Covariant equations for the tetraquark and more

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We derive covariant equations for a system of two quarks and two antiquarks where the effect of quark-antiquark annihilation is taken into account. In our approach, only pairwise interactions are retained, while all possibilities of overcounting are excluded by (i) keeping terms in the kernel that are consistent with a meson-meson and diquark-antidiquark substructure, and (ii) introducing four-body equations with a novel structure that specifically avoids the generation of overcounted terms. The resulting tetraquark bound state equations are given for the case of general two-body interactions, and for the specific case of separable interactions that lead to a description of the tetraquark in terms of meson-meson and diquark-antidiquark degrees of freedom where the effects of quark-antiquark annihilation are included. The inclusion of $2q2\bar{q}$- and $q\bar{q}$-channel coupling in our approach enables a wide variety of applications of our equations to other processes within the $2q2\bar{q}$ system, and to other two-particle plus two-antiparticle systems.

I. INTRODUCTION

The present paper is motivated by recent studies by Heupel, Eichmann, Popovici and Fischer (HEPF) [1,2] of tetraquark bound states in the framework of covariant four-body equations based on continuum quantum field theory (QFT). These authors described the underlying two-quark two-antiquark ($2q2\bar{q}$) dynamics in terms of meson, diquark, and antidiquark degrees of freedom, and for this purpose used the four-body equations of Khvedelidze and Kvinikhidze (KK) [3]. The equations of KK are exact in the pair-interaction approximation, but are valid, strictly speaking, only for systems like $4q$ where annihilation does not take place; as such, they can describe the $2q2\bar{q}$ system only if the effect of $q\bar{q}$ pair annihilation is neglected. The purpose of the present paper is to derive equations for the system of two quarks and two antiquarks where $q\bar{q}$ pair annihilation is taken into account.

In the context of continuum QFT, the derivation of covariant equations for few-body systems is an important but nontrivial task, as one routinely encounters the notorious problem of overcounting of Feynman diagrams. Such overcounting problems have been solved in the last two decades for a number of cases [3–8]. One of these is the case of four-quark equations where even the seemingly natural task of summing pair-interaction kernels leads to overcounting [1,3]. Likewise, overcounting provides a major challenge when formulating $2q2\bar{q}$ equations where

$q\bar{q}$ annihilation is taken into account. In this paper we have solved the overcounting problem by (i) keeping terms in the kernel that are consistent with the meson-meson (MM) and diquark-antidiquark ($D\bar{D}$) coupled channels approach of HEPF, and (ii) introducing new coupled channel four-body equations whose form is explicitly constructed to avoid overcounting. In this way we have derived $2q2\bar{q}$ equations whose kernels encode the process of $q\bar{q}$ annihilation.

Although our equations, Eqs. (30), do not depend on the form of the two-body interaction, the case of separable interactions corresponding to bound-state mesons, diquarks, and antidiquarks is of special interest as it provides a description of the tetraquark in terms MM and $D\bar{D}$ degrees of freedom, similar to that of HEPF, yet where $q\bar{q}$ annihilation is taken into account through inclusion of processes like those illustrated in Fig. 1. Our final equations, Eqs. (42), not only allow one to assess the contribution of nonexotic $q\bar{q}$ states to the makeup of tetraquarks, but they can more generally serve as a tool for identifying the tetraquark, meson molecule, or hybrid states; and given the covariant field-theoretical setting, they can offer better insights into the underlying dynamics of the strong interaction.

Apart from tetraquarks, our equations can describe the scattering processes of usual mesons ($q\bar{q}$ bound states). They are also suitable for the construction of scattering amplitudes corresponding to all possible processes in the system of two nucleons and two antinucleons. The results of the paper are also useful for detailed studies of the nonexotic $q\bar{q}$ bound states [9]; namely, the two-body $q\bar{q}$ Bethe-Salpeter kernel can be constructed as a solution of

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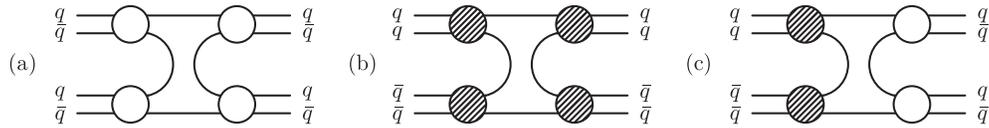[†]boris.blankleider@flinders.edu.au

FIG. 1. Examples of terms involving $q\bar{q}$ annihilation contributing to the tetraquark amplitude within a model involving only meson, diquark and antidiquark constituents: (a) MM scattering, (b) $D\bar{D}$ scattering, (c) $D\bar{D}{\leftarrow}MM$ transition. Such terms are not taken into account when the covariant four-body equations of KK [3] are used to describe the $2q2\bar{q}$ system.

the $2q2\bar{q}$ equations, corresponding to an infinite sum of physically meaningful Feynman diagrams in the $q\bar{q}$ kernel. Analogous studies could also be considered in the system of two electrons and two positrons.

II. DERIVATION

In this section we present a derivation of four-body equations for the $2q2\bar{q}$ system where quark annihilation is taken into account. For clarity of presentation, we treat the quarks as distinguishable, as all the necessary antisymmetrization can be performed at the end of the derivation (Green functions and t matrices need to be summed with respect to permutations of either initial or final state quark or antiquark quantum numbers using antisymmetrizing factors of -1).

The four-body Green function G and the corresponding t matrix X, defined by

$$G = G_0 + G_0 X G_0, \qquad (1)$$

satisfy Dyson equations which relate them to the four-body interaction kernel K:

$$G = G_0 + G_0 K G, \qquad (2a)$$

$$X = K + K G_0 X, \qquad (2b)$$

where G_0 is the free four-body Green function. Following the notation of Ref. [1], we assign labels 1,2 to the quarks and 3,4 to the antiquarks. The kernel K can be formally expressed as

$$K = K_2 + K_3, \qquad (3)$$

where K_2 consists of only pairwise interactions, and K_3 consists of all other contributions, necessarily involving three- and four-body forces. One can then write K_2 as a sum of three terms whose structure is illustrated in Fig. 2, and correspondingly expressed as

FIG. 2. Structure of the four-body kernel K_2 where only two-body forces are included. Each of the summed diagrams corresponds to the term $K_{aa'}$ of Eq. (4), where index a (a') is given by the numerical labels of the two top (bottom) quark (q) or antiquark ($\bar{q}$) lines. Note that the precise mathematical meaning of each diagram is given by Eq. (10).

$$K_2 = \sum_{aa'} K_{aa'}, \qquad (4)$$

where the index $a \in \{12, 13, 14, 23, 24, 34\}$ enumerates six possible pairs of particles, and the double index $aa' \in \{(12, 34), (13, 24), (14, 23)\}$ enumerates three possible two pairs of particles. Thus $K_{aa'}$ describes the part of the four-body kernel where all interactions are switched off except those within the pairs a and a'.

