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Center symmetry and area laws

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$SU(N_c)$ gauge theories containing matter fields may be invariant under transformations of some subgroup of the $\mathbb{Z}_{N_c}$ center; the maximum such subgroup is $\mathbb{Z}_p$, with p depending on N_c and the representations of the various matter fields in the theory. Confining $SU(N_c)$ gauge theories in either $3+1$ or $2+1$ space-time dimensions and with matter fields in any representation have string tensions for representation R given by $\sigma_R = \sigma_f \frac{p_R(p-p_R)g(p_R(p-p_R))}{(p-1)g(p-1)}$ with $p_R = n_R \text{mod}(p)$, where σ_f is the string tension for the fundamental representation, g is a positive finite function and n_R is the n-ality of R. This implies that a necessary condition for a theory in this class to have an area law is invariance of the theory under a nontrivial subgroup of the center. Significantly, these results depend on p regardless of the value of N_c.

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The nature of confinement in QCD and related gauge theories is quite subtle and remains a subject of considerable interest [1]. QCD lacks an order parameter for confinement. However, various cousins of QCD have well-defined order parameters. In $SU(N_c)$ Yang-Mills theory, the Polyakov loop in Euclidean space serves as an order parameter [3]. It is connected to center symmetry: the Euclidean action is invariant under center transformations while the Polyakov loop is not [2]. The presence of quark fields in QCD spoils the invariance of QCD under center transformations. Yang-Mills theory has another set of indicators of confinement that are lacking in QCD, namely area laws for Wilson loops [4] associated with the various representations of the group. A Wilson loop for a representation R, σ_R, is characterized by its string tension

$$\sigma_R \equiv \lim_{L\to\infty, T\to\infty} \frac{\log(\langle W^R \rangle)}{LT}$$
$$W^R \equiv \text{tr}\left(\text{P} \times \exp\left(i\int_C \sum_i A_i T_i^R \cdot dx\right)\right) \qquad (1)$$

where W is the Wilson loop operator, C is a closed rectangular path of length L in a spatial direction and T in the temporal direction, P indicates path ordering, A_i is the ith gluon field and T_i^R is the ith generator in representation R. A nonzero string tension is the defining characteristic of an area law for the Wilson loop. Over the years, there has been considerable interest in the dependence of the string tension on representation for $SU(N_c)$ Yang-Mills theory and related theories [5–17]. While, there has been controversy as to the detailed form of the dependance on the representation, with Casimir scaling or a sine law scaling being two popular conjectures, that the ratio of string tensions in different representations depends only on the n-ality of R and N_c, and not on the representation itself appears to be generally accepted. The belief is that the long distance behavior is determined by the formation of a "k-string" with k being the n-ality of the representation; gluon screening allows all representations with the same n-ality to connect to the same k-string. In QCD, quarks spoil the area law, forcing all string tensions to zero: physically, the area law breaks down in QCD because quark-antiquark pairs can screen the color charges [2].

Center symmetry and an area law for the Wilson loop have long been associated. 't Hooft's classic paper [18] introducing the notion of center vortices, did so to explain the area law. However, the connection between an area law for the Wilson loop in a gauge theory and center symmetry is subtle since a Wilson loop in an infinite space is neutral under center transformations. This paper explores the connection between the area law for the Wilson loop and center symmetry, for $SU(N_c)$ gauge theories generally i.e. beyond pure Yang-Mills.

To explore the connection, a large class of theories, $SU(N_c)$ gauge theories in $3+1$ or $2+1$ space-time dimensions with matter fields in all allowable representations, is considered. In $3+1$ dimensions, theories with matter fields in large representations are not ultraviolet complete. However, in $2+1$ dimensions, matter fields in all representations are possible [19], greatly enlarging the class. The connection is explored by studying relations among the string tensions which characterized area laws for various representations and relating these to center symmetry. This large class is interesting since it contains theories where matter fields spoil some, but not all, of the $\mathbb{Z}_{N_c}$ center symmetry, retaining invariance under a $\mathbb{Z}_p$ subgroup of the center. These matter fields also affect Wilson loops, properties of which that are sensitive only to p, the amount of center symmetry that survives are of interest. Recall that a hint of a deep connection between the area law and center symmetry is the fact that in QCD, the

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same thing which spoils center symmetry also spoils the area law, namely the quarks. The issue explored here, is whether matter fields that spoil only part of center symmetry have effects on area laws that are predictable solely from the amount of center symmetry preserved. For example: what properties of large Wilson loops are shared by an SU(12) gauge theory in $2+1$ space-time dimensions with quarks in a three-index symmetric representation, an SU(15) gauge theory in $2+1$ dimensions with quarks in both the 12-index antisymmetric representation and 9-index symmetric representation, and a pure Yang-Mills theory for SU(3) in $3+1$ dimension, all of which have a maximum $\mathbb{Z}_3$ symmetry?

