_4 and I_3 containing products of the four- and three-delta functions are always 0 since the arguments of δ's cannot vanish simultaneously.

Among the terms I_2 containing the product of the two-delta functions, and for the same reasons as in the previous case, only the one containing $\delta(k'_0 - a_-)\delta(k'_0 + a_-)$ gives a nonzero contribution to the integral when $k'_0 = \pm a_- = 0$, that is, when $\varepsilon_{k'} = \varepsilon_{k_s}$. This contribution reads

$$\begin{aligned}I_2 &= -i\pi^2\int_0^\infty k'^2 dk'\int_{-\infty}^{+\infty} dk'_0\delta(k'_0 - a_-)\delta(k'_0 + a_-)\frac{f(k_0,k,k'_0,k')}{k'^2_0 - a_+^2},\\ &= +i\pi^2\int_0^\infty dk'\delta[2(\varepsilon_{k'} - \varepsilon_{k_s})]\frac{1}{4\epsilon_{k'}\epsilon_{k_s}} f(k_0,k,0,k') = \frac{i\pi^2 k_s}{8\varepsilon_{k_s}} W_0(k_0,k,0,k_s)F_0(0,k_s),\end{aligned} \tag{15}$$

where we have used

$$\frac{1}{a_-^2 - a_+^2} = -\frac{1}{4\epsilon_{k'}\epsilon_{k_s}}, \tag{16}$$

$$\delta[\varepsilon_{k'} - \varepsilon_{k_s}] = \frac{\epsilon_{k_s}}{k_s}\delta(k' - k_s). \tag{17}$$

The sum of the four terms from (13) containing a one-delta function reads

$$\begin{aligned}I_1 = -i&\int_0^\infty k'^2 dk'\int_{-\infty}^\infty dk'_0 f(k_0,k,k'_0,k')\\ &\times\left[-i\pi\delta(k'_0 - a_+)\frac{1}{k_0'^2 - a_-^2}\mathrm{PV}\frac{1}{k'_0 + a_+} + i\pi\delta(k'_0 + a_+)\frac{1}{k_0'^2 - a_-^2}\mathrm{PV}\frac{1}{k'_0 - a_+}\right.\\ &\left.-i\pi\delta(k'_0 - a_-)\frac{1}{k_0'^2 - a_+^2}\mathrm{PV}\frac{1}{k'_0 + a_-} + i\pi\delta(k'_0 + a_-)\frac{1}{k_0'^2 - a_+^2}\mathrm{PV}\frac{1}{k'_0 - a_-}\right].\end{aligned} \tag{18}$$

After integrating over dk'_0, the two integrals over dk' remain:

$$I_1 = \frac{\pi}{4\varepsilon_{k_s}} \mathrm{PV} \int_0^\infty \frac{k'^2 dk'}{2\varepsilon_{k'} a_-} [f(k_0, k, k'_0 = a_-, k') + f(k_0, k, k'_0 = -a_-, k')] - \frac{\pi}{4\varepsilon_{k_s}} \int_0^\infty \frac{k'^2 dk'}{2\varepsilon_{k'} a_+} [f(k_0, k, k'_0 = a_+, k') + f(k_0, k, k'_0 = -a_+, k')]. \quad (19)$$

Since the W_0 kernel is symmetric with respect to the change of sign of the variables k_0 and k'_0

$$W_0(-k_0, k, -k'_0, k') = W_0(k_0, k, k'_0, k'),$$

the solution $F_0(k_0, k)$ is also symmetric, that is, $F_0(-k_0, k) = F_0(k_0, k)$. By inserting this relation in (19), one can see that the symmetrized value of the kernel W_0 with respect to the variable k'_0

$$W_0^S(k_0, k, k'_0, k') = W_0(k_0, k, k'_0, k') + W_0(k_0, k, -k'_0, k') \quad (20)$$

appears naturally in the formulation. After substituting (14) and (20) into (19), one gets

$$I_1 = \frac{\pi}{4\varepsilon_{k_s}} \mathrm{PV} \int_0^\infty \frac{k'^2 dk'}{2\varepsilon_{k'} a_-} W_0^S(k_0, k, a_-, k') F_0(a_-, k') - \frac{\pi}{4\varepsilon_{k_s}} \int_0^\infty \frac{k'^2 dk'}{2\varepsilon_{k'} a_+} W_0^S(k_0, k, a_+, k') F_0(a_+, k'). \quad (21)$$

The first integrand in (21) is singular due to the factor $a_- = \varepsilon_{k'} - \varepsilon_{k_s}$ in the denominator, which vanishes at $k' = k_s$. It must be understood in the sense of principal value. The integral is well defined but, because of the singularity of the integrand at $k' = k_s$, explicitly manifested in the form

$$\frac{1}{a_-} = \frac{1}{\varepsilon_{k'} - \varepsilon_{k_s}} = \frac{\varepsilon_{k'} + \varepsilon_{k_s}}{k'^2 - k_s^2},$$

it requires an additional treatment to be transformed in a nonsingular form. The singularity is eliminated using the subtraction technique, that is,

$$\mathrm{PV} \int_0^\infty \frac{h(k') dk'}{k'^2 - a^2} = \int_0^\infty dk' \left[\frac{h(k') - h(a)}{k'^2 - a^2}\right], \quad (22)$$

based on the identity

$$\mathrm{PV} \int_0^\infty dk' \frac{1}{k'^2 - a^2} = 0, \quad \text{if } a \neq 0. \quad (23)$$

The condition $a \neq 0$ prevents us from setting $k_s = 0$ in our equation.

The second integrand in (21) is nonsingular and it does not require any additional treatment.

Let us finally consider in (13) the term which does not contain any delta function. Its contribution to the right-hand side of Eq. (7) can be represented as

$$I_0 = -i \int_0^\infty k'^2 dk' \mathrm{PV} \int_{-\infty}^\infty \frac{f(k_0, k, k'_0, k') dk'_0}{(k'_0 - a_-)(k'_0 + a_-)(k'_0 - a_+)(k'_0 + a_+)}$$
$$= -i \int_0^\infty \frac{k'^2 dk'}{4\varepsilon_{k_s}\varepsilon_{k'}} \mathrm{PV} \int_{-\infty}^\infty dk'_0 f(k_0, k, k'_0, k') \left[\frac{1}{(k'^2_0 - a_+^2)} - \frac{1}{(k'^2_0 - a_-^2)}\right] = I_0^+ + I_0^-.$$

The singularity $\frac{1}{(k'^2_0 - a_+^2)}$ in I_0^+ is regularized by using the same subtraction technique (22) as previously:

$$I_0^+ = -i \int_0^\infty \frac{k'^2 dk'}{4\varepsilon_{k_s}\varepsilon_{k'}} \mathrm{PV} \int_{-\infty}^\infty dk'_0 \frac{f(k_0, k, k'_0, k')}{(k'^2_0 - a_+^2)}$$
$$= -i \int_0^\infty \frac{k'^2 dk'}{4\varepsilon_{k_s}\varepsilon_{k'}} \mathrm{PV} \int_{-\infty}^\infty dk'_0 \left[\frac{f(k_0, k, k'_0, k')}{(k'^2_0 - a_+^2)} - \frac{f(k_0, k, a_+, k')}{(k'^2_0 - a_+^2)}\right].$$

The singularity $\frac{1}{(k'^2_0 - a_-^2)}$ in I_0^- requires some care since $a_- = \varepsilon_{k'} - \varepsilon_{k_s}$ vanishes when $k' = k_s$ while the relation (22) is valid only if $a \neq 0$. Notice that (23) diverges if $a = 0$ and therefore cannot be applied by the simple replacement $k' \to k'_0$. We use instead the relation

$$\mathrm{PV} \int_{-\infty}^\infty \frac{dx}{x^2 - a^2} = \frac{\pi^2}{2} \delta(a), \quad (24)$$

which has been derived in Appendix B. For $a = a_- = \varepsilon_{k'} - \varepsilon_{k_s}$, it takes the form

$$\mathrm{PV}\int_{-\infty}^{\infty}\frac{dk_0'}{[k_0'^2-(\varepsilon_{k'}-\varepsilon_{k_s})^2]}=\frac{\pi^2}{2}\delta(\varepsilon_{k'}-\varepsilon_{k_s}). \tag{25}$$

The subtraction formula (22) must now be replaced by

$$\begin{aligned}I_0^-&=i\int_0^\infty\frac{k'^2dk'}{4\varepsilon_{k_s}\varepsilon_{k'}}\mathrm{PV}\int_{-\infty}^{\infty}dk_0'\frac{f(k_0,k,k_0',k')}{k_0'^2-a_-^2}\\&=i\int_0^\infty\frac{k'^2dk'}{4\varepsilon_{k_s}\varepsilon_{k'}}\int_{-\infty}^{\infty}dk_0'\left[\frac{f(k_0,k,k_0',k')}{k_0'^2-a_-^2}-\frac{f(k_0,k,a_-,k')}{k_0'^2-a_-^2}\right]+\frac{i\pi^2}{2}\int_0^\infty\frac{k'^2dk'}{4\varepsilon_{k_s}\varepsilon_{k'}}\delta(\varepsilon_{k'}-\varepsilon_{k_s})f(k_0,k,a_-,k').\end{aligned} \tag{26}$$

The integrand of the first term in (26)—inside the square brackets—is again regular.

The last term is transformed using relation (17) and performing the integral over k' into

$$\frac{i\pi^2}{2}\int_0^\infty\frac{k'^2dk'}{4\varepsilon_{k_s}\varepsilon_{k'}}\delta(\varepsilon_{k'}-\varepsilon_{k_s})f(k_0,k,a_-,k')=\frac{i\pi^2{k_s}^2}{8\varepsilon_{k_s}}f(k_0,k,a_-,k_s)=\frac{i{k_s}^2}{8\varepsilon_{k_s}^2}W_0(k_0,k,0,k_s)F_0(0,k_s). \tag{27}$$

It gives exactly the same contribution as the two-delta term (15). The sum of these two contributions results in multiplying the coefficient in (15) by a factor of 2.

