PHYSICAL REVIEW D **90,** 046008 (2014)

Higher spin contributions to holographic fluid dynamics in AdS_5/CFT_4

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(Received 8 April 2013; published 26 August 2014)

We calculate the graviton's β function in the AdS string-theoretic sigma model, perturbed by vertex operators for Vasiliev's higher spin gauge fields in AdS_5. The result is given by $\beta_{mn} = R_{mn} + 4T_{mn}(g, u)$ (with the AdS radius set to 1 and the graviton polarized along the AdS_5 boundary), with the matter stress-energy tensor given by that of conformal holographic fluid in $d = 4$, evaluated at the temperature given by $T = \frac{1}{\pi}$. The stress-energy tensor is given by $T_{mn} = g_{mn} + 4u_m u_n + \sum_N T^{(N)}_{mn}$ where u is the vector excitation satisfying $u^2 = -1$ and N is the order of the gradient expansion in the dissipative part of the tensor. We calculate the contributions up to $N = 2$. The higher spin excitations contribute to the β function, ensuring the overall Weyl covariance of the matter stress tensor. We conjecture that the structure of gradient expansion in $d = 4$ conformal hydrodynamics at higher orders is controlled by the higher spin operator algebra in AdS_5.

DOI: 10.1103/PhysRevD.90.046008 PACS numbers: 04.50.-h, 11.25.Mj

I. INTRODUCTION

AdS/CFT correspondence is known to be an efficient tool to investigate dynamics of strongly coupled conformal field theories, such as nonlinear fluid dynamics. For example, the equations of hydrodynamics can be obtained by deforming the solutions of gravity with negative cosmological constant and requiring that the deformations asymptotically satisfy the Einstein equations. The AdS/hydrodynamics correspondence particularly was used to calculate various transport coefficients in holographic fluid leading to remarkable predictions such as the ratio of entropy density to sheer viscosity in conformal fluid [1–13]. The equations of conformal hydrodynamics can altogether be cast in the form of the "conservation law":

$$\nabla_m T^{mn} = 0 \tag{1}$$

where

$$T^{mn} = \sum_{N=0}^{\infty} T^{mn(N)} \tag{2}$$

where

$$T^{mn(0)} = \frac{1}{3}\epsilon(g^{mn} + 4u^m u^n) \tag{3}$$

is the ideal fluid part (with $\epsilon \sim T^4$ being the energy density satisfying $\epsilon = 3P$ where P is the pressure and T is the temperature),

$$\begin{aligned} T^{mn(1)} &= -\eta\rho^{mn} - \zeta\Pi^{mn}\vec{\nabla}\vec{u} \\ \Pi^{mn} &= \eta^{mn} + u^m u^n \\ \rho^{mn} &= \Pi^{mp}\Pi^{nq}\nabla_{(p}u_{q)} - \frac{2}{3}\Pi^{mn}\Pi^{pq}\nabla_p u_q \end{aligned} \tag{4}$$

being the viscous part (with η and ζ being the shear and the bulk viscosities proportional to the third power of the temperature; bulk term is absent in case of conformal invariance) and terms with $N \geq 2$ representing the dissipative corrections to the Navier-Stokes equation, traceless and transverse, satisfying $T^{mn(N)}u_m = 0$, which are of order N in the derivatives of u and become significant if the mean free path is comparable to the characteristic wavelength in the fluid [5–7].

Thus the full stress-energy tensor in hydrodynamics involves the derivative (gradient) expansion in the velocity with each expansion order producing new transport coefficients. For example, the second order terms result in five new transport coefficients in conformal hydrodynamics. At present, there exist various approaches to generate the derivative expansion (4) in the dual gravity theories. Strictly speaking, none of these approaches has complete control over the expansion (4) and the systematic calculation of the relative transport coefficients from dual gravity models is still problematic, especially beyond the second order hydrodynamics [14,15]. Many gravity models describing the holographic fluids generally involve the Gauss-Bonnet terms that are of higher order in the curvature and the resulting transport coefficients particularly depend on the Gauss-Bonnet coupling. These theories typically have issues with unitarity and causality which signals that, in general, they may not be fundamental but rather effective theories, with certain physical degrees of freedom, such as higher spins, integrated out. For this reason, string theory (which naturally includes higher spin modes) appears to be a

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particularly promising framework to approach the AdS/hydrodynamics duality and to test the transport coefficients at higher orders. In this paper we analyze the problem of AdS/hydrodynamics correspondence from the string theory side, by computing the graviton's conformal β function in the sigma model for AdS_5 noncritical string theory, with the graviton polarized along the $d = 4$ AdS boundary. The string model that we use is the Ramond-Neveu-Schwarz (RNS) string theory perturbed by vertex operators describing gravitational perturbations around the AdS_5 background and higher spin gauge fields in Vasiliev's framelike formalism. The low-energy limit of this model is given by the MMSW (MacDowell-Mansouri-Stelle-West) gravity [16–18] coupled to Vasiliev's higher spin gauge fields [19–22] and the vacuum solution of the low-energy theory is given by the AdS geometry [23]. Our main result (checked up to $N = 2$ level, with higher order checks now being in progress) is that the beta function of the graviton is given by

$$\beta_{mn} = R_{mn} + 4g_{mn} + 4T_{mn}$$
$$T_{mn} = g_{mn} + 4u_m u_n + \sum_{N=1}^{\infty} T_{mn}^{(N)} \qquad (5)$$

where $T_{mn}^{(N)}$ are the terms in the derivative expansion of the stress-energy tensor in $d = 4$ hydrodynamics. In other words, the low-energy equations in the AdS string model are given by the Einstein equations with cosmological term and the matter, with the latter described by the hydrodynamical stress-energy tensor. Here g_{mn} and u_m are the massless excitations described by spin 2 and 1 vertex operators in the AdS string model, in closed and open string sectors accordingly. The spin 1 operators (related to transvection isometry generators in AdS space [23]) serve as sources of the velocity vector in this model. As for the open string vertex operators for the massless higher spins, in this paper, instead of coupling them to generic Vasiliev's higher spin gauge fields, we consider the special case of coupling these operators to polynomial combinations of u_m, constructed to satisfy the same linearized on-shell [Becchi-Rouet-Stora-Tyutin (BRST)-invariance] conditions as the underlying higher spins. As a result, in the leading order of α', the structure of the higher order corrections to β_{mn} (polynomial in u) is determined by the structure constants of the operator algebra of the higher spin vertex operators (this operator algebra, in turn, fully controls the cubic couplings for generic higher spins). In the leading α' order, only the three-point correlation functions on the world sheet contribute to the graviton's β function. Our main result is that the matter stress tensor appearing in the β function reproduces the derivative expansion (4) in the stress tensor of the conformal fluid at the temperature $T = \frac{1}{\pi}$, which is checked up to the order of $N = 2$. Since the temperature transforms covariantly under Weyl rescalings, this result implies that the AdS string theory computation reproduces the stress tensor of the conformal fluid at a particular temperature gauge. We find that, at the order of $N = 2$ and higher, the β function receives nontrivial contributions from higher spin vertex operators. These contributions are crucial to ensure the conformal covariance of the stress tensor. In particular, at the $N = 2$ level the graviton's β function is contributed by the $\langle 2 - 3 - 3 \rangle$ correlator on the disc, while at higher orders operators of spin 4 and higher also enter the game, so the holographic derivative expansion (4) is controlled by operator algebra of higher spin vertices in the limit of $\alpha' \to 0$. The rest of the paper is organized as follows: In Sec. 2, we explain the basic vertex operator setup of the sigma model, whose low-energy limit describes the AdS gravity coupled to higher spins in the framelike formalism. In Sec. 3, we perform the computations of the $\langle 1 - 1 - 2 \rangle$ and $\langle 1 - 3 - 3 \rangle$ correlators, contributing to the graviton's β function and reproducing the holographic expansion (4) up to the second order. In the concluding section, we comment on the structure of the higher order terms related to higher spin contributions and discuss physical implication of our results.

II. ADS STRING σ MODEL: VERTEX OPERATORS AND 2d WEYL INVARIANCE

In this section we review the construction of the string-theoretic sigma model [23] with some modifications that will be used in our calculation of the graviton's beta function. Technically, the sigma model that we use in calculations in this work is similar but not identical to the one constructed in our previous works (e.g. see [23]) as it will combine vertex operators for both Fronsdal-like objects (such as the vertex operator for a graviton describing perturbations around AdS vacuum) and those of Vasiliev's type (describing framelike higher spin excitations around an AdS vacuum solution of the low-energy equations of motion).

The AdS string sigma model considered in [23] was based purely on vertex operators for framelike gauge fields (rather than those of Fronsdal type) and was described by the generating functional

$$Z(e_m^a, \omega_m^{ab}, \Omega_m^{A_1\ldots A_{s-1}|B_1\ldots B_t}) = \int D[X, \psi, \varphi, \lambda, \text{ghosts}] \exp\Big\{ -S_{\text{RNS}} + e_m^a F^m \bar{L}_a + \omega_m^{ab}(p)\Big(F_b^m \bar{L}_a - \frac{1}{2} F_{ab} \bar{L}^m\Big) + \text{c.c.} + \sum_{s\geq 3; 0\leq t\leq s-1} \Omega_m^{A_1\ldots A_{s-1}|B_1\ldots B_t} V^m_{A_1\ldots A_{s-1}|B_1\ldots B_t} \Big\} \qquad (6)$$

where

$$\begin{aligned} S_{\rm RNS} &= S_{\rm matter} + S_{bc} + S_{\beta\gamma} + S_{\rm Liouville} \\ S_{\rm matter} &= -\frac{1}{4\pi}\int d^2z(\partial X_m\bar\partial X^m + \psi_m\bar\partial\psi^m + \bar\psi_m\partial\bar\psi^m) \\ S_{bc} &= \frac{1}{2\pi}\int d^2z(b\bar\partial c + \bar b\partial\bar c) \\ S_{\beta\gamma} &= \frac{1}{2\pi}\int d^2z(\beta\bar\partial\gamma + \bar\beta\partial\bar\gamma) \\ S_{\rm Liouville} &= -\frac{1}{4\pi}\int d^2z(\partial\varphi\bar\partial\varphi + \bar\partial\lambda\lambda + \partial\bar\lambda\bar\lambda \\ &\quad + \mu_0 e^{B\varphi}(\lambda\bar\lambda + F)) \end{aligned} \tag{7}$$

where $S_{\rm RNS}$ is the full d-dimensional RNS superstring action; $X^m\,(m = 0,\ldots d-1)$ are the space-time coordinates; φ, λ, F are components of super-Liouville field and the Liouville background charge is

$$Q = B + B^{-1} = \sqrt{\frac{9-d}{2}}. \tag{8}$$

Next, e^a_m and ω^{ab}_m are vielbein and spin connection gauge fields generated by closed string vertex operators whose holomorphic and antiholomorphic components are given by

$$\begin{aligned} F_m &= -2K_{U_1}\circ\int dz\lambda\psi_m e^{ipX}(z) \\ U_1 &= \lambda\psi_m e^{ipX} + \frac{i}{2}\gamma\lambda((\vec p\,\vec\psi)\psi_m - p_m P^{(1)}_{\phi-\chi})e^{ipX} \end{aligned} \tag{9}$$

or manifestly

$$F_m = -2\int dz\left\{\lambda\psi_m(1 - 4\partial cce^{2\chi-2\phi}) + 2ce^{\chi-\phi}\left(\lambda\partial X_m - \partial\varphi\psi_m + q\psi_m P^{(1)}_{\phi-\chi} - \frac{i}{2}((\vec p\,\vec\psi)\psi_m - p_m P^{(1)}_{\phi-\chi})\right)\right\}e^{ipX}(z). \tag{10}$$

Next,

$$\bar L^a = \int d\bar z e^{-3\bar\phi}\left\{\bar\lambda\bar\partial^2X^a - 2\bar\partial\bar\lambda\bar\partial X^a + ip^a\left(\frac{1}{2}\bar\partial^2\bar\lambda + \frac{1}{q}\bar\partial\bar\varphi\bar\partial\bar\lambda - \frac{1}{2}\bar\lambda\left(\bar\partial\bar\varphi)^2 + (1+3q^2)\bar\lambda\left(3\bar\partial\bar\psi_b\bar\psi^b - \frac{1}{2q}\bar\partial^2\bar\varphi\right)\right)\right\}e^{ipX} \tag{11}$$

at the minimal negative picture -3 representation and

$$\bar L^a = K\circ\int d\bar z e^{\bar\phi}\left\{\bar\lambda\bar\partial^2X^a - 2\bar\partial\bar\lambda\bar\partial X^a + ip^a\left(\frac{1}{2}\bar\partial^2\bar\lambda + \frac{1}{q}\bar\partial\bar\varphi\bar\partial\bar\lambda - \frac{1}{2}\bar\lambda\left(\bar\partial\bar\varphi)^2 + (1+3q^2)\bar\lambda\left(3\bar\partial\bar\psi_b\bar\psi^b - \frac{1}{2q}\bar\partial^2\bar\varphi\right)\right)\right\}e^{ipX} \tag{12}$$

at the minimal positive picture $+1$ representation (similarly for its holomorphic counterpart L^a). Here and elsewhere below the normalizations of vertex operators are chosen so they lead to standard normalizations of corresponding kinetic terms in low-energy effective action. Then,

$$F_{ma} = F^{(1)}_{ma} + F^{(2)}_{ma} + F^{(3)}_{ma} \tag{13}$$

where

$$\begin{aligned} F^{(1)}_{ma} &= -4qK_{U_2}\circ\int dzce^{\chi-\phi}\lambda\psi_m\psi_a \\ U_2 &= [Q - Q_3, ce^{\chi-\phi}\lambda\psi_m\psi_a e^{ipX}] \\ &\quad - \frac{i}{2}c\lambda((\vec p\,\vec\psi)\psi_a\psi_m - p_m\psi_a P^{(1)}_{\phi-\chi})e^{ipX}(z) \end{aligned} \tag{14}$$

$$\begin{aligned} F^{(2)}_{ma} &= K\circ\int dz\psi_m\psi_a e^{ipX} \\ &= -4\left\{Q, \int dzce^{2\chi-2\phi}e^{ipX}\psi_m\psi_a(z)\right\} \end{aligned} \tag{15}$$

and

$$F^{(3)}_{ma} = K\circ\int dze^{\phi}(\psi_{[m}\partial^2X_{a]} - 2\partial\psi_{[m}\partial X_{a]})e^{ipX}(z). \tag{16}$$