The principal result of this paper is that in any confining theory in this class, σ_R, the string tension in representation R, is given by

$$\sigma_R = \sigma_f \frac{x_R g(x_R)}{(p-1)g(p-1)} \quad \text{with} \qquad x_R = p_R(p-p_R) \quad \text{where } p_R = n_R \text{mod}(p) \tag{2}$$

the maximum subgroup of the center under which the theory is invariant, $\mathbb{Z}_p$ determines p, $g(x)$ is a positive finite function on the domain $0 \le x < p^2/4$ that may depend on the theory, n_R is the n-ality of R and σ_f is the string tension for the fundamental representation. The key point is that the relation depends on p, the amount of center symmetry, rather than N_c, the number of colors.

This result follows from two properties of $\mathrm{SU}(N_c)$ gauge theories. Consider a gauge theory with m fields (gluons plus matter fields), with the first field carrying n-ality n_1, the second field carrying n-ality n_2 etc. Property (i) is that the maximum subgroup of the center under which the theory is invariant is $\mathbb{Z}_p$ with

$$p = \gcd(N_c, n_1, n_2, \ldots, n_m) \tag{3}$$

where gcd is the greatest common divisor. Property (ii) is that there exists a way to combine fields in the theory into a composite in a representation R, if n_R, the n-ality of representation, is an integer multiple of p as given Eq. (3). Significantly, the same value of p appears in both properties (i) and (ii). Ultimately this relates the amount of center symmetry in a theory to the theory's ability to screen color charge in a given representation. After introducing a few basic ideas, these properties will be proved and from them the principal result and some corollaries derived.

The n-ality of a representation is the number of boxes in the Young tableau specifying the representation—modulo N_c [2]. Thus, the Clebsch-Gordan decomposition of the product of operators in two representations with n-aility n_1 and n_2 contains only representations with n-anlity equal to (n_1+n_2) mod(N_c). Any representation with fixed n-ality can be obtained from any other representation with the same n-ality by combining it with some number of adjoint representations using appropriate Clebsch-Gordan coefficients. Thus by adding gluons to a combination of fields in a given representation, a combination of fields in any representation with the same n-ality can be obtained.

The center group associated with a Lie group contains those elements of the Lie group which commute with all elements of the group [20]. For $\mathrm{SU}(N_c)$, the center is $\mathbb{Z}_{N_c}$ and contains N_c elements given by $C_j = z_j \mathbf{1}$ where $\mathbf{1}$ is the $N_c \times N_c$ identity matrix and $z_j = \exp(i\frac{2\pi j}{N_c})$ with $j = 0, 1, 2, \ldots N_c - 1$. Center invariance for a gauge theory has a relative simple formulation on the lattice [2] but the connection between that formulation and the continuum is a bit subtle. Here the analysis is based on an equivalent formulation [21] directly based on the continuum version of the theory in Euclidean space. The formulation depends on the space having a finite extent, β, in the temporal direction and with periodic boundary conditions for bosons and antiperiodic for fermions ones. This setup corresponds to working at finite temperature [22]. Zero temperature physics can be studied by taking the zero temperature limit at the end of the problem. A center transformation on the gauge field has the following form:

$$A_\mu \to A'_\mu \equiv \Omega A_\mu \Omega^\dagger - g\Omega\partial_\mu\Omega^\dagger \tag{4}$$

$$\text{with} \quad \Omega(\vec{x}, \beta + t) = C\Omega(\vec{x}, t) \tag{5}$$

where Ω is an element of the gauge group at any point in space-time, $\mathbf{1}$ is the identity element, g is the gauge coupling and C is a nontrivial element of the center. Equation (4) is of the form of a local gauge transformation and leaves the Yang-Mills Lagrangian density invariant at every space-time point. It is not a true gauge transformation since in a gauge transformation Ω satisfies periodic boundary conditions: $\Omega(\vec{x}, \beta + t) = \Omega(\vec{x}, t)$. However, while Ω is not periodic, if A_μ satisfies periodic boundary conditions with $A_\mu(\vec{x}, 0) = A_\mu(\vec{x}, \beta)$ then so does A'_μ. For pure Yang-Mills theory, the only fields are the gluons whose boundary conditions are preserved by all center transformations. The transformations are thus allowable within the theory and leave the action invariant; Yang-Mills theory is center invariant.