Since, as noticed above, the solution $F_0(k_0,k)$ is symmetric with respect to $k_0\to -k_0$, Eq. (7) can be reduced to the interval $k_0\in[0,\infty]$. After reducing the integral term to the same interval in k_0' and introducing the symmetric kernel W_S given by (20), we finally obtain the S-wave equation that we aimed to solve and that does not contain the pole singularities:

$$\begin{aligned}F_0(k_0,k)&=F_0^B(k_0,k)+\frac{i\pi^2k_s}{8\varepsilon_{k_s}}W_0^S(k_0,k,0,k_s)F_0(0,k_s)\\&+\frac{\pi}{2M}\int_0^\infty\frac{dk'}{\varepsilon_{k'}(2\varepsilon_{k'}-M)}\left[k'^2W_0^S(k_0,k,a_-,k')F_0(|a_-|,k')-\frac{2{k_s}^2\varepsilon_{k'}}{\varepsilon_{k'}+\varepsilon_{k_s}}W_0^S(k_0,k,0,k_s)F_0(0,k_s)\right]\\&-\frac{\pi}{2M}\int_0^\infty\frac{k'^2dk'}{\varepsilon_{k'}(2\varepsilon_{k'}+M)}W_0^S(k_0,k,a_+,k')F_0(a_+,k')\\&+\frac{i}{2M}\int_0^\infty\frac{k'^2dk'}{\varepsilon_{k'}}\int_0^\infty dk_0'\left[\frac{W_0^S(k_0,k,k_0',k')F_0(k_0',k')-W_0^S(k_0,k,a_-,k')F_0(|a_-|,k')}{k_0'^2-a_-^2}\right]\\&-\frac{i}{2M}\int_0^\infty\frac{k'^2dk'}{\varepsilon_{k'}}\int_0^\infty dk_0'\left[\frac{W_0^S(k_0,k,k_0',k')F_0(k_0',k')-W_0^S(k_0,k,a_+,k')F_0(a_+,k')}{k_0'^2-a_+^2}\right].\end{aligned} \tag{28}$$

Notice the appearance of the absolute value in the argument $|a_-|$ and that the relation $W_0^S(k_0,k,0,k_s)=2W_0(k_0,k,0,k_s)$ accounts for the factor 8 in the denominator of the nonintegral term. We also remind readers that $M=2\varepsilon_{k_s}$.

The origin of the different terms appearing in (28) is quite clear.

(1) The nonintegral term (second term in the first line) is a sum of two equal contributions in (13): the first one comes from the two-δ-function term I_2 [Eq. (15)], and the second one comes from the term without δ functions, precisely the last term in the subtraction (26) given by Eq. (27).

(2) The one-dimensional integral terms (second and third lines) result from the contribution I_1 [Eq. (21)], i.e., from the four contributions of one-δ-function terms—$\delta(k_0'\pm a_-)$ in the second line and $\delta(k_0'\pm a_+)$ in the third line—after integration over k_0'.

(3) The last two lines come from the product of four principal values (no δ function) and the contribution I_0 [Eq. (24)]; however, they also come without the term (27), which is incorporated into the nonintegral term in the first line of Eq. (28).

The differences appearing in squared brackets (second, fourth, and fifth lines) correspond to the subtractions (22) and (26) used to remove the pole singularities: $2\varepsilon_{k'}=M$ in the second line, $k_0'=a_-$ in the fourth line, and $k_0'=a_+$ in the fifth line, respectively.

III. KERNEL SINGULARITIES

In view of the numerical integration of (28), it is useful to know the precise positions of the singularities both in the kernel and in the Born term.

The above considerations were devoted to treat the poles of the free constituent propagators. However, these singularities are not the only ones. The propagator of the exchanged particle, i.e., the kernel (4), also has two poles which, after partial wave decomposition, turn into the logarithmic singularities of the W_0 kernel (11). Although the log singularities can be integrated numerically by "brute force," to improve precision, it is useful to treat them, too. Their positions are found analytically, both in k'_0 and k' variables.

Let us first consider the singularities on k'_0. It follows from (11) that W_0 is singular when $|\eta(k_0,k,k'_0,k')| = 1$, where η is defined by (12). Solving two equations $\eta = \pm 1$ relative to k'_0, we find the four singularities of W_0:

$$k'_0 = k_0 + \sqrt{\mu^2 + (k \pm k')^2},$$
$$k'_0 = k_0 - \sqrt{\mu^2 + (k \pm k')^2}. \tag{29}$$

The symmetric kernel W_0^S has four additional singularities when $|\eta(k_0,k,-k'_0,k')| = 1$. Together with (29), it means that W_0^S is singular at the eight points:

$$k'_0 = k_0 + \sqrt{\mu^2 + (k \pm k')^2},$$
$$k'_0 = k_0 - \sqrt{\mu^2 + (k \pm k')^2},$$
$$k'_0 = -k_0 + \sqrt{\mu^2 + (k \pm k')^2},$$
$$k'_0 = -k_0 - \sqrt{\mu^2 + (k \pm k')^2}. \tag{30}$$

However, since Eq. (28) was reduced to the interval $0 < k'_0 < \infty$, one should take into account only the singularities on the positive axis $k'_0 > 0$. This is equivalent to taking the absolute value of (30); that is, W_S is singular at the four k'_0 values:

$$k'_0 = |k_0 + \sqrt{\mu^2 + (k \pm k')^2}|,$$
$$k'_0 = |k_0 - \sqrt{\mu^2 + (k \pm k')^2}|.$$

Their positions depend on the integration variable k' (moving singularities) as well on the external momenta k_0 and k.

The numerical integration over k'_0 is split into as many intervals as needed in order to contain a single singularity in only one of its borders. The integral over each of these intervals is made safely by choosing an appropriate change of variable.

The kernel singularities in the k' variable manifest themselves only in the second and third lines of Eq. (28). They are also given by the solutions, with respect to k', of $\eta(k_0,k,k'_0,k') = \pm 1$ and $\eta(k_0,k,-k'_0,k') = \pm 1$ for the particular values $k'_0 = a_\pm$. For the symmetrized W_0^S kernel, this gives the positions detailed in what follows.

The term with $k'_0 = a_-$, written in the second line of Eq. (28), is singular at

$$k'_- = \frac{\epsilon k Q_+ \pm d_+\sqrt{Q_+^2 - 4m^2(d_+^2 - k^2)}}{2(d_+^2 - k^2)}, \qquad \epsilon = \pm 1,$$

where

$$Q_+ = d_+^2 - k^2 + m^2 - \mu^2, \quad d_+ = k_0 + \frac{M}{2}, \quad M = 2\varepsilon_{k_s}.$$

That is, the integrand of the second term versus k' has the four singularities denoted by $k_{-,1,2,3,4}$:

$$k'_{-,1} = \frac{+kQ_+ + d_+\sqrt{Q_+^2 - 4m^2(d_+^2 - k^2)}}{2(d_+^2 - k^2)},$$
$$k'_{-,2} = \frac{+kQ_+ - d_+\sqrt{Q_+^2 - 4m^2(d_+^2 - k^2)}}{2(d_+^2 - k^2)},$$
$$k'_{-,3} = \frac{-kQ_+ + d_+\sqrt{Q_+^2 - 4m^2(d_+^2 - k^2)}}{2(d_+^2 - k^2)},$$
$$k'_{-,4} = \frac{-kQ_+ - d_+\sqrt{Q_+^2 - 4m^2(d_+^2 - k^2)}}{2(d_+^2 - k^2)}. \tag{31}$$

The term with $k'_0 = a_+$, written in the third line of Eq. (28), is singular at

$$k'_+ = \frac{\epsilon k Q_- \pm |d_-|\sqrt{Q_-^2 - 4m^2(d_-^2 - k^2)}}{2(d_-^2 - k^2)}, \qquad \epsilon = \pm 1,$$

where

$$Q_- = d_-^2 - k^2 + m^2 - \mu^2, \qquad d_- = k_0 - \frac{M}{2}.$$

That is, the integrand has the four singularities $k'_{+,1,2,3,4}$. Their positions are obtained from Eqs. (31) by the replacement $Q_+ \to Q_-$, $d_+ \to d_-$.

Since the integration domain of the k' variable is positive, one should take into account only the real and positive values of the above singularities.

A. Born term

The S-wave Born term (8) is singular both in the k_0 and k variables. The singularities are logarithmic in its real part, and there are Heaviside-like discontinuities in the

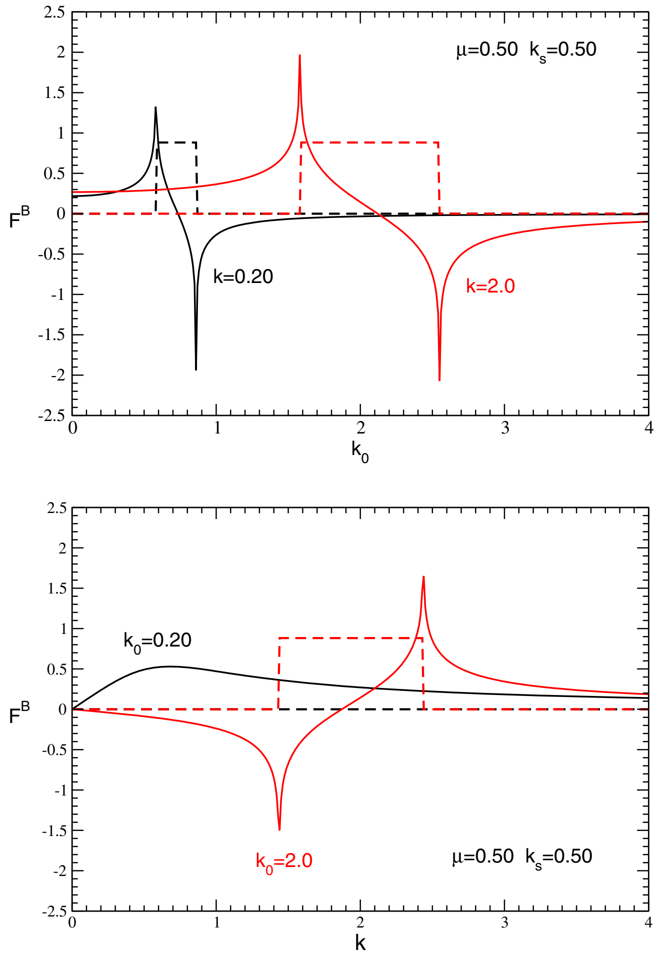

FIG. 3 (color online). Singularities of the S-wave Born term F_0^B as a function of k_0 for two different values of $k = 0.20$ and $k = 2.0$ (upper panel) and as a function of k for two different values of $k_0 = 0.20$ and $k_0 = 2.0$ (lower panel).

imaginary one. Their positions can be found from the condition $|\eta(k_0, k, k'_0 = 0, k' = k_s)| = 1$, where η is defined in (12).