In order to express $K_{aa'}$ in terms of the two-body kernels K_a, we make use of the Green function $G_{aa'}$ that corresponds to the sum of all Feynman diagrams where the pair a is disconnected from the pair a':

$$G_{aa'} = G_a G_{a'}, \qquad (5)$$

where G_a is the two-body Green function for particle pair a. Both G_a and the corresponding two-body t matrix X_a, defined by

$$G_a = G_a^0 + G_a^0 X_a G_a^0, \qquad (6)$$

satisfy Dyson equations which relate them to the two-body interaction kernel K_a:

$$G_a = G_a^0 + G_a^0 K_a G_a, \qquad (7a)$$

$$X_a = K_a + K_a G_a^0 X_a, \qquad (7b)$$

where G_a^0 is the free Green function for pair a. Similarly, both $G_{aa'}$ and the corresponding four-body t matrix $X_{aa'}$, defined by

$$G_{aa'} = G_0 + G_0 X_{aa'} G_0, \qquad (8)$$

satisfy Dyson equations which relate them to the four-body interaction kernel $K_{aa'}$:

$$G_{aa'} = G_0 + G_0 K_{aa'} G_{aa'}, \qquad (9a)$$

$$X_{aa'} = K_{aa'} + K_{aa'} G_0 X_{aa'}. \qquad (9b)$$

Then using Eqs. (5), (7a), and (9a) one obtains[1]

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[1]In four-body expressions we shall suppress factors of G_a^{0-1} associated with noninteracting pairs in the a channel. For example, Eq. (10) is shorthand for $K_{aa'} = K_a G_{a'}^{0-1} + K_{a'} G_a^{0-1} - K_a K_{a'}$
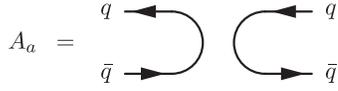
FIG. 3. Disconnected part of the $q\bar{q}$ t matrix X_a. Left arrows indicate particles, as labeled, while right arrows indicate the corresponding antiparticles.

$$K_{aa'} = K_a + K_{a'} - K_a K_{a'}. \tag{10}$$

In the case of qq or $\bar{q}\bar{q}$ channels ($\alpha = 12, 34$), K_a is the sum of two-body irreducible diagrams, all of which are connected. However, in the case of a $q\bar{q}$ channel ($\alpha = 13, 14, 23, 24$), K_a also contains a disconnected part which corresponds to the annihilation (creation) of the $q\bar{q}$ pairs into (from) vacuum in the initial (final) states. This disconnected part of K_a can be derived from Eq. (7a) given that the same disconnectedness is present in G_a in the form of the product of two single quark Green functions corresponding to the independent propagation of q and $\bar{q}$ in the t channel (this disconnected part of G_a should not be confused with the free Green function G_a^0, which corresponds to the independent propagation of q and $\bar{q}$ in the s channel). In other words, the $q\bar{q}$ t matrix X_a has a disconnected part A_a which consists of two u-turned quark lines corresponding to the annihilation (creation) of the $q\bar{q}$ pairs in the initial (final) state, as illustrated in Fig. 3. Such disconnected parts are present in the $2q2\bar{q}$ system, and are not taken into account by the four-body equations of Ref. [3] (these equations were developed to describe four-body systems like $4q$, where there are no annihilation channels).

A. Exact $2q2\bar{q}$ equations

Exact $2q2\bar{q}$ equations in the pairwise approximation can be obtained by analogy with the derivation for the covariant pion-two-nucleon (πNN) system [4] where inclusion of pion absorption leads to a corresponding overcounting problem, as noted above. The procedure is to first expose two-body $q\bar{q}$ cuts in the four-body Green function, as was done in Eq. (31) of Ref. [4] where NN cuts were exposed. The remaining $q\bar{q}$ irreducible part of the $2q2\bar{q}$ Green function will then satisfy the $4q$ equations of Refs. [1,3]. Details of this derivation will be presented elsewhere.

B. Approximate $2q2\bar{q}$ equations

Here we derive $2q2\bar{q}$ equations by modifying KK's $4q$ equations in such a way that disconnected two-body kernels A_a are included without any occurrence of overcounting. Although this way may not be efficient for derivation of exact $2q2\bar{q}$ equations, it suits well the nature of the approximations used in Ref. [1] where only two-meson (MM) and diquark-antidiquark ($D\bar{D}$) states are exposed in the equations. It is worth noting that the

approach taken here is very different from the one used in Ref. [4] to derive the πNN equations.

We begin with Eqs. (2)–(10). The difference from the $4q$ case is that the $q\bar{q}$ kernels, K_a, contain disconnected parts which correspond to the annihilation of $q\bar{q}$ pairs into vacuum. Inclusion of these disconnected parts leads to an important difference between the $2q2\bar{q}$ formulation and the one for the $4q$ system: the "pair-interaction approximation" where the full four-body kernel K is equated with the pairwise kernel K_2 of Eq. (4), by itself, does not make sense in the $2q2\bar{q}$ case due to a double-counting problem in the corresponding Green function. In the exact $2q2\bar{q}$ equations described above, three- and four-body force counterterms need to be included in order to cancel the double-counted terms generated by iteration of the pair-interaction kernels—that is why discarding three- and four-body forces is not allowed in this setting. Here we show another way of avoiding this double counting: we work out how to keep only that part of the pair-interaction kernel which is physically meaningful on the one hand side, and that does not generate double-counted terms on the other.

We consider the two-body correlations t matrix $X_{aa'}$ defined by Eq. (8). Using Eqs. (5), (6), and (8), one can express $X_{aa'}$ in terms of the two-body t matrices X_a and $X_{a'}$ as

$$X_{aa'} = X_a + X_{a'} + X_a X_{a'}. \tag{11}$$

Writing

$$X_a = T_a + A_a, \tag{12}$$

where T_a and A_a are the connected and disconnected parts of X_a, respectively, we note that it is T_a which corresponds to the physical two-body scattering amplitude, while the disconnected part A_a contributes to the physical four-body amplitude where it describes $q\bar{q}$ annihilation into vacuum. The explicit possibilities for X_a are

$$X_{12} = T_{12},$$
$$X_{34} = T_{34},$$
$$X_{13} = T_{13} + A_{13},$$
$$X_{14} = T_{14} + A_{14},$$
$$X_{23} = T_{23} + A_{23},$$
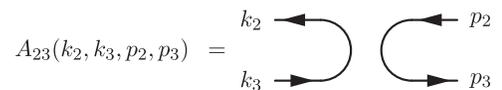$$X_{24} = T_{24} + A_{24}. \tag{13}$$

FIG. 4. The amplitude A_{23} (disconnected part of the $q\bar{q}$ t matrix X_{23}). With the initial (final) quark assigned momentum label p_2 (k_2), and corresponding antiquark assigned momentum $-p_3$ ($-k_3$), so that $p_2 - p_3 = 0$, the expression for A_{23} is given as in Eq. (14).

Note that A_a is nonzero only in $q\bar{q}$ subspace; amplitude A_{23}, for example, is illustrated in Fig. 4 and explicitly given by

$$A_{23}(k_2, k_3, p_2, p_3) = -\delta(p_2 - p_3)S^{-1}(k_2)S^{-1}(p_2), \quad (14)$$

where the momenta are assigned to the quark line direction, so that p_2 (k_2) are the momenta of the incoming (outgoing) quarks, and $-p_3$ ($-k_3$) are the momenta of the corresponding antiquarks. More specifically, one has the following quantum field theoretic definitions of the Green function quantities in the 23 channel:

$$G_{23}(k_2, k_3, p_2, p_3) = \int e^{i(k_2 y_2 - k_3 y_3 - p_2 x_2 + p_3 x_3)} \langle 0|Tq(y_2)\bar{q}(y_3)\bar{q}(x_2)q(x_3)|0\rangle dy_2 dy_3 dx_2 dx_3$$
$$= G_{23}^0 + G_{23}^0(T_{23} + A_{23})G_{23}^0, \quad (15a)$$

$$G_{23}^0(k_2, k_3, p_2, p_3) = \int e^{i(k_2 y_2 - p_2 x_2)} \langle 0|Tq(y_2)\bar{q}(x_2)|0\rangle dy_2 dx_2 \int e^{i(-k_3 y_3 + p_3 x_3)} \langle 0|Tq(x_3)\bar{q}(y_3)|0\rangle dy_3 dx_3$$
$$= S(p_2)\delta(k_2 - p_2)S(p_3)\delta(k_3 - p_3), \quad (15b)$$

$$[G_{23}^0 A_{23} G_{23}^0](k_2, k_3, p_2, p_3) = -\int e^{i(k_2 y_2 - k_3 y_3)} \langle 0|Tq(y_2)\bar{q}(y_3)|0\rangle dy_2 dx_2 \int e^{i(-p_2 x_2 + p_3 x_3)} \langle 0|Tq(x_3)\bar{q}(x_2)|0\rangle dy_3 dx_3$$
$$= -S(k_2)\delta(k_2 - k_3)S(p_2)\delta(p_2 - p_3). \quad (15c)$$

Note that the minus sign in the definition of A_{23} is due to the Wick theorem, as can be seen easily from the fact that A_{23} can be obtained from G_{23}^0 by switching outgoing quark ends, thus entailing a sign change.

