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Nonlocal halo bias with and without massive neutrinos

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Understanding the biasing between the clustering properties of halos and the underlying dark matter distribution is important for extracting cosmological information from ongoing and upcoming galaxy surveys. While on sufficiently large scales the halo overdensity is a local function of the mass density fluctuations, on smaller scales the gravitational evolution generates nonlocal terms in the halo density field. We characterize the magnitude of these contributions at third order in perturbation theory by identifying the coefficients of the nonlocal invariant operators, and extend our calculation to include nonlocal (Lagrangian) terms induced by a peak constraint. We apply our results to describe the scale dependence of halo bias in cosmologies with massive neutrinos. The inclusion of gravity-induced nonlocal terms and, especially, a Lagrangian k^2 contribution is essential to reproduce the numerical data accurately. We use the peak-background split to derive the numerical values of the various bias coefficients from the excursion set peak mass function. For neutrino masses in the range $0 \leq \sum_i m_{\nu_i} \leq 0.6$ eV, we are able to fit the data with a precision of a few percents up to $k = 0.3h\,\mathrm{Mpc}^{-1}$ without any free parameter.

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I. INTRODUCTION

Cosmological parameter estimation from galaxy clustering data is hampered by galaxy biasing, i.e. by the fact that galaxies do not perfectly trace the underlying mass distribution. Various theoretical arguments and outcomes of numerical simulations suggest that, on sufficiently large scales, the galaxy overdensity $\delta_{\mathrm{h}}(\vec{x},\tau)$ can be written as a generic function $f[\delta(\vec{x},\tau)]$ of the mass density perturbation $\delta(\vec{x},\tau)$. This function can be Taylor-expanded, with the unknown coefficients in the series defining the so-called bias parameters

$$\delta_{\mathrm{h}}(\vec{x},\tau) = f[\delta(\vec{x},\tau)] = b_1(\tau)\delta(\vec{x},\tau) + \frac{b_2(\tau)}{2}(\delta^2(\vec{x},\tau) - \langle\delta^2(\vec{x},\tau)\rangle) + \cdots. \tag{1.1}$$

We use the subscript (h), which stands for halos, because galaxies form within dark matter (DM) halos and, therefore, understanding the clustering properties of the halos is a key step towards an accurate description of galaxy biasing. Furthermore, this is a simpler problem since DM halos collapse under the action of gravity solely.

The local model [1] described in, e.g., (1.1) is however incomplete: there is indeed no *a priori* reason why the halo density contrast should be only a local function of the matter density contrast. Indeed, already at second order in perturbation theory, the gravitational evolution generates a term quadratic in the tidal tensor and, therefore, nonlocal in the density field. This term is absent in the initial conditions. This point was made in Refs. [2–5] for the matter density contrast and subsequently investigated in the context of halo bias in Refs. [6–13]. In Refs. [8,13] in particular, it was pointed out that the symmetries (essentially extended Galilean and Lifshitz symmetries) present in the dynamical equations for the halo and DM systems allow to construct a set of invariant operators which should appear in the halo bias expansion, precisely because they are allowed by the symmetries of the problem. These invariants lead to nonlocal bias contributions, which numerical simulations have already detected at the quadratic level and found to agree well with the prediction of perturbation theory [11,12]. In this paper we compute the nonlocal bias coefficients in the basis of invariant nonlocal bias operators at third order in perturbation theory by using the Lagrangian bias parameters generated within the peak-background split model.

Although the magnitude of the nonlocal bias terms is small relative to the linear halo bias, upcoming large scale structure data will be sensitive to them. In particular, the impact of the nonlocal bias, if not accounted for, could mimic the k-dependent suppression of the growth rate in cosmologies with massive neutrinos [14–19] (see [20,21] for detailed reviews of the subject). Their influence on the spatial distribution of DM halos has been recently scrutinized in a series of papers [22–24] using large N-body simulations that incorporate massive neutrinos as an extra set of particles (see also the recent work of [25]). Massive neutrinos generate a scale-dependent bias in the power spectrum of DM halos. As we will see, this effect is somewhat degenerate with the signature left by the various nonlocal bias terms, which must be taken into account in order to reproduce the N-body data with good accuracy (i.e. at the $\leq 5\%$ level).