In the k_0 variable (see the upper part of Fig. 3), there are two singularities at the points

$$k_0(k) = \sqrt{(k \pm k_s)^2 + \mu^2}$$

for any value of k.

In the variable k (see the bottom part of Fig. 3), the singularities (in its definition domain $k > 0$) are given by the same equation rewritten in the form

$$k(k_0) = |k_s \pm \sqrt{k_0^2 - \mu^2}|$$

and exist only for $k_0 > \mu$.

The Minkowski BS amplitude F_0 has many nonanalyticities due to its Born term F_0^B and to the interaction kernel, including inelastic threshold effects. However, the only singularities (infinite values and discontinuities) in the physical domain of its arguments are those originated by the Born term F_0^B itself. Their existence makes it difficult to represent F_0 on a basis of regular functions in view of a numerical solution of Eq. (7). To circumvent this problem, we factorize out the Born amplitude by making the replacement

$$F_0 = F_0^B \chi f_0, \tag{32}$$

where f_0 is a smoother function obeying the BS transformed equation

$$F_0 = F_0^B + K F_0 \Rightarrow f_0 = \frac{1}{\chi} + \frac{1}{\chi F_0^B} K F_0^B \chi f_0.$$

χ is an arbitrary but suitable function introduced to provide a convenient inhomogeneous term. After that, the singularities of the F_0 are casted into the kernel and integrated using the same procedure as above. We obtain in this way a nonsingular equation for a nonsingular function f_0 which can be solved by standard methods.

The off-mass-shell BS amplitude F_0 in Minkowski space can thus be safely computed.

IV. EXTRACTING SCATTERING OBSERVABLES

The amplitude $F(k, k_s; p)$ satisfying the BS equation (3) is related to the S matrix by

$$S = 1 + i(2\pi)^4 \delta^{(4)}(p - p_f) F(k, k_s; p).$$

The unitarity condition for the S matrix $S^\dagger S = 1$ is rewritten in terms of the amplitude $F^{\rm on}(k, k_s; p)$ (which is on the mass shell $k_1^2 = k_2^2 = k_{1s}^2 = k_{2s}^2 = m^2$) as follows:

$$i(F^{\rm on\dagger} - F^{\rm on}) = \int F^{\rm on\dagger} F^{\rm on} (2\pi)^4 \delta^{(4)}(p - k_1 - k_2) \frac{d^3 k_1}{(2\pi)^3 2\varepsilon_1} \frac{d^3 k_2}{(2\pi)^3 2\varepsilon_2}. \tag{33}$$

The sum over intermediate states in the product $S^\dagger S$ is understood as integration with the measure given in (33). After substituting the partial wave decomposition (5) into Eq. (33), the latter obtains the form

$$i(F^{\rm on *}_L - F^{\rm on}_L) = \frac{2k_s}{\varepsilon_{k_s}} |F^{\rm on}_L|^2, \tag{34}$$

where $F^{\rm on}_L \equiv F_L(k_0 = 0, k = k_s)$ is the on-shell amplitude. The function satisfying Eq. (34) is represented as

$$F^{\rm on}_L = \frac{\varepsilon_{k_s}}{k_s} \exp(i\delta_l) \sin\delta_l \tag{35}$$

with arbitrary real δ_l. Solving Eq. (35) relative to δ_l, we find that the on-shell amplitude determines the phase shift according to

$$\delta_L(k_s) = \frac{1}{2i}\log\left(1 + \frac{2ik_s}{\varepsilon_{k_s}} F_L^{\rm on}\right). \tag{36}$$

V. EUCLIDEAN SCATTERING AMPLITUDE

Equation (28) is free from the poles of the constituent propagators and after the appropriate treatment of the logarithmic singularities provides the desired solution for the off-shell amplitude in Minkowski space. This equation is, however, rather cumbersome, even in the simplest case of two scalar particles in the S wave we are considering, and looks rather different from the initial BS equation (3). It would thus be of the highest interest to have at our disposal an independent test of the numerical solutions.

The test we have performed is based on a Euclidean version of the initial BS equation for the scattering amplitude in Minkowski space (3), formally written as

$$F^M = F^{M,B} + K^M F^M. \tag{37}$$

We can define the so-called Euclidean BS scattering amplitude (see Fig. 4) by

$$F^E(k_4, k) = F^M(k_0 = ik_4, k).$$

By applying the Wick rotation $k_0 = ik_4$ to (3), we will derive in what follows the equation satisfied by F^E. We will introduce all along this section the index M or E to distinguish between both amplitude quantities.

Some preliminary remarks are in order.

(1) Contrary to the Minkowski case, the off-shell Euclidean amplitude cannot be used to compute physical observables like electromagnetic form factors even in the bound-state case. The reason is the impossibility to make the Wick rotation in the form factor integral [6,8].

(2) The on-mass-shell condition for $k_0 = 0$ corresponds to $k_4 = 0$. Therefore, both amplitudes, although obeying different equations, should coincide on the mass shell $F^M(k_0 = 0, k_s) = F^E(k_4 = 0, k_s)$ and should thus provide the same phase shifts. This property will be used to check our Minkowski results.

(3) In the case of the scattering states, the Wick rotation cannot be performed in a naive way to Eq. (37) by simply replacing $k_0 \to k_4 = -ik_0$. This important point will be developed below.

A. Rotating the integration contour

Let us consider the pole positions appearing in Eq. (7) and given by (9). The integration domain $k' < k_s$ corresponds to $\varepsilon_{k'} < \varepsilon_{k_s}$ and hence to $\mathrm{Re}[k_0'^{(2)}] > 0$ and $\mathrm{Re}[k_0'^{(3)}] < 0$. The positions of these singularities are illustrated in Fig. 5. The contour cannot be anticlockwise rotated without taking into account the residues at the poles $k_0'^{(2)}$ and $k_0'^{(3)}$. Notice that for the bound-state problem, the value ε_{k_s} is replaced by $\frac{M}{2}$. Since for any k' we have $\varepsilon_{k'} > \frac{M}{2}$, when performing the Wick rotation in the bound-state equation, there is no crossed singularity and so no additional contribution.

Therefore, for the bound-state case, the positions of singularities allow us to safely rotate the contour, as it is shown in Fig. 4. In this way, one gets the equation for the Euclidean BS amplitude F^E.

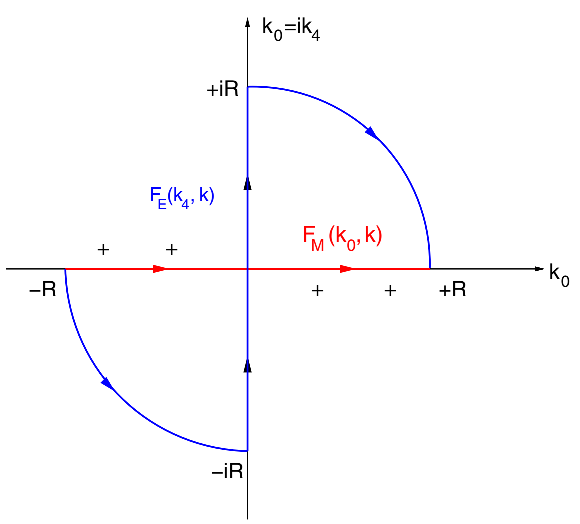

FIG. 4 (color online). Minkowski (F_M) and Euclidean (F_E) BS amplitudes in the complex k_0 plane with (in crosses) bound-state singularities.

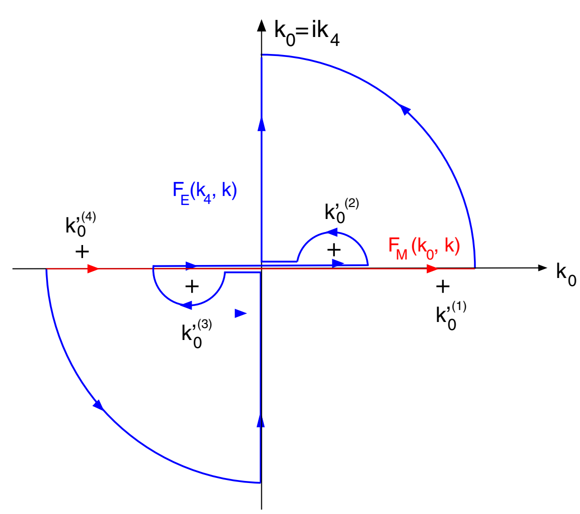

FIG. 5 (color online). Singularities of the propagators for the scattering state and the integration contour after rotation in the complex plane k_0, if $k' < k_s$.

On the contrary, for the scattering state case, the rotated contour in the complex plane of k'_0 (see Fig. 5) crosses two of the four pole singularities $k_0'^{(2)}$ and $k_0'^{(3)}$ displayed in the integrand of Eq. (3). They are on the real axes k'_0 at the points $k'_0 = \pm(\varepsilon_{k_s} - \varepsilon_{k'})$. The residues in these poles (to be added to the integral terms) are still expressed in terms of the Minkowski off-shell amplitude:

$$\tilde{F}^M_L(k') \equiv F^M_L(k'_0 = \varepsilon_{k_s} - \varepsilon_{k'}, k'). \tag{38}$$

Thus, when transforming Eq. (37) into a Euclidean, we obtain not a Euclidean equation but a nonsingular equation which indeed contains on the left-hand side the Euclidean BS amplitude F^E but, however, in the right-hand side, under the integrals, contains both the Euclidean amplitude F^E and the particular Minkowski amplitude $\tilde{F}^M_L(k)$ defined in (38). One can similarly derive another nonsingular equation which contains in the left-hand side the particular Minkowski amplitude $\tilde{F}^M_L(k)$ and in the right-hand side, under the integrals, again both the Euclidean amplitude F^E and the particular Minkowski amplitude $\tilde{F}^M_L(k)$. In this way, we derive the system of two equations which couples the Euclidean amplitude $F^E_L(k_4, k)$ to the particular Minkowski off-shell amplitude $\tilde{F}^M_L(k)$. A similar derivation is described in Ref. [27].