The covariant equations of KK were derived for four-body systems, like $4q$, where the pair interactions are described by connected two-body t matrices T_a, that is, for the case where the disconnected parts A_a are equal to zero for all two-body channels a. Using the model where only two-body correlations are included, $K = K_2$ (which obviously has no double-counting problems arising from disconnected two-body kernels), KK showed that the resultant four-body t matrix, denoted by T, can be expressed as

$$T = \sum_{aa'} \mathcal{T}_{aa'}, \quad (16)$$

where

$$\mathcal{T}_{aa'} = T_{aa'} + T_{aa'}G_0(\mathcal{T}_{bb'} + \mathcal{T}_{cc'}), \quad aa' \neq bb' \neq cc' \neq aa'. \quad (17)$$

In Eq. (17), the amplitude $T_{aa'}$ is the t matrix corresponding to Green function $G_{aa'}$; that is, $T_{aa'} \equiv X_{aa'}$ for the special case where all $A_a = 0$, in which case Eq. (11) is written as

$$T_{aa'} = T_a + T_{a'} + T_a T_{a'}. \quad (18)$$

In order to derive covariant equations for the four-body system $2q2\bar{q}$ where $q\bar{q}$ annihilation is included, we shall start with the KK equations, Eqs. (16)–(18), and examine the consequence of including the disconnected parts A_a by simply making the replacements $T_a \to X_a = T_a + A_a$; that is, we examine the consequences of writing the full amplitude X, defined in Eq. (1), as

$$X = \sum_{aa'} \mathcal{X}_{aa'}, \quad (19)$$

where

$$\mathcal{X}_{aa'} = X_{aa'} + X_{aa'}G_0(\mathcal{X}_{bb'} + \mathcal{X}_{cc'}), \quad aa' \neq bb' \neq cc' \neq aa', \quad (20)$$

with amplitude $X_{aa'}$ being given by Eq. (11). For this purpose it is useful to introduce amplitudes $A_{aa'}$ defined by

$$X_{aa'} = T_{aa'} + A_{aa'}, \quad (21)$$

where $T_{aa'}$ is defined by Eq. (18), so that

$$A_{aa'} = A_a + A_{a'} + T_a A_{a'} + A_a T_{a'} + A_a A_{a'}. \quad (22)$$

From the outset, we shall discard the product of disconnected terms $A_a A_{a'}$ (consisting of $A_{13}A_{24}$ and $A_{14}A_{23}$), as they do not contribute to the physically meaningful part of the four-body t matrix. Nevertheless, it is apparent (as demonstrated below) that Eq. (20) is still problematic as it suffers from double-counting problems. Here we propose to handle the overcounting problem by making an approximation that is consistent with the one used in Ref. [1];

namely, we shall neglect terms in Eqs. (18) and (22) that are linear in T_a, so that[2]

$$T_{aa'} \to T_a T_{a'}, \tag{23a}$$

$$A_{aa'} \to A_a + A_{a'}. \tag{23b}$$

In this approximation the two-body correlation t matrices $X_{aa'}$ are modeled as

$$X_{aa'} \to T_a T_{a'} + A_a + A_{a'}, \tag{24}$$

or specifically,

$$X_{12,34} \to T_{12} T_{34}, \tag{25a}$$

$$X_{13,24} \to T_{13} T_{24} + A_{13} + A_{24}, \tag{25b}$$

$$X_{14,23} \to T_{14} T_{23} + A_{14} + A_{23}. \tag{25c}$$

The approximation of Eq. (23a) was used in Ref. [1], and is based on the physically motivated assumption of the tetraquark being mainly a bound state of two mesons or of diquark-antidiquark pairs. Thus only two $q\bar{q}$ pair interaction t matrices ($T_{13,24}$ and $T_{14,23}$) are modified with respect to the paper of Ref. [1] through the addition of disconnected parts A_a. Analysis of the double-counting problem inherent in Eq. (20) provides additional support for the approximations of Eqs. (25). For example, consider the term $T_a A_{a'}$ which appears in Eq. (22). Such a term is illustrated in Fig. 5(a) for the case of $T_{14}A_{23}$ which would arise as part of the inhomogeneous term $X_{14,23}$ of Eq. (20). Yet already in the second iteration of Eq. (20) there would be the term $X_{14,23} G_0 X_{12,34} G_0 X_{14,23}$ giving rise to an amplitude $A_{23} T_{12} T_{34} A_{23}$, illustrated in Fig. 5(b), which is already contained $T_{14}A_{23}$. Thus, instead of introducing three- and four-body forces to compensate double-counted terms, as prescribed by the exact approach, the approximations of Eqs. (25) provide an alternative description where these compensating forces are effectively taken into account without going beyond a pair-interaction model.

However, even with the approximations of Eqs. (25) implemented, Eq. (20) still generates double-counted terms. Namely, the term $A_{13}A_{14}$ generated in the first iteration can be obtained from A_{13} by switching antiquark labels 3 and 4 in the initial state, as illustrated in Fig. 6; yet just this switching will be produced by antisymmetrization of the solution of Eq. (20). Such troublesome double-counted terms can be avoided by modifying Eq. (20) in such a way that will not allow the kernels A_{13} and A_{14} to
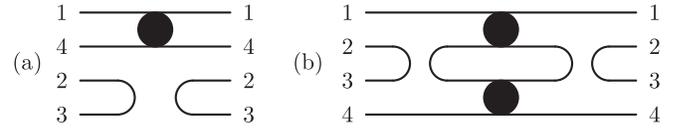
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[2]In the Appendix we discuss the overcounting problem from a broader perspective where we show how more general equations $2q2\bar{q}$ can be derived for the case where the linear terms T_a are retained.

FIG. 5. Example of overcounting resulting from a replacement $T_a \to X_a = T_a + A_a$ in the KK equations: (a) amplitude $T_{14}A_{23}$ contained in the inhomogeneous term $X_{14,23}$ of Eq. (20), (b) amplitude $A_{23}T_{12}T_{34}A_{23}$ contained in the second iteration term $X_{14,23}G_0 X_{12,34} G_0 X_{13,23}$. The amplitude in (b) is already contained in the amplitude in (a).

meet through the process of iteration. To this end we split the component amplitudes $\mathcal{X}_{aa'}$ into two parts

$$\mathcal{X}_{aa'} = \mathcal{T}_{aa'} + \mathcal{A}_{aa'} \tag{26}$$

and introduce the following coupled equations that replace those of Eq. (20):

$$\mathcal{A}_{aa'} = A_{aa'} + A_{aa'} G_0 (\mathcal{T}_{bb'} + \mathcal{T}_{cc'}), \tag{27a}$$

$$\mathcal{T}_{aa'} = T_{aa'} + T_{aa'} G_0 (\mathcal{X}_{bb'} + \mathcal{X}_{cc'}), \tag{27b}$$

where $aa' \neq bb' \neq cc' \neq aa'$. Equations (27) provide the sought-after description of the covariant $2q2\bar{q}$ system where $q\bar{q}$ annihilation is taken into account in a way that is free from overcounting, and that is consistent with the two-meson and diquark-antidiquark model of Ref. [1].