Here the homotopy transform of an operator V $K\circ V$ is defined according to

$$\begin{aligned} K\circ V &= T + \frac{(-1)^N}{N!}\oint\frac{dz}{2i\pi}(z-w)^N :K\partial^N W:(z) \\ &\quad + \frac{1}{N!}\oint\frac{dz}{2i\pi}\partial_z^{N+1}[(z-w)^N K(z)]K\{Q_{\rm brst}, U\} \end{aligned} \tag{17}$$

where w is some arbitrary point on the world sheet, U and W are the operators defined according to

$$[Q_{\rm brst}, V(z)] = \partial U(z) + W(z), \tag{18}$$

$$K = ce^{2\chi-2\phi} \tag{19}$$

is the homotopy operator satisfying $\{Q_{\rm brst}, K\} = 1$ and N is the leading order of the operator product

$$K(z_1)W(z_2) \sim (z_1 - z_2)^N Y(z_2) + O((z_1 - z_2)^{N+1}). \tag{20}$$

The *partial* homotopy transform $T \to L = K_\Upsilon\circ T$ of an operator T based on Υ is defined according to

$$L(w) = K_\Upsilon \circ T = T + \frac{(-1)^N}{N!}\oint \frac{dz}{2i\pi}(z-w)^N : K\partial^N \Upsilon : (z) + \frac{1}{N!}\oint \frac{dz}{2i\pi}\partial_z^{N+1}[(z-w)^N K(z)]K\{Q_{\mathrm{brst}}, U\} \qquad (21)$$

where N is the leading order of the operator product expansion (OPE) of K and Υ. Particularly, if $[Q_{\mathrm{brst}}, T] = \oint \Upsilon$, the partial homotopy transform obviously coincides with the usual homotopy transform. Finally, $V^m_{a_1 \ldots a_{s-1}|b_1 \ldots b_t}; 0 \leq t \leq s-1$ are the open string vertex operators for emission of gauge fields of spin s which, in Vasiliev's approach, are described (for each s) by a collection of two-row fields $\Omega^{s-1|t} \equiv \Omega_m^{a_1 \ldots a_{s-1}|b_1 \ldots b_t}$. In this approach, only the $\Omega^{s-1|0}$ field is dynamical while those with nonzero t values can be expressed in terms of order t derivatives of the dynamical field: $\Omega^{s-1|t} \sim \partial^{(t)}\Omega^{s-1|0}$ through generalized zero torsion constraints (e.g. see [19–22,24]). In string theory, these constraints are realized in terms of ghost cohomology conditions on the higher spin vertex operators [25]. The BRST-invariance constraints on the higher spin vertex operators (6) lead to linearized on-shell constraints on the framelike fields while BRST nontriviality conditions lead to gauge symmetry transformation by these fields; the world sheet correlators of the appropriate vertex operators multiplied by the corresponding space-time fields are then invariant by construction [26]. The vertex operators in the generating functional (6) can be classified in terms of ghost cohomologies $H_n \sim H_{-n-2}; n \geq 0$. For example, the spin 2 operators for vielbein and connection gauge fields are the elements of $H_0 \otimes \bar{H}_1 + \mathrm{c.c.}$ (with H and $\bar{H}$) referring to holomorphic and antiholomorphic parts) while the class of higher spin operators V_s of $s \geq 3$ that we are considering is restricted to open string vertex operators at nonzero cohomologies; typically, $V_s \in H_n$ with $s-2 \geq n \geq 2s-2$ (this includes both dynamical and the extra fields that sit at different cohomologies, with the dynamical field occupying the lowest order positive cohomology). In the previous works [25,27] we analyzed the low-energy limit of the model (6) showing that, in the leading order in e and ω in the absence of the open string excitations (spin 1 and higher spins), its low-energy equations of motion are given by

$$d\omega + \omega \wedge \omega - e \wedge e = 0 \qquad (22)$$

whose vacuum solution is given by AdS geometry (here and elsewhere, unless specified otherwise we set the AdS radius $\rho_{\mathrm{AdS}} = 1$).

All the vertex operators (6) are related to underlying global symmetries of space-time. In particular, at the limit of momentum zero, the L^a operators entering expressions for vertex operators of vielbein are related to transvection generators in the isometry algebra of AdS_d while F^{mn} operators are related to the rotational part of this isometry algebra. $V^m_{a_1 \ldots a_{s-1}|b_1 \ldots b_t}$ operators, in turn, are related to the higher spin currents, or the generators of the higher spin symmetry algebra which, to put it roughly, is the infinite-dimensional algebra related to the universal enveloping of the isometry algebra. The higher spin algebra is thus realized in superstring theory as the operator algebra of the appropriate higher spin states, whose structure constants are given by the relevant three-point correlators, or the leading order contributions to the conformal beta functions. In the present paper, we investigate the correlators contributing to the β function of the graviton. Our interpretation of the spin 1 and the higher spin vertex operators is, however, different from the one of the previous papers [23,25]. Instead of interpreting the vertex operators as the emission vertices for fundamental particles, we consider them as sources of various polynomials in the vector field u^m (with the polynomial degree obviously related to the spin value) with the structure of the polynomials determined by the on-shell conditions on the corresponding operators. The polynomials in u, constructed in such a manner, correspond to special configurations of higher spin fields, solving Pauli-Fierz conditions in the on-shell limit. In the present paper we restrict ourselves to these particular solutions although it would certainly be important to generalize the calculations presented in this paper to higher spin vertex operators of more general structure.

The idea is that, in the limit of $\alpha' \to 0$, the polynomial contributions to the β functions and the derivative expansion (2) are controlled by the appropriate structure constants in operator algebra of the higher spin vertex operators for framelike gauge fields (which, in turn, naturally realize higher spin algebra in a certain basis).

The constraint $u^2 = -1$ particularly follows from the on-shell conditions, allowing us to interpret u^m as the velocity vector in some underlying fluid. Then the β-function equations of the graviton are realized as the Einstein equations with the cosmological term and with the matter, with the matter stress tensor being that of the hydrodynamics. Our claim is that the derivative expansion in holographic $d = 4$ hydrodynamics is determined, in the leading order, by the higher spin algebra in AdS_5 (calculated in a string theory approach), with the higher order dissipative terms controlled by the derivative structure of higher spin correlators.

III. GRAVITON IN THE FRAMELIKE SIGMA MODEL AND TWO-DIMENSIONAL WEYL INVARIANCE OF THE OPERATORS

As was explained above, the first building block that we shall need in our construction is the graviton vertex operator describing metric perturbations around the AdS vacuum, as opposed to operators for vielbeins and spin connections present in (6). Similarly to the flat space case (where the graviton operator is an object bilinear in flat space translation operators), the vertex operator for the graviton that we are looking for has to be an object bilinear in AdS_5 isometry generators (transvections), with the BRST constraints imposing appropriate on-shell conditions and gauge transformations. According to (6) there are two types of such operators—those of L type and those of F type. **The bilinears of mixed $L-F$ type correspond to vielbeins and connection gauge fields (elements of $[H_0 \otimes \bar{H}_1]$ cohomology) so the suitable candidates are either of $F-F$ type in $[H_0 \otimes \bar{H}_0]$ cohomology or $L-L$ type in $[H_1 \otimes \bar{H}_1]$. The objects of $F-F$ type, however, clearly do not reproduce proper on-shell conditions and have excessive gauge symmetry; therefore the appropriate candidate for the graviton operator is the one in $[H_1 \otimes \bar{H}_1] \sim [H_{-3} \otimes \bar{H}_{-3}]$, with the explicit expression given by

$$
\begin{aligned}
V_{\text{grav}}^{H_{-3}\otimes H_{-3}} = G_{mn}(p)c\bar{c}e^{-3\phi-3\bar{\phi}}\Big\{&\bar{\lambda}\bar{\partial}^2X^m - 2\bar{\partial}\bar{\lambda}\bar{\partial}X^m \\
&+ ip^m\left(\frac{1}{2}\bar{\partial}^2\bar{\lambda} + \frac{1}{q}\partial\varphi\partial\lambda - \frac{1}{2}\lambda(\partial\varphi)^2 + (1+3q^2)\lambda\left(3\partial\psi_p\psi^p - \frac{1}{2q}\partial^2\varphi\right)\right)\Big\}\Big\{\bar{\lambda}\bar{\partial}^2X^n - 2\bar{\partial}\bar{\lambda}\bar{\partial}X^n \\
&+ ip^n\left(\frac{1}{2}\bar{\partial}^2\bar{\lambda} + \frac{1}{q}\bar{\partial}\bar{\varphi}\bar{\partial}\bar{\lambda} - \frac{1}{2}\bar{\lambda}(\bar{\partial}\bar{\varphi})^2 + (1+3q^2)\bar{\lambda}\left(3\bar{\partial}\bar{\psi}_q\bar{\psi}^q - \frac{1}{2q}\bar{\partial}^2\bar{\varphi}\right)\right)\Big\}e^{ipX}
\end{aligned}
\tag{23}
$$

at minimal negative picture -3 unintegrated representation and

$$
\begin{aligned}
V = G_{mn}(p)K\bar{K}\circ\int d^2ze^{\phi+\bar{\phi}}\Big\{&\bar{\lambda}\bar{\partial}^2X^m - 2\bar{\partial}\bar{\lambda}\bar{\partial}X^m y \\
&+ ip^m\left(\frac{1}{2}\bar{\partial}^2\bar{\lambda} + \frac{1}{q}\partial\varphi\partial\lambda - \frac{1}{2}\lambda(\partial\varphi)^2 + (1+3q^2)\lambda\left(3\partial\psi_p\psi^p - \frac{1}{2q}\partial^2\varphi\right)\right)\Big\}\Big\{\bar{\lambda}\bar{\partial}^2X^n - 2\bar{\partial}\bar{\lambda}\bar{\partial}X^n \\
&+ ip^n\left(\frac{1}{2}\bar{\partial}^2\bar{\lambda} + \frac{1}{q}\bar{\partial}\bar{\varphi}\bar{\partial}\bar{\lambda} - \frac{1}{2}\bar{\lambda}(\bar{\partial}\bar{\varphi})^2 + (1+3q^2)\bar{\lambda}\left(3\bar{\partial}\bar{\psi}_q\bar{\psi}^q - \frac{1}{2q}\bar{\partial}^2\bar{\varphi}\right)\right)\Big\}e^{ipX}
\end{aligned}
\tag{24}
$$

at minimal positive picture $+1$ representation (note that the operators at positive cohomologies are always integrated). The antiholomorphic $\bar{K}$ transformation is defined similarly to the holomorphic one (17). The transformation $G^{mn} \to G^{mn} + p^{(m}\Lambda^{n)}$ shifts (24) by BRST-exact part. The leading order contribution to the graviton's beta function is the result of the Weyl invariance constraints on the operator (24). These constraints can be conveniently deduced from the OPE:

$$
\sim \int d^2z\int d^2wT_{z\bar{z}}(z,\bar{z})V_{\text{grav}}(w,\bar{w}) \tag{25}
$$

by expanding around the midpoint and evaluating the coefficient in front of $\sim\frac{V_{\text{grav}}(\frac{z+w}{2},\frac{\bar{z}+\bar{w}}{2})}{|z-w|^2}$ (note that the trace $T_{z\bar{z}}$ of the stress-energy tensor, generating the Weyl transformation, is nonzero off shell or, equivalently, in the underlying ϵ expansion). For a usual graviton operator $\sim G_{mn}(p)\int d^2w\partial X^m\bar{\partial}X^ne^{ipX}(w,\bar{w})$ in the bosonic string this procedure leads, after simple calculation, to the standard β-function contribution, quadratic in momentum, given by the linearized part of the Ricci tensor plus the second derivative of the dilaton $\sim R_{mn}^{\text{lin}} - 2p_mp_n\Phi$ with $\Phi \sim tr(G_{mn})$. The calculation, leading to the identical result, is similar in superstring theory. The graviton operator should then be taken at canonical ghost picture [unintegrated $b-c$ picture and $(-1,-1)$ $\beta-\gamma$ ghost picture], so $V_{\text{grav}} = c\bar{c}e^{-\phi-\bar{\phi}}\psi^m\psi^ne^{ipX}$ and the relevant terms in the stress tensor are

$$
\begin{aligned}
T_{z\bar{z}} &\equiv T_{z\bar{z}}^{\text{matter}} + T_{z\bar{z}}^{b-c} + T_{z\bar{z}}^{\beta-\gamma} \\
&= \frac{1}{2}(-\partial X_m\bar{\partial}X^m - \bar{\partial}\psi_m\psi^m - \partial\bar{\psi}_m\bar{\psi}^m \\
&\quad + \partial\sigma\bar{\partial}\sigma + \partial\chi\bar{\partial}\chi - \partial\phi\bar{\partial}\phi).
\end{aligned}
\tag{26}
$$