An additional test is to check that the off-shell Minkowski amplitude $F^M_L(k_0, k)$ obtained by solving Eq. (7) coincides, for the particular value $k_0 = \pm(\varepsilon_{k_s} - \varepsilon_k)$, with the independent solution $\tilde{F}^M_L(k)$ of the system of equations. Furthermore, we can check that all three on-mass amplitudes (for the S waves, in particular) coincide with each other:

$$F^M_0(k_0 = 0, k = k_s) = F^E_0(k_4 = 0, k = k_s) = \tilde{F}^M_0(k_s)$$

and therefore give the same phase shifts.

Below, in this section, we sketch the derivation of this system of two equations which couples the amplitudes $F^E(k_4, k, z)$ and $\tilde{F}^M(k, z)$. In this derivation, we will consider the case when the amplitudes are not decomposed in the partial waves. The two equations for the S-wave amplitudes $F^E_0(k_4, k)$ and $\tilde{F}^M_0(k)$ solved numerically will be given in Appendix C.

We should also analyze the position of singularities in the kernel (4). They are at the points

$$k_0'^{\pm} = k_0 \pm \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2} \mp i\epsilon.$$

When we rotate the contour around $k'_0 = 0$, we must distinguish two cases. If $|k_0| < \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2}$, the singularity

$$k_0'^{+} = k_0 + \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2} - i\epsilon \tag{39}$$

is in the fourth quadrant and

$$k_0'^{-} = k_0 - \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2} + i\epsilon \tag{40}$$

in the second one. In this case, the contour can be rotated anticlockwise without crossing the singularities. If $|k_0| > \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2}$, both singularities are in the same half-plane (e.g., at the right half-plane, if $k_0 > 0$) and the contour cannot be rotated.

However, Wick rotation must be performed in both the k'_0 and k_0 variables simultaneously. That is, by rotating the integration contour in k'_0 by an angle ϕ, we also change the variable $k_0 \to k_0 \exp(i\phi)$. Then, the positions of singularities (39) and (40) are also rotated. As it can be easily checked, $k_0'^{-}$ is rotated faster than the contour and $k_0'^{+}$ is rotated slower; therefore, they move away from the contour and the contour rotation can be safely done. Its final position, after rotation by $\phi = \pi/2$, is $-i\infty < k'_0 < i\infty$, whereas the final value of k_0 turns into ik_0. The singularities of the propagator (4) do not prevent the Wick rotation in both variables.

For the amplitude $F(\vec{k}, k_0 = \varepsilon_{k_s} - \varepsilon_k; \vec{k}_s)$, the condition $|k_0| < \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2}$ turns into

$$\varepsilon_{k_s} - \varepsilon_{k'} < \sqrt{(\vec{k} - \vec{k}')^2 + \mu^2}.$$

Since we integrate over $\vec{k}'$, the maximal value of the left-hand side is $\varepsilon_{k_s} - m$ and the minimal value of the right-hand side is μ. Hence, this inequality is violated if $\varepsilon_{k_s} - m > \mu$, that is, when $\varepsilon_{k_s} > m + \mu \to \sqrt{s} > 2m + 2\mu$, i.e., above the two-meson creation threshold. The singularities of the one-boson exchange kernel allow the contour rotation only in this kinematical domain. Above that, additional contributions should be taken into account. The same conclusion was found in Ref. [27]. In our solution of the Euclidean equation, we will not exceed the two-meson creation threshold.

B. Euclidean equation

We start with performing the Wick rotation shown in Fig. 5 to Eq. (3). In the c.m. frame $\vec{p} = 0$, it is transformed into

$$F^E(k_4, \vec{k}; \vec{k}_s) = V^B(k_4, \vec{k}; \vec{k}_s) + \int \frac{d^4k'}{(2\pi)^4} \frac{V(k_4, \vec{k}'; k'_4, \vec{k}') F^E(k'_4, \vec{k}'; \vec{k}_s)}{({k'_4}^2 + a_-^2)({k'_4}^2 + a_+^2)} + S, \tag{41}$$

where

$$V(k_4,\vec{k};k'_4,\vec{k}') = \frac{16\pi m^2\alpha}{(k_4-k'_4)^2+(\vec{k}-\vec{k}')^2+\mu^2}, \tag{42}$$

$$V^B(k_4,\vec{k};\vec{k}_s) = V(k_4,\vec{k};k'_4=0,\vec{k}'=\vec{k}_s), \tag{43}$$

and S denotes the contribution due to the two singularities shown in Fig. 5. This contribution, existing only if $\varepsilon_{k_s}-\varepsilon_{k'}>0$, is given by the sum of two residues $S=S_1+S_2$; S_1 is the contribution from $k'_0=(\varepsilon_{k_s}-\varepsilon_{k'})+i\epsilon$ multiplied by $(2\pi i)$ and S_2 the one from $k'_0=-(\varepsilon_{k_s}-\varepsilon_{k'})-i\epsilon$ multiplied by $(-2\pi i)$.

Contribution S_1 has the form

$$S_1(k_0) = \frac{\pi g^2}{4(2\pi)^4}\frac{\tilde{F}^M(k',z')}{\varepsilon_{k_s}\varepsilon_{k'}\left[-a_-+i\epsilon\right]\left[-a_-+\sqrt{(\vec{k}'-\vec{k})^2+\mu^2}-k_0\right]\left[-a_--\sqrt{(\vec{k}'-\vec{k})^2+\mu^2}-k_0+i\epsilon\right]}, \tag{44}$$

whereas S_2 is given by $S_2(k_0)=S_1(-k_0)$. The sum S_1+S_2 is symmetric relative to $k_0\to -k_0$, as it should be. At this point, it is interesting to keep these expressions for $S_{1,2}$ with k_0 not replaced by ik_4. The reason will become clear later, when, deriving another equation, we will substitute $k_0=\varepsilon_{k_s}-\varepsilon_k$. Above the $2m+\mu$ inelastic threshold, these factors give an imaginary contribution, making the elastic phase shift complex. The above form of $S_{1,2}$ is convenient to find this imaginary part.

Setting $k_0=ik_4$, the preceding expression reads

$$S(ik_4) = \frac{g^2\pi}{(2\pi)^4}\int_{k'<k_s} d^3k'\,\frac{\tilde{F}^M(k',z')}{2\varepsilon_{k'}\varepsilon_{k_s}(a_--i\epsilon)}\frac{\left[k_4^2-(a_--\sqrt{(\vec{k}'-\vec{k})^2+\mu^2})(a_-+\sqrt{(\vec{k}'-\vec{k})^2+\mu^2})\right]}{\left[k_4^2+(a_--\sqrt{(\vec{k}'-\vec{k})^2+\mu^2})^2\right]\left[k_4^2+(a_-+\sqrt{(\vec{k}'-\vec{k})^2+\mu^2})^2\right]}. \tag{45}$$

Equation (41) is not singular. The factor $1/[{k'_4}^2+a_-^2(k')]$ in the integrand of (41) is singular when $k'_4=0$ and $k'=k_s$ simultaneously, but this singularity is, in fact, canceled by a similar term in S given by (45). It is, however, convenient to cancel these two singularities explicitly and analytically and to obtain a regular resulting expression. The transformations are elementary but lengthy and will not be carried out in detail. Below are indicated the main steps.

We make a subtraction in the integrand and add the subtracted term:

$$\begin{aligned} F^E(k,k_4,z) = {} & V^B(k_4,\vec{k},\vec{k}_s) + \frac{1}{(2\pi)^4}\int d^3k'\int_{-\infty}^{\infty} dk'_4 V(k_4,\vec{k};k'_4,\vec{k}')\left\{\frac{F^E(k'_4,k',z')}{({k'_4}^2+a_-^2)({k'_4}^2+a_+^2)} - \frac{F^E(0,k',z')}{({k'_4}^2+a_-^2)a_+^2}\right\} \\ & + \frac{1}{(2\pi)^4}\int d^3k'\int_{-\infty}^{\infty} dk'_4\frac{V(k_4,\vec{k};k'_4,\vec{k}')F^E(0,k',z')}{({k'_4}^2+a_-^2)a_+^2} + S[\tilde{F}^M]. \end{aligned} \tag{46}$$

There is no singularity in the difference. The subtracted term is not unique. Our choice was motivated in order to obtain an analytic result for the integral over dk'_4 in the last line of Eq. (46).

The term which we add is singular: $k'=k_s$. We perform an additional subtraction to eliminate this singularity and again add the subtracted term. This additional term is, of course, again singular but analytic. Its contribution, after integration in the limits $0<k'<k_s-\delta$ and $k_s+\delta<k'<\infty$ and at $\delta\to 0$, is $\sim\log(\delta/m)$. It is exactly canceled analytically by a similar term in the singular part of S. The resulting S-wave equation is regular and given in Appendix C, Eq. (C1).

Equation (46) and, equivalently (after the cancellation of singularities), Eq. (C1) relate the Euclidean amplitude $F^E(k_4,k,z)$ and the Minkowski one $\tilde{F}^M(k,z)$ appearing in S. To determine both amplitudes, we should obtain an additional equation.

This new equation is still obtained by performing a Wick rotation $k'_0=ik'_4$ to (3). However, instead of taking $k_0=ik_4$, we set $k_0=\varepsilon_{k_s}-\varepsilon_k$ for $k<k_s$. As discussed at the end of Sec. V, for this particular value of k_0, the kernel singularities do not prevent the Wick rotation below the two-meson creation threshold. In this way, we get the following equation (symbolically):

$$\tilde{F}_0^M(k,z) = \text{right-hand side of Eq. (46) at } [k_4=i(\varepsilon_{k_s}-\varepsilon_k)], \tag{47}$$

that is, the right-hand side term of Eq. (46) taken at the value $k_4=i(\varepsilon_{k_s}-\varepsilon_k)$. The corresponding explicit S-wave equation is given in Appendix C, Eq. (C8).

VI. RESULTS

We present in this section the results of solving the BS equation in Minkowski space (28) and the coupled Euclidean-Minkowski system of Eqs. (C1) and (C8). Some details of the numerical methods used are given in Appendix A.

We have first computed the bound-state solutions, denoted by $\Gamma_0(k_0, k)$, by dropping the inhomogeneous term $F_0^B(k_0, k)$ in (28) and setting $M = 2m - B$. The binding energies B thus obtained coincide, within four-digit accuracy, with the ones calculated in our previous work [4].