The corresponding tetraquark bound state equations are

$$\Psi = \sum_{aa'} \Psi_{aa'}, \tag{28}$$

where

$$\Psi_{aa'} = \Psi_{aa'}^T + \Psi_{aa'}^A \tag{29}$$

and

$$\Psi_{aa'}^A = G_0 A_{aa'} (\Psi_{bb'}^T + \Psi_{cc'}^T), \tag{30a}$$

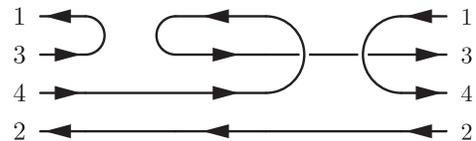$$\Psi_{aa'}^T = G_0 T_{aa'} (\Psi_{bb'} + \Psi_{cc'}), \tag{30b}$$

FIG. 6. Example of overcounting inherent in Eq. (20). Illustrated is amplitude $A_{13}A_{14}$, which is generated in the first iteration of Eq. (20), but that is also obtained from A_{13} when the initial state antiquark labels 3 and 4 are interchanged through antisymmetrization.

where $aa' \neq bb' \neq cc' \neq aa'$. To save on notation, we shall suppress writing factors of G_0. Then from Eqs. (30) one obtains the following closed-form equation for $\Psi_{aa'}^T$:

$$\Psi_{aa'}^T = T_{aa'}[(1 + A_{cc'})\Psi_{bb'}^T + (1 + A_{bb'})\Psi_{cc'}^T + (A_{bb'} + A_{cc'})\Psi_{aa'}^T]. \tag{31}$$

One should note that the kernels $T_{aa'}$ are not compact as they contain singular δ functions corresponding to the pair $a(a')$ total 4-momentum conservation. Similarly, the kernels $T_{aa'}A_{cc'}$ are not compact either as they involve δ functions restricting the total momentum of some $q\bar{q}$ pairs to zero. One should therefore iterate Eq. (31) once to cast it in the form where the kernels are compact. The procedure of compactification is simpler if one uses the separable approximation for two-body t matrices:

$$T_a = -\Gamma_a D_a \bar{\Gamma}_a, \tag{32}$$

where D_a is the propagator for the bound particle in channel a (diquark, antidiquark, or meson), and Γ ($\bar{\Gamma}$) is the vertex function for the particle's disintegration into (formation from) its quark or antiquark constituents.[3] Showing explicit dependence on momentum variables, Eq. (32) can be expressed as

$$T_a(p_1'p_2', p_1p_2) = -\Gamma_a(p_1'p_2')D_a(P)\bar{\Gamma}_a(p_1p_2), \tag{33}$$

where $P = p_1 + p_2$ is the total off-mass-shell momentum of the bound particle. Substituting into Eq. (31) leads to the factorization of the $2q2\bar{q}$ bound state wave function as

$$\Psi_{aa'}^T = \Gamma_a D_a \Gamma_{a'} D_{a'} \Phi_{aa'}, \tag{34a}$$

$$\Phi_{aa'} = \bar{\Gamma}_a \bar{\Gamma}_{a'}[(1 + A_{cc'})\Psi_{bb'}^T + (1 + A_{bb'})\Psi_{cc'}^T + (A_{bb'} + A_{cc'})\Psi_{aa'}^T], \tag{34b}$$

where $\Phi_{aa'}$ are the components of the $2q2\bar{q}$ bound state vertex function in MM and $D\bar{D}$ space; i.e., $\Phi_{13,24}$ ($\Phi_{12,34}$) is the $MM\theta$ ($D\bar{D}\theta$) covariant vertex. As functions of momenta, Eq. (34a) can be written as

$$\Psi_{aa'}^T(p, q, q', P) = \Gamma_a(q, Q)D_a(Q)\Gamma_{a'}$$
$$\times (q', Q')D_{a'}(Q')\Phi_{aa'}(p, P), \tag{35}$$

where P is the $2q2\bar{q}$ bound state total momentum, p is the relative momentum between its respective constituents, q, q' are the relative momenta of the (anti)diquarks and mesons, and Q, Q' are their off-mass-shell momenta. Using Eq. (34a) in Eq. (34b) one obtains a closed set of equations for the bound state vertex functions:

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[3]Note that our definitions of Γ and $\bar{\Gamma}$ are the ones often used for separable potentials, but differ from the ones used in Ref. [1].

$$\Phi_{aa'} = \bar{\Gamma}_a \bar{\Gamma}_{a'}(1 + A_{cc'})\Gamma_b \Gamma_{b'} D_b D_{b'} \Phi_{bb'}$$
$$+ \bar{\Gamma}_a \bar{\Gamma}_{a'}(1 + A_{bb'})\Gamma_c \Gamma_{c'} D_c D_{c'} \Phi_{cc'}$$
$$+ \bar{\Gamma}_a \bar{\Gamma}_{a'}(A_{bb'} + A_{cc'})\Gamma_a \Gamma_{a'} D_a D_{a'} \Phi_{aa'}, \tag{36}$$

where $aa' \neq bb' \neq cc' \neq aa'$. Although formally a set of three coupled equations, consideration of antisymmetry reduces Eq. (36) to a set of two equations for the two components, MM and $D\bar{D}$, of the tetraquark. To show this, we define the component vertex functions as

$$\Phi_D = \Phi_{12,34}, \tag{37a}$$

$$\Phi_M = \Phi_{13,24} = -\Phi_{14,23}. \tag{37b}$$

The relation $\Phi_{13,24} = -\Phi_{14,23}$ for the MM component of the tetraquark follows from the antisymmetry of the diquark and antidiquark wave functions with respect to permutation of the quarks' quantum numbers: $\Gamma_{12} = -\Gamma_{21}$ and $\Gamma_{34} = -\Gamma_{43}$. This antisymmetry property relates $MM \leftarrow D\bar{D}$ transition kernels to each other,

$$\bar{\Gamma}_{13}\bar{\Gamma}_{24}(1 + A_{14,23})\Gamma_{12}\Gamma_{34} = -\bar{\Gamma}_{14}\bar{\Gamma}_{23}(1 + A_{13,24})\Gamma_{12}\Gamma_{34}, \tag{38}$$

which in turn can be used in Eq. (36) to show $\Phi_{13,24} = -\Phi_{14,23}$.

The two equations for Φ_M and Φ_D, mentioned above, consist of two lines of Eq. (36), one corresponding to $aa' = 13, 24$, $bb' = 12, 34$, and $cc' = 14, 23$ (for which $A_{bb'} = 0$), and another corresponding to $aa' = 12, 34$, $bb' = 13, 24$, and $cc' = 14, 23$:

$$\Phi_M = (\bar{\Gamma}_{13}\bar{\Gamma}_{24}A_{14,23}\Gamma_{13}\Gamma_{24} - \bar{\Gamma}_{13}\bar{\Gamma}_{24}\Gamma_{14}\Gamma_{23})MM\Phi_M$$
$$+ \bar{\Gamma}_{13}\bar{\Gamma}_{24}(1 + A_{14,23})\Gamma_{12}\Gamma_{34}D\bar{D}\Phi_D, \tag{39a}$$

$$\Phi_D = 2\bar{\Gamma}_{12}\bar{\Gamma}_{34}(1 + A_{14,23})\Gamma_{13}\Gamma_{24}MM\Phi_M$$
$$+ 4\bar{\Gamma}_{12}\bar{\Gamma}_{34}A_{23}\Gamma_{12}\Gamma_{34}D\bar{D}\Phi_D, \tag{39b}$$

where we have used $A_{14,23} = A_{14} + A_{23}$ and the following relations analogous to Eq. (38):

$$\bar{\Gamma}_{12}\bar{\Gamma}_{34}(1 + A_{14,23})\Gamma_{13}\Gamma_{24} - \bar{\Gamma}_{12}\bar{\Gamma}_{34}(1 + A_{13,24})\Gamma_{14}\Gamma_{23}$$
$$= 2\bar{\Gamma}_{12}\bar{\Gamma}_{34}(1 + A_{14,23})\Gamma_{13}\Gamma_{24}, \tag{40a}$$

$$\bar{\Gamma}_{12}\bar{\Gamma}_{34}(A_{13,24} + A_{14,23})\Gamma_{12}\Gamma_{34} = 2\bar{\Gamma}_{12}\bar{\Gamma}_{34}A_{13,24}\Gamma_{12}\Gamma_{34}$$
$$= 4\bar{\Gamma}_{12}\bar{\Gamma}_{34}A_{23}\Gamma_{12}\Gamma_{34}. \tag{40b}$$

