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We investigate the effect of a finite volume on the critical behavior of the theory of the strong interaction, Quantum Chromodynamics (QCD), by means of a quark-meson model for $N_f = 2$ quark flavors. In particular, we analyze the effect of a finite volume on the location of the critical point in the phase diagram existing in our model. In our analysis, we take into account the effect of long-range fluctuations with the aid of renormalization group techniques. We find that these quantum and thermal fluctuations, absent in mean-field studies, play an important role for the dynamics in a finite volume. We show that the critical point is shifted towards smaller temperatures and larger values of the quark chemical potential if the volume size is decreased. This behavior persists for antiperiodic as well as periodic boundary conditions for the quark fields as used in many lattice QCD simulations.

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I. INTRODUCTION

The detection of the critical end point in the phase diagram at finite temperature and quark chemical potential of the theory of the strong interaction (QCD) is an inherently difficult problem. In fact, it may even be the case that there is no first-order chiral phase transition at large chemical potential [1–7] and the critical end point (CEP) of the chiral phase boundary does not exist at all. The complications arising in the search for the critical point are manifold. For example, one of the most important fully nonperturbative theoretical tools for the exploration of the QCD phase diagram, namely lattice QCD simulations, suffers from the so-called sign problem when applied to finite quark chemical potential. Although powerful techniques have been developed to circumvent this problem for smaller chemical potentials [8–19], ranging from reweighting techniques over Taylor expansions to imaginary-valued quark chemical potentials, these studies are still restricted to a finite simulation volume; see, e.g., Refs. [20–23] for reviews.

The effects of a finite volume on the curvature of the chiral phase transition line at small chemical potentials can be determined from lattice QCD results [17,18,24–27]. Recently, the effects of a finite volume $V = L^3$ as well as of long-range fluctuations have been analyzed by means of the quark-meson model which serves as a low-energy QCD model [28,29]. It was found that depending on the pion mass the curvature becomes continuously smaller for decreasing volume sizes and periodic boundary conditions for the quark fields in the spatial directions. Decreasing the

length of the box below $m_\pi L \lesssim 2$, the curvature then tends to increase again and even exceeds its infinite-volume value. This very specific behavior can be traced back to the implementation of periodic boundary conditions. For antiperiodic boundary conditions, the quarks do not have a spatial zero mode and the curvature is a monotonically decreasing function of the volume size. These results for the curvature are in accordance with the behavior of low-energy observables, such as the quark condensate, as a function of the volume size at zero temperature [30,31]. In any case, the volume dependence of the curvature suggests also that the (chiral) CEP may be shifted in a finite volume. From the above discussion, for instance, it appears most likely that the critical point is shifted towards larger values of the chemical potential and smaller temperatures in case of periodic boundary conditions and $2 \lesssim m_\pi L < \infty$. However, it may also very well completely disappear in the small-volume limit. The discussion of the volume dependence of the location of the CEP also raises a conceptual issue, namely how to define a CEP associated with a diverging susceptibility in a finite system. We shall discuss this in more detail below.

In the present work, we shall focus on the behavior of the QCD critical end point as a function of the volume size, with an emphasis on quark fields with periodic boundary conditions in spatial directions. We expect that this setup is the most useful one to help in further guiding present and future lattice simulations of the QCD phase diagram. For our study, we employ the quark-meson model. This model has already proven to be useful in our previous studies of

chiral dynamics at finite temperature or quark chemical potential in a finite volume [28,29,32,33]. We expect that such a model setup is very useful to provide a framework for understanding better the mechanisms of chiral symmetry breaking in finite systems. The volume dependence of the chiral CEP has been studied before with the aid of quark-meson-type models in mean-field approximations [34–38]. In the present work, we go beyond the mean-field approximation and include the fluctuations of the meson fields. We have found that the fluctuations of bosonic fields, in particular fluctuations of the order parameter, are of utmost importance for the physics of such models in a finite volume [28,29,33,39], in addition to the effects of fermionic fluctuations. For a continuous symmetry such as the chiral flavor symmetry, fluctuations of Goldstone modes restore the symmetry in a finite-volume system in the absence of explicit symmetry breaking. This implies that the phase with spontaneously broken chiral symmetry does not exist at all in the chiral limit [40]. Therefore it is reasonable to expect that the chiral phase transition temperature decreases when the current quark mass (pion mass) is decreased for a given fixed volume size. This has also been found in a previous renormalization group study including fluctuation effects associated with the Goldstone modes; see Ref. [32]. Note that a mean-field study cannot capture this effect correctly due to the absence of the Goldstone-mode fluctuations. In fact, it is possible to obtain a finite chiral phase transition temperature in mean-field theory in a finite-volume system, even in the chiral limit. On the other hand, if our present study of the shift of the CEP in a finite volume is in accordance with previous mean-field studies in a given parameter range defined by the dimensionless quantity $m_\pi L$, then our results can be considered as a hint that fluctuation effects are at least subdominant in this $m_\pi L$ regime for some observables.