The OPE of V_{grav} with $T_{z\bar{z}}^{\text{matter}}$ then contributes the term $\sim p^2G_{mn}$ to the graviton's beta function (which is the gauge-fixed linearized part of the Ricci tensor, with the gauge condition $\sim p^mG_{mn} = 0$), while the contribution stemming from the OPE with $T_{z\bar{z}}^{b-c}$ cancels the one from the OPE with $T_{z\bar{z}}^{\beta-\gamma}$ since $\partial\sigma\bar{\partial}\sigma(z,\bar{z})c\bar{c}(w,\bar{w}) \sim \frac{1}{|z-w|^2}c\bar{c}(w,\bar{w})$,

$\partial\phi\bar{\partial}\phi(z,\bar{z})e^{-\phi-\bar{\phi}}(w,\bar{w}) \sim \frac{1}{|z-w|^2}e^{-\phi-\bar{\phi}}(w,\bar{w})$ and σ and ϕ terms of $T_{z\bar{z}}$ have opposite signs. It is this cancellation that ensures the absence of "cosmological terms" in the β function of the graviton with the conventional vertex operator leading to Einstein gravity around the flat vacuum. In the case of the vertex operator (24), the OPE of $T^{\mathrm{matter}}_{z\bar{z}}$ with $V^{H_{-3}\otimes H_{-3}}_{\mathrm{grav}}$ still results in the appearance of the linearized Ricci tensor. However, since this operator is the element of $H_{-3}\otimes\bar{H}_{-3}$, and its canonical ϕ-ghost picture is $(-3,-3)$ [23], the contributions from $T^{b-c}_{z\bar{z}}$ and $T^{\beta-\gamma}_{z\bar{z}}$ no longer cancel each other as

$$(T^{b-c}_{z\bar{z}}+T^{\beta-\gamma}_{z\bar{z}})(z,\bar{z})V^{H_{-3}\otimes H_{-3}}_{\mathrm{grav}}(w,\bar{w}) \sim \frac{\frac{1}{2}(1-3^2)V^{H_{-3}\otimes H_{-3}}_{\mathrm{grav}}}{|z-w|^2} \tag{27}$$

leading to the cosmological term proportional to $\sim 4G_{mn}$ in the β function. Thus the Weyl invariance condition brings the piece proportional to $\sim R^{\mathrm{linearized}}_{mn}+4g_{mn}$ to the β function (assuming that the dilaton is switched off). The higher order (quadratic) terms in β_{mn} are given by the appropriate three-point functions. In the next section we shall analyze these terms by computing the corresponding three-point correlators on the disc.

IV. GRAVITON'S β FUNCTION: QUADRATIC CONTRIBUTIONS

We start with the analysis of $\langle 1-1-2\rangle$ and $\langle 3-3-2\rangle$ correlators on the disc. These correlators give rise to contributions of zero and second powers in momentum, particularly producing terms corresponding to the stress tensor of ideal fluid and second order hydrodynamics (in this paper we disregard the higher order contributions, such as those of the quartic order). The first order terms stem from Weyl invariance constraints on the operators while the third order is produced by $\langle 2-2-3\rangle$ and $\langle 2-2-1\rangle$ disc correlators. (In this paper, however, we do not consider the third order terms.) We start with the $\langle 1-1-2\rangle$ contribution. The spin 1 vertex operator is the element of H_1, given by

$$V_{s=1} = u_m L^m(p) = K\circ\int dz e^{\bar{\phi}}\Big\{\lambda\partial^2X^a - 2\partial\lambda\partial X^a + ip^a\Big(\frac{1}{2}\partial^2\lambda+\frac{1}{q}\partial\varphi\partial\lambda-\frac{1}{2}\lambda(\partial\varphi)^2+(1+3q^2)\lambda\Big(3\partial\psi_b\psi^b-\frac{1}{2q}\partial^2\varphi\Big)\Big)\Big\}e^{ipX}. \tag{28}$$

To ensure the overall ϕ-ghost number balance (-2 on the disc) it is convenient to take the graviton's operator unintegrated at $(-3,-3)$ picture representations while transforming both of the integrated spin 1 operators to picture 2. The full expression for $V_{s=1}$ at picture 2 is complicated; however we do not need all the terms but only those contributing to the three-point $\langle 2-1-1\rangle$ correlator according to ghost number selection rules. The picture 2 operator contains three classes of such terms—those proportional to the $e^{2\phi}$ ghost factor, those proportional to $be^{3\phi-\chi}$ and those proportional to ce^{χ}, so the nonvanishing ghost correlators are proportional to the exponential factors $\sim\langle e^{-3\phi-3\bar{\phi}}(0)ce^{\chi+\phi}(\tau_1)be^{3\phi-\chi}(\tau_2)\rangle$ and $\sim\langle e^{-3\phi-3\bar{\phi}}(0)\times e^{2\phi}(\tau_1)e^{2\phi}(\tau_2)\rangle$ where τ_1 and τ_2 are the locations of the $s=1$ operators. Straightforward evaluation of the picture-changing transformation of (24), however, shows that the overall coefficient in front of the terms proportional to $be^{3\phi+\chi}$ vanishes, so it is only the second ghost structure $\sim\langle e^{-3\phi-3\bar{\phi}}(0)e^{2\phi}(\tau_1)e^{2\phi}(\tau_2)\rangle$ that is relevant to the correlator. Thus we only need the part of $V_{s=1}$ at picture 2 proportional to $e^{2\phi}$; straightforward application of picture-changing and homotopy transformations lead to the following expression for the relevant part of $V_{s=1}$:

$$V_{s=1}(z;p) = u^m(p)\sum_{k=1}^{5}\frac{1}{192\times(5-k)!}\int d\tau(w-\tau)^4 e^{2\phi+ipX}P^{(4)}_{2\phi-2\chi-\sigma}P^{(5-k)}_{\phi-\chi}L^{(k)}_m(\lambda,\varphi,X,\psi,\tau) \tag{29}$$

where $P^{(N)}_{a_1\phi+a_2\chi+a_3\chi}$ ($a_{1,2,3}$ are numbers) are the conformal dimension N ghost polynomials whose definition and properties are discussed in [25], the space-time vectors $L^{(k)}_m$; $k=1,\ldots,5$ are the conformal dimension k operators consisting of the matter fields, whose manifest expressions are given by

$$
\begin{aligned}
L_m^{(k)} = &\left\{\frac{1}{(k-1)!}\partial^{(k-1)}\psi_m\lambda - \frac{1}{(k-2)!}\partial^{(k-2)}\psi_m\partial\lambda(1-\delta_1^k)\right.\\
&-\frac{i}{(k-3)!}\vec{p}\partial^{(k-3)}\vec{\psi}F_m a(1-\delta_1^k)a(1-\delta_2^k)\\
&+3ip_m(1+3Q^2)\lambda(1-\delta_1^k)\left[\frac{1}{(k-2)!}\partial^{(k-1)}\vec{X}\vec{\psi} - \frac{1}{(k-3)!}\partial^{(k-2)}\vec{X}\partial\vec{\psi}(1-\delta_2^k)\right]\\
&-\frac{i}{2(k-1)!}p_m\partial^{(k)}\varphi + (1-\delta_1^k)\left[-\frac{1}{(k-2)!}\partial^{(p-1)}\varphi\partial X_m + \frac{ip_m}{2Q}\partial^{(p-1)}\varphi\partial\varphi\right]\\
&+(1-\delta_1^k)(1-\delta_2^k)\left[\frac{1}{2(k-3)!}\partial^{(k-2)}\varphi\partial^2 X_m\right.\\
&\left.+\frac{ip_m}{4}\partial^{(p-2)}\varphi\left((\partial\varphi)^2 + 2(1+3Q^2)\left(\partial\vec{\psi}\vec{\psi} - \frac{1}{2Q}\partial^2\varphi\right)\right)\right]\\
&+ip_m\frac{(1-\delta_1^k)}{(k-2)!}\left[\frac{1}{2Q}\left[\partial^{(k-2)}\lambda\partial\lambda - \frac{1}{2}\partial^{(k-2)}\lambda\lambda\partial\varphi - \frac{1+3Q^2}{4Q(k-1)}\partial^{(p-1)}\lambda\lambda\right] + \delta_k^1\right]\\
&-\left[\frac{3}{2}(1+3Q^2)p_m\lambda(\vec{p}\vec{\psi}) - (2+3Q^2)ip_m\partial\varphi - 2Q\partial X_m\right]\\
&\left.+\delta_k^2\left[\frac{3}{2}(1+3Q^2)p_m\lambda(\vec{p}\partial\vec{\psi}) + \frac{Q}{2}\partial^2 X_m - \frac{iQp_m}{4}(\partial\varphi)^2\right]\right\}e^{ipX}. \qquad (30)
\end{aligned}
$$

Although the expression (29) for the integrated picture 2 $V_{s=1}$ depends on an arbitrary point z on the world sheet, this dependence is irrelevant in correlation functions since all the w derivatives of (29) are BRST exact. For this reason, w can be chosen arbitrarily in the integral (29).

We are now prepared to analyze the three-point $\langle 2-1-1\rangle$ amplitude on the disc. The unintegrated $V_{s=2}$ vertex is convenient to place at the disc's origin, that is, at the zero point. The calculation strategy is similar to the one described in [27]. It is convenient to map a disc to a half-plane using the conformal transformation:

$$
z \to f(z) = \frac{i}{2}\frac{z+i}{z-i} \qquad (31)
$$

and to calculate the three-point correlator on the plane. The integrals over the disc boundary are then transformed into integrals over the real line. On the half-plane, it is convenient to choose $w_1 = w_2 = \frac{i}{2}$ in τ_1 and τ_2 integrals for the open string vertices. Having calculated the half-plane correlators, we shall further conformally map it back to the disc and evaluate the integrals (which essentially will become the angular integrals). Under the transformation (31) the left part of the $V_{s=2}$ vertex operator is mapped to $z_1 = \frac{i}{2}$ while the right part is mapped to $z_2 = -\frac{i}{2}$. The ghost factors of the correlator for each term in the sum over k_1, k_2 [stemming from the summation over k in (29)] are given by

$$
\begin{aligned}
A_{\text{ghost}}^{(k_1,k_2)}(p,k,q) &= \left\langle ce^{-3\phi}\left(\frac{i}{2}\right)ce^{-3\phi}\left(-\frac{i}{2}\right)e^{2\phi}P^{(4)}_{2\phi-2\chi-\sigma}P^{(5-k_1)}_{\phi-\chi}(\tau_1)e^{2\phi}P^{(4)}_{2\phi-2\chi-\sigma}P^{(5-k_2)}_{\phi-\chi}(\tau_2)\right\rangle\\
&= \left|\frac{i}{2}-\tau_1\right|^{12}\left|\frac{i}{2}-\tau_2\right|^{12} \times H^{(5-k_1)}_{-3;-3;2}\left(\tau_1\Big|\frac{i}{2},-\frac{i}{2},\tau_2\right)H^{(5-k_2)}_{-3;-3;2}\left(\tau_2\Big|\frac{i}{2},-\frac{i}{2},\tau_1\right)\\
&\quad\times\left[H^{(4)}_{-5;-5;4}\left(\tau_1\Big|\frac{i}{2},-\frac{i}{2},\tau_2\right)H^{(4)}_{-5;-5;4}\left(\tau_2\Big|\frac{i}{2},-\frac{i}{2},\tau_1\right)\right.\\
&\quad\left.+12(\tau_1-\tau_2)^2 H^{(3)}_{-5;-5;4}\left(\tau_1\Big|\frac{i}{2},-\frac{i}{2},\tau_2\right)H^{(3)}_{-5;-5;4}\left(\tau_2\Big|\frac{i}{2},-\frac{i}{2},\tau_1\right)\right] \qquad (32)
\end{aligned}
$$

where the functions $H^{(N)}_{a_1,\ldots a_N}(\tau|\tau_1,\ldots\tau_N)$ are defined according to

$$H^{(N)}_{a_1,\ldots a_N}(\tau|\tau_1,\ldots\tau_N) = N! \sum_{N|m_1,\ldots,m_N}^{m_1+\ldots+m_N=N} \prod_{j=1,m_j\neq 0}^{N} \frac{1}{m_j P_{m_j}!} \sum_{i=1}^{N} \frac{a_i}{(\tau_i-\tau)^{m_j}}. \quad (33)$$