With respect to meson quantum numbers, Eqs. (39) admit both symmetric and antisymmetric solutions because the kernels of Eqs. (39) do not change when the meson quantum numbers are swapped in initial and final states

simultaneously. To exclude the antisymmetric solutions, Eqs. (39) should be symmetrized with respect to meson quantum numbers. Such symmetrization and the replacements (renormalization)

$$\Gamma_{12} \to \frac{1}{\sqrt{2}}\Gamma_{12}, \qquad \bar{\Gamma}_{12} \to \frac{1}{\sqrt{2}}\bar{\Gamma}_{12}, \qquad \Gamma_{34} \to \frac{1}{\sqrt{2}}\Gamma_{34},$$

$$\bar{\Gamma}_{34} \to \frac{1}{\sqrt{2}}\bar{\Gamma}_{34}, \qquad \Phi_D \to 2\Phi_D \qquad (41)$$

cast Eqs. (39) into a form where the symmetry with respect to the indistinguishable meson legs is manifest:

$$\Phi_M = \sum_P (\bar{\Gamma}_{13}\bar{\Gamma}_{24}A_{14,23}\Gamma_{13}\Gamma_{24} - \bar{\Gamma}_{13}\bar{\Gamma}_{24}\Gamma_{14}\Gamma_{23})\frac{MM}{2}\Phi_M$$

$$+ \bar{\Gamma}_{13}\bar{\Gamma}_{24}(1 + A_{14,23})\Gamma_{12}\Gamma_{34}D\bar{D}\Phi_D, \qquad (42a)$$

$$\Phi_D = \bar{\Gamma}_{12}\bar{\Gamma}_{34}(1 + A_{14,23})\Gamma_{13}\Gamma_{24}\frac{MM}{2}\Phi_M$$

$$+ \bar{\Gamma}_{12}\bar{\Gamma}_{34}A_{23}\Gamma_{12}\Gamma_{34}D\bar{D}\Phi_D, \qquad (42b)$$

where $\sum_P$ stands for the sum over meson legs' permutation in either the initial or final state. Note the combinatorial normalization factor $1/2$ at each intermediate state of two indistinguishable mesons. To understand the renormalization in Eq. (41), we note that the symmetric Eqs. (42) could be obtained in the above derivation if one would renormalize the ansatz Eq. (33) for qq and $\bar{q}\bar{q}$ amplitudes by factor $1/2$; for example, in the qq case

$$T_{12}(p_1'p_2', p_1p_2) = -\frac{1}{2}\Gamma_{12}(p_1'p_2')D(P)\bar{\Gamma}_{12}(p_1p_2). \quad (43)$$

The Γ_{12} extracted from Eq. (43), owing to the factor $1/2$, is the correctly normalized vertex function of a diquark composed of indistinguishable quarks, in that $2T_{12}$ contains all diagrams of scattering of indistinguishable quarks; for example, in the one gluon exchange approximation, $2T_{12}$ corresponds to the symmetric sum of two single gluon exchange diagrams. If, instead, the ansatz of Eq. (33) and the corresponding Eqs. (39) are slightly more convenient (as mentioned above), it is only because, for example, in the one gluon exchange approximation kernel, the vertex Γ_{qq} is related to $\Gamma_{q\bar{q}}$ only by the substitution of an antiquark with a quark leg without a factor. The reason is that the quark-quark and quark-antiquark scattering amplitudes T_{12} and T_{13} satisfy the same equation, $T = K + KT$, where K corresponds to a single diagram of one gluon exchange.

C. Double-counting problem

Although the physically transparent form of our final equations for the vertex functions Φ_M and Φ_D, Eqs. (42), should dispel any concerns that some important parts may still be missing or some parts still overcounted, there exists

a rigorous way to check this. To formulate exact QFT equations for few-body systems like πNN and $2q2\bar{q}$ where some particles can be absorbed by others (e.g. π by N), or pairs of particles can undergo annihilation (e.g. $q\bar{q}$), one starts with the general structure of the full few-body Green function that, in the case of the $2q2\bar{q}$ system, is manifested by the relation

$$G^{(4)} = G_{ir}^{(4)} + G_{ir}^{(4-2)}G_0^{(2)-1}G^{(2)}G_0^{(2)-1}G_{ir}^{(2-4)}, \qquad (44)$$

where $G^{(2)}$ is the full two-body $q\bar{q}$ Green function, and $G_{ir}^{(4)}$ is the $q\bar{q}$ irreducible part of the full $2q2\bar{q}$ Green function $G^{(4)}$; further, $G_{ir}^{(2-4)}$ ($G_{ir}^{(4-2)}$) is the sum of all $q\bar{q}$ irreducible diagrams of the Green function corresponding to the transition $q\bar{q} \leftarrow 2q2\bar{q}$ ($2q2\bar{q} \leftarrow q\bar{q}$). The main task is then to express $G_{ir}^{(2-4)}$ and $G_{ir}^{(4-2)}$ in terms of $G_{ir}^{(4)}$. To be consistent with the problem setting (which is to derive equations coupling the $q\bar{q}$ and $2q2\bar{q}$ channels), $G^{(2)}$ also should be expressed in terms of $G_{ir}^{(4)}$ (thereby exposing the $2q2\bar{q}$ intermediate states in $G^{(2)}$). For $G_{ir}^{(2-4)}$, this would normally be accomplished by isolating the last possible $2q2\bar{q}$ cut in $G_{ir}^{(2-4)}$, thereby splitting this amplitude into two parts: $G_{ir}^{(4)}$ to the right of the cut, and a $q\bar{q} \leftarrow 2q2\bar{q}$ amplitude that is both $q\bar{q}$ and $2q2\bar{q}$ irreducible to the left of the cut. The problem is that such a "last $2q2\bar{q}$ cut" is not unique, and special procedures need to be implemented to avoid consequent overcounting. In just this way, QFT few-body equations were derived for the πNN problem in Ref. [4].

We now show that our final equations, Eqs. (42), can be cast into the form specified by Eq. (44). To begin, we rewrite Eqs. (42) in the form

$$\Phi_M = (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S\left(A_{23} - \frac{P_{34}}{2}\right)(\Gamma_{13}\Gamma_{24})^S\frac{MM}{2}\Phi_M$$

$$+ (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S\left(\frac{1}{2} + A_{23}\right)\Gamma_{12}\Gamma_{34}D\bar{D}\Phi_D, \qquad (45a)$$

$$\Phi_D = \bar{\Gamma}_{12}\bar{\Gamma}_{34}\left(\frac{1}{2} + A_{23}\right)(\Gamma_{13}\Gamma_{24})^S\frac{MM}{2}\Phi_M$$

$$+ \bar{\Gamma}_{12}\bar{\Gamma}_{34}A_{23}\Gamma_{12}\Gamma_{34}D\bar{D}\Phi_D, \qquad (45b)$$

where $(\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S$ denotes the wave function of two indistinguishable mesons, so that

$$(\Gamma_{13}\Gamma_{24})^S = \Gamma_{13}^p\Gamma_{24}^k + \Gamma_{13}^k\Gamma_{24}^p, \qquad (46)$$

where p and k are the meson momenta, and P_{34} stands for permutation of antiquark legs 3 and 4, so that