In the next Sec. II we briefly introduce the renormalization group (RG) setup which underlies our study. The results for the location of the critical point as a function of the volume size are then discussed in Sec. III. Finally, our conclusions and a brief outlook are given in Sec. IV.

II. RENORMALIZATION GROUP SETUP

For our investigation of critical behavior in a finite volume, we employ the quark-meson model for $N_f = 2$ flavors. In our present work, we shall use the following ansatz/truncation for the RG scale-dependent effective action:

$$\Gamma_k[\bar{q}, q, \phi] = \int d^4x \left\{ \bar{q}(i\partial + ig(\sigma + i\vec{\tau} \cdot \vec{\pi}\gamma_5) + i\gamma_0\mu)q + \frac{1}{2}(\partial_\mu\phi)^2 + U_k(\phi^2) - c\sigma \right\}, \quad (1)$$

where $\phi^T = (\sigma, \vec{\pi})$ has $N_f^2 = 4$ components and q represents $N_f = 2$ quark flavors. The scale k denotes the RG scale

which is introduced below. The initial condition for our RG flow study of this effective action is given by the so-called classical or “microscopic” action defined at the scale $k = \Lambda$. This scale is determined by the validity of a hadronic representation of QCD. The mesonic potential at the UV scale is parametrized by two couplings, m_Λ^2 and λ_Λ ,

$$U_\Lambda(\phi^2) = \frac{1}{2}m_\Lambda^2\phi^2 + \frac{1}{4}\lambda_\Lambda(\phi^2)^2. \quad (2)$$

The current quark masses are determined by the term linear in the field $\sigma \sim \bar{q}q$ which also renders the pion fields $\vec{\pi}$ massive in the low-energy limit. The bosonic couplings m_Λ and λ_Λ , the Yukawa coupling g , and the $O(4)$ -symmetry breaking term c are parameters of our model which are used to fit a given set of low-energy observables; see the discussion below. In our model, the order parameter for chiral symmetry breaking is given by the (vacuum) expectation value of the scalar field $\langle\sigma\rangle$ which can be identified with the pion decay constant f_π . Note that we do not take into account a possible running of the Yukawa coupling g , as done in, e.g., Refs. [41–47]. The analysis of these studies suggests that it is justified to neglect the running of this coupling in our study, at least for our first RG study of the phase diagram in a finite volume. For more quantitative predictions of the shift of the CEP, of course, this running has to be taken into account eventually.

In the following, we study the RG flow of the order-parameter potential U in leading order of the derivative expansion. We do not take into account corrections arising from a nontrivial RG flow of the wave-function renormalizations of the quark and meson fields.¹ Since we are not aiming at a quantitatively accurate determination of either the absolute location of the QCD critical point or of the critical behavior as measured by critical exponents, we consider this to be a justified approximation. However, we would like to emphasize that our present approximation already includes effects beyond the mean-field limit. In fact, it has been found that the critical exponents it yields at the thermal phase transition in the quark-meson model already agree with the exact values on the 2% level²; see, e.g., Refs. [33,42,43,53–55]. This observation can be traced back to the small anomalous dimensions associated with these wave-function renormalizations found in the model; see, e.g., Refs. [42,43,55]. A more detailed discussion of the relation of the present approximation to the mean-field approximation in terms of a derivative expansion and a

¹To be specific, we consider the wave-function renormalizations Z_ϕ and Z_ψ of the mesons and the quarks to be constant, respectively: $Z_\psi = Z_\phi = 1$. This implies that the anomalous dimensions associated with these fields are zero, $\eta_\psi = \eta_\phi = 0$.

²Note that a detailed error analysis is difficult and is mostly done by varying and/or optimizing the regularization scheme, by taking into account higher orders in the derivative expansion, and by taking into account the full momentum dependence of a given set of n -point functions; see, e.g., Refs. [48–52].

so-called large- N_c expansion of the quantum effective action Γ can be found in Refs. [43,44,46,56,57].