Here $\{m_1,\ldots m_N\}; m_1 < m_2\ldots < m_N$ are the partitions of number N of length N including zeros, P_{m_j} for $m_j \neq 0$ are the multiplicities at which given m_j enter the partition; and by definition $P_0 \equiv P_{m_j=0} \equiv 0$, $P_0! = 1$ no matter how many zeros enter the partition. For example, the partition $10 = 0+0+0+0+0+1+1+2+3+3$ would read as $m_1 = 0; m_2 = 1$, $m_3 = 2$, $m_4 = 3$ with $P_{m_1} = P_0 = 0$; $P_{m_2} \equiv P_1 = 2$; $P_{m_3} \equiv P_2 = 1$; $P_{m_4} \equiv P_3 = 2$. Therefore the overall $\langle 2-1-1\rangle$ correlator on the half-plane is given by

$$\begin{aligned}\left\langle V_{s=2}\left(\frac{i}{2},-\frac{i}{2}\right)V_{s=1}\left(\frac{i}{2}\right)V_{s=1}\left(\frac{i}{2}\right)\right\rangle &= g^{m_1m_2}(p)u^{n_1}(q_1)u^{n_2}(q_2)\\ &\times \sum_{k_1=1}^{5}\sum_{k_2=1}^{5}\frac{1}{192^2\times(5-k_1)!(5-k_2)!}\\ &\times \int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\infty}d\tau_2\left(\frac{i}{2}-\tau_1\right)^4\left(\frac{i}{2}+\tau_2\right)^4\left[\frac{(\frac{i}{2}+\tau_1)(\frac{i}{2}+\tau_2)}{(\frac{i}{2}-\tau_1)(\frac{i}{2}-\tau_2)}\right]^{k_1k_2}\\ &\times \left|\frac{i}{2}-\tau_1\right|^{12}\left|\frac{i}{2}-\tau_2\right|^{12}H^{(5-k_1)}_{-3;-3;2}\left(\tau_1\Big|\frac{i}{2},-\frac{i}{2},\tau_2\right)H^{(5-k_2)}_{-3;-3;2}\left(\tau_2\Big|\frac{i}{2},-\frac{i}{2},\tau_1\right)\\ &\times\left[H^{(4)}_{-5;-5;4}\left(\tau_1\Big|\frac{i}{2},-\frac{i}{2},\tau_2\right)H^{(4)}_{-5;-5;4}\left(\tau_2\Big|\frac{i}{2},-\frac{i}{2},\tau_1\right)\right.\\ &\left.+12(\tau_1-\tau_2)^2H^{(3)}_{-5;-5;4}\left(\tau_1\Big|\frac{i}{2},-\frac{i}{2},\tau_2\right)H^{(3)}_{-5;-5;4}\left(\tau_2\Big|\frac{i}{2},-\frac{i}{2},\tau_1\right)\right]\\ &\times\left\langle F_{m_1}\left(p;\frac{i}{2}\right)F_{m_2}\left(p;-\frac{i}{2}\right)L^{(k_1)}_{n_1}(q_1;\tau_1)L^{(k_2)}_{n_2}(q_2;\tau_2)\right\rangle. \quad (34)\end{aligned}$$

The final step is to evaluate the matter part of the correlator in (34), given by $\langle F_{m_1}(p;\frac{i}{2})F_{m_2}(p;-\frac{i}{2})L^{(k_1)}_{n_1}\times(q_1;\tau_1)L^{(k_2)}_{n_2}(q_2;\tau_2)\rangle$, with the expressions for $L^{(k)}_n$ given in (30). It is convenient to define the following functions:

$$\begin{aligned}R^{(a)}_m(y|(x_1,p_1);(x_2,p_2),(x_3,p_3)) &= (-1)^a(a-1)!\left[\frac{ip_{1m}}{(y-x_1)^a}+\frac{ip_{2m}}{(y-x_2)^a}+\frac{ip_{3m}}{(y-x_3)^a}\right]K^{(a_1,a_2,a_3,a_4)}_{\lambda}(z_1,z_2,\tau_1,\tau_2)\\ &\equiv \langle\partial^{(a_1)}\lambda(z_1)\partial^{(a_2)}\lambda(z_2)\partial^{(a_3)}\lambda(\tau_1)\partial^{(a_4)}\lambda(\tau_2)\rangle\\ &= \frac{(-1)^{a_1+a_3}(a_1+a_2)!(a_3+a_4)!}{(z_1-z_2)^{a_1+a_2+1}(\tau_1-\tau_2)^{a_3+a_4+1}}+\frac{(-1)^{a_1+a_2}(a_1+a_3)!(a_2+a_4)!}{(z_1-\tau_1)^{a_1+a_3+1}(z_2-\tau_2)^{a_2+a_4+1}}\\ &\quad+\frac{(-1)^{a_1+a_2}(a_1+a_3)!(a_2+a_4)!}{(z_1-\tau_2)^{a_1+a_4+1}(z_2-\tau_1)^{a_2+a_3+1}}\\ &S^{m_1m_2n_1n_2}_{[a,b,c,d]}(z_1,z_2,\tau_1,\tau_2)\\ &= \frac{(-1)^{a+c}\eta^{m_1m_2}\eta^{n_1n_2}}{(z_1-z_2)^{a+b}(\tau_1-\tau_2)^{c+d}}+\frac{(-1)^{a+b}\eta^{m_1n_1}\eta^{m_2n_2}}{(z_1-\tau_1)^{a+c}(z_2-\tau_2)^{b+d}}\\ &\quad+\frac{(-1)^{a+b}\eta^{m_1n_2}\eta^{m_2n_1}}{(z_1-\tau_2)^{a+d}(z_2-\tau_1)^{b+c}}. \quad (35)\end{aligned}$$

Then the straightforward computation gives (with $z_1 = \bar{z}_2 = \frac{i}{2}$)

$$\left\langle F_{m_1}\left(p;\frac{i}{2}\right)F_{m_2}\left(p;-\frac{i}{2}\right)L^{(k_1)}_{n_1}(q_1;\tau_1)L^{(k_2)}_{n_2}(q_2;\tau_2)\right\rangle = \sum_{l=1}^{27}Z^{(l)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) \quad (36)$$

with the explicit expressions for the matter interaction tensors $Z^{(l)}_{mnpq}$ $(l = 1, \ldots, 27)$ given by (A1)–(A27) in the Appendix.

This concludes the computation of the $\langle 2-1-1\rangle$ correlator, contributing to the graviton's β function. Next, consider the contributions from spin 3 excitations, that stem from the $\langle 2-3-3\rangle$ correlator. The spin 3 vertex operators are given by

$$V^{(-3)}_{s=3} = H_{mab}(q)ce^{-3\phi}\psi^m\partial X^a\partial X^b e^{iqX}(z) \tag{37}$$

at negative unintegrated cohomology H_{-3} representation and

$$V^{(+1)}_{s=3} = H_{mab}(q)K\circ\oint dze^{\phi}\psi^m\partial X^a\partial X^b e^{iqX}(z) \tag{38}$$

at positive H_1 cohomology representation. The on-shell conditions on the spin 3 field H_{mab} are given by

$$q^a H_{mab} = 0 \tag{39}$$

$$\eta^{ab}H_{mab} = 0 \tag{40}$$

and

$$\eta^{ma}H_{mab} = 0. \tag{41}$$

As was noted above, instead of considering general configurations of H_{mab}, we are looking for polynomial combinations of u coupling to the vertex operators (37), (38) and satisfying the on-shell conditions (39)–(41) to ensure the BRST properties of the operators. The only suitable combination satisfying (39)–(41) is given by

$$H_{mab}(p) = \int d^4k\int d^4q u_m(k+q)u_a(k-p)u_b(q-p) + \frac{1}{2}\delta_{ab}u_m(p) - \frac{1}{2}(\delta_{ma}u_b(p) + \delta_{mb}u_a(p)). \tag{42}$$

In order to satisfy (39)–(41) u_a must furthermore satisfy $u_a u^a = -1$ with zero vorticity condition $p_{[a}u_{b]}(p) = 0$ and incompressibility $p_a u^a(p) = 0$.

(Note, however, that the zero vorticity and incompressibility conditions must only be imposed in the on-shell limit; in the full β function of the graviton these conditions are not satisfied as the β function is the object which is essentially off shell.)

On the other hand, the $u^2 = -1$ constraint can also be obtained from the vanishing on the β function for the spin 1 operator (29) which, in the leading order, can be computed to give $\beta^a_{u_a} \sim u_b(g^{ab} + u^a u^b)$.

So as the β function is the object that must be calculated off shell, in the calculations below we shall keep the terms that are both nontransverse and have nonzero vorticities, as they only vanish in the on-shell limit.

As in the $\langle 2-1-1\rangle$ computation, it is convenient to take the graviton's operator unintegrated at canonical $(-3,-3)$ picture, locating it at the disc's origin (accordingly, at $z = \frac{i}{2}$ on the half-plane). As for spin 3 operators, located at the boundary of the disc (accordingly, on the real line after the transformation to the half-plane) they both should therefore be taken integrated at picture $+2$. Instead transforming the operator (38) to picture 2 by picture-changing transform, it is more convenient to consider the operator

$$\begin{aligned} V^{2|1} &= 2\omega^{a_1a_2|b}_n V^n_{a_1a_2|b}(p) \\ V^n_{a_1a_2|b}(p) &= K\circ\oint e^{2\phi}(-2\partial\psi^m\psi_b\partial X_{(a_1}\partial^2X_{a_2)} \\ &\quad - 2\partial\psi^m\partial\psi_b\partial X_{a_1}\partial X_{a_2} \\ &\quad + \psi^m\partial^2\psi_b\partial X_{a_1}\partial X_{a_2})e^{ipX} \end{aligned} \tag{43}$$

with $V^{2|1}$ being a vertex operator for the spin 3 *extra* field $\omega^{2|1}$ in Vasiliev's framelike formalism [25]. This extra field is related to the dynamical metriclike field $\omega^{2|0} \equiv H_{nab}$ of (38) up to BRST-exact terms through the cohomology constraint [25] given by

$$\omega^{ab|c}_n(p) = 2p^c H^{ab}_n(p) - p^a H^{bc}_n(p) - p^b H^{ac}_m(p). \tag{44}$$

In addition, it is straightforward to check that the Weyl invariance of $V^{2|1}$ also requires

$$\omega^{ab|c}_c = 0 \tag{45}$$

which can also be seen directly from the primary field constraint on $V^{2|1}$ at the dual -4 picture. In particular, this implies that the graviton's β function can be shifted according to

$$\beta^{mn} \to \beta^{mn} + \mathrm{const}\times\omega^{mn|c}_c \tag{46}$$

since such a shift corresponds to the same on-shell limit and, in this limit, does not violate the conformal invariance on the world sheet. Next, given (42) and (44), the vanishing $\omega^{ab|c}_c$ condition (45) leads to

$$-p_{(a}u_{b)} + \int d^4k\int d^4q u^m(k+q)p_m u_a(k-p)u_b(q-p) = 0 \tag{47}$$

or, in the position space,

$$\partial_{(a}u_{b)} + u_{(a}(\vec{u}\,\vec{\partial})u_{b)} = 0. \tag{48}$$

Note that, with the $u^2 = -1$ constraint the left-hand side of (48) can be cast as the traceless tensor, transverse with respect to u^a:

$$\omega_c^{ab|c} \sim \Pi^{ac}\Pi^{bd}\partial_{(c}u_{d)} - \frac{1}{3}\Pi^{ab}(\partial_c u^c)$$

which is nothing but the first-derivative dissipative term in the hydrodynamical stress tensor.

The straightforward calculation of the $\langle 2-3-3\rangle$ correlator then gives

$$\begin{aligned}
&\left\langle V_{s=2}\left(\frac{i}{2},-\frac{i}{2}\right)V_{s=3}\left(\frac{i}{2}\right)V_{s=3}\left(\frac{i}{2}\right)\right\rangle \\
&\quad = g^{m_1m_2}(p)(2q_1^{c_1}H^{a_1b_1n_1}(q_1) - q_1^{a_1}H^{b_1c_1n_1}(q_1) - q_1^{b_1}H^{a_1c_1n_1}(q_1)) \\
&\quad\times(2q_2^{c_2}H^{a_1b_1n_1}(q_2) - q_2^{a_2}H^{b_2c_2n_2}(q_2) - q_2^{b_2}H^{a_2c_2n_2}(q_2)) \\
&\quad\times\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\infty}d\tau_2\left(\frac{i}{2}-\tau_1\right)^{2-q_1q_2}\left(\frac{i}{2}+\tau_2\right)^{2-q_1q_2} \\
&\quad\times\left(\frac{i}{2}-\tau_2\right)^{6-q_1q_2}\left(\frac{i}{2}+\tau_1\right)^{6-q_1q_2}(\tau_1-\tau_2)^{q_1q_2-4}\times\Big[H^{(4)}_{-5;-5;4}(\tau_1|z_1,z_2,\tau_2)H^{(4)}_{-5;-5;4}(\tau_2|z_1,z_2,\tau_1) \\
&\quad+\frac{12}{(\tau_1-\tau_2)^2}H^{(3)}_{-5;-5;4}(\tau_1|z_1,z_2,\tau_2)H^{(3)}_{-5;-5;4}(\tau_2|z_1,z_2,\tau_1)\Big] \\
&\quad\times\sum_{\alpha_1,\alpha_2=1}^{2}(\delta^2_{\alpha_1}-2\delta_{\alpha_1^1})(\delta^2_{\alpha_2}-2\delta_{\alpha_2^1})(-4\delta_1^{\beta_1}\delta_0^{\gamma_1}\delta_1^{\rho_1}\delta_2^{\lambda_1}-4\delta_1^{\beta_1}\delta_1^{\gamma_1}\delta_1^{\rho_1}\delta_1^{\lambda_1}+2\delta_0^{\beta_1}\delta_2^{\gamma_1}\delta_1^{\rho_1}\delta_1^{\lambda_1}) \\
&\quad\times(-4\delta_1^{\beta_2}\delta_0^{\gamma_2}\delta_1^{\rho_2}\delta_2^{\lambda_2}-4\delta_1^{\beta_2}\delta_1^{\gamma_2}\delta_1^{\rho_2}\delta_1^{\lambda_2}+2\delta_0^{\beta_2}\delta_2^{\gamma_2}\delta_1^{\rho_2}\delta_1^{\lambda_2}) \\
&\quad\times\frac{(-1)^{\alpha_2+\beta_1+\gamma_1+\lambda_1}(4-\alpha_1-\alpha_2)}{(z_1-z_2)^{5-\alpha_1-\alpha_2}}\times\frac{\eta_{n_1n_2}\eta_{c_1c_2}(\beta_1+\beta_2)!(\gamma_1+\gamma_2)!-\eta_{n_1c_2}\eta_{n_2c_1}}{(\tau_1-\tau_2)^{\beta_1+\beta_2+\gamma_1+\gamma_2+2}} \\
&\quad\times\left[\frac{\eta_{m_1a_1}\eta_{m_2a_2}\eta_{b_1b_2}(\alpha_1+\rho_1-1)!(\alpha_2+\rho_2-1)!(\lambda_1+\lambda_2-1)!}{(z_1-\tau_1)^{\alpha_1+\rho_1}(z_2-\tau_2)^{\alpha_2+\rho_2}(\tau_1-\tau_2)^{\lambda_1+\lambda_2}}\right. \\
&\quad\left.+\frac{\eta_{m_1a_2}\eta_{m_2a_1}\eta_{b_1b_2}(\alpha_1+\rho_2-1)!(\alpha_2+\rho_1-1)!(\lambda_1+\lambda_2-1)!}{(z_1-\tau_2)^{\alpha_2+\rho_1}(z_2-\tau_1)^{\alpha_1+\rho_2}(\tau_1-\tau_2)^{\lambda_1+\lambda_2}}\right].
\end{aligned} \tag{49}$$