The paper is organized as follows. In Sec. II, we present our computation of the nonlocal bias at third order in perturbation theory as well as the calculation of the Lagrangian bias parameters through the peak-background split model. In Sec. III we discuss the halo bias in the presence of massive neutrinos and argue that contributions from nonlocal Lagrangian and gravity bias must be accounted for in order to fit the numerical data. We conclude in Sec. IV.

II. NONLOCAL BIAS UP TO THIRD ORDER

In this section we extend the analysis of [8,11,12] and compute the nonlocal bias coefficients up to third order in perturbation theory. Our starting point is the evolution over cosmic time and in Eulerian space of the halo progenitors—the so-called proto-halos—until their virialization. The basic idea is that, while their shapes and topology change as a function of time (smaller substructures gradually merge to form the final halo), their center of mass moves along a well-defined trajectory determined by the surrounding mass density field [26]. Therefore, unlike virialized halos that undergo merging, by construction proto-halos always preserve their identity. Their total number is therefore conserved over time, such that we can write a continuity equation for their number density

$$\dot{\delta}_{\rm h}(\vec{x},\tau) + \vec{\nabla}\cdot[(1+\delta_{\rm h}(\vec{x},\tau))\vec{v}(\vec{x},\tau)] = 0. \tag{2.1}$$

We may subtract from it the DM mass conservation equation of motion

$$\dot{\delta}(\vec{x},\tau) + \vec{\nabla}\cdot[(1+\delta(\vec{x},\tau))\vec{v}(\vec{x},\tau)] = 0, \tag{2.2}$$

where we have assumed unbiased halo velocity (we will relax this assumption later), to get

$$\dot{\delta}_{\rm h}(\vec{x},\tau) - \dot{\delta}(\vec{x},\tau) + \vec{\nabla}\cdot[(\delta_{\rm h}(\vec{x},\tau)-\delta(\vec{x},\tau))\vec{v}(\vec{x},\tau)] = 0. \tag{2.3}$$

This is the fundamental equation which we will solve order by order in perturbation theory, and which gives rise to a nonlocal bias expansion. For the sake of simplicity, we will restrict ourselves to a matter-dominated Universe and denote by $\vec{x}$ the comoving spatial coordinates, $\tau = \int {\rm d}t/a$ the conformal time where a is the scale factor in the Freidmann-Robertson-Walker (FRW) metric, and $\mathcal{H} = {\rm d}\ln a/{\rm d}\tau$ the conformal expansion rate. In addition, $\delta(\vec{x},\tau) = (\rho(\vec{x},\tau)/\bar{\rho} - 1)$ is the overdensity over the mean matter density $\bar{\rho}$, $\delta_{\rm h}(\vec{x},\tau)$ stands for the halo counterpart, and $\vec{v}(\vec{x},\tau)$ is the common peculiar velocity. In the following we will also denote by $\Phi(\vec{x},\tau)$ the gravitational potential induced by density fluctuations. We begin with a review of the first- and second-order calculations before deriving the third-order nonlocal biases.

A. First order

At first order from Eq. (2.3) we get

$$\dot{\delta}^{(1)}_{\rm h}(\vec{x},\tau) - \dot{\delta}^{(1)}(\vec{x},\tau) = 0, \tag{2.4}$$

or

$$\delta^{(1)}_{\rm h}(\vec{x},\tau) = \delta^{(1)}_{\rm h}(\vec{x},\tau_i) + \delta^{(1)}(\vec{x},\tau) - \delta^{(1)}(\vec{x},\tau_i), \tag{2.5}$$

where τ_i is some initial time. We assume that the initial bias expansion is local and depends only on the linear DM density contrast through the (Lagrangian) bias coefficients