Before we turn to the discussion of our actual results for the CEP, we would like to briefly introduce a few more details on our approach. In the following we employ the Wetterich equation for the derivation of the RG flow equation for the order-parameter potential U [58]:

$$\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \{ [\Gamma_k^{(2)} + R_k]^{-1} \partial_t R_k \}. \quad (3)$$

This flow equation describes the change of the quantum effective action Γ under variation of the RG scale k , or the RG “time” $t = \ln(k/\Lambda)$, and therefore allows us to interpolate between the initial action $S \simeq \Gamma_\Lambda$ at the UV scale Λ and the quantum effective action $\Gamma \equiv \Gamma_{k \rightarrow 0}$ in the limit $k \rightarrow 0$. To regularize the theory, a regulator function R_k is included. For our finite-temperature and finite-volume studies, it is convenient to choose a dimensionally reduced regulator function, i.e. a so-called spatial regulator which only regularizes the spatial momentum modes [59,60]. Our particular choice for the regulator function is a close relative of the optimized regulator function [61–64]. We would like to add that the use of the dimensionally reduced regulator at finite temperature does not cause any conceptual problem in the local potential approximation underlying the present work. At next-to-leading order in the derivative expansion, complications arise from the fact that this class of regulators breaks explicitly Poincaré invariance. However, this can then be handled by suitably adjusting the initial conditions of the RG flow [44].

Inserting the truncation (1) into Eq. (3), we obtain the flow equation for the RG-scale dependent order-parameter potential U_k in a finite cubic volume $V = L^3$; see Refs. [29,33]:

$$\begin{aligned} \partial_t U_k(\phi^2) = k^5 & \left[\frac{3}{E_\pi} \left(\frac{1}{2} + n_B(E_\pi) \right) \mathcal{B}_p(kL) \right. \\ & + \frac{1}{E_\sigma} \left(\frac{1}{2} + n_B(E_\sigma) \right) \mathcal{B}_p(kL) \\ & - \frac{2N_c N_f}{E_q} (1 - n_F(E_q, \mu) \\ & \left. - n_F(E_q, -\mu)) \mathcal{B}_\ell(kL) \right], \quad (4) \end{aligned}$$

where quasiparticle energies are defined as

$$E_i = \sqrt{k^2 + M_i^2}, \quad i \in \{\pi, \sigma, q\}. \quad (5)$$

The effective squared meson masses are determined by derivatives of the potential U_k ,

$$M_\pi^2 = 2 \frac{\partial U_k}{\partial \phi^2}, \quad M_\sigma^2 = 2 \frac{\partial U_k}{\partial \phi^2} + 4\phi^2 \frac{\partial^2 U_k}{\partial (\phi^2)^2}, \quad (6)$$

whereas the quark mass is directly related to the Yukawa coupling g ,

$$M_q^2 = g^2 \phi^2. \quad (7)$$

Note that, strictly speaking, the quantities M_π , M_σ , and M_q are the physical masses of the associated particles only if evaluated in the ground state of the theory. The thermal dynamics of the theory is controlled by the bosonic and fermionic occupation numbers,

$$n_B(E) = \frac{1}{e^{E/T} - 1}, \quad n_F(E, \mu) = \frac{1}{e^{(E-\mu)/T} + 1}, \quad (8)$$

whereas the volume dependence is governed by the mode counting functions \mathcal{B}_ℓ :

$$\mathcal{B}_\ell(kL) = \frac{1}{(kL)^3} \sum_{\vec{n} \in \mathbb{Z}^3} \theta((kL)^2 - (2n + \delta_{\text{ap}, \ell})^2 \pi^2). \quad (9)$$

Here, \vec{n} labels a three-dimensional vector of integers and the label ℓ refers to antiperiodic (ap) and periodic (p) boundary conditions, respectively. Note that, due to our choice for the regulator function, the spatial and thermal contributions in the flow equation factorize, which facilitates the numerical evaluation [59].

From a phenomenological point of view, the mode counting functions already reflect the fact that the dynamics of the theory in the small-volume limit is only governed by the spatial zero mode in the case of periodic boundary conditions. For antiperiodic boundary conditions, on the other hand, we observe that these functions tend to zero for $kL \rightarrow 0$. Thus, the fields effectively become static and condensation of quark-antiquark pairs is suppressed. In the infinite-volume limit ($L \rightarrow \infty$), we find that these functions approach a finite number which depends only on the chosen regularization scheme. We recover the known flow equation for the chiral order-parameter potential of the quark-meson model in this limit [65,66].

The flow equation (4) for the order-parameter potential is a partial differential equation in the RG scale k , where $0 \leq k \leq \Lambda$, and the field variable ϕ . In the present study, we solve it by discretizing the field variable ϕ ; see, e.g., Ref. [66] for details. For a review on results from such a direct numerical solution of this type of flow equation for the potential and, in particular, for results concerning the phase diagram of QCD low-energy models in the infinite-volume limit, we refer the reader to Ref. [67].