For theories containing matter fields, the matter transforms under center transformations in the same way as under gauge transformations: quarks in the fundamental representation $q \to q' \equiv \Omega q$ while quarks in the adjoint representation transform according to $q^{\mathrm{adj}} \to q'^{\mathrm{adj}} \equiv \Omega q^{\mathrm{adj}} \Omega^\dagger$, etc. If such a transformation is allowable given the boundary conditions, it leaves the action invariant as it is of the form of a gauge transformation. However, it need not be allowable. Consider a theory containing a field ψ with n-ality n_ψ. Under center transformations associated with a particular element of the center $\mathbb{Z}_j = z_j \mathbf{1}$ in which $\psi \to \psi'$, the following identity must hold:

$$\frac{\psi(\vec{x},t)}{\psi(\vec{x},t+\beta)} = z_j^{n_\psi} \frac{\psi'(\vec{x},t)}{\psi'(\vec{x},t+\beta)}. \tag{6}$$

From the boundary conditions, $\psi_n(\vec{x},t)/\psi_n(\vec{x},t+\beta)$ is fixed to be ± 1 depending on whether the field is a bosonic or fermionic. If ψ satisfies the boundary conditions, then ψ' only satisfies the boundary conditions if

$$z_j^{n_\psi} = e^{i\frac{2\pi j n_\psi}{N_c}} = 1. \tag{7}$$

Unless Eq. (7) holds for all fields in the theory, the center transformation is not allowable. If $n_\psi = 0$ for all matter fields, as happens if they are in the adjoint representation, then Eq. (7) is always satisfied; the theory is center symmetric. If the theory contains a field with $n_\psi = 1$, as happens in a theory with quarks in the fundamental representation, then Eq. (7) is *never* satisfied except for $j = 0$, which is trivial since the "transformation" is just the identity. Such theories are not center symmetric.

If the theory has matter fields with n-ality different from zero or one, the situation is more interesting. First consider a theory with one type of matter field with n-ality, n_ψ. If n_ψ and N_c are relatively prime, then no values of j other than zero satisfy Eq. (7). Conversely, if n_ψ and N_c are not relatively prime, then Eq. (7) is satisfied if, and only if, $j = lN_c/p$, where $p = \gcd(N_c, n_\psi)$ and l is a non-negative integer less than p: transformations associated with the $\mathbb{Z}_p$ subgroup of the $\mathbb{Z}_{N_c}$ center group are allowable. In the most general case, there are multiple fields in the theory, gluons and some number of matter fields. The first matter field carries n-ality n_1; the second field carries n-ality n_2; etc. For a would-be center transformation to be allowable; it must separately preserve the boundary conditions for *all* of the matter fields as well as for the gluons. Thus the set of allowable transformations is for the group whose elements are simultaneously elements of $\mathbb{Z}_{p_1}, \mathbb{Z}_{p_2}, \ldots$ up through $\mathbb{Z}_{p_m}$ with $p_i = \gcd(N_c, n_i)$. This group is $\mathbb{Z}_p$ with p given by Eq. (3).

Having established property (i), consider property (ii). Even if a theory has no matter field in representation R, it may still contain matter in that representation. Multiple fields in a theory can combine into a configuration in representation R. For example, in Yang-Mills theory for the exceptional group G_2, three gluon fields (in the adjoint) can combine into the fundamental [23]. An $\mathrm{SU}(N_c)$ gauge theory in $3+1$ or $2+1$ space-time dimensions will have matter in representation R if, and only if, matter fields can be combined to yield a representation with n-ality n_R. Consider, the most general $\mathrm{SU}(N_c)$ theory which has m fields carrying n-ality $n_1, n_2, \ldots n_m$ and ask whether fields in the theory can be combined into representation R. The issue amounts to whether fields can be combined into n_R where the n-alities add when fields are combine together. Thus, the possible n-alities are $n = \sum_{i=0}^m l_i n_i \bmod(N_c)$ where l_i are integers. This implies that a sufficient condition for fields to combine into representation R is that there exists a set of integer l_i which satisfy the equation

$$\sum_{i=0}^m l_i n_i \mathrm{mod}(N_c) = n_R. \tag{8}$$

However, an elementary result from number theory, a generalization of Bézout's identity [24], implies that if

$$r \times \gcd(n_1, n_2, \ldots, n_m, N_c) = n_R \tag{9}$$

where r is a positive integer, then there exists a set of integers $l_1, l_2, \ldots, l_m, L$ such that $LN_c + \sum_{i=0}^m l_i m_i = n_R$ which is equivalent to Eq. (8). Thus, fields in the theory can be combined into representation R if $n_R = rp$ for some positive integer r where p is given by Eq. (3), establishing property (ii). Superficially properties (i) and (ii) deal with quite different things: the amount of center symmetry and the representations of matter which the theory possesses. However, they are connected in that they both depend on p as given by Eq. (3); the connection is number theoretic in origin.