This concludes the computation of the integrand of the $2-3-3$ correlator contributing to the graviton's β function. The next step is to perform the integrations of the lengthy expressions (A1)–(A27) and (49) in τ_1 and τ_2. All the integrals entering the $\langle 2-1-1\rangle$ and $\langle 2-3-3\rangle$ amplitudes (A1)–(A27), (49) have the form

$$I(\alpha_1,\alpha_2;\beta_1,\beta_2;\gamma;L) = (z_1-z_2)^L\int_{-\infty}^{\infty}d\tau_1\int_{-\infty}^{\tau_1}d\tau_2(\tau_1-z_1)^{\alpha_1}(\tau_1-z_2)^{\alpha_2}(\tau_2-z_1)^{\beta_1}(\tau_2-z_2)^{\beta_2}(\tau_1-\tau_2)^{\gamma} \tag{50}$$

with the powers in the integrands given by

$$\begin{aligned}
\alpha_{1,2} &= -2q_1q_2 + M_{1,2} \\
\beta_{1,2} &= -2q_1q_2 + N_{1,2} \\
\gamma &= k_1k_2 + P
\end{aligned} \tag{51}$$

where L, M_1, M_2, N_1, N_2, P are various combinations of the integer numbers following from the manifest expressions (A1)–(A27), (49), so the overall $\langle 2-1-1\rangle$ and $\langle 2-3-3\rangle$ amplitudes are given by the appropriate summations

$$\langle 2-s-s\rangle|_{s=1,3} \sim \sum_{L,M_1,M_2,N_1,N_2,P} I(a_1,a_2;b_1,b_2;c;d). \tag{52}$$

The contributions of these sums to the AdS graviton's β function in the limit $\alpha' \to 0$ are then determined by the coefficients in front of the simple pole $\sim (q_1 q_2)^{-1}$ produced by the integrations. Although the integral (50) looks like a complicated one of the hypergeometric type, things simplify drastically if we use the overall conformal invariance of the amplitudes (34), (49) allowing us to map the half-plane expressions back to the disc. The integrals over τ_1 and τ_2 are then conformally mapped to the double angular integral over φ_1, φ_2 with the angular variable $0 \leq \varphi \leq 2\pi$ parametrizing the boundary of the disc. With the conformal transformation (31), it is easy to check that the relation between φ and τ is

$$\tau = \frac{1}{2}\tan\left(\frac{\varphi}{2} + \frac{\pi}{4}\right). \tag{53}$$

The resulting angular integrals turn out to be remarkably simpler. Namely, simple computation using the conformal transformations of (31), (53) and changing the angular variable according to $\frac{\varphi}{2} + \frac{\pi}{4} \to \varphi$ gives

$$\begin{aligned} I(\alpha_1,\alpha_2;\beta_1,\beta_2;\gamma;L) = (-1)^{\frac{L}{2}} 2^{-\alpha_1-\alpha_2-\beta_1-\beta_2-\gamma} &\times [F(\alpha_1+\alpha_2|\beta_1+\beta_2|\gamma) + F(\alpha_1+\alpha_2+2|\beta_1+\beta_2|\gamma) \\ &+ F(\alpha_1+\alpha_2|\beta_1+\beta_2+2|\gamma) + F(\alpha_1+\alpha_2|\beta_1+\beta_2|\gamma) \end{aligned} \tag{54}$$

where

$$F(\alpha|\beta|\gamma) = \int_0^\pi d\varphi_1 \int_0^{\varphi_1} d\varphi_2 \tan^\alpha\varphi_1 \tan^\beta\varphi_2 (\tan\varphi_1 - \tan\varphi_2)^\gamma. \tag{55}$$

Integrating one obtains

$$F(\alpha|\beta|\gamma) = i\pi\left[\frac{(\delta^{\alpha+\beta}_{-\gamma-2} + \delta^{\alpha+\beta}_{-\gamma-4})\Gamma(\beta+1)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+2)} + \frac{(\delta^{\alpha+\beta}_{-\gamma-4} + \delta^{\alpha+\beta}_{-\gamma-6})\Gamma(\beta+3)\Gamma(\gamma+1)}{\Gamma(\beta+\gamma+4)}\right]. \tag{56}$$

This is precisely the pole structure we are looking for. Using Mathematica, it is now straightforward to simplify the integrands of (A1)–(A27), (49), to substitute the appropriate values of $I(\alpha_1,\alpha_2;\beta_1,\beta_2;\gamma;L)$ for each of the integrals we are using and to compute the coefficient in front of the pole $(k_1k_2)^{-1}$ in the field theory limit $\alpha' \to 0$. The final result is that the contributions of spin 1 and spin 3 excitations to the β function of the graviton are given by

$$\beta^{mn}_{\langle 2-1-1\rangle} + \beta^{mn}_{\langle 2-3-3\rangle} = \Lambda\frac{dg^{mn}(p)}{d\Lambda} = 32\int d^4q u^m(q-p)u^n(q+p) - \frac{1}{2}\sum_{j=1}^{10} T_j^{mn} \tag{57}$$

where

$$\begin{aligned} T_1^{mn} = 3\{&\delta_a^{b_1}\delta_c^{b_2}\delta_m^{b_3} - \delta_c^{b_1}\delta_a^{b_2}\delta_m^{b_3} - \delta_a^{b_1}\delta_m^{b_2}\delta_c^{b_3} + \delta_c^{b_1}\delta_m^{b_2}\delta_a^{b_3} - \delta_m^{b_1}\delta_c^{b_2}\delta_a^{b_3} + \delta_m^{b_1}\delta_a^{b_2}\delta_c^{b_3}\} \\ &\times\left\{\int d^4k\int d^4q_1\int d^4q_2 u^a(k-p)(q_1+q_2)^{(c}u^{n)}(q_1+q_2)u_{b_1}(q_1-k-p)(q_2-k-p)_{b_2}u_{b_3}(q_2-k-p)\right. \\ &+\int d^4k_1\int d^4k_2\int d^4k_3\int d^4q_1\int d^4q_2(q_1+q_2)_d u^a(k_1+k_2)u^d(k_2-p) \\ &\left.\times u^{(c}(k_3-k_2+p)u^{n)}(q_1+q_2)u_{b_1}(q_1-k_3-k_2+p)(q_2-k_3-k_2+p)_{b_2}u_{b_3}(q_2-k_3-k_2+p)\right\} \end{aligned} \tag{58}$$

$$\begin{aligned}T_2^{mn} &= 64\eta_{st}\int d^4k\left\{\omega_c^{ms|c}(k-p)\omega_d^{nt|d}(k+p)-\frac{1}{3}\eta^{mn}\omega_c^{ps|c}(k-p)\omega_d^{ps|d}(k+p)\right\}\\ &+\left(\frac{20}{3}-12Q^2-\frac{8}{Q^2}\right)\eta^{mn}\int d^4k[(k-p)_a(k+p)^a u_b(k-p)u^b(k+p)\\ &+(k-p)_a(k+p)_b u^b(q-p)u^a(k-p)]\\ &-\frac{64}{3}\int d^4k\int d^4q_1\int d^4q_2 u^m(k-p)u^n(q_1+q_2)\\ &\times[(q_2-k-p)_a(q_1-k-p)_b u^a(q_1-k-p)u^b(q_2-k-p)\\ &+(q_2-k-p)_a(q_1-k-p)^a u_b(q_1-k-p)u^b(q_2-k-p)]\end{aligned} \tag{59}$$

$$T_3^{mn} = 32\int d^4k(k-p)_a u^a(k-p)\omega_c^{mn|c}(k+p) \tag{60}$$

$$\begin{aligned}T_4^{mn} &= 96\int d^4k\int d^4q_1\int d^4q_2(k-p)_a(q_1+q_2)_b u^m(k-p)u^n(q_1+q_2)u^a(q_1-k-p)u^b(q_2-k-p)\\ &-32\eta^{mn}\int d^4k\int d^4q_1\int d^4q_2(k-p)_a(q_1+q_2)_b u_p(k-p)u^p(q_1+q_2)u^a(q_1-k-p)u^b(q_2-k-p)\\ &-16\eta^{mn}\int d^4k_1\int d^4k_2\int d^4k_3\int d^4q_1\int d^4q_2 u^m(k_1+k_2)u^n(k_2-p)\\ &\times(k_3-k_2+p)_a(q_1+q_2)_b u_p(k_3-k_2+p)u^p(q_1+q_2)\\ &\times u^a(q_1-k_3-k_2+p)u^b(q_2-k_3-k_2+p)\end{aligned} \tag{61}$$

$$\begin{aligned}T_5^{mn} &= 4\left(3Q^2-1-\frac{2}{Q^2}\right)\int d^4k u^p(p-k)(p+k)_p\left(\frac{3}{2}(p+k)_m u_n(p+k)\right.\\ &\left.+\frac{3}{2}(p+k)^n u^m(p+k)-\eta^{mn}(p+k)^a u_a(p+k)\right)\end{aligned} \tag{62}$$

$$\begin{aligned}T_6^{mn} &= 12\int d^4k\int d^4q_1\int d^4q_2\{(q_2-k+p)_c u^n(k+p)u_b(q_1+q_2)u^c(q_1-k+p)\\ &\times(u^b(q_2-k+p)(q_2-k+p)^m+u^m(q_2-k+p)(q_2-k+p)^b)\}+\mathrm{perm}\{m\leftrightarrow n\}\end{aligned} \tag{63}$$

$$\begin{aligned}T_7^{mn} &= -16\int d^4k\int d^4q_1\int d^4q_2(u^m(k+p)u^n(q_1+q_2)+u^n(k+p)u^m(q_1+q_2))\\ &\times u^c(q_1-k+p)u^a(q_2-k+p)(q_2-k+p)_a(q_2-k+p)_c\end{aligned} \tag{64}$$

$$\begin{aligned}T_8^{mn} &= -16\eta^{mn}\int d^4k\int d^4q_1\int d^4q_2 u^a(k+p)u^b(q_1+q_2)\\ &\times u^c(q_1-k+p)u_b(q_2-k+p)(q_2-k+p)_a(q_2-k+p)_c\end{aligned} \tag{65}$$

$$\begin{aligned}T_9^{mn} &= 48\int d^4k_1\int d^4k_2\int d^4k_3\int d^4q_1\int d^4q_2\\ &\times\{u^{(m}(q_1+q_2+k_1)u^{n)}(q_1+q_2-k_1)u^a(p-q_1-k_2)u^b(p-q_1+k_2)\\ &\times u^p(q_2-p-k_3)u_b(q_2-p+k_3)(q_2-p+k_3)_a(q_2-p+k_3)_p\}\end{aligned} \tag{66}$$

$$\begin{aligned}T_{10}^{mn} &= 16\int d^4k_1\int d^4k_2\int d^4k_3\int d^4q_1\int d^4q_2\\ &\times\{u^{(m}(q_1+q_2+k_1)u^{n)}(p-q_1-k_2)u^a(q_1+q_2-k_1)u^b(p-q_1+q_2)\\ &\times u^p(q_2-p-k_3)u_b(q_2-p+k_3)(q_2-p+k_3)_a(q_2-p+k_3)_p\}.\end{aligned} \tag{67}$$

As it is clear from (57)–(67), the overall result for the β function of the graviton, polarized along the $d = 4$ boundary, generally depends on the value Q of the Liouville background charge (which in turn can be expressed in terms of the central charge $c_{\rm Liouv} = 1 + 3Q^2$). Therefore in general, the trace of the β function is nonzero. Transforming (58)–(67) to the position space and using the $u^2 = -1$ condition it is straightforward to check that the overall trace of (57)–(67) vanishes for $Q = \sqrt{2}$ which precisely is the case for $d + 1 = 5$, where the trace of spin 1 contributions is canceled by that of spin 3. In this case the answer has a natural interpretation in terms of holographic fluid. This concludes the computation of the graviton's β function in the AdS string sigma model, up to terms quadratic in momentum. Combining (57)–(67) and (27) one finds that vanishing of the beta function (57) leads to low-energy equations of motion in space-time, equivalent to equations of gravity with the matter, described by the stress-energy tensor of four-dimensional conformal fluid. To see the relevance of this matter stress tensor to holographic hydrodynamics, one has to shift β^{mn} according to

$$\beta^{mn} \to \tilde{\beta}^{mn} = \beta^{mn} + 16i\omega_c^{mn|c} \tag{68}$$

with $\omega_p^{mn|p}$ given by (42), (44). As was explained above, such a shift does not change the on-shell limit of the theory due to the Weyl invariance constraint (45) on the spin 3 vertex operator. The resulting stress-energy tensor for the matter then simply describes the conformally invariant second order hydrodynamics at the temperature $T = \pi^{-1}$ with five extra transport coefficients in the second order. The relative values of the transport coefficients are becoming remarkably close to those obtained in the AdS_5 gravity computations [5], with less then 10% discrepancy, albeit at a specific temperature in string theory calculations performed in this work. Note that in conformal second order hydrodynamics both the stress-energy tensor and the temperature transform covariantly under the 4d Weyl rescalings: $g_{mn} \to e^{\rho} g_{mn}$ according to $T^{mn} \to e^{-3\rho} T^{mn}$ and $T \to e^{-\frac{\rho}{2}} T$ so the temperature can always be fixed by appropriate Weyl transformation. In other words, the holographic second order hydrodynamics appears in a particular gauge which, in a sense, is not surprising, as it is generally the case in string theory calculations. In the next concluding section we shall discuss the implications of the main result (57) and particularly outline the calculations that still need to be done.