For our numerical study of the phase diagram, we need to fix the parameters of our model, namely m_Λ , λ_Λ , the Yukawa coupling g , the UV scale Λ , and the symmetry breaking parameter c . As already mentioned above, we use

these parameters to fit a given set of low-energy observables, namely the pion mass m_π , the constituent quark mass m_q , and the pion decay constant f_π . Clearly, the set of parameters determined in this way is by no means unique. Loosely speaking, this parameter ambiguity leaves its trace in the phase structure of the quark-meson model. As has been pointed out in Ref. [6], the chiral CEP can be shifted almost arbitrarily by varying the mass of the sigma meson. Here, we therefore pursue the following strategy: we choose a set of initial conditions such that we find a chiral CEP at a certain position in the infinite-volume phase diagram. Since we are not aiming at a quantitative determination of the position of the CEP, the absolute values of its coordinates play only a secondary role. Once the parameters have been fixed, we vary the box size and follow the shift of the CEP. We have checked for various sets of parameters that the qualitative behavior of the CEP as a function of the volume size indeed appears to be “universal.” To be specific, in the calculation presented here, we have used $\Lambda = 1$ GeV, $m_\Lambda = 0.881$ GeV, $\lambda_\Lambda = 0$, and $c = 5 \times 10^{-4}$ GeV³ for the model parameters. At zero temperature, this yields $m_q = 318$ MeV, $m_\sigma = 461$ MeV, and $m_\pi = 75$ MeV. From a phenomenological point of view, the pion mass is clearly too small. Nevertheless, we have chosen this value since it currently appears to be the smallest accessible value for the pion mass in lattice simulations [68] and we expect that the effect of the long-range fluctuations in a finite volume is most pronounced for small pion masses. While the results for a pion mass of $m_\pi \approx 140$ MeV are similar, this choice leads to a qualitatively clearer picture of the finite-volume effects.³

III. CRITICAL END POINT AND FINITE-VOLUME EFFECTS

Let us now discuss effects of a finite volume on the location of the chiral CEP. The CEP is associated with a residual $Z(2)$ symmetry. For finite pion masses, it is uniquely defined in the infinite-volume limit by a diverging (longitudinal) susceptibility $\chi_\sigma \sim \partial\sigma_0/\partial c$, where σ_0 is the order parameter for chiral symmetry breaking⁴ and c is a measure for the explicit symmetry breaking; see Eq. (1). In our model study, χ_σ assumes a simple form; see, e.g., Refs. [33,69]:

$$\chi_\sigma = \frac{1}{M_\sigma^2}, \quad (10)$$

where M_σ^2 is defined in Eq. (6). The corresponding transverse susceptibility associated with the Goldstone modes, i.e. pion modes, does not provide us with new information for the present study, since it is directly related to the chiral order parameter.

With the parameter set given in the previous section, we find a diverging (longitudinal) susceptibility associated with a chiral CEP at

$$\left(\mu_{\text{CEP}}^{(\infty)}, T_{\text{CEP}}^{(\infty)}\right) \approx (298.6 \pm 1 \text{ MeV}, 35.0 \pm 1 \text{ MeV}); \quad (11)$$

see also the top left panel of Fig. 1. The errors arise from an uncertainty in our numerical determination of the coordinates of the CEP and are identical for both our infinite volume study and our finite-volume studies. These errors do not account for systematic errors associated with, e.g., our truncation for the effective action. In particular, we would like to stress again that the results for the location of the CEP in quark-meson- or Nambu–Jona-Lasinio-type (NJL) models in general suffer from a parameter ambiguity [6]. However, this ambiguity affects the present study very little, since we are only interested in the shift of the position of the CEP in a finite volume relative to its infinite-volume coordinates.

As we have already indicated above, the fluctuations of the meson fields play a prominent role in studies of the chiral critical behavior in a finite volume. This is a consequence of the appearance of the pions as the light Goldstone modes in the breakdown of the chiral symmetry. In absence of an explicit symmetry breaking term, these fluctuations restore the chiral symmetry in a finite volume in the long-range (infrared) limit [40]. A further complication related to the finite volume in a study of critical behavior is to find a proper definition of a CEP which is in the infinite-volume limit associated with a diverging susceptibility. In a finite volume, the susceptibilities are expected to be bounded from above. To be more precise, the magnitude of the susceptibility is assumed to scale with the volume size like

$$\chi_\sigma \sim L^2 \sim V^{\frac{2}{3}}. \quad (12)$$

This already follows from a simple dimensional analysis.⁵ Thus, the susceptibilities are necessarily finite in a finite volume. In order to study the volume dependence of the position of the CEP, we therefore first require a definition of

³Note that, if we tune the parameters such that the pion mass eventually approaches its phenomenological value, $m_\pi \approx 140$ MeV for vanishing temperature and chemical potential, we find that the CEP in infinite volume is already shifted to very small values of the temperatures. For finite volumes, we find that the CEP is then shifted in the same way as for smaller pion masses, see our discussion below. However, the fact that this takes place at very low temperatures makes a numerical study presently very cumbersome.