$$P_{34}\Gamma_{13}^p\Gamma_{24}^k = \Gamma_{14}^p\Gamma_{23}^k. \qquad (47)$$

The set of Eqs. (45) can be written in matrix form as

$$\Phi = V G_0^M \Phi, \tag{48}$$

where

$$\Phi(p,k) = \begin{pmatrix} \Phi_M(p,k) \\ \Phi_D(p,k) \end{pmatrix}, \quad G_0^M = \begin{pmatrix} \frac{1}{2} MM & 0 \\ 0 & D\bar{D} \end{pmatrix}, \tag{49}$$

and V is a 2×2 matrix kernel corresponding to all four transitions between MM and $D\bar{D}$ states. One can express V as a sum $V = V_{q\bar{q}} + V_{2q2\bar{q}}$, where $V_{q\bar{q}}$ and $V_{2q2\bar{q}}$ are the parts of the kernel corresponding to $q\bar{q}$ and $2q2\bar{q}$ s-channel exchanges, respectively:

$$V_{q\bar{q}} = \begin{pmatrix} (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S A_{23} (\Gamma_{13}\Gamma_{24})^S & (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S A_{23} \Gamma_{12}\Gamma_{34} \\ \bar{\Gamma}_{12}\bar{\Gamma}_{34} A_{23} (\Gamma_{13}\Gamma_{24})^S & \bar{\Gamma}_{12}\bar{\Gamma}_{34} A_{23} \Gamma_{12}\Gamma_{34} \end{pmatrix}, \tag{50a}$$

$$V_{2q2\bar{q}} = \frac{1}{2} \begin{pmatrix} -(\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S G_0 P_{34} (\Gamma_{13}\Gamma_{24})^S & (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S G_0 \Gamma_{12}\Gamma_{34} \\ \bar{\Gamma}_{12}\bar{\Gamma}_{34} G_0 (\Gamma_{13}\Gamma_{24})^S & 0 \end{pmatrix}. \tag{50b}$$

Note that the propagator for four noninteracting quarks, G_0, is shown explicitly in Eq. (50b) whereas in Eqs. (45) it is omitted for notational convenience. These kernels are illustrated diagrammatically in Fig. 7.

The inhomogeneous equation for the MM-$D\bar{D}$ Green function G corresponding to the homogeneous Eq. (48) is

$$G = G_0^M + G_0^M (V_{q\bar{q}} + V_{2q2\bar{q}}) G. \tag{51}$$

It can be written in the form

$$G = G_{ir} + G_{ir} V_{q\bar{q}} G, \tag{52}$$

where G_{ir} is the sum of all $q\bar{q}$ irreducible terms in Green function G, and itself satisfies the equation

$$G_{ir} = G_0^M + G_0^M V_{2q2\bar{q}} G_{ir}. \tag{53}$$

Using Eq. (15c), which we shall write in the current four-body context as $A_{23} = -S_{23} G_{14}^0 S_{23}$, Eq. (50a) can be written as

$$V_{q\bar{q}} = -\begin{pmatrix} (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S S_{23} G_{14}^0 S_{23} (\Gamma_{13}\Gamma_{24})^S & (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S S_{23} G_{14}^0 S_{23} \Gamma_{12}\Gamma_{34} \\ \bar{\Gamma}_{12}\bar{\Gamma}_{34} S_{23} G_{14}^0 S_{23} (\Gamma_{13}\Gamma_{24})^S & \bar{\Gamma}_{12}\bar{\Gamma}_{34} S_{23} G_{14}^0 S_{23} \Gamma_{12}\Gamma_{34} \end{pmatrix} = -\begin{pmatrix} N_{MM} G_{14}^0 \bar{N}_{MM} & N_{MM} G_{14}^0 \bar{N}_{D\bar{D}} \\ N_{D\bar{D}} G_{14}^0 \bar{N}_{MM} & N_{D\bar{D}} G_{14}^0 \bar{N}_{D\bar{D}} \end{pmatrix}, \tag{54}$$

where the repeated indices 2 and 3, which stand for quantum numbers of the second quark and the third antiquark, are mute; i.e., they are summation indices. Therefore, for example, the expression

$$N_{MM} = (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S S_{23} \tag{55}$$

is the amplitude of the transition of quark 1 and antiquark 4 to two mesons, and the propagator S_{23} corresponds to the internal exchanged quark line. Similarly

$$N_{D\bar{D}} = \bar{\Gamma}_{12}\bar{\Gamma}_{34} S_{23} \tag{56}$$

is the amplitude of the transition of quark 1 and antiquark 4 to a diquark-antidiquark pair. The kernel of Eq. (54) thus consists of terms of the form $N_i(p',k') G_{14}^0 \bar{N}_j(p,k)$ as illustrated in Fig. 7(a), where initial and final MM or $D\bar{D}$ states are separated only by a two-body $q\bar{q}$ intermediate state. The matrix of Eq. (54) can be written in a compact symbolic form as a direct product of column (N) and row ($\bar{N}$) matrices:

$$V_{q\bar{q}} = -N G_{q\bar{q}}^0 \bar{N}, \tag{57}$$

where $G_{q\bar{q}}^0 = G_{14}^0$, and

$$N = \begin{pmatrix} N_{MM} \\ N_{D\bar{D}} \end{pmatrix} = \begin{pmatrix} (\bar{\Gamma}_{13}\bar{\Gamma}_{24})^S S_{23} \\ \bar{\Gamma}_{12}\bar{\Gamma}_{34} S_{23} \end{pmatrix}, \tag{58a}$$

$$\bar{N} = (\bar{N}_{MM} \quad \bar{N}_{D\bar{D}}) = (S_{23}(\Gamma_{13}\Gamma_{24})^S \quad S_{23}\Gamma_{12}\Gamma_{34}). \tag{58b}$$

Then from Eqs. (52) and (53) we get

$$G = G_{ir} + G_{ir} N G_{q\bar{q}} \bar{N} G_{ir}, \tag{59}$$

where $G_{q\bar{q}}$ is the $q\bar{q}$ Green function which contains all $q\bar{q}$ intermediate states and is itself determined by equation

$$G_{q\bar{q}} = G_{q\bar{q}}^0 + G_{q\bar{q}}^0 (\bar{N} G_{ir} N) G_{q\bar{q}}. \tag{60}$$

Here $\bar{N} G_{ir} N$ is the $q\bar{q}$ interaction potential. In Eq. (59) the $q\bar{q}$ cuts ($G_{q\bar{q}}^0$) are exposed via Green function $G_{q\bar{q}}$, as specified by Eq. (60). With Eq. (59), we have obtained the realization of Eq. (44) for the particular case of the

separable approximation of Eq. (33). It is interesting to note that in this approximate case, we have been able to derive equations for the $2q2\bar{q}$ system where $q\bar{q}$ annihilation is included, but without having to face the above mentioned ambiguity of the last $2q2\bar{q}$ cut in $G^{(2-4)}$. The point is that the term $\bar{N}G_{ir}$ appearing in Eq. (59) contains within it just such a last $2q2\bar{q}$ cut that is not unique, yet $\bar{N}$ is determined unambiguously in Eq. (59). Careful analysis of this approximate model may thus lead us to the solution of this well-known problem of the general case; in particular, it may be possible to deduce a criterion which helps one to make an unambiguous choice of the very last cut, such that the double-counting problem is avoided. Finally, we note that the decomposition of Eq. (59) allows one to see whether something is overcounted, or what may be missing, in the initial approximate description provided by Eqs. (30), and how it may be improved.

III. DISCUSSION

In this paper we have derived covariant equations for the $2q2\bar{q}$ system where $q\bar{q}$ annihilation is taken into account. This has been achieved in a model where the kernel consists only of terms that allow for a description in terms of MM and $D\bar{D}$ degrees of freedom in the case where the separable approximations of Eq. (33) are used for the two-body interactions. We find it encouraging that the parts of the kernel that are neglected in this model [namely, T_a, $T_aA_{a'}$, and $A_aA_{a'}$; see Eqs. (23)] are just the ones that would cause

overcounting if retained. The parts of the kernel retained in the model (namely, $T_aT_{a'}$ and A_a) can still cause overcounting when two disconnected terms, A_{13} and A_{14}, are allowed to meet through iteration. To stop this from happening, we have introduced four-body equations with a novel structure designed specifically to prevent this type of overcounting.