V. CONCLUSIONS AND DISCUSSION

In case of $d = 4$, $Q_{d+1=5} = \sqrt{2}$ the two-derivative piece of the matter stress tensor in the graviton's β function becomes traceless and can be interpreted in terms of two-derivative corrections to conformal hydrodynamics in $d = 4$. Transforming to the position space, it is straightforward to relate the contributions to the graviton's β function to corresponding terms in the gradient expansion in conformal hydrodynamics. The contributions related to the Weyl invariance constraints on the graviton's operator combined with contributions from the $\langle 2 - 1 - 1 \rangle$ correlator of order zero in momentum result in the ideal conformal fluid terms in the β function, proportional to $g^{mn} + 4u^m u^n$. Shifting the β function by the trace of $\omega^{2|1}$ spin 3 extra field, $\sim\omega_p^{mn|p}$, which vanishing on shell follows from the Weyl invariance constraints on the spin 3 operator (45), leads to the leading order dissipative term, containing one derivative due to the ghost cohomology/zero torsion constraint (A2) relating extra fields to the dynamical field in Vasiliev's formalism. Finally, the contributions (58)–(67) given by $T_i^{mn}(i = 1, \ldots, 10)$ are quadratic in momentum and stem from the $\langle 2 - 3 - 3 \rangle$ correlator combined with the appropriate terms from the $\langle 2 - 1 - 1 \rangle$ correlator. These terms describe the two-derivative dissipative corrections in the second order hydrodynamics [5,6,15]. The spin 3 contribution is crucial to ensure the vanishing trace of the matter tensor. Note that, at least in the approximation considered in this paper (up to second order) there are no contributions from the mixed $\langle 2 - 1 - 3 \rangle$ correlator, as all the relevant terms in this correlator are cubic in λ and vanishing for this reason. Transforming to the position space, it is straightforward to identify T_i^{mn} with the corresponding two-derivative structures in the second order hydrodynamics, related to five new transport coefficients for the conformal fluid, appearing in the second order. Namely,

$$\begin{aligned}
&T_1^{mn} \sim \epsilon^{bb_1b_2b_3}\epsilon^{abc(m}\rho_c^{n)} u_a u_{b_1}\partial_{b_2}u_{b_3}\\
&T_2^{mn} \sim 3\rho^{ma}\rho_a^n - \eta^{mn}\rho^{ab}\rho_{ab}\\
&T_3^{mn} \sim \rho^{mn}\partial_a u^a\\
&T_4^{mn} \sim 3(\vec{u}\vec{\partial})u^m(\vec{u}\vec{\partial})u^n - (\eta^{mn} + u^m u^n)(\vec{u}\vec{\partial})u_a(\vec{u}\vec{\partial})u^a\\
&\sum_{i=5}^{10} T_i^{mn} \sim (3\Pi^{ma}\Pi^{nb} - \Pi^{mn}\Pi^{ab})(\vec{u}\vec{\partial})(\partial_a u_b + \partial_b u_a)
\end{aligned} \tag{69}$$

with $\rho^{ab} \sim \omega_c^{ab|c}$.

These structures are all well known to appear in the second order of the gradient expansion of the conformal fluid. They correspond to T_{2a}, T_{2b}, T_{2c}, T_{2d} and T_{2e} terms, considered in [5].

The correlators considered in this paper, as well as those related to graviton interactions with operators of higher spin values, will also contribute the higher derivative contributions (with three and more derivatives) that were not addressed in this work. At this stage, many more higher spin correlators should enter the game, possibly including those with mixed symmetries and those coming from the closed string sector. As in the two-derivative case, however, the conformal symmetry significantly reduces the number of terms and new transport coefficients at higher orders. It is not clear at present if higher order corrections to the gradient expansion in conformal hydrodynamics can be described in terms of contributions from two-row Vasiliev's framelike fields or if more mixed symmetry degrees of freedom are needed. The latter almost certainly produce the structures that are present in the third and higher order hydrodynamics but violate the $4d$ conformal symmetry; however the question is whether the contributions from the two-row fields are sufficient to describe the conformal limit. To answer these questions we need to have better understanding of the general expansion structure of higher order hydrodynamics. Our main conjecture, based on the leading order results of this paper, suggests that, in general, the gradient expansion in conformal hydrodynamics in $d = 4$ is controlled by the higher spin correlators in string theory and, in the leading α' order, the derivative structure of the gradient expansion must be holographically related to that of higher spin vertices and to the structure constants of higher spin algebra in AdS_5, with the orders of the expansion roughly corresponding to the total spin value carried by the higher spin vertices. It would be particularly interesting to explore the relation of the gradient expansion at higher orders to well-known structures of the cubic and quartic vertices for higher spins [24,28–37] which presumably should exist in the limit of $\alpha' \to 0$. If the higher spin interpretation of the gradient expansion in hydrodynamics, investigated in this paper in the string theory context, is still correct at higher orders, the higher spin algebra in $d = 5$ would provide a powerful tool allowing us to control the transport coefficients in higher order hydrodynamics. Another important problem to investigate is the role of α' corrections in this expansion and their holographic interpretation. This may lead to new nontrivial and intriguing symmetries relating the expansion structures and transport coefficients at different orders and understanding these symmetries in terms of higher spin quantization. The work on these and other issues is currently in progress and we hope to be able to present our results soon in future works.

ACKNOWLEDGMENTS

It is a pleasure to thank Bum-Hoon Lee, Shiraz Minwalla, Hermann Nicolai and Misha Vasiliev for useful discussions and comments. One of us (D. P.) would like to express his gratitude to Hermann Nicolai for kind hospitality and to the Albert Einstein Institute (AEI) in Potsdam where part of this work has been completed. D. P. also would like to thank the organizers of the 2nd Solvay Workshop on Higher Spin Gauge Theories at Solvay Institute at University of Brussels and the participants of this workshop for stimulating discussions. This work was supported by a National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University with Grant No. 2005-0049409. S. L. and D. P. also acknowledge the support of the NRF Grant No. 2012-004581.

APPENDIX CALCULATION OF THE MATTER INTERACTION TENSORS

In this appendix we present explicit expressions for the matter interaction tensors $Z^{(i)}_{mnpq}(i = 1, \ldots, 27)$ entering the overall expression (36) of the matter part of the $\langle 1 - 1 - 2\rangle$ correlator contributing to the graviton's β function. The straightforward computation of these tensors gives

$$\begin{aligned}
Z^{(1)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = {}& \sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})(\delta^0_{b_1}-\delta^1_{b_1})(\delta^0_{b_2}-\delta^1_{b_2})\\
&\times\left\{\left[\frac{(-1)^{a_1+a_2+b_2+k_1}\eta_{n_1n_2}}{(k_1-b_1-1)!(k_2-b_2-1)!(\tau_1-\tau_2)^{p_1+p_2-b_1-b_2+1}}\left(\frac{(a_1+a_2-1)!\eta_{m_1m_2}}{(z_1-z_2)^{a_1+a_2}}\right.\right.\right.\\
&\left.+R^{(a_1)}_{m_1}(z_1|(z_2,p);(\tau_1,q_1),(\tau_2,q_2))R^{(a_2)}_{m_2}(z_2|(z_1,p);(\tau_1,q_1),(\tau_2,q_2))\right)\\
&\times\left(\frac{(4-a_1-a_2)!(b_1+b_2)!}{(z_1-z_2)^{5-a_1-a_2}(\tau_1-\tau_2)^{b_1+b_2+1}}-\frac{(2-a_1+b_1)!(2-a_2+b_2)!}{(z_1-\tau_1)^{3-a_1-b_2}(z_2-\tau_2)^{-a_2+b_1+3}}\right.\\
&\left.\left.+\frac{(2-a_1+b_2)!(2-a_2+b_1)!}{(z_1-\tau_1)^{3-a_1-b_1}(z_2-\tau_2)^{-a_2+b_2+3}}\right)\right]-(1-\delta^1_{k_1})(1-\delta^2_{k_2})\frac{iq_2^{n_1}}{(k_2-3)!}\\
&\times\frac{(-1)^{k_1-b_1}(k_1+k_2-b_1-3)!K^{(2-a_1,2-a_2,b_1,2-b_2)}_{\lambda}(z_1,z_2,\tau_1,\tau_2)}{(\tau_1-\tau_2)^{k_1+k_2-b_1-2}}\\
&\times\left(\frac{(-1)^{a_1}(a_1+a_2-1)!\eta_{m_1m_2}}{(z_1-z_2)^{a_1+a_2}}R^{(b_2)}_{n_2}(\tau_2|(z_1,p);(z_2,p);(\tau_1,q_1))\right.\\
&+\frac{(-1)^{a_1}(a_1+b_2-1)!\eta_{m_1n_2}}{(z_1-\tau_2)^{a_1+b_2}}R^{(a_2)}_{m_2}(\tau_1|(z_1,p);(z_2,p);(\tau_2,q_2))\\
&\left.\left.+\frac{(-1)^{a_2}(a_2+b_2-1)!\eta_{m_2n_2}}{(z_2-\tau_2)^{a_2+b_2}}R^{(a_1)}_{m_2}(z_1|(z_2,p);(\tau_1,q_1);(\tau_2,q_2))\right)\right\}.
\end{aligned}\tag{A1}$$

Next,

$$Z^{(2)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = (1-\delta^0_{k_1})\sum_{a_1,a_2=1}^{2}\sum_{b_1=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})(\delta^0_{b_1}-\delta^1_{b_1}(1-\delta^1_{p_1})) \times \frac{(-1)^{a_1+b_1+k_1+k_2}(3+9Q^2)\eta_{m_1m_2}q_{2n_1}q_{2n_2}K_\lambda^{2-a_1;2-a_2;b_1;0}(z_1,z_2,\tau_1,\tau_2)}{2(k_1-b_1-1)!(\tau_1-\tau_2)^{k_1+k_2-1-b_1}(z_1-z_2)^{a_1+a_2}}. \quad \text{(A2)}$$

Next,

$$Z^{(3)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = (1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})\sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2}) \times (\delta^0_{b_2}-\delta^1_{b_2}(1-\delta^2_{k_2}))(\delta^0_{b_1}-\delta^1_{b_1}(1-\delta^1_{k_1})) \times \frac{3iq_{2n_2}(1+3Q^2)(-1)^{k_1-b_1}K_\lambda^{2-a_1;2-a_2;b_1;0}(z_1,z_2,\tau_1,\tau_2)}{(k_1-b_1-1)!(k_2-b_2-2)!(\tau_1-\tau_2)^{k_1-b_1+b_2}} \times \left[\frac{(-1)^{a_1}\eta_{m_1m_2}R^{(k_2-b_2-1)}_{n_1}(\tau_2|(p,z_1),(p,z_2),(q_1,\tau_1))}{(z_1-z_2)^{a_1+a_2}} + \frac{(-1)^{a_1}\eta_{m_1n_1}R^{(a_2)}_{m_2}(z_2|(p,z_1),(q_1,\tau_1),(q_2,\tau_2))}{(z_1-z_2)^{a_1+k_2-b_2-1}} + \frac{(-1)^{a_2}\eta_{m_2n_1}R^{(a_1)}_{m_1}(z_1|(p,z_2),(q_1,\tau_1),(q_2,\tau_2))}{(z_1-z_2)^{a_2+k_2-b_2-1}}\right]. \quad \text{(A3)}$$

Next,

$$Z^{(4)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = (1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1})(1-\delta^1_{k_2})(1-\delta^2_{k_1})(1-\delta^2_{k_2}) \times \sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})(\delta^2_{b_1}-2\delta^1_{b_1})(\delta^2_{b_2}-2\delta^1_{b_2}) \times \frac{(-1)^{k_1}(q_1q_2)K_\lambda^{2-a_1;2-a_2;2-b_1;2-b_2}(z_1,z_2,\tau_1,\tau_2)S^{[a_1;a_2;b_1;b_2]}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2)}{(k_1-3)!(k_2-3)!(\tau_1-\tau_2)^{k_1+k_2-5}}. \quad \text{(A4)}$$

Next,

$$Z^{(5)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = 3(1+3Q^2)q_1^nq_{2n_2}(1-\delta^0_{k_2})\sum_{a_1,a_2=1}^{2}\sum_{b_1=1}^{2}\sum_{b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})(\delta^2_{b_1}-2\delta^1_{b_1}) \times (\delta^0_{b_2}-\delta^1_{b_2}(1-\delta^2_{k_2}))\frac{(-1)^{k_1}K_\lambda^{2-a_1;2-a_2;2-b_1;0}(z_1,z_2,\tau_1,\tau_2)S^{[a_1;a_2;b_1;k_2-b_2-1]}_{m_1m_2n_1n}(z_1,z_2,\tau_1,\tau_2)}{(k_1-3)!(k_2-2-b_2)(\tau_1-\tau_2)^{k_1+b_2-2}}. \quad \text{(A5)}$$