⁴Note that σ_0 can be identified with the pion decay constant in our quark-meson model. Moreover, we have $\sigma_0 \sim \langle \bar{\psi}\psi \rangle$.

⁵The relation between the maximal value of the susceptibility and the box size $\chi_\sigma \sim L^{\gamma/\nu}$ may receive corrections, depending on the values of the critical exponents. In particular, since $\gamma/\nu = 2 - \eta$, simple dimensional analysis gives the correct result only in the absence of an anomalous dimension. This holds in our present approximation.

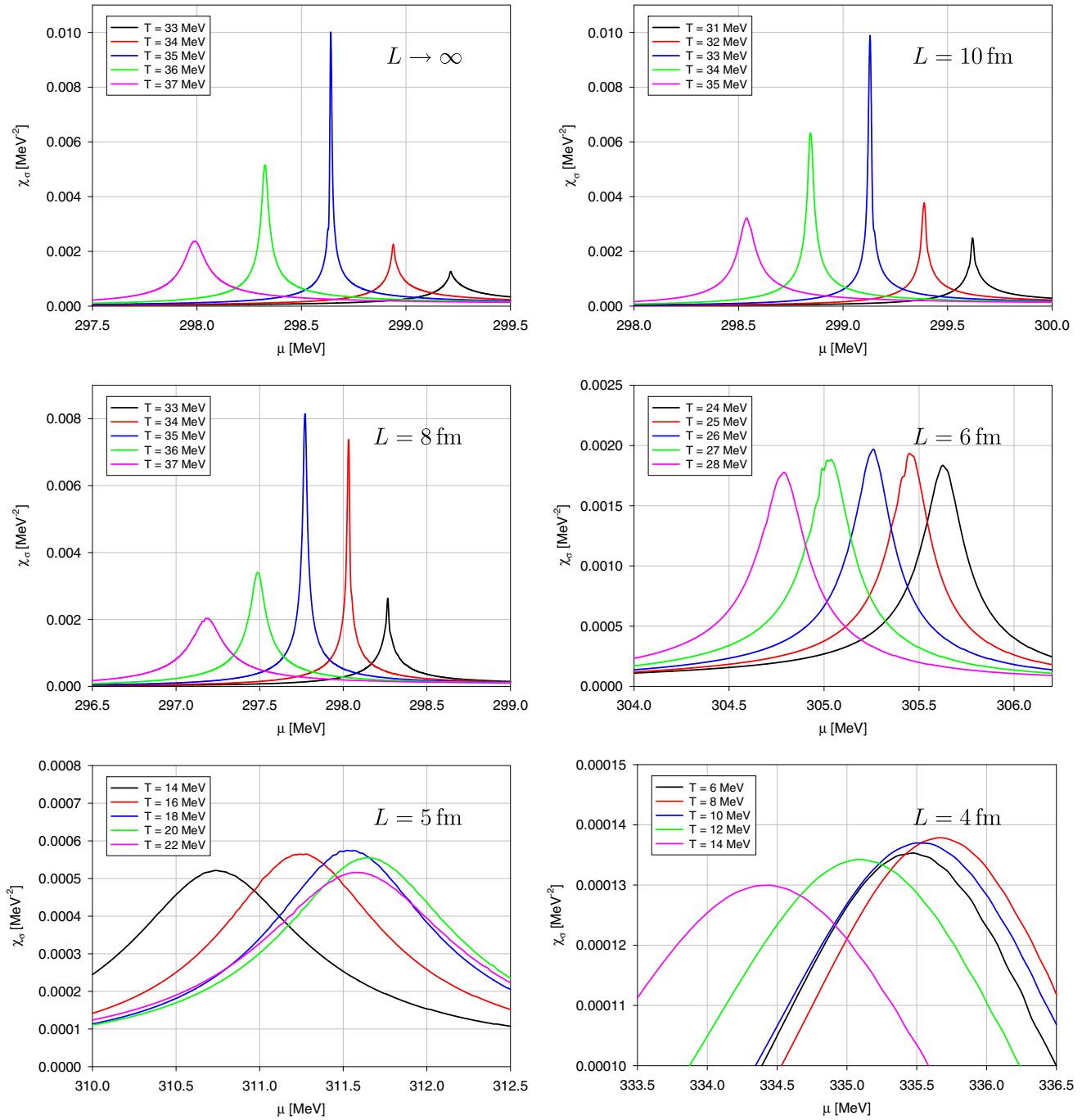


FIG. 1 (color online). Chiral (longitudinal) susceptibilities $\chi_\sigma = \frac{1}{M_\sigma^2}$ as a function of the quark chemical potential μ for various temperatures and volume sizes ranging from $L \rightarrow \infty$ from the top left panel to $L = 4$ fm in the bottom right panel as obtained for periodic boundary conditions for the quark fields in spatial directions.

the CEP in a finite volume. For a given finite volume $V = L^3$, our discussion suggests that the CEP can be defined as the point in the (μ, T) plane with the coordinates $(\mu_{\text{CEP}}^{\text{max}}, T_{\text{CEP}}^{\text{max}})$ where the maximum value of the susceptibility χ_σ^{max} is attained:

$$\chi_\sigma^{\text{max}}(\mu_{\text{CEP}}^{\text{max}}(L), T_{\text{CEP}}^{\text{max}}(L)) = \max_{(\mu, T)} \chi_\sigma(\mu, T, L). \quad (13)$$

In the large-volume limit, we then have

$$\lim_{L \rightarrow \infty} (\mu_{\text{CEP}}^{\text{max}}(L), T_{\text{CEP}}^{\text{max}}(L)) = (\mu_{\text{CEP}}^{(\infty)}, T_{\text{CEP}}^{(\infty)}), \quad (14)$$

as it should be. In the following we shall use the coordinates $(\mu_{\text{CEP}}^{\text{max}}, T_{\text{CEP}}^{\text{max}})$ to locate the CEP in a finite volume, with the additional constraint that the

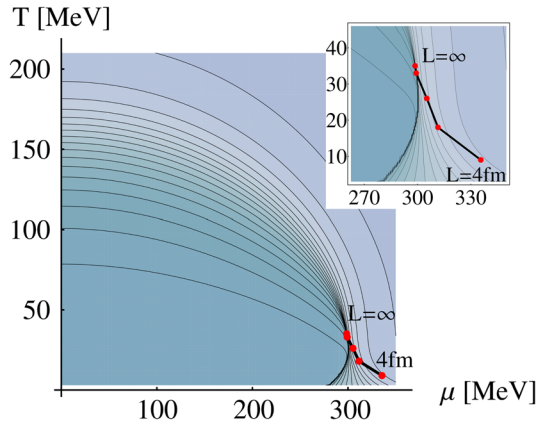