For the general case of two-body interactions, our equations for the tetraquark bound state are given by Eqs. (30). For two-body separable interactions, as specified by Eq. (33), our equations are expressed in terms of MM and $D\bar{D}$ degrees of freedom, and presented first for distinguishable particles, Eq. (36). Taking into account the antisymmetry of identical quarks and antiquarks, but without reference to the symmetry of the two meson states, the equations reduce from three coupled equations down to two, as given by Eqs. (39). Finally, with the mesons symmetrized, we obtain Eqs. (48).

Our Eqs. (48) reduce to those of Ref. [1] if we eliminate the effect of $q\bar{q}$ annihilation by setting $V_{q\bar{q}} = 0$, or equivalently, by setting $A_{aa'} = 0$ in Eqs. (39). The kernels involving $A_{aa'}$ correspond to quark box diagrams, as in Fig. 7(a), where two-body $q\bar{q}$ intermediate states are incorporated. In this way the two-body $q\bar{q}$ component contributions are buried in the kernels of Eqs. (39), even though they are written in terms of only meson and diquark degrees of freedom. Adding these box diagrams does not complicate the tetraquark equations of Ref. [1] in the sense they are one-loop diagrams, just like the kernels in Ref. [1].
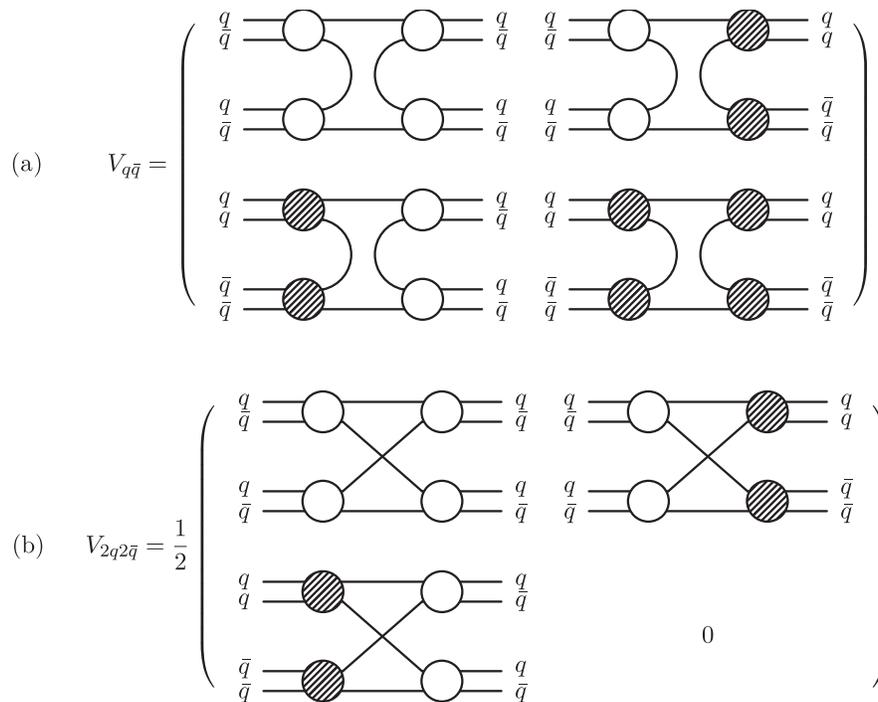

FIG. 7. Diagrammatic representation of the kernels of Eqs. (50): (a) kernels with $q\bar{q}$ intermediate states, as given explicitly in Eq. (54), (b) kernels with $2q2\bar{q}$ intermediate states, as given explicitly by Eq. (50b).

The complication is that one gets two equations instead of one. Also, including the box diagrams makes the model complete up to the one-loop level since all other effects involve two and more loops in the kernel.

It is worth pointing out that the MM-$D\bar{D}$ picture of a tetraquark follows from the separable approximation for the input two-body scattering amplitudes, and that the addition of the box diagrams is not something that is beyond this approximation: one only adds some disconnected parts in $q\bar{q}$ channels to make the equations applicable to the $2q2\bar{q}$ system. This addition restores missing topologies and is not part of the dynamics: all the dynamics is encoded in the two-body scattering amplitudes T_a.

With our equations, it will be possible to ascertain the importance of $q\bar{q}$ annihilation in the description of the tetraquark, in a quantitative way.

ACKNOWLEDGMENT

APPENDIX: OVERCOUNTING IN EQ. (20)

In this Appendix we explain in more detail how the overcounting problem encountered in Eq. (20) is solved. The simplest double-counted term appears already in the first iteration of Eq. (20), resulting in iterated disconnected terms A_{ij}:
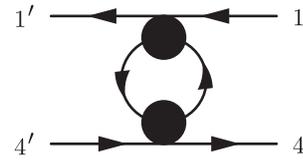

FIG. 8.   Form of the interaction between quark 1 and antiquark 4 inside the amplitude $A_{23}T_{12}T_{34}A_{23}$.

$$X_{13,24}(X_{14,23} + X_{12,34}) = A_{13}A_{14} + \cdots \quad (A1)$$

whose double-counting property was illustrated in Fig. 6. Similarly, we illustrated in Fig. 5 how the second iteration $X_{14,23}X_{12,34}X_{14,23}$ generates the term $A_{23}T_{12}T_{34}A_{23}$ which is already contained in the amplitude $T_{14}A_{23}$ which forms part of the inhomogeneous term $X_{14,23}$. Terms of the type $T_aA_{a'}$ were discarded in the approximate kernels of Eqs. (25) just to avoid such double counting.

However, it is noteworthy that the term $A_{23}T_{12}T_{34}A_{23}$, by itself, involves double counting in the case of exact T_{12} and T_{34}. This is evident from Fig. 5(b) where the interaction between quark 1 and antiquark 4 is of the form given in Fig. 8. Overcounting will occur if the amplitudes T_{12} and T_{34} contain t-channel exchanges of interacting $q\bar{q}$ pairs. On the other hand, in the often used rainbow-ladder approximation for the two-body t matrices (which is a factorization assumption for the t matrices in the s channel) no terms are double counted. Indeed

$$A_{23}T_{12}T_{34}A_{23} = A_{23}(p_2', p_3', p_2, p_3) \int (dk)S(k_2', k_3')T_{12}(p_1'k_2', p_1k_2)T_{34}(k_3'p_4', k_3p_4)S(k_2, k_3), \quad (A2)$$

where $S(k_2, k_3) = \delta(k_2 - k_3)S(k_2)$ is a quark propagator, the integration over four momenta, $(dk) = dk_2'dk_3'dk_2dk_3$, is reduced to a one-loop 4-momentum integral upon the use of 4-momentum conservation δ functions, including one coming from the scattering amplitude, $T_{12}(p_1'k_2', p_1k_2) = T_{12}(p_1'k_2', p_1k_2)\delta(p_1' + k_2' - p_1 - k_2)$. The rainbow-ladder approximation implies the factorization in the s channel:

$$T_{12}(p_1'p_2', p_1p_2) = -\Gamma(p_1'p_2')D(P)\bar{\Gamma}(p_1p_2),$$
$$P = p_1 + p_2. \quad (A3)$$

Note that the same factorization approximation for the two-body t matrices in the t channel,

$$T_{12}(p_1'p_2', p_1p_2) = -\Gamma(p_1'p_1)M(p_1' - p_1)\bar{\Gamma}(p_2'p_2), \quad (A4)$$

would lead to a double counting in Eq. (A2). Indeed, this double counting can be seen in the unphysical second-order pole at $(p_1' - p_1)^2 = m_M^2$.