Next,

$$Z^{(6)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -9(1+3Q^2)q_{1n_1}q_{2n_2}(1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1})(1-\delta^1_{k_2}) \times \sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})(\delta^0_{b_1}-\delta^1_{b_1}(1-\delta^2_{k_1}))(\delta^0_{b_2}-\delta^1_{b_2}(1-\delta^2_{k_2})) \times \frac{(-1)^{b_1}(b_1+b_2)!K_\lambda^{2-a_1;2-a_2;1;1}(z_1,z_2,\tau_1,\tau_2)\eta^{mn}S^{[a_1;a_2;k_1-b_1-1;k_2-b_2-1]}_{m_1m_2mn}(z_1,z_2,\tau_1,\tau_2)}{(k_1-b_1-2)!(k_2-b_2-2)!(\tau_1-\tau_2)^{b_1+b_2+1}} \quad \text{(A6)}$$

$$Z^{(7)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -\frac{1}{4}q_{1n_1}q_{2n_2}\sum_{a_1,a_2=1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2}) \times \frac{(-1)^{k_1}(a_1+a_2-1)!(4-a_1-a_2)!(k_1+k_2-1)!}{(k_1-1)!(k_2-1)!(z_1-z_2)^5(\tau_1-\tau_2)^{p_1+p_2}} \tag{A7}$$

$$Z^{(8)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \frac{1}{2}q_{1n_1}q_{2n_2}\frac{(-1)^{k_1}k_1(1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})(1-\delta^2_{k_2})}{(k_2-2)!(\tau_1-\tau_2)^{k_1+1}} \times \sum_{a_1,a_2=1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})\frac{(-1)^{a_2}(a_1+a_2-1)!}{(z_1-z_2)^{a_1+a_2}} \times\left[\frac{(k_2-a_2)!(2-a_1)!}{(z_1-\tau_2)^{k_2-a_2+1}(z_2-\tau_2)^{3-a_1}}-\frac{(k_2-a_1)!(2-a_2)!}{(z_1-\tau_2)^{k_2-a_1+1}(z_2-\tau_2)^{3-a_2}}\right] \tag{A8}$$

$$\begin{aligned} &Z^{(9)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) \\ &\quad = (1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2}) \\ &\qquad \times\sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})\left(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}\right)\left(-\delta^0_{b_2}+\frac{1}{2}\delta^1_{b_2}\right) \\ &\qquad \times\frac{(-1)^{k_1+a_1+b_1}(k_1+k_2-b_1-b_2-3)!(4-a_1-a_2)!}{(z_1-z_2)^{5-a_1-a_2}(\tau_1-\tau_2)^{k_1+k_2-b_1-b_2-2}} \\ &\qquad \times\left\{S^{a_1;a_2 1+b_1;1+b_2}_{m_1m_2n_1n_2}+\frac{(-1)^{a_1}(a_1+b_1)!\eta_{m_1n_1}R^{(a_2)}_{m_2}(z_2|(p,z_1);(q_1,\tau_1);(q_2,\tau_2))R^{(1+b_2)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_1-\tau_1)^{a_1+b_1+1}}\right. \\ &\qquad +\frac{(-1)^{a_2}(a_2+b_1)!\eta_{m_2n_1}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2,\tau_2))R^{(1+b_2)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_2-\tau_1)^{a_2+b_1+1}} \\ &\qquad \left.+\frac{(-1)^{a_1}(a_1+b_2)!\eta_{m_1n_2}R^{(a_2)}_{m_2}(z_2|(p,z_1);(q_1,\tau_1);(q_2,\tau_2))R^{(1+b_1)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{(z_1-\tau_2)^{a_1+b_2+1}}\right\} \end{aligned} \tag{A9}$$

$$\begin{aligned} Z^{(10)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) &= (1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})\frac{iq_{2n_2}}{2Q(k_1-b_1-2)!(k_2-2)!} \\ &\quad\times\sum_{a_1,a_2=1}^{2}\sum_{b_1=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})\left(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}\right) \\ &\quad\times\left[\frac{(k_1-b_1-1)!}{(\tau_1-\tau_2)^{k_1-b_1}}\times\frac{(k_2-a_1)!(2-a_2)!}{(z_1-\tau_2)^{k_2-a_1+1}(z_2-\tau_2)^{3-a_2}}-\frac{(k_2-a_2)!(2-a_1)!}{(z_2-\tau_2)^{k_2-a_2+1}(z_1-\tau_2)^{3-a_1}}\right] \\ &\quad\times\left[\frac{(-1)^{a_2+1}(a_1+b_1)!\eta_{m_1n_1}R^{(a_2)}_{m_2}(z_2|(p,z_1);(q_1,\tau_1);(q_2,\tau_2))}{(z_1-\tau_1)^{a_1+b_1+1}}\right. \\ &\quad\left.+\frac{(-1)^{a_1+1}(a_1+b_1)!\eta_{m_2n_1}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2,\tau_2))}{(z_2-\tau_1)^{a_2+b_1+1}}\right] \end{aligned} \tag{A10}$$

$$Z^{(11)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -(1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})\frac{iq_{2n_2}(2+3Q^2)}{(k_1-b_1-2)!(k_2-2)!}$$
$$\times\sum_{a_1,a_2=1}^{2}\sum_{b_1=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})\left(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}\right)$$
$$\times\left[\frac{(-1)^{k_1-b_1}(k_1-b_1-1)!}{(\tau_1-\tau_2)^{k_1-b_1}}\times\frac{(-1)^{a_1}(4-a_1-a_2)!}{(z_1-z_2)^{5-a_1-a_2}}\right]$$
$$\times\left[\frac{(-1)^{a_1}(a_1+b_1)!\eta_{m_1n_1}R^{(a_2)}_{m_2}(z_2|(p,z_1);(q_1,\tau_1);(q_2,\tau_2))}{(z_1-\tau_1)^{a_1+b_1+1}}\right.$$
$$\left.+\frac{(-1)^{a_2}(a_1+b_1)!\eta_{m_2n_1}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2,\tau_2))}{(z_2-\tau_1)^{a_2+b_1+1}}\right] \tag{A11}$$

$$Z^{(12)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \frac{ip_{m_2}}{2}\eta_{n_1n_2}(1-\delta^0_{k_1})(1-\delta^0_{k_2})$$
$$\times\sum_{a_1}^{2}\sum_{b_1,b_2=0}^{1}\frac{(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^0_{b_1}-\delta^1_{b_1}(1-\delta^1_{p_1}))(\delta^0_{b_2}-\delta^1_{b_2}(1-\delta^1_{p_2}))}{(k_1-b_1-1)!(k_2-b_2-1)!}$$
$$\times(-1)^{k_1-b_1}(k_1+k_2-b_1-b_2-2)!\frac{K^{2-a_1;2;b_1;b_2}_{\lambda}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2,\tau_2))}{(\tau_1-\tau_2)^{k_1+k_2-b_1-b_2-1}} \tag{A12}$$

$$Z^{(13)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \frac{1}{2}q_{2n_1}p_{m_2}\eta_{m_1n_2}(1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})(1-\delta^2_{k_2})$$
$$\times\sum_{a_1}^{2}\sum_{b_1,b_2=0}^{1}\frac{(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^2_{a_2}-2\delta^1_{a_2})(\delta^0_{b_1}-\delta^1_{b_1}(1-\delta^1_{p_1}))(\delta^0_{b_2}-\delta^1_{b_2}(1-\delta^1_{p_2}))}{(k_1-b_1-1)!(k_2-3)!}$$
$$\times(-1)^{a_1+b_1+k_1}(a_1+a_2-1)!\frac{K^{2-a_1;2;b_1;2-a_2}_{\lambda}(z_1,z_2,\tau_1,\tau_2)(k_1+k_2-b_1-3)!}{(z_1-\tau_2)^{a_1+a_2}(\tau_1-\tau_2)^{k_1+k_2-b_1-2}} \tag{A13}$$

$$Z^{(14)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -\frac{3}{2}(1+3Q^2)q_{2n_2}p_{m_2}\eta_{m_1n_1}(1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})$$
$$\times\sum_{a_1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})(\delta^0_{b_1}-\delta^1_{b_1}(1-\delta^1_{p_1}))(\delta^0_{b_2}-\delta^1_{b_2}(1-\delta^1_{p_2}))$$
$$\times(-1)^{a_1+b_1+k_1}(a_1+k_2-b_2-2)!\frac{K^{2-a_1;2;b_1;0}_{\lambda}(z_1,z_2,\tau_1,\tau_2)(k_1+b_2-b_1)!}{(z_1-\tau_2)^{a_1+k_2-b_2-1}(\tau_1-\tau_2)^{k_1+b_2-b_1+1}} \tag{A14}$$

$$Z^{(15)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = p_{m_2}q_{1n_1}\eta_{m_1n_2}\frac{(1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_2})}{(k_1-1)!}$$
$$\times\sum_{a_1}^{2}\sum_{b_1=0}^{1}\frac{(\delta^2_{a_1}-2\delta^1_{a_1})(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}(1-\delta^1_{k_1}))}{(k_2-b_2-2)!}\frac{(a_1+b_2)!}{(z_1-\tau_2)^{a_1+b_2+1}}$$
$$\times\left[\frac{1}{2}\frac{(2-a_1)!k_1!(k_2-1-b_2)!}{(z_1-z_2)^{3-a_1}(z_2-\tau_1)^{k_1+1}(z_2-\tau_2)^{k_2-b_2}}\right.$$
$$\left.+\frac{1}{4}(-1)^{k_1+1}\frac{(4-a_1)!(k_1+k_2-2-b_2)!}{(z_1-z_2)^{5-a_1}(\tau_1-\tau_2)^{k_1+k_2-b_2-1}}\right] \tag{A15}$$

$$Z^{(16)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \frac{p_{m_1}q_{1n_1}\eta_{m_2n_1}}{(k_1-1)!}\sum_{a_1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})(-1)^{a_1+1}\frac{(3-a_1)!k_1!}{(z_1-z_2)^{4-a_1}(z_2-\tau_1)^{k_1+1}} \times\left[\frac{1}{2}\delta^2_{k_2}\frac{(a_1+1)!}{(z_1-\tau_2)^{a_1+2}}-2\delta^1_{k_2}\frac{a_1!}{(z_1-\tau_2)^{a_1+1}}\right] \tag{A16}$$

$$Z^{(17)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = (1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1})(1-\delta^1_{k_2})(-ip_{m_2}) \times\sum_{a_1=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})\left(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}(1-\delta^2_{k_1})\right)\left(-\delta^0_{b_2}+\frac{1}{2}\delta^1_{b_2}(1-\delta^2_{k_2})\right) \times\frac{(k_1-1-b_1)(k_1-1-b_2)}{(z_2-\tau_1)^{k_1-b_1}(z_1-\tau_2)^{k_2-b_2}}\frac{\eta_{m_1n_1}(a_1+b_1)!R^{(1+b_2)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_1-\tau_1)^{a_1+b_1+1}} +\frac{\eta_{m_1n_2}(a_1+b_2)!R^{(1+b_1)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{(z_2-\tau_2)^{a_1+b_2+1}} \tag{A17}$$

$$Z^{(18)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -p_{m_2}q_{2n_2}\eta_{m_1n_1}(1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1})(1-\delta^1_{k_2}) \times\sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})\left(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}(1-\delta^2_{k_1})\right) \times\frac{(-1)^{a_1}(a_1+b_1)!}{(k_1-2-b_1)!(z_1-\tau_1)^{a_1+b_1+1}}\frac{1}{Q^2}\left(-\delta^1_{a_2}+\frac{1+3Q^2}{2}\delta^2_{a_2}\right)\left(\frac{1}{2}\delta^0_{b_2}+\frac{1}{8}\delta^1_{b_2}(1-\delta^2_{k_2})\right) \times\left\{\frac{(-1)^{a_1+a_2+k_1+b_1}(4-a_1-a_2)!}{(k_1-b_1-2)!(z_1-z_2)^{5-a_1-a_2}}\left[\frac{(k_2+a_2-b_2-2)!(k_1-b_1+b_2-1)!}{(z_2-\tau_2)^{k_2+a_2-b_2-1}(\tau_1-\tau_2)^{k_1-b_1+b_2}} +\frac{(a_2+b_2)!(k_1+k_2-b_1-b_2-3)!}{(z_2-\tau_2)^{a_2+b_2+1}(\tau_1-\tau_2)^{k_1+k_2-b_1-b_2-2}}\right] +\frac{(-1)^{a_1+k_1+b_1}(1-\delta^2_{k_2})}{8(k_2-3)!(z_1-z_2)^{3-a_1}}\left[\frac{2(k_1+k_2-b_1-4)!}{(z_2-\tau_2)^4(\tau_1-\tau_2)^{k_1+k_2-b_1-3}}+\frac{2(k_2-2)!(k_1-b_1-1)!}{(z_2-\tau_2)^{k_2+1}(\tau_1-\tau_2)^{k_1-b_1}}\right] +\frac{3(1+3Q^2)(-1)^{a_1+b_1+k_1}(1-\delta^2_{k_2})(k_1+k_2-b_1-4)!}{(k_2-3)!(z_1-z_2)^{3-a_1}(z_2-\tau_2)^4(\tau_1-\tau_2)^{k_1+k_2-b_1-3}}\right\} \tag{A18}$$