FIG. 2 (color online). Contour plot of the magnitude of the chiral order parameter $\sigma_0 = f_\pi$ in the plane spanned by temperature and quark chemical potential for L towards infinity: The order parameter decreases from dark to light shading. The inlaid (red) dots show the behavior of the CEP [associated with a $Z(2)$ symmetry] as a function of the volume size for periodic boundary conditions for the quark fields in spatial directions. We observe that the CEP is shifted towards larger values of the chemical potential but smaller values of the temperature for smaller volumes.

so-determined CEP has to be continuously connected to the infinite-volume CEP, and thus to the chiral crossover, under a change of the volume size, as also shown in Fig. 2. This requirement becomes necessary for volume sizes of $L \lesssim 5$ fm and periodic boundary conditions where we observe that a second CEP develops in the phase diagram at smaller chemical potential and temperatures, with values of the susceptibility that may exceed those of the CEP at larger chemical potential.⁶

In Fig. 1, we show our results for the longitudinal susceptibility χ_σ for various temperatures T and volume sizes (ranging from $L \rightarrow \infty$ from the top left panel to $L = 4$ fm in the bottom right panel) as a function of the quark chemical potential μ . In the infinite-volume limit, we indeed observe that the susceptibility along the crossover line increases when we increase the quark chemical potential until it diverges at the CEP. Comparing the panels for the various box sizes, we also find that the maximum values of the susceptibilities decrease with smaller box size, in accordance with the expectations. In the first panel of Fig. 1, we show the susceptibility for $L \rightarrow \infty$ close but not exactly at the CEP. The maximum value is limited by $\chi_\sigma^{\max} \sim \frac{1}{k_{\text{IR}}^2}$ since, for technical reasons, the RG flow is evaluated only for $k \rightarrow k_{\text{IR}}$ with a small but finite value $k_{\text{IR}} \approx 20$ MeV. Most importantly, however, we observe that the position of the CEP is, within the errors, shifted towards smaller values of the temperature and larger values

⁶This second CEP has been studied and analyzed in, e.g., Ref. [66] but is not at the heart of the present study.