An interesting issue is the appearance of an imaginary part in $\Gamma_0(k_0, k)$ when normalized, for instance, by $\Gamma(0,0) = 1$. In spite of this real normalization condition, $\Gamma_0(k_0, k)$ will become imaginary, depending on the kinematical domain of its arguments corresponding to the virtual meson creation. Written in terms of variables k_i, defined in (1), they read

$$k_1^2 > (m+\mu)^2 \quad \text{or} \quad k_2^2 > (m+\mu)^2, \tag{48}$$

which, in terms of the variables (k_0, k) in the center-of-mass frame, become

$$\left(\frac{M}{2} + k_0\right)^2 - \vec{k}^2 > (m+\mu)^2,$$
$$\left(\frac{M}{2} - k_0\right)^2 - \vec{k}^2 > (m+\mu)^2.$$

Γ obtains an imaginary part if one of these two preceding conditions is fulfilled. In the (k_0, k) plane and for positive values of k_0, this gives the locus

$$k_0(k) = -\frac{M}{2} + \sqrt{k^2 + (m+\mu)^2}. \tag{49}$$

Above this curve (represented in Fig. 6), the imaginary part of $\Gamma \neq 0$.

We display in Fig. 7 the k_0 dependence of the imaginary part of the amplitude $\Gamma(k_0, k)$ obtained in our calculations for different values of k. It corresponds to the parameters $\alpha = 1.44$, $\mu = 0.50$, and $B = 0.01$. We can thus check that a vanishing imaginary part of Γ appears at the k_0 values given by Eq. (49). Note also the difficulty in reproducing a sharp nonanalytic threshold behavior in terms of smooth functions even if they are as flexible as splines. The small oscillations in the vicinity of the threshold are artifacts of our spline basis. They can be reduced by increasing the number of basis elements.

The scattering amplitude $F_0(k_0, k)$ in Minkowski space has been calculated for $L = 0$ states, and the corresponding phase shifts have been extracted according to (36).

The BS relativistic formalism accounts naturally for the meson creation in the scattering process, when the available kinetic energy allows it. The inelasticity threshold corresponding to n-particle creation is given by

$$k_s^{(n)} = m\sqrt{\left(\frac{\mu}{m}\right)n + \frac{1}{4}\left(\frac{\mu}{m}\right)^2 n^2}. \tag{50}$$

Below the first inelastic threshold $k_s^{(1)} = \sqrt{m\mu + \mu^2/4}$, the phase shifts are real. This unitarity condition is not automatically fulfilled in our approach but appears as a consequence of handling the correct solution and provides a stringent test of the numerical method. Above $k_s^{(1)}$, the phase shift obtains an imaginary part which behaves like

$$\text{Im}[\delta_0] \sim (k_s - k_s^{(1)})^2 \tag{51}$$

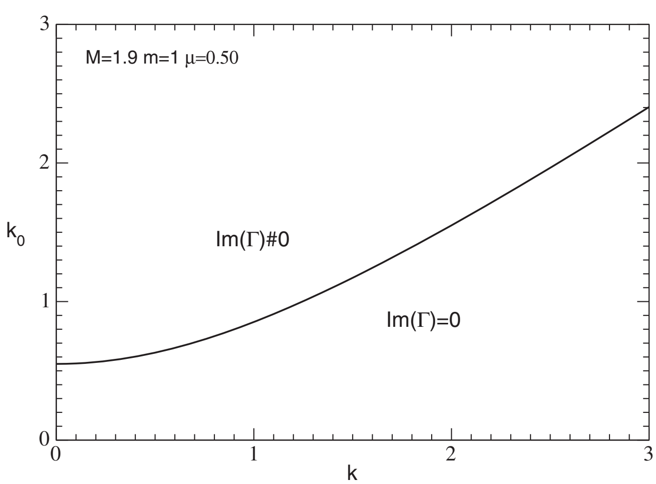

FIG. 6. Locus for $\text{Im}(\Gamma) = 0$ in the (k_0, k) plane for $m = 1$, $M = 1.90$, and $\mu = 0.5$. Below this curve, $\text{Im}(\Gamma) = 0$, and above, $\text{Im}(\Gamma) \neq 0$.

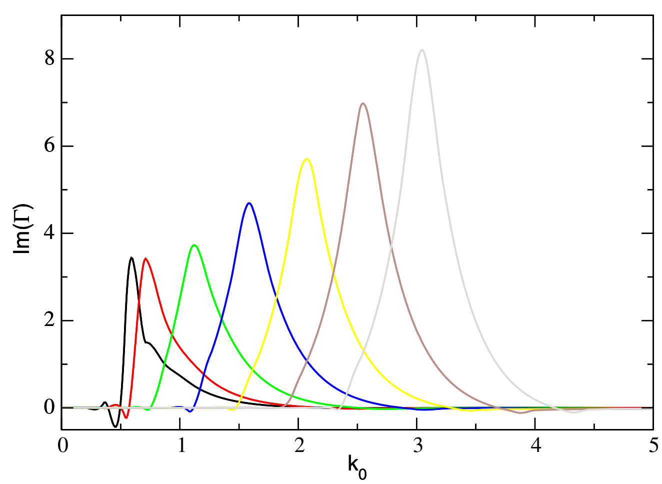

FIG. 7 (color online). $\text{Im}[\Gamma(k_0, k)]$ as a function of k_0 for different values of k. It corresponds to $\alpha = 1.44$, $B = 0.01$, and $\mu = 0.5$.

TABLE I. Real and imaginary parts of the phase shift (degrees) calculated by solving Eq. (28) versus incident momentum k_s for $\alpha = 1.2$ and $\mu = 0.5$. The corresponding first inelastic threshold is $k_s^{(1)} = 0.75$.

k_s	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.3	1.5	2.0	2.5	3.0
Re$[\delta_0]$	124	99.9	77.8	65.1	56.2	49.3	43.9	39.4	35.7	32.5	29.7	22.8	19.3	13.3	9.78	7.78
Im$[\delta_0]$	0	0	0	0	0	0	0	0	0.033	0.221	0.453	0.848	0.852	0.578	0.333	0.203

in the threshold vicinity. Higher inelasticity thresholds, corresponding to the creation of two, three, etc., intermediate mesons at $k_s^{(n)}$, are also taken into account in our calculations.

Some selected results corresponding to $\alpha = 1.2$ and $\mu = 0.5$ are listed in Table I. We have also solved the system of equations, derived in Sec. V and Appendix C [Eqs. (C1) and (C8)], coupling the Euclidean amplitude to the Minkowski one at the particular value $k_0 = \varepsilon_{k_s} - \varepsilon_k$. The phase shifts found by these two independent methods are consistent with each other within the accuracy given in this table. They are also rather close to the ones found in Ref. [27].

Figure 8 (upper panel) shows the real phase shifts calculated with BS (solid line) as a function of the scattering momentum k_s. They are compared to the nonrelativistic (NR) values (dashed lines) provided by the Schrödinger equation with the Yukawa potential. For this value of α, there exists a bound state and, according to the Levinson theorem, the phase shift starts at 180°. One can see that the difference between relativistic and nonrelativistic results is considerable, even for a relatively small incident momentum.

The lower panel shows the imaginary part of the phase shift. It appears starting from the first inelastic meson-production threshold $k_s^{(1)} = 0.75$ and displays the expected quadratic behavior (51). Simultaneously, the modulus squared of the S matrix (dashed line) starts differing from unity. The results of this figure contain the contributions of the second $k_s^{(2)} = 1.118$ meson creation threshold, the third one $k_s^{(3)} = 1.435$, etc., up to the eight-meson creation threshold $k_s^{(8)} = 2.828$. Notice that, as mentioned in Sec. V B, the applicability of the method coupling the Euclidean and Minkowski amplitudes is limited in its present formulation to the second inelastic threshold while our direct Minkowski-space approach can go through.

The low energy parameters were computed directly at $k_s = 0$ and found to be consistent with a quadratic fit to the effective range function $k \cot \delta_0(k) = -\frac{1}{a_0} + \frac{1}{2} r_0 k^2$. The BS scattering length a_0 as a function of the coupling constant α is given in Fig. 9 for $\mu = 0.50$. It is compared to the NR values. The singularities correspond to the appearance of the first bound state at $\alpha_0 = 1.02$ for BS and $\alpha_0 = 0.840$ for NR. One can see that the differences between a relativistic and a nonrelativistic treatment of the same problem are not of kinematical origin since even for processes involving zero energy, they can be substantially large, especially in the presence of a bound state. It is worth noticing that only in the limit $\alpha \to 0$ are the two curves tangent to each other, and in this region, the results are given by the Born approximation

$$a_0^B = -\frac{1}{\mu} \frac{m}{\mu} \alpha, \tag{52}$$

which is the same for the NR and the BS equations. Beyond this region, both dynamics are not compatible. This

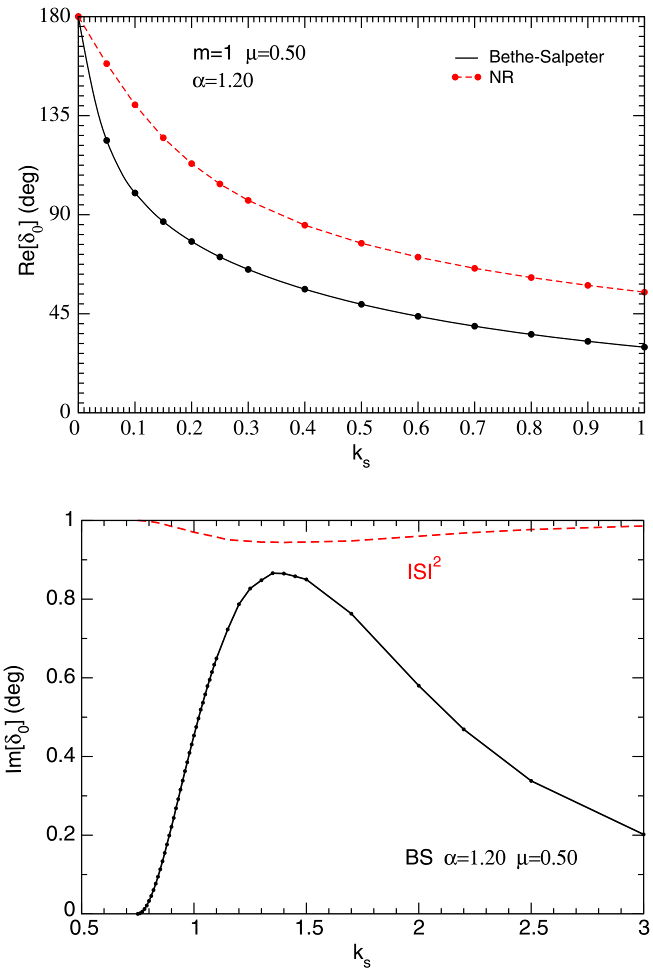

FIG. 8 (color online). Upper panel: Real phase shift (degrees) for $\alpha = 1.2$ and $\mu = 0.50$ calculated via the BS equation (solid line) compared to the nonrelativistic results (dashed line). Lower panel: Imaginary phase shift (degrees) calculated via the BS equation (solid line) and the value $|S|^2 = \exp(-2\mathrm{Im}(\delta))$ (dashed line).