$$Z^{(19)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -p_{m_2}q_{2n_2}\eta_{m_1n_1}(1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1}) \times\sum_{a_1,a_2=1}^{2}\sum_{b_1,b_2=0}^{1}\frac{(\delta^2_{a_1}-2\delta^1_{a_1})(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}(1-\delta^2_{k_1}))}{(k_1-2-b_1)!}\left(-\frac{1+3Q^2}{4Q}\delta^0_{b_2}+\frac{1}{2Q}\delta^1_{b_2}(1-\delta^2_{k_2})\right) \times\left(\frac{1}{Q}\delta^1_{a_2}-\frac{1+3Q^2}{2Q}\delta^2_{a_2}\right)\left\{\frac{(a_1+b_1)!(a_2+k_1-b_1-2)!}{(z_1-\tau_1)^{a_1+b_1+1}(z_2-\tau_1)^{a_2+k_1-b_1-1}} \times\left[\frac{(1-a_1+k_2-b_2)!(2-a_2+b_2)!}{(z_1-\tau_2)^{2-a_1+k_2-b_2}(z_2-\tau_2)^{3+b_2-a_2}}-\frac{(1-a_2+k_2-b_2)!(2-a_1+b_2)!}{(z_1-\tau_2)^{2-a_2+k_2-b_2}(z_2-\tau_2)^{3+b_2-a_1}}\right] +\frac{(-1)^{a_1}(1-\delta^1_{k_2})(k_1-b_1-1)!}{(z_2-\tau_1)^{k_1-b_1}(z_2-\tau_2)^2}\left[\frac{(k_2-a_1)!}{(z_1-\tau_2)^{k_2-a_1+1}(z_2-\tau_2)}-\frac{(k_2-2)!}{(z_1-\tau_2)^{3-a_1}(z_2-\tau_2)^{k_2-1}}\right]\right\} \tag{A19}$$

$$
\begin{aligned}
Z^{(20)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) &= (1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1})\delta^1_{k_2}\\
&\times\sum_{a_1=1}^{2}\sum_{b_1=0}^{1}\frac{(\delta^2_{a_1}-2\delta^1_{a_1})(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}(1-\delta^2_{k_1}))}{(k_1-2-b_1)!}\bigg\{(2+3Q^2)p_{m_2}q_{2n_2}\eta_{m_1n_1}\\
&\times\frac{(-1)^{a_1}(a_1+b_1)!}{2(z_1-\tau_1)^{a_1+b_1+1}}\bigg[\frac{(-1)^{a_1+b_1+k_1}(4-a_1)!(k_1-b_1-1)!}{(z_1-z_2)^{5-a_1}(\tau_1-\tau_2)^{k_1-b_1}}\\
&-2\frac{(-1)^{a_1+1}(k_1-b_1-1)!}{(z_1-z_2)^{3-a_1}(z_2-\tau_1)^{k_1-b_1}(z_2-\tau_2)^2}\bigg]\\
&+\sum_{a_2=1}^{2}(-2\delta^1_{a_2}+(1+3Q^2)\delta_{a_2^2})(ip_{m_2})\frac{(-1)^{a_1+a_2+1}(4-a_1-a_2)!(k_1-b_1+a_2-2)!}{(z_1-z_2)^{5-a_1-a_2}(z_2-\tau_1)^{k_1-b_1+a_2-1}}\\
&\times\bigg[\frac{(-1)^{a_1}(a_1+b_1)!\eta_{m_1n_1}R^{(1)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_1-\tau_1)^{a_1+b_1+1}}\\
&+\frac{(-1)^{a_1}a_1!\eta_{m_1n_2}R^{(1+b_1)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{(z_1-\tau_2)^{a_1+1}}\\
&+\frac{(-1)^{1+b_1}(1+b_1)!\eta_{n_1n_2}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2\tau_2))}{(\tau_1-\tau_2)^{b_1+2}}\bigg]\bigg\}
\end{aligned}
\tag{A20}
$$

$$
\begin{aligned}
Z^{(21)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) &= (1-\delta^0_{k_1})(1-\delta^0_{k_2})(1-\delta^1_{k_1})\delta^2_{k_2}\\
&\times\sum_{a_1,a_2=1}^{2}\sum_{b_1=0}^{1}\frac{(\delta^2_{a_1}-2\delta^1_{a_1})(-\delta^0_{b_1}+\frac{1}{2}\delta^1_{b_1}(1-\delta^2_{k_1}))}{(k_1-2-b_1)!}\\
&\times\frac{(\delta^1_{a_2}-\frac{1}{2}(1+3Q^2)\delta_{a_2^2})(ip_{m_2})(-1)^{a_1+1}(4-a_1-a_2)!}{(z_1-z_2)^{5-a_1-a_2}}\\
&\times\bigg\{\frac{(-1)^{a_2}(a_2+k_1-b_1-2)!}{(z_2-\tau_1)^{a_2+k_1-b_1-1}}\bigg[\frac{(-1)^{a_1}(a_1+b_1)!\eta_{m_1n_1}R^{(2)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{2(z_1-\tau_1)^{a_2+k_1-b_1-1}}\\
&+\frac{(-1)^{a_1}(a_1+1)!\eta_{m_1n_2}R^{(1+b_1)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{2(z_1-\tau_2)^{a_1+2}}\\
&+\frac{(-1)^{1+a_1}(b_1+2)!\eta_{m_1n_2}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2\tau_2))}{2(\tau_1-\tau_2)^{b_1+3}}\bigg]\\
&+\frac{iq_{2n_2}\eta_{m_1n_1}(-1)^{a_1+a_2+b_1+k_1}(a_1+b_1)!a_2!(k_1-b_1-1)!}{2(z_1-\tau_1)^{a_1+b_1+1}(z_2-\tau_2)^{a_2+1}(\tau_1-\tau_2)^{k_1-b_1}}\bigg\}
\end{aligned}
\tag{A21}
$$

$$
\begin{aligned}
Z^{(22)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) &= (1-\delta^0_{k_1})(1-\delta^1_{k_1})\sum_{a_1=1}^{2}\sum_{b_1=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})\left(\frac{\delta^0_{b_1}}{2Q}+\frac{(1+3Q^2)(1-\delta^2_{k_1})\delta^1_{b_1}}{4Q}\right)\\
&\times\frac{p_{m_2}q_{1n_1}\eta_{m_1n_2}(-1)^{a_1}(k_1-1-b_1)!(1+b_1)!}{(z_1-z_2)^{3-a_1}(z_2-\tau_1)^{k_1}}\times\left[\frac{Q\delta^2_{k_2}(a_1+1)!}{2(z_1-\tau_2)^{a_1+2}}-\frac{2Q\delta^1_{k_2}a_1!}{(z_1-\tau_2)^{a_1+1}}\right]
\end{aligned}
\tag{A22}
$$

$$
\begin{aligned}
Z^{(23)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) &= -(1-\delta^0_{k_1})(1-\delta^1_{k_1})(1-\delta^2_{k_1})\frac{3(1+3Q^2)}{2(k_1-3)!}\times\frac{\eta_{m_1n_2}p_{m_2}q_{1n_1}}{(z_2-\tau_1)^4}\\
&\times\sum_{a_1=1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})\frac{(-1)^{a_1+1}}{(z_1-z_2)^{3-a_1}}\left[\frac{(-1)^{a_1}Q\delta^2_{k_2}(a_1+1)!}{2(z_1-\tau_2)^{a_1+2}}-\frac{2Q\delta^1_{k_2}a_1!}{(z_1-\tau_2)^{a_1+1}}\right]
\end{aligned}
\tag{A23}
$$

$$Z^{(24)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = -p_{m_2}q_{1n_1}\eta_{m_1n_2}\times\sum_{a_1=1}^{2}\sum_{b_1=0}^{1}(\delta^2_{a_1}-2\delta^1_{a_1})$$
$$\times\left\{\frac{(-\frac{1+3Q^2}{4Q}\delta^0_{b_1}+\frac{1}{2Q}(1-\delta^1_{k_1})\delta^1_{b_1})}{2(k_1-b_1-1)!}\left[-\frac{Q\delta^2_{k_2}(a_1+1)!}{2(z_1-\tau_2)^{a_1+2}}+(-1)^{a_1}\frac{2Q\delta^1_{k_2}a_1!}{(z_1-\tau_2)^{a_1+1}}\right]\right.$$
$$\times\left[\frac{(k_1-a_1-b_1+1)!(b_1+2)!}{(z_1-\tau_1)^{k_1-a_1-b_1+2}(z_2-\tau_1)^{3+b_1}}-\frac{(2-a_1+b_1)!(k_1+1-b_1)!}{(z_1-\tau_1)^{3-a_1+b_1}(z_2-\tau_1)^{p_1+2-b_1}}\right]$$
$$+\sum_{a_2=0}^{2}\left[\left(\frac{\delta^1_{a_2}}{Q}-\frac{1+3Q^2}{2Q}\delta^2_{a_2}\right)\times\frac{1-\delta^1_{p_1}}{2(k_1-2)!}\times\frac{a_2!}{(\tau_1-\tau_2)^{a_2+1}}\right]$$
$$\times\left[-\frac{Q\delta^2_{k_2}(a_1+1)!}{2(z_1-\tau_2)^{a_1+2}}+(-1)^{a_1}\frac{2Q\delta^1_{k_2}a_1!}{(z_1-\tau_2)^{a_1+1}}\right]$$
$$\times\left[-\frac{Q\delta^2_{k_2}(a_1+1)!}{2(z_1-\tau_2)^{a_1+2}}+(-1)^{a_1}\frac{2Q\delta^1_{k_2}a_1!}{(z_1-\tau_2)^{a_1+1}}\right]$$
$$\left.\times\left[\frac{(k_1-a_1)!}{(z_1-\tau_1)^{k_1-a_1+1}(z_2-\tau_1)^{3-a_2}}-\frac{(k_1-a_2)!}{(z_1-\tau_1)^{3-a_1}(z_2-\tau_1)^{k_1+1-a_2}}\right]\right\} \tag{A24}$$

$$Z^{(25)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \delta^1_{k_1}\delta^1_{k_2}\sum_{a_1,a_2=1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})(2\delta^2_{a_2}-(1+3Q^2)\delta^1_{a_2})$$
$$\times\left[-(1+3Q^2)p_{m_2}q_{1n_1}\eta_{m_1n_2}\frac{(-1)^{a_2+1}a_1!a_2!(4-a_1-a_2)!}{(z_1-z_2)^{5-a_1-a_2}(z_1-\tau_2)^{a_1+1}(z_2-\tau_1)^{a_2+1}}\right.$$
$$+2Q^2(-1)^{a_1+1}(ip_{m_2})\times\frac{(4-a_1)!}{(z_1-z_2)^{5-a_1}}$$
$$\times\left(\frac{(-1)^{a_1}a_1!\eta_{m_1n_1}R^{(1)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_1-\tau_1)^{a_1+1}}\right.$$
$$+\frac{(-1)^{a_1}a_1!\eta_{m_1n_2}R^{(1)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{(z_1-\tau_2)^{a_1+1}}$$
$$\left.\left.-\frac{\eta_{n_1n_2}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2\tau_2))}{(\tau_1-\tau_2)^2}\right)\right] \tag{A25}$$

$$Z^{(26)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \delta^2_{k_1}\delta^2_{k_2}\sum_{a_1=1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})(ip_{m_2})\times\frac{(4-a_1)!}{(z_1-z_2)^{5-a_1}}$$
$$\times\left(\frac{(-1)^{a_1}(a_1+1)!\eta_{m_1n_1}R^{(2)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_1-\tau_1)^{a_1+2}}\right.$$
$$+\frac{(-1)^{a_1}a_1!\eta_{m_1n_2}R^{(2)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{(z_1-\tau_2)^{a_1+2}}$$
$$\left.+\frac{6\eta_{n_1n_2}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2\tau_2))}{(\tau_1-\tau_2)^4}\right)+\frac{p_{m_2}q_{2n_2}\eta_{m_1n_1}(a_1+1)!}{(z_1-\tau_1)^{a_1+2}(z_1-z_2)^{3-a_1}(z_2-\tau_2)^4} \tag{A26}$$

$$
\begin{aligned}
Z^{(27)}_{m_1m_2n_1n_2}(z_1,z_2,\tau_1,\tau_2|p,q_1,q_2) = \delta^1_{k_1}\delta^2_{k_2}\sum_{a_1=1}^{2}(\delta^2_{a_1}-2\delta^1_{a_1})\Bigg\{&\frac{ip_{m_2}Q^2(-1)^{a_1+1}(4-a_1)!}{2(z_1-z_2)^{5-a_1}}\\
&\times\Bigg(\frac{(-1)^{a_1}a_1!\eta_{m_1n_1}R^{(2)}_{n_2}(\tau_2|(p,z_1);(p,z_2);(q_1,\tau_1))}{(z_1-\tau_1)^{a_1+1}}\\
&+\frac{(-1)^{a_1}(a_1+1)!\eta_{m_1n_2}R^{(1)}_{n_1}(\tau_1|(p,z_1);(p,z_2);(q_2,\tau_2))}{(z_1-\tau_2)^{a_1+2}}\\
&-\frac{\eta_{n_1n_2}R^{(a_1)}_{m_1}(z_1|(p,z_2);(q_1,\tau_1);(q_2\tau_2))}{(\tau_1-\tau_2)^2}\Bigg)-\frac{Qp_{m_2}q_{2n_2}\eta_{m_1n_1}a_1!}{2(z_1-z_2)^{3-a_1}(z_1-\tau_1)^{a_1+1}(z_2-\tau_2)^4}\\
&+Qp_{m_2}q_{1n_1}\eta_{m_1n_2}\sum_{a_2=1}^{2}\left(\frac{\delta^1_{a_2}}{Q}-\frac{1+3Q^2}{2Q}\delta^2_{a_2}\right)\\
&\times\frac{(-1)^{a_2+1}(a_1+1)!a_2!(4-a_1-a_2)!}{(z_1-z_2)^{5-a_1-a_2}(z_1-\tau_2)^{a_1+2}(z_2-\tau_1)^{a_2+1}}\Bigg\}. \qquad \text{(A27)}
\end{aligned}
$$

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