The physical picture for area laws in various representations in Yang-Mills theory is the k-string [2,5–17]. This is the lowest lying flux tube configuration for a color source carrying n-ality k. Only the n-ality, k, matters as gluon screening can shift the representation of the color charge to one with the same n-ality, allowing the system to relax to the lowest energy flux tube with n-ality k. (Note, that the argument that only n-ality matters is valid in $3+1$ and $2+1$ space-time dimensions but not in $1+1$ dimension where there are no dynamical gluons. However, it applies to theories in $1+1$ dimensions which have matter fields in the adjoint.) Moreover, the string tension for representations with n-ality k and those with $N_c - k$ are identical since $N_c - k$ is equivalent to $-k$ and simply amounts to switching all color charges to their conjugates (e.g. the fundamental to the antifundamental) which clearly couple to the same k-string. The string tension for n-ality zero representations vanishes since in these cases the color charge can be fully screened.

One expects the k-string picture to be valid beyond Yang-Mills and to hold for confining theories in the large class of theories considered here. Clearly in this larger class, $\sigma_R = 0$ if $n_R = 0$ since in this case, the color charge can be fully screened, just as in pure Yang-Mills. The principal physics difference between this larger class and Yang-Mills is that matter fields can also screen color sources. As a consequence:

(i) σ_R, the string tension of representation depends only on $n_R \mathrm{mod}(p)$ where p is given in Eq. (3). This follows from property (ii) which means that screening can change the n-ality by an integral multiple on p. Representations whose n-ality differs by an integral multiple of p couple to the same k-string and have the same string tension.

(ii) $\sigma_R = \sigma_{R'}$ if $p_R = (p - p'_R)$. This follows from the fact that N_c is an integer multiple of p, the fact that string tension only depends on the $n_R \mathrm{mod}(p)$ and from charge conjugation which implies that the string tension for representations with n-ality k and $N_c - k$.

These facts are fully encoded in Eq. (2) provided $g(x)$ is positive and finite and p is given by Eq. (3). Since, property (i) implies that $\mathbb{Z}_p$ is the maximum subgroup of the center for the theory, where p is also given in Eq. (2), the principal result of this paper relating the maximum subgroup of the center to properties of the string tension has been established. It is important to stress that Eq. (2) depends on the "p-ality," i.e. the n-ality $\mathrm{mod}(p)$. In effect, things depend on p regardless of N_c: the center group under which the theory is invariant, rather than N_c, determines which string tensions are identical.

Four significant corollaries follow from this result:

(1) For theories with a maximum subgroup of $\mathbb{Z}_2$ or $\mathbb{Z}_3$, all representations which have the a nonzero string tension have the same string tension. This follows from the structure of x_R; for $p = 2$ all representations have either $x_R = 0$ or $x_R = 1$, while for $p = 3$, $x_R = 0$ or $x_R = 2$.

(2) A necessary condition for any theory in the class to have a nonzero string tension for representation R is for n_R not to be an integer multiple of p. This follows from the fact that $x = 0$ whenever n_R is an integer multiple of p.

(3) A necessary condition for a theory in the class to have an area law for Wilson loops for the fundamental representation is for the theory to be invariant under a nontrivial subgroup of the center. This follows since the trivial subgroup has $p = 1$ which implies that $x_R = 0$, and hence a vanishing string tension, for all representations.

(4) A necessary condition for any theory in the class to have an area law for Wilson loops for all representations with nonzero n-ality is for the theory to be invariant under the full $\mathbb{Z}_{N_c}$ center group. This follows directly from corollary 2.

Again, it should be stressed that all of these corollaries depend the size of the center group rather than N_c. Corollary 3 is particularly significant. It indicates that the connection between area laws and center symmetry is profound—invariance under some nontrivial center transformations is necessary for area laws to exist in this large class of gauge theories. While this is in accord with the prevailing "folklore" of the field, it is gratifying to see formally both that it holds quite generally, and why.

Corollary 1 shows that theories with $\mathbb{Z}_2$ or $\mathbb{Z}_3$ as the maximum subgroup of the center have the ratio of the string tension in any representation to that of the fundamental fixed to be unity or zero. Remarkably this is regardless of any other details of the theory including N_c, the dimension of space-time or the precise matter content of the theory. It is interesting to speculate on whether theories with $p > 3$ also have universal behavior with σ_R/σ_f, fixed entirely by p_R and p independently of all other details. If true, the dependence of the ratio on p_R and p must be the same as the dependence on n_R and N_c respectively as in super Yang-Mills theory, since that is in the class. This would imply a sine law as SYM is known to have this behavior [2] and would fix g to be $g(x) = A\cos(\frac{\pi}{2}\sqrt{1 - \frac{4x}{p^2}})$ where A is an arbitrary constant. However, at present one does not know whether $g(x_R)$ is universal. Perhaps, future numerical lattice studies can shed light on the issue.

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[1] For a pedagogical review of many of the issues which arise in this paper see Ref. [2].

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