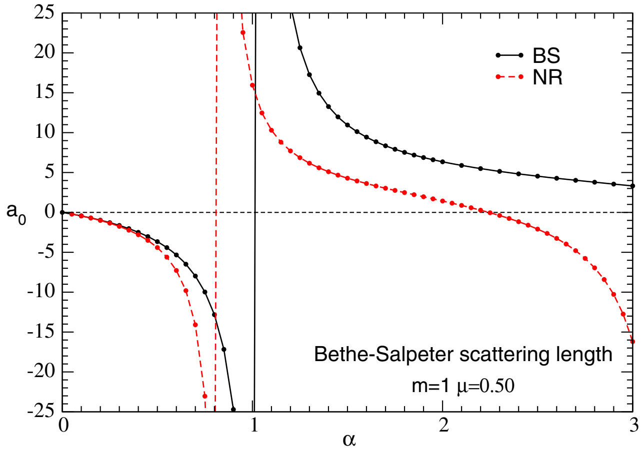

FIG. 9 (color online). BS scattering length a_0 versus the coupling constant α (solid lines), compared to the nonrelativistic results (dashed lines) for $\mu = 0.5$.

nonmatching between NR and relativistic equations was already pointed out in Refs. [32,33] when computing the binding energies B of a two-scalar and two-fermion system in the limit $B \to 0$ with different relativistic approaches. A recent work [16] devoted to this problem proposes the construction of an equivalent nonrelativistic potential using the technique of geometrical spectral inversion. It would be interesting to check the robustness of this equivalence by extending it to the scattering states and to form factors.

Some numerical values of the scattering length for different values of the exchanged mass μ were given in Ref. [10], Table I. It can be checked by direct inspection that the scaling properties of the nonrelativistic equation [34], in particular, the relation between the scattering length corresponding to different values of μ and the coupling constants

$$a_0\left(\frac{\mu}{m}, \alpha\right) = \frac{1}{\mu} a_0\left(1, \frac{\alpha}{\frac{\mu}{m}}\right) \tag{53}$$

are no longer valid except in the Born approximation region.

We display in Fig. 10 the factorized part of the off-shell scattering amplitude f_0, defined in (32), as a function k_0 for different values of k. It corresponds to the parameters $\alpha = 1.2$, $\mu = 0.5$, and $k_s = 1.0$. The upper panel displays its real part and the lower panel the imaginary one. Its k dependence is shown in Fig. 11 for different values of k_0. The regular amplitudes f_0 are those effectively computed in our approach. As one can see, these functions are no longer singular, although they present several sharp structures and cusps both on its real and imaginary parts. The corresponding three-dimensional plots $F_0(k_0, k)$ for the values $\alpha = 0.5$, $\mu = 0.5$, and $k_s = 0.5$ are given in Fig. 4 of Ref. [10].

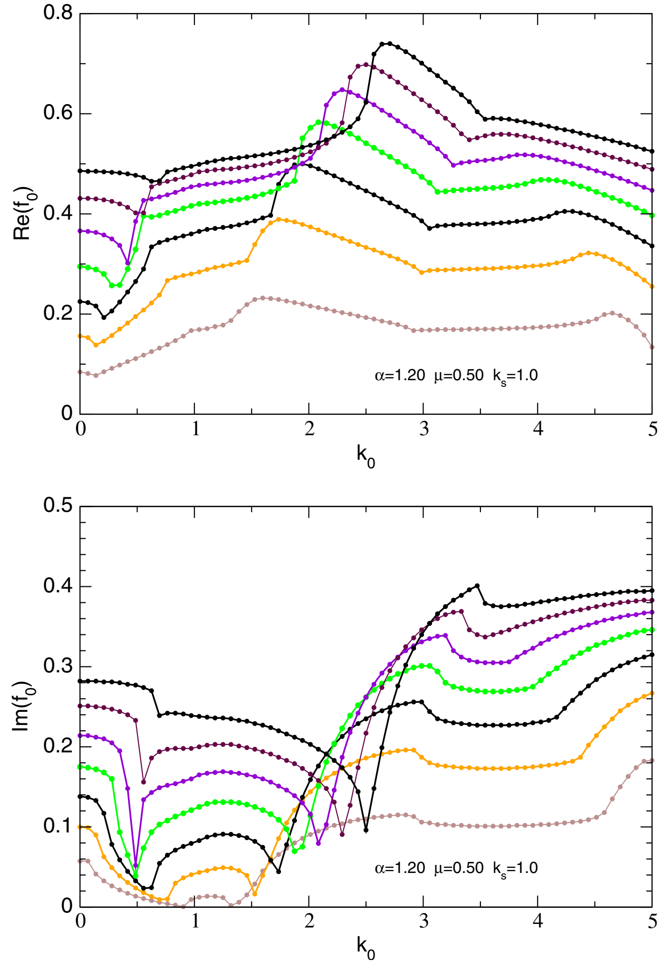

FIG. 10 (color online). Real (upper panel) and imaginary (lower panel) parts of the factorized off-shell scattering amplitude f_0 defined in Eq. (32) for $\alpha = 1.2$ and $\mu = 0.50$ versus k_0 for different fixed values of k.

The off-shell scattering amplitudes F_0 computed using our direct method in Minkowski space have been tested using also the results of the coupled Euclidean-Minkowski equation derived in Sec. V. As it was explained in Sec. V, these equations couple the Euclidean amplitude to the Minkowski one for the particular off-shell value $k_0 = \varepsilon_{k_s} - \varepsilon_k$, denoted by $\tilde{F}_0(k)$. The comparison of the off-shell amplitude $\tilde{F}_0(k)$ found by this method with the solution of Eq. (28) $F_0(k) = F^M(k_0 = \varepsilon_{k_s} - \varepsilon_k, k)$ for the same argument k_0 provides an independent test of our direct method based on Eq. (28).

Two illustrative examples are shown in Fig. 12 for different values of the parameters α, μ, and k_s. The solid lines denote the real parts of the amplitudes $F_0(k)$ (in black) and $\tilde{F}_0(k)$ (solid red line), whereas the dashed lines denote their imaginary parts.

The upper panel corresponds to $\alpha = 0.5$, $\mu = 0.50$, and $k_s = 0.5$. The results of $F_0(k)$ and $\tilde{F}_0(k)$ are not visibly

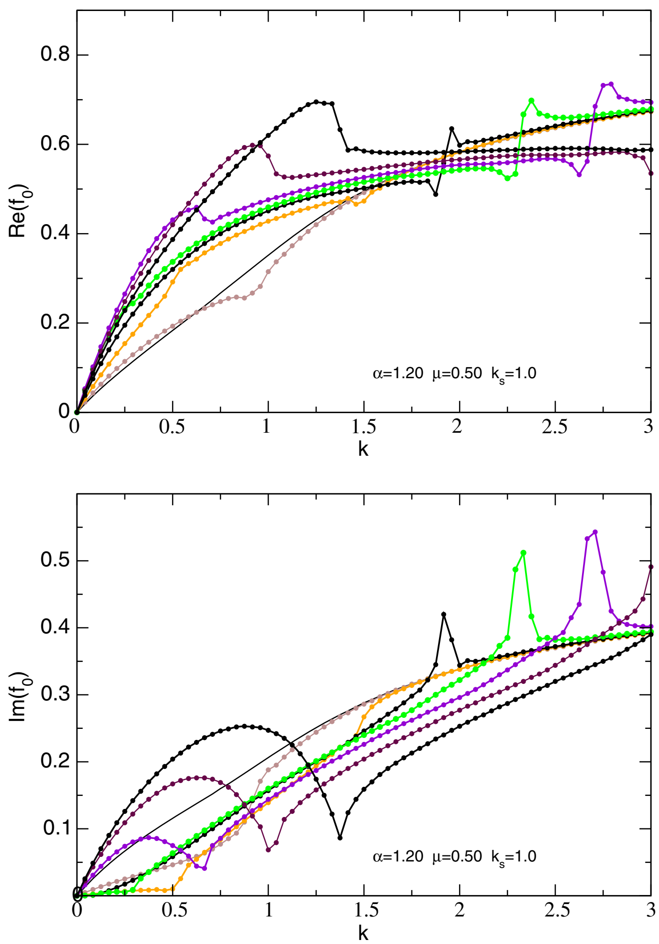

FIG. 11 (color online). Real (upper panel) and imaginary (lower panel) parts of the factorized off-shell scattering amplitude f_0 defined in Eq. (32) for $\alpha = 1.2$ and $\mu = 0.50$ versus k for different fixed values of k_0.