$$\begin{aligned}\delta_{\rm h}(\vec{x},\tau_i) &= \sum_\ell \frac{b^{\rm L}_\ell(\tau_i)}{\ell!}(\delta^{(1)}(\vec{x},\tau_i))^\ell \\ &= \sum_\ell \frac{b^{\rm L}_\ell(\tau)}{\ell!}(\delta^{(1)}(\vec{x},\tau))^\ell \\ &\simeq b^{\rm L}_1(\tau)\delta^{(1)}(\vec{x},\tau) + \frac{1}{2}b^{\rm L}_2(\tau)(\delta^{(1)}(\vec{x},\tau))^2 \\ &\quad + \frac{1}{3!}b^{\rm L}_3(\tau)(\delta^{(1)}(\vec{x},\tau))^3 + \cdots,\end{aligned} \tag{2.6}$$

where $b^{\rm L}_\ell(\tau) = b^{\rm L}_\ell(\tau_i)(a(\tau_i)/a(\tau))^\ell$. Using $\delta^{(1)}_{\rm h}(\vec{x},\tau_i) = b^{\rm L}_1(\tau)\delta(\vec{x},\tau)$, we obtain the standard result

$$\delta^{(1)}_{\rm h}(\vec{x},\tau) \simeq (1+b^{\rm L}_1(\tau))\delta^{(1)}(\vec{x},\tau). \tag{2.7}$$

B. Second order

At second order we may use the first-order result to write Eq. (2.3) in the form

$$\begin{aligned}&\dot{\delta}^{(2)}_{\rm h}(\vec{x},\tau) - \dot{\delta}^{(2)}(\vec{x},\tau) \\ &\quad + \vec{\nabla}\cdot[(\delta^{(1)}_{\rm h}(\vec{x},\tau) - \delta^{(1)}(\vec{x},\tau))\vec{v}^{(1)}(\vec{x},\tau)] = 0,\end{aligned} \tag{2.8}$$

which is solved by

$$
\begin{aligned}
\delta_{\rm h}^{(2)}(\vec{x},\tau) &= \delta_{\rm h}^{(2)}(\vec{x},\tau_i) + \delta^{(2)}(\vec{x},\tau) - \int^{\tau} {\rm d}\eta\, b_1^{\rm L}(\eta)\vec{\nabla}\cdot[\delta^{(1)}(\vec{x},\eta)\vec{v}^{(1)}(\vec{x},\eta)] \\
&\simeq \frac{1}{2} b_2^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^2 + \delta^{(2)}(\vec{x},\tau) - \int^{\tau} {\rm d}\eta\, b_1^{\rm L}(\eta)\vec{\nabla}\cdot[\delta^{(1)}(\vec{x},\eta)\vec{v}^{(1)}(\vec{x},\eta)] \\
&= \frac{1}{2} b_2^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^2 + \delta^{(2)}(\vec{x},\tau) - \int^{\tau} {\rm d}\eta\, b_1^{\rm L}(\eta)\vec{\nabla}\delta^{(1)}(\vec{x},\eta)\cdot\vec{v}^{(1)}(\vec{x},\eta) + \int^{\tau} {\rm d}\eta\, b_1^{\rm L}(\eta)\delta^{(1)}(\vec{x},\eta)\dot{\delta}^{(1)}(\vec{x},\eta) \\
&= \frac{1}{2} b_2^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^2 + \delta^{(2)}(\vec{x},\tau) - \frac{\tau}{2} b_1^{\rm L}(\tau)\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau) + b_1^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^2, \qquad (2.9)
\end{aligned}
$$

or

$$
\delta_{\rm h}^{(2)}(\vec{x},\tau) = -\frac{1}{\mathcal{H}} b_1^{\rm L}(\tau)\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau) + \frac{1}{2} b_2^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^2 + \delta^{(2)}(\vec{x},\tau) + b_1^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^2. \qquad (2.10)
$$