TABLE I. Relative shifts $\Delta\mu_{\text{CEP}} = (\mu_{\text{CEP}}^{\max}/\mu_{\text{CEP}}^{(\infty)} - 1)$ and $\Delta T_{\text{CEP}} = (T_{\text{CEP}}^{\max}/T_{\text{CEP}}^{(\infty)} - 1)$ of the CEP coordinates $(\mu_{\text{CEP}}^{\max}, T_{\text{CEP}}^{\max})$ in a finite volume $V = L^3$ compared to its position $(\mu_{\text{CEP}}^{(\infty)}, T_{\text{CEP}}^{(\infty)})$ in the infinite-volume limit. Here, we have chosen $m_\pi = 75$ MeV which refers to the pion mass in the infinite volume limit for $(\mu, T) = (0, 0)$. For the quantum fields, we have chosen periodic boundary conditions in spatial directions. The errors arise from the uncertainty in determining the position of the maximum value of the susceptibility; see main text for details.

L (fm)	$m_\pi L$	$\Delta\mu_{\text{CEP}}$	ΔT_{CEP}
10	3.75	0.00(1)	-0.06(6)
8	3.00	0.00(1)	0.00(6)
6	2.25	0.02(1)	-0.26(5)
5	1.88	0.04(1)	-0.49(4)
4	1.50	0.12(1)	-0.74(4)

of the chemical potential for decreasing box size.⁷ To be specific, the CEP is shifted from $(\mu_{\text{CEP}}^{(\infty)}, T_{\text{CEP}}^{(\infty)}) \approx (298.6 \pm 1 \text{ MeV}, 35.0 \pm 1 \text{ MeV})$ for $L \rightarrow \infty$ to $(\mu_{\text{CEP}}^{\max}, T_{\text{CEP}}^{\max}) = (335.6 \pm 1 \text{ MeV}, 9.0 \pm 1 \text{ MeV})$ for $L = 4$ fm. In other words, μ_{CEP}^{\max} increases by about 10%, whereas T_{CEP}^{\max} decreases by about 75% in the considered range of volume sizes; see also Table I for values of the relative shift of the CEP for several other volume sizes. Thus, the temperature coordinate of the CEP appears to be affected more strongly by the finite volume than the one associated with the quark chemical potential. In Fig. 2, we illustrate the results for the shift of the coordinates of the CEP in our model study. For $L \lesssim 4$ fm, we do not find a CEP any more, at least none compatible with our definition. In any case, we have checked that the direction of the finite-volume shift of the CEP does not depend on our choice for the model parameters. The latter affects our results only on a quantitative level.

Our results show that the position of the CEP depends strongly on the box size. The fact that the position of the CEP is shifted towards larger values of the chemical potential and lower values of the temperature might be considered helpful from the point of view of lattice QCD simulations. If the observed shift of the CEP in our model study carries over to lattice QCD simulations, then it implies that, in a finite volume, the CEP is pushed into a domain of the phase diagram which is very difficult to access with such simulations.

Our results also indicate that comparatively large volume sizes are required to effectively resolve the position of the CEP in the infinite-volume limit. To be specific, we find that the position of the CEP in the infinite-volume limit is only reached for $L \gtrsim 10$ fm. However, this statement

⁷Note that we have cross-checked our results for the location of the CEP as obtained from an analysis of the susceptibility by directly studying the shape of the chiral order-parameter potential.

depends on our choice for the model parameters which ultimately determines the position of the CEP in the continuum. The scale L is in a broad sense in competition with the two other scales T and μ which appear also in the quark propagator. All three scales can be considered as cutoff scales for the low-momentum modes of the quark fields. If therefore

$$1/L \ll M_{\min}, \quad (15)$$

where

$$M_{\min} = \min(\mu_{\text{CEP}}^{(\infty)}, T_{\text{CEP}}^{(\infty)}), \quad (16)$$

then we expect that the position of the CEP remains effectively unchanged, since L does not impose an additional cutoff.⁸ If, on the other hand, $1/L$ is of the order of M_{\min} ,

$$1/L \sim \mathcal{O}(M_{\min}), \quad (17)$$

then we expect finite-volume effects on the CEP to become important. Since the actual position of the CEP depends on our choice for the parameter set, the smallest value of L , for which a significant shift of the position of the CEP is found, is clearly parameter dependent. From the point of view of lattice QCD simulations, these considerations imply that the shift of the CEP due to the presence of a finite volume is only relevant for small box sizes, if the possibly existing CEP in the QCD phase diagram is located at smaller values of the chemical potential and larger values of the temperature. In other words, we expect that in this case the infinite-volume position of the CEP can be reached already for small box sizes.