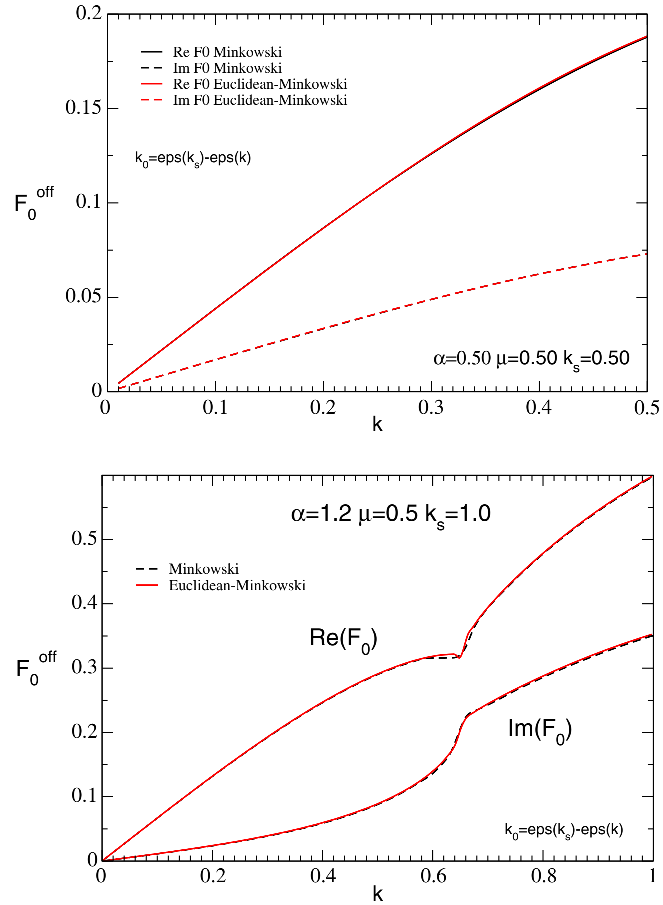

FIG. 12 (color online). Comparison between the Minkowski amplitude F_0 (black lines) and $\tilde{F}_0$ (red lines) obtained by solving the Euclidean-Minkowski coupled equations for the particular off-shell value $k_0 = \varepsilon_{k_s} - \varepsilon_k$ (see Sec. V). The solid lines denote the real parts of the amplitudes, and the dashed lines denote their imaginary parts. The upper panel corresponds to $\alpha = 0.5$, $\mu = 0.50$, and $k_s = 0.5$, and the lower panel corresponds to $\alpha = 1.2$, $\mu = 0.50$, and $k_s = 1.0$.

distinguishable, their relative difference being at the level of 10^{-3}–10^{-4}.

The lower panel corresponds to the parameters $\alpha = 1.2$, $\mu = 0.50$, and $k_s = 1.0$. In this case, a cusplike structure develops around $k \approx 0.65$. The Euclidean-Minkowski formulation shed some light on the origin of this cusp, which actually comes from the last term $\tilde{g}_i$ in Eq. (C8). According to its definition in Eq. (C9), this term does not contribute below the first meson creation threshold, that is, if $\sqrt{s} = 2\varepsilon_{k_s} < 2m + \mu$, since in this case, the argument of the θ function is always negative.

Above the threshold, the amplitude $\tilde{g}(k)$ is also 0 for $k > k_c$, where k_c is a critical value given by

$$k_c = \frac{1}{2\sqrt{s}}\sqrt{(s-\mu^2)(s-(2m+\mu)^2)}. \tag{54}$$

When $k < k_c$, the argument of the θ function versus the integration variable k' can be positive, which allows a nonzero value of $\tilde{g}_i$. If $k \to k_c$ from below, the integration domain over k' due to the θ function shrinks to

$$k'_c - c\sqrt{k_c - k} \le k' \le k'_c + c\sqrt{k_c - k},$$

where k'_c and c are independent of k and k'. One then has, in this limit,

$$\tilde{g}_i(k) \propto \sqrt{k_c - k}.$$

This is the reason of the cusp behavior, with an infinite derivative at $k = k_c$, which manifests itself in the lower panel of Fig. 12. For $m = 1$, $\mu = 0.5$, and $k_s = 1$, the critical value, given by Eq. (54), is $k_c = 0.651$, in agreement with the position of the cusp seen in our results. We have also checked from our numerical solutions that the cusp position moves as a function of k_s, according to

Eq. (54). It is interesting to point out that from the analysis of our purely Minkowski BS equation (28), the origin of this cusp is not evident at all. It is, however, remarkably manifested in the corresponding numerical solution seen in Fig. 12. We would like to emphasize that this cusplike structure described analytically above is only one example of the many structures seen in the results displayed in Figs. 10 and 11, although their analysis was beyond the scope of the present work.

The small differences near the cusp between the real parts of $F_0(k)$ and $\tilde{F}_0(k)$ are purely numerical and indicate the difficulty in reproducing sharp behaviors in terms of smooth functions. They can be reduced by increasing the number of grid points.

The results presented in Fig. 12 confirm the validity and accuracy of our direct Minkowski-space calculations.

VII. CONCLUSION

We present a new method to solve the Bethe-Salpeter equation in Minkowski space. Contrary to the preceding approaches devoted to this problem, this method does not make use of the Nakanishi integral representation of the amplitude but it is based on a direct solution of the equation taking properly into account the many singularities. A regular equation is finally obtained and solved numerically by standard methods.

It has been successfully applied to bound and scattering states. The Bethe-Salpeter off-shell scattering amplitude in Minkowski space has been computed for the first time.

Applying the Wick rotation to the original Bethe-Salpeter equation, an independent system of equations coupling the Euclidean amplitude to the Minkowski one for a particular off-shell value has been derived. It provides an independent test for our approach.

Coming on the mass shell, the elastic phase shifts and low energy parameters were accurately computed. They considerably differ, even at zero energy, from the non-relativistic ones. Above the meson creation threshold, an imaginary part of the phase shift appears and has also been calculated.

The results presented here were limited to the S wave in the spinless case and the ladder kernel but they can be extended to any partial wave.

The off-shell Bethe-Salpeter scattering amplitude thus obtained has been further used to calculate the transition form factor [14]. In its full off-shell form, it can be used as an input in the three-body Bethe-Salpeter Faddeev equations.

ACKNOWLEDGMENTS

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APPENDIX A: NUMERICAL METHODS

We consider a generic two-dimensional integral equation

$$\begin{aligned} f(x,y) = f^B(x,y) &+ \int_0^\infty dy' V(x,y,a_{y'},y') f(a_{y'},y') + \int_0^\infty dy' V(x,y,b_{y'},y') f(b_{y'},y') \\ &+ \int_0^\infty dy' \int_0^\infty dx' \frac{V(x,y,x',y')f(x',y') - V(x,y,a_{y'},y')f(a_{y'},y')}{x'^2 - a_{y'}^2} \\ &+ \int_0^\infty dy' \int_0^\infty dx' \frac{V(x,y,x',y')f(x',y') - V(x,y,b_{y'},y')f(b_{y'},y')}{x'^2 - b_{y'}^2}. \end{aligned} \tag{A1}$$

The solution f is searched in the compact domain $[0,x_m] \times [0,y_m]$ in the form

$$f(x,y) = \sum_{ij} c_{ij} S_i(x) S_j(y), \tag{A2}$$

where c_{ij} are unknown coefficients to be determined and S_i is a basis of spline functions (see, for instance, Ref. [6]). They are defined, respectively, in $[0,x_m]$ and $[0,y_m]$ and are cubic piecewise in each of the N_x (N_y) intervals in which $[0,x_m]$ ($[0,y_m]$) is divided.

The expansion (A2) is supposed to be valid on a set of selected points $\{\bar{x}_i\} \times \{\bar{y}_j\}$ with $i = 0, 2N_x + 1$ and $j = 0, 2N_y + 1$ suitably chosen in order to maximize the accuracy of the solution.

By inserting (A2) into (A1), one is led with a linear system of equations

$$\sum_{i'j'} U_{ij,i'j'} c_{i'j'} = f^B_{ij} + \sum_{i'j'} A_{ij,i'j'} c_{i'j'} \tag{A3}$$

with an inhomogeneous term given by

$$f^B_{ij} = f^B(\bar{x}_i, \bar{y}_j) \tag{A4}$$

and the matrices U and A being

$$U_{ij,i'j'} = S_{i'}(\bar{x}_i)S_{j'}(\bar{y}_j) \tag{A5}$$

and

$$\begin{aligned} A_{ij,i'j'} = & \int_0^{y_{\max}} dy' V(\bar{x}_i,\bar{y}_j,a,y')S_{i'}(a_{y'})S_{j'}(y') + \int_0^{y_{\max}} dy' V(\bar{x}_i,\bar{y}_j,b,y')S_{i'}(b)S_{j'}(y') \\ & + \int_0^{y_{\max}} dy' S_{j'}(y') \int_0^{x_{\max}} dx' \frac{V(\bar{x}_i,\bar{y}_j,x',y')S_{i'}(x') - V(\bar{x}_i,\bar{y}_j,a_{y'},y')S_{i'}(a_{y'})}{x'^2 - a_{y'}^2} \\ & + \int_0^{y_{\max}} dy' S_{j'}(y') \int_0^{x_{\max}} dx' \frac{V(\bar{x}_i,\bar{y}_j,x',y')S_{i'}(x') - V(\bar{x}_i,\bar{y}_j,b_{y'},y')S_{i'}(b_{y'})}{x'^2 - b_{y'}^2}. \end{aligned} \tag{A6}$$

An interesting property of the splines used is the fact that the functions $S_{2i}(x)$ and $S_{2i+1}(x)$ have a support limited to the two consecutive intervals $[x_{i-1}, x_i] \cup [x_i, x_{i+1}]$ and vanish elsewhere. This reduces considerably the computation of the matrix elements.

After removing the many singularities following the techniques explained in Sec. II, all integrands appearing in (A6) are regular functions and the integrations can be performed using the standard Gauss quadrature methods. The number collocation points on each dimension equal the number of spline bases, and the validation procedure allows us to determine the coefficients c_{ij} of the expansion (A2). One is finally led to solve a complex linear system denoted symbolically by

$$(U - A)c = f^b,$$

with dimension $d = (2N_x + 1)(2N_y + 1)$.

When dealing with a finite integration domain in both variables k' and k'_0, some care must be taken to use the subtraction technique (22). Indeed this relation—used for k' as well as for k'_0 integrations—is based on the identity (23) which is valid only in an infinite domain and that must be properly adapted. Thus, for a generic variable $z = x, y$ integrated over a finite domain $z \in [0, L]$, the relation

$$\int_0^\infty \frac{dz}{z^2 - a^2} = 0$$

must be replaced by

$$\int_0^L \frac{dz}{z^2 - a^2} + \frac{1}{2a} \log\left|\frac{L+a}{L-a}\right| = 0. \tag{A7}$$

The integral term in (A7) is used to eliminate the singularities on the finite interval $[0, L]$, whereas the logarithmic term represents a finite volume correction.