To perform the time integrals, we have used the scalings provided by the first-order quantities (in matter domination)

$$
\begin{aligned}
v_i^{(1)}(\vec{x},\tau) &= -\frac{\tau}{3}\partial_i\varphi(\vec{x}) = -\frac{2}{3\mathcal{H}}\partial_i\varphi(\vec{x}), \\
\delta^{(1)}(\vec{x},\tau) &= \frac{\tau^2}{6}\nabla^2\varphi(\vec{x}) = \frac{2}{3\mathcal{H}^2}\nabla^2\varphi(\vec{x}), \qquad (2.11)
\end{aligned}
$$

where $\varphi(\vec{x})$ is the initial condition for the gravitational potential $\Phi(\vec{x},\tau)$. In order to elaborate further the second-order halo density contrast, we remind the reader that [2,27]

$$
\begin{aligned}
\delta^{(2)}(\vec{x},\tau) &= \frac{\tau^4}{2\cdot 126}[5(\nabla^2\varphi(\vec{x}))^2 + 2\partial_k\partial_p\varphi(\vec{x})\partial^k\partial^p\varphi(\vec{x}) + 7\partial^i\varphi(\vec{x})\nabla^2\partial_i\varphi(\vec{x})] \\
&= \frac{5}{7}(\delta^{(1)}(\vec{x},\tau))^2 + \frac{8}{63\mathcal{H}^4}\partial_k\partial_p\varphi(\vec{x})\partial^k\partial^p\varphi(\vec{x}) - \frac{1}{\mathcal{H}}\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau) \qquad (2.12)
\end{aligned}
$$

and define the nonlocal bias operator

$$
s_{ij}(\vec{x},\tau) = \frac{2}{3\mathcal{H}^2}\partial_i\partial_j\Phi(\vec{x},\tau) - \frac{1}{3}\delta_{ij}\delta(\vec{x},\tau), \qquad (2.13)
$$

to get

$$
\partial_k\partial_p\varphi(\vec{x})\partial^k\partial^p\varphi(\vec{x}) = \frac{9\mathcal{H}^4}{4}(s^{(1)}(\vec{x},\tau))^2 + \frac{3\mathcal{H}^4}{4}(\delta^{(1)}(\vec{x},\tau))^2. \qquad (2.14)
$$

From this expression we deduce

$$
\delta^{(2)}(\vec{x},\tau) = \frac{17}{21}(\delta^{(1)}(\vec{x},\tau))^2 + \frac{2}{7}(s^{(1)}(\vec{x},\tau))^2 - \frac{1}{\mathcal{H}}\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau). \qquad (2.15)
$$

We finally arrive at

$$
\delta_{\rm h}^{(2)}(\vec{x},\tau) \simeq (1 + b_1^{\rm L}(\tau))\delta^{(2)}(\vec{x},\tau) + \left(\frac{1}{2}b_2^{\rm L}(\tau) + \frac{4}{21}b_1^{\rm L}(\tau)\right)(\delta^{(1)}(\vec{x},\tau))^2 - \frac{2}{7}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2. \qquad (2.16)
$$

This result reproduces exactly the one derived in Refs. [6,12] and shows that at second order in perturbation theory the halo overdensity is not a local function of the underlying matter overdensity.

C. Third order

We now proceed to the original part of the computation at third order. The equation to solve is

$$\dot{\delta}_{\rm h}^{(3)}(\vec{x},\tau)-\dot{\delta}^{(3)}(\vec{x},\tau)+\vec{\nabla}\cdot[(\delta_{\rm h}^{(1)}(\vec{x},\tau)-\delta^{(1)}(\vec{x},\tau))\vec{v}^{(2)}(\vec{x},\tau)]+\vec{\nabla}\cdot[(\delta_{\rm h}^{(2)}(\vec{x},\tau)-\delta^{(2)}(\vec{x},\tau))\vec{v}^{(1)}(\vec{x},\tau)]=0, \tag{2.17}$$

where [27]