Let us finally speculate about phenomenological implications of our results. The role of finite-size effects in the search for the CEP in heavy-ion collision experiments has been previously discussed in, e.g., Refs. [34–37]. Here, we would like to simply comment on how the boundary conditions for the quark fields in spatial directions affect our results. In addition to periodic boundary conditions, we have also studied the shift of the CEP in a finite volume in the presence of antiperiodic boundary conditions for the quarks. We also find in this case that the CEP is shifted to lower temperatures but larger values of the chemical potential and eventually disappears for $L \lesssim 4$ fm. Although we expect neither periodic nor antiperiodic boundary conditions to be at work in the expanding fireball in a heavy-ion collision experiment and although we also do not claim that the observed direction of the shift of the CEP in a finite volume holds for general boundary

conditions and general geometries of the volume, the observed finite-volume shift of the CEP may at least be considered as a possible scenario that could take place in the experiments. Depending on the experimental setup, traces of the CEP in the experimental data may therefore be found at different coordinates (μ, T) .

IV. CONCLUSIONS AND OUTLOOK

Using nonperturbative functional RG techniques, we have computed the shift of the CEP of a quark-meson model in a finite volume for periodic as well as antiperiodic boundary conditions for the quark fields in spatial directions. In our study, we have also included fluctuations of the meson fields by means of a derivative expansion of the effective action. The effect of these fluctuations has not been taken into account in recent mean-field studies [34–38]. However, they play an important role in finite-volume systems, as they tend to restore the chiral symmetry in a finite volume in the limit of small quark masses.

The model underlying our studies does not contain gluonic degrees of freedom and it is not confining. However, it can be considered as an effective low-energy model for dynamical chiral symmetry breaking which allows us to analyze the effects of a finite volume on the chiral dynamics in simple terms. We have found that the CEP in a finite volume is shifted towards smaller temperatures and larger values of the chemical potential when the volume is decreased, in accordance with earlier mean-field NJL model studies [34–37]. The volume size below which the CEP is significantly shifted depends on the actual position of the CEP in the infinite-volume limit. The same holds for the volume size below which the CEP disappears. Interestingly, we have found that these qualitative aspects are present independent of our choice for the spatial boundary conditions (periodic or antiperiodic) for the quark fields. The fact that the shift of the CEP towards lower temperatures and larger chemical potentials may already set in for comparatively large volume sizes (depending on the coordinates of the CEP in the infinite-volume limit) could be a hint towards a further complication in the search for the CEP with lattice QCD simulations.

While our present work focuses on chiral aspects of the QCD phase diagram, the inclusion of dynamical gauge degrees of freedom in this RG study following Ref. [70] (see Refs. [71,72] for reviews) would represent an interesting extension. This opens the possibility to study the interplay of chiral and confining dynamics in a finite volume. A considerable volume dependence is indeed also expected in the gauge sector of the theory; see, e.g., Refs. [73,74]. However, the extension of the present low-energy model to a so-called Polyakov-loop extended low-energy model (see, e.g., Refs. [75–89]) appears to be a first natural step towards a fully dynamical RG study of the QCD phase diagram in a finite volume. For a first mean-field study of finite-volume effects with a Polyakov-loop

⁸Note that the pion mass scale and the scale associated with the pion decay constant have been implicitly taken into account in this analysis as these quantities are parameters of the model which essentially determine the location of the CEP in the infinite volume limit.

extended low-energy QCD model, we refer the reader to Ref. [38]. The corresponding RG flow equation, which also takes effects of meson fluctuations into account, can be found in Ref. [29]. In addition to the inclusion of confinement effects, it might be worthwhile to study the effect of further bosonic composites, such as diquarks, on the finite-volume shift of the CEP since it has been found in infinite-volume studies that they may affect the structure of the phase diagram and the location of the CEP significantly; see, e.g., Refs. [81,90,91].

In summary, our present investigation already shows that there is a qualitative effect of a finite volume on the structure of the chiral phase diagram, which can be measured in terms of the coordinates of the CEP. The observed shift of the CEP in a finite volume could be useful to further guide present and future studies of the QCD

phase diagram with lattice simulations as well as the experimental search for the CEP, and therefore it may help us to better understand the dynamics underlying strongly interacting matter.

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