APPENDIX B: DERIVING EQ. (25)

Let us consider the integral

$$I(y) = \mathrm{PV} \int_{-\infty}^{\infty} \frac{dx}{x^2 - y^2} \tag{B1}$$

appearing in (25). Since it is 0 if $y \neq 0$ [see Eq. (23)] and diverges if $y = 0$, we expect it to be proportional to the delta function $\delta(y)$. We replace it by the regularized integral

$$I_\epsilon(y) = \int_{-\infty}^{\infty} \frac{(x^2 - y^2)dx}{(x^2 - y^2)^2 + \epsilon^2}, \tag{B2}$$

which tends to $I(y)$ when $\epsilon \to 0$. Calculating this integral, we find

$$I_\epsilon(y) = \frac{\pi\sqrt{\sqrt{y^4 + \epsilon^4} - y^2}}{\sqrt{2}\sqrt{y^4 + \epsilon^4}}.$$

When $\epsilon \to 0$, this is a very sharp function in the vicinity of $y = 0$ with the value $I_\epsilon(0) = \pi/(\sqrt{2\epsilon}) \to \infty$. It represents a delta function. The integral

$$\int_{-\infty}^{\infty} I_\epsilon(y)dy = \frac{\pi^2}{2}$$

gives the normalization coefficient. We conclude that

$$I(y) = \mathrm{PV} \int_{-\infty}^{\infty} \frac{dx}{x^2 - y^2} = \frac{\pi^2}{2}\delta(y). \tag{B3}$$

APPENDIX C: COUPLED EUCLIDEAN-MINKOWSKI SYSTEM OF EQUATIONS FOR THE S-WAVE AMPLITUDES

In Sec. V, making a Wick rotation in the BS equation, we derived the system of equations coupling the Euclidean amplitude $F^E(k_4,k,z)$ and the Minkowski one for the particular off-shell k_0 value $\tilde{F}^M(k,z)=F^M(k_0=\varepsilon_{k_s}-\varepsilon_k,k,z)$ with $k\in[0,k_s]$. To underline the main steps, this was done without partial wave decomposition. Here, we give the equations for the S wave, which we solved numerically. We remind readers that the partial waves F_L^E and F_L^M are defined by Eq. (6).

The first equation reads

$$F_0^E(k_4,k)=\frac{1}{16\pi}V_0^B(k_4,k)+\frac{1}{4\pi^3}\int_0^\infty k'^2dk'\int_0^\infty dk'_4\frac{V_s(k_4,k;k'_4,k')}{(k_4'^2+a_-^2)}\left\{\frac{F_0^E(k',k'_4)}{(k_4'^2+a_+^2)}-\frac{F_0^E(k'_4=0,k')}{a_+^2}\right\}$$
$$+\frac{1}{4\pi^3}\int_0^\infty k'^2dk'\pi\left\{\frac{F_0^E(k'_4=0,k')}{|a_-|a_+^2}V_1(k_4,k,k')-\frac{k_s}{k'}\frac{\varepsilon_{k_s}^3F_0^E(k'_4=0,k_s)}{2\varepsilon_{k'}^4|a_-|a_+}V_0^B(k_4,k)\right\}+h+g, \quad \text{(C1)}$$

where the kernels

$$V_0^B(k_4,k)=\frac{4\pi m^2\alpha}{kk_s}\log\frac{k_4^2+\mu^2+(k+k_s)^2}{k_4^2+\mu^2+(k-k_s)^2}, \quad \text{(C2)}$$

$$V_s(k_4,k;k'_4,k')=\frac{4\pi m^2\alpha}{kk'}\log\frac{[k_4^2+k_4'^2+\mu^2+(k+k')^2]^2-4k_4^2k_4'^2}{[k_4^2+k_4'^2+\mu^2+(k-k')^2]^2-4k_4^2k_4'^2}, \quad \text{(C3)}$$

$$V_1(k_4,k,k')=\frac{4\pi m^2\alpha}{kk'}\log\frac{[|a_-|+\sqrt{(k+k')^2+\mu^2}]^2+k_4^2}{[|a_-|+\sqrt{(k-k')^2+\mu^2}]^2+k_4^2}. \quad \text{(C4)}$$

The amplitude $\tilde{F}_0^M(k)$ enters in the terms h and h.

The term h is given by

$$h(k_4,k)=\frac{1}{16\pi^2}F_0^E(k'_4=0,k=k_s)V_0^B(k_4,k)\frac{k_s}{\varepsilon_{k_s}}\left(\frac{\varepsilon_{k_s}^2}{m^2}+4\log\frac{\varepsilon_{k_s}}{m}-2-\log 4\right)$$
$$+\frac{1}{8\pi^2}\tilde{F}_0^M(k=k_s)V_0^B(k_4,k)\frac{1}{\varepsilon_{k_s}}\left(2m\arctan\frac{k_s}{m}-k_s\log\frac{2k_s}{m}\right). \quad \text{(C5)}$$

It does not contain the integrals, and it contains the on-shell Minkowski and Euclidean amplitudes, which coincide with each other.

The term g is given by

$$g(k_4,k)=\frac{1}{8\pi^2}\int_0^{k_s}k'^2dk'\left\{\frac{\tilde{F}_0^M(k')}{\varepsilon_{k'}\varepsilon_{k_s}a_-}V_2(k_4,k,k')-\frac{k_s}{k'}\frac{2\varepsilon_{k_s}\tilde{F}_0^M(k=k_s)}{\varepsilon_{k'}^2a_+a_-}V_0^B(k_4,k)\right\}+\frac{i}{16\pi}\frac{k_s}{\varepsilon_{k_s}}\tilde{F}_0^M(k=k_s)V_0^B(k_4,k). \quad \text{(C6)}$$

It contains on the kernel $V_2(k_4,k,k')$

$$V_2(k_4,k,k')=\frac{2\pi m^2\alpha}{kk'}\log\left|\frac{[k_4^2+a_-^2+\mu^2+(k+k')^2]^2-4a_-^2[\mu^2+(k+k')^2]}{[k_4^2+a_-^2+\mu^2+(k-k')^2]^2-4a_-^2[\mu^2+(k-k')^2]}\right|. \quad \text{(C7)}$$

Notice that

$$V_1(k_4,k,k'=k_s)=V_2(k_4,k,k'=k_s)=V_0^B(k_4,k),$$

where $V_0^B(k,k_4)$ is defined in (C2).

This completes the full definition of Eq. (C1). Because of subtraction, which we have under any integral, these integrals are nonsingular.

The second equation has the following form:

$$
\begin{aligned}
\tilde{F}_0^M(k) &= \frac{1}{16\pi} V_0^B(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k) \\
&+ \frac{1}{4\pi^3}\int_0^\infty k'^2 dk' \int_0^\infty dk'_4 \frac{V_s(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k; k', k'_4)}{(k_4'^2 + a_-^2)} \left\{ \frac{F_0^E(k'_4, k')}{(k_4'^2 + a_+^2)} - \frac{F_0^E(k'_4 = 0, k')}{a_+^2} \right\} \\
&+ \frac{\pi}{4\pi^3}\int_0^\infty k'^2 dk' \left\{ \frac{F_0^E(k'_4 = 0, k')}{|a_-|a_+^2} V_1(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k, k') \frac{k_s \varepsilon_{k_s}^3 F_0^E(k'_4 = 0, k_s)}{2k' \varepsilon_{k'}^4 |a_-| a_+} V_0^B(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k) \right\} \\
&+ \tilde{h} + \tilde{g} + \tilde{g}_i.
\end{aligned} \tag{C8}
$$

In contrast to Eq. (C1), Eq. (C8) contains the term

$$
\tilde{g}_i(k) = \frac{ig^2}{64\pi}\int_0^{k_s} \frac{k'^2 dk'}{kk'\varepsilon_{k_s}\varepsilon_{k'} a_-} \tilde{F}_0^M(k') \theta\left(1 - \frac{|(2\varepsilon_{k_s} - \varepsilon_{k'} - \varepsilon_k)^2 - k'^2 - k^2 - \mu^2|}{2kk'}\right). \tag{C9}
$$

Because of restriction given by the theta function, one can show that this term is identically 0 below the one-meson creation threshold $2m + \mu$. Namely, it provides the value $\mathrm{Im}[\delta]$ above the threshold: if we omit it, we find always $\mathrm{Im}[\delta] = 0$. The denominator $a_- = \varepsilon_{k'} - \varepsilon_{k_s}$ never crosses 0 since the theta function does not allow that (it restricts the domain where $a_- \neq 0$). Kernels V_0^B, V_s, V_1, and V_2 are correspondingly defined by Eqs. (C2), (C3), (C4), and (C7) above.

The quantity $\tilde{h}$ reads

$$
\begin{aligned}
\tilde{h}(k) &= \frac{1}{16\pi^2} F_0^E(k_4 = 0, k = k_s) V_0^B(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k) \frac{k_s}{\varepsilon_{k_s}} \left(\frac{\varepsilon_{k_s}^2}{m^2} + 4 \log\frac{\varepsilon_{k_s}}{m} - 2 - \log 4 \right) \\
&+ \frac{1}{8\pi^2} \tilde{F}_0^M(k = k_s) V_0^B(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k) \frac{1}{\varepsilon_{k_s}} \left(2m \arctan\frac{k_s}{m} - k_s \log\frac{2k_s}{m} \right).
\end{aligned} \tag{C10}
$$

The term $\tilde{g}$ has the form

$$
\begin{aligned}
\tilde{g}(k) &= \frac{1}{8\pi^2}\int_0^{k_s} k'^2 dk' \left\{ \frac{\tilde{F}_0^M(k')}{\varepsilon_{k'}\varepsilon_{k_s} a_-} V_2(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k, k') - \frac{k_s}{k'} \frac{2\varepsilon_{k_s} \tilde{F}_0^M(k = k_s)}{\varepsilon_{k'}^2 a_- a_+} V_0^B(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k) \right\} \\
&+ \frac{i}{16\pi} \frac{k_s}{\varepsilon_{k_s}} \tilde{F}_0^M(k = k_s) V_0^B(k_4 = i(\varepsilon_{k_s} - \varepsilon_k), k).
\end{aligned} \tag{C11}
$$

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