Because of the overcounting just discussed, we drop both the $A_aA_{a'}$ and $T_aA_{a'}$ terms from the kernel $A_{aa'}$ as defined by Eq. (22); in this way, we specify all the kernels as

$$X_{aa'} = T_{aa'} + A_{aa'}, \quad (A5a)$$

$$T_{aa'} = T_a + T_{a'} + T_aT_{a'}, \quad (A5b)$$

$$A_{aa'} = A_a + A_{a'}. \quad (A5c)$$

Specifically, the kernels $X_{aa'}$ are given in the stated approximation as

$$X_{12,34} = T_{12} + T_{34} + T_{12}T_{34}, \quad (A6a)$$

$$X_{13,24} = T_{13} + T_{24} + T_{13}T_{24} + A_{13} + A_{24}, \quad (A6b)$$

$$X_{14,23} = T_{14} + T_{23} + T_{14}T_{23} + A_{14} + A_{23}. \quad (A6c)$$

These approximate kernels still generate double counting once they are iterated via Eq. (20). In particular, iteration

leads to (i) the term $A_{13}A_{14}$ which can be obtained from A_{13} by switching antiquark 3 and 4 legs in the initial state, but this term will be produced by antisymmetrization of the solution of Eq. (20), and (ii) the part $A_{23}T_{12}A_{23}$ of the second iteration, $A_{23}X_{12,34}A_{23}$, which leads to an A_{23} type term with an overdressed first quark line. Such overcounting can be avoided by modifying Eq. (20) in such a way that will prevent troublesome pairs of kernels, like A_{13} and A_{14}, A_{23} and T_{12}, etc., ever meeting each other when the equations are iterated.

To this end we split the kernels $T_{aa'}$ into two parts, specified as

$$T_{aa'} = T_{aa'}^1 + T_{aa'}^2, \tag{A7a}$$

$$T_{aa'}^1 = T_a + T_{a'}, \tag{A7b}$$

$$T_{aa'}^2 = T_a T_{a'}, \tag{A7c}$$

and correspondingly, express the amplitude $\mathcal{X}_{aa'}$ of Eq. (20) as

$$\mathcal{X}_{aa'} = \mathcal{T}_{aa'} + \mathcal{A}_{aa'}, \tag{A8a}$$

$$\mathcal{T}_{aa'} = \mathcal{T}_{aa'}^1 + \mathcal{T}_{aa'}^2. \tag{A8b}$$

With these definitions, the $2q2\bar{q}$ amplitude X is given by

$$X = \sum_{aa'} \mathcal{X}_{aa'}, \tag{A9}$$

where the modified equations for the components $\mathcal{X}_{aa'}$ are given by

$$\mathcal{A}_{aa'} = A_{aa'} + A_{aa'}(\mathcal{T}_{bb'}^2 + \mathcal{T}_{cc'}^2), \tag{A10a}$$

$$\mathcal{T}_{aa'}^1 = T_{aa'}^1 + T_{aa'}^1(\mathcal{T}_{bb'} + \mathcal{T}_{cc'}), \tag{A10b}$$

$$\mathcal{T}_{aa'}^2 = T_{aa'}^2 + T_{aa'}^2(\mathcal{X}_{bb'} + \mathcal{X}_{cc'}), \tag{A10c}$$

where $aa' \neq bb' \neq cc' \neq aa'$. Equations (A10) are obtained by expressing Eq. (20) symbolically in terms of the above component amplitudes as

$$\mathcal{A} = A + A(\underline{\mathcal{T}^1} + \mathcal{T}^2 + \underline{\mathcal{A}}), \tag{A11a}$$

$$\mathcal{T}^1 = T^1 + T^1(\mathcal{T}^1 + \mathcal{T}^2 + \underline{\mathcal{A}}), \tag{A11b}$$

$$\mathcal{T}^2 = T^2 + T^2(\mathcal{T}^1 + \mathcal{T}^2 + \mathcal{A}) \tag{A11c}$$

and discarding the underlined terms. The term $A\mathcal{A}$ is discarded because it generates the problematic term $A_{13}A_{14}$ discussed above. The term $A\mathcal{T}^1$ is discarded because all the terms it generates in the second iteration of Eqs. (A11),

$$A\mathcal{T}^1 \to A\mathcal{T}^1(\mathcal{T}^1 + \mathcal{T}^2 + \mathcal{A}) \to A\mathcal{T}^1(T^1 + T^2 + A), \tag{A12}$$

suffer double counting. For example, the part $A_{23}T_{12}A_{23}$ of $A\mathcal{T}^1A$ is a A_{23} type term with an overdressed first quark line. The terms $A\mathcal{T}^1(T^1 + T^2)$ involve subdiagrams with a product of two two-body scattering amplitudes, $T^{(2)}G_0^{(2)}T^{(2)}$, similar to that of Fig. 8. The term $\mathcal{T}^1\mathcal{A} \sim T_{14}A_{23}$ is discarded because the same term is partially obtained in $A\mathcal{T}^2 \to A\mathcal{T}^2A \sim A_{23}T_{12}T_{34}A_{23}$, as discussed above.

Equations for the bound state wave function Ψ, corresponding to Eqs. (A10), are derived by taking the residue of X at the pole in the energy plane corresponding to the mass of the tetraquark. Defining the wave function components corresponding to the amplitudes of Eqs. (A8) as

$$\Psi = \sum_{aa'} \Psi_{aa'}, \tag{A13}$$

where

$$\Psi_{aa'} = \Psi_{aa'}^T + \Psi_{aa'}^A, \tag{A14a}$$

$$\Psi_{aa'}^T = \Psi_{aa'}^1 + \Psi_{aa'}^2, \tag{A14b}$$

the bound state equations corresponding to Eqs. (A10) are

$$\Psi_{aa'}^A = A_{aa'}(\Psi_{bb'}^2 + \Psi_{cc'}^2), \tag{A15a}$$

$$\Psi_{aa'}^1 = T_{aa'}^1(\Psi_{bb'}^T + \Psi_{cc'}^T), \tag{A15b}$$

$$\Psi_{aa'}^2 = T_{aa'}^2(\Psi_{bb'} + \Psi_{cc'}), \tag{A15c}$$

where $aa' \neq bb' \neq cc' \neq aa'$.

As we mentioned above, Eq. (A15) will be considered in full in a later publication, while here we consider a simpler approximation, corresponding to setting $T_{aa'}^1 = 0$, in which case the kernels are given explicitly by Eqs. (25). The equations for the $2q2\bar{q}$ amplitude X are then obtained from Eqs. (A8)–(A10), by setting $\mathcal{T}_{aa'}^1 = 0$, and therefore $\mathcal{T}_{aa'}^2 = \mathcal{T}_{aa'}$:

$$X = \sum_{aa'} \mathcal{X}_{aa'}, \tag{A16a}$$

$$\mathcal{X}_{aa'} = \mathcal{T}_{aa'} + \mathcal{A}_{aa'}, \tag{A16b}$$

where

$$\mathcal{A}_{aa'} = A_{aa'} + A_{aa'}(\mathcal{T}_{bb'} + \mathcal{T}_{cc'}), \tag{A17a}$$

$$\mathcal{T}_{aa'} = T_{aa'} + T_{aa'}(\mathcal{X}_{bb'} + \mathcal{X}_{cc'}), \tag{A17b}$$

and $aa' \neq bb' \neq cc' \neq aa'$. Similarly, equations for the bound state wave function are

$$\Psi = \sum_{aa'} \Psi_{aa'}, \tag{A18a}$$

$$\Psi_{aa'} = \Psi_{aa'}^T + \Psi_{aa'}^A, \tag{A18b}$$

where

$$\Psi_{aa'}^A = A_{aa'}(\Psi_{bb'}^T + \Psi_{cc'}^T), \tag{A19a}$$

$$\Psi_{aa'}^T = T_{aa'}(\Psi_{bb'} + \Psi_{cc'}), \tag{A19b}$$

and $aa' \neq bb' \neq cc' \neq aa'$.

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