$$\begin{aligned}\vec{v}_i^{(2)}(\vec{x},\tau)&=\frac{\tau^3}{18}\left[-\partial_i\partial_j\varphi(\vec{x})\partial^j\varphi(\vec{x})-\frac{3}{7}\frac{\partial_i}{\nabla^2}((\nabla^2\varphi(\vec{x}))^2-\partial_k\partial_p\varphi(\vec{x})\partial^k\partial^p\varphi(\vec{x}))\right]\\ \vec{\nabla}\cdot\vec{v}_i^{(2)}(\vec{x},\tau)&=\frac{\tau^3}{9}\left[-\partial_j\nabla^2\varphi(\vec{x})\partial^j\varphi(\vec{x})-\partial_i\partial_j\varphi(\vec{x})\partial^i\partial^j\varphi(\vec{x})-\frac{3}{7}((\nabla^2\varphi(\vec{x}))^2-\partial_k\partial_p\varphi(\vec{x})\partial^k\partial^p\varphi(\vec{x}))\right]\\ &=\frac{\tau^3}{18}\left[-\partial_j\nabla^2\varphi(\vec{x})\partial^j\varphi(\vec{x})-\frac{4}{7}\partial_i\partial_j\varphi(\vec{x})\partial^i\partial^j\varphi(\vec{x})-\frac{3}{7}(\nabla^2\varphi(\vec{x}))^2\right].\end{aligned} \tag{2.18}$$

Equation (2.17) gives

$$\begin{aligned}\delta_{\rm h}^{(3)}(\vec{x},\tau)&=\delta_{\rm h}^{(3)}(\vec{x},\tau_i)+\delta^{(3)}(\vec{x},\tau)-\int^\tau d\eta b_1^{\rm L}(\eta)\vec{\nabla}\cdot[\delta^{(1)}(\vec{x},\eta)\vec{v}^{(2)}(\vec{x},\eta)]-\int^\tau d\eta b_1^{\rm L}(\eta)\vec{\nabla}\cdot[\delta^{(2)}(\vec{x},\eta)\vec{v}^{(1)}(\vec{x},\eta)]\\ &\quad-\int^\tau d\eta\vec{\nabla}\cdot\left[\left(\left(\frac{1}{2}b_2^{\rm L}(\eta)+\frac{4}{21}b_1^{\rm L}(\eta)\right)(\delta^{(1)}(\vec{x},\eta))^2-\frac{2}{7}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2\right)\vec{v}^{(1)}(\vec{x},\eta)\right],\end{aligned} \tag{2.19}$$

or

$$\begin{aligned}\delta_{\rm h}^{(3)}(\vec{x},\tau)&=\frac{1}{3!}b_3^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^3+\delta^{(3)}(\vec{x},\tau)-\int^\tau d\eta b_1^{\rm L}(\eta)\vec{\nabla}\delta^{(1)}(\vec{x},\eta)\cdot\vec{v}^{(2)}(\vec{x},\eta)-\int^\tau d\eta b_1^{\rm L}(\eta)\delta^{(1)}(\vec{x},\eta)\vec{\nabla}\cdot\vec{v}^{(2)}(\vec{x},\eta)\\ &\quad-\int^\tau d\eta b_1^{\rm L}(\eta)\vec{\nabla}\delta^{(2)}(\vec{x},\eta)\cdot\vec{v}^{(1)}(\vec{x},\eta)-\int^\tau d\eta b_1^{\rm L}(\eta)\delta^{(2)}(\vec{x},\eta)\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\eta)\\ &\quad-\int^\tau d\eta\left(\frac{1}{2}b_2^{\rm L}(\eta)+\frac{4}{21}b_1^{\rm L}(\eta)\right)\vec{\nabla}[(\delta^{(1)}(\vec{x},\eta))^2]\cdot\vec{v}^{(1)}(\vec{x},\eta)\\ &\quad-\int^\tau d\eta\left(\frac{1}{2}b_2^{\rm L}(\eta)+\frac{4}{21}b_1^{\rm L}(\eta)\right)(\delta^{(1)}(\vec{x},\eta))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\eta)\\ &\quad+\frac{2}{7}\int^\tau d\eta b_1^{\rm L}(\eta)\vec{\nabla}(s^{(1)}(\vec{x},\eta))^2\cdot\vec{v}^{(1)}(\vec{x},\eta)+\frac{2}{7}\int^\tau d\eta b_1^{\rm L}(\eta)(s^{(1)}(\vec{x},\eta))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\eta).\end{aligned} \tag{2.20}$$

We can integrate over time to obtain

$$\begin{aligned}\delta_{\rm h}^{(3)}(\vec{x},\tau)&=\frac{1}{3!}b_3^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^3+\delta^{(3)}(\vec{x},\tau)-\frac{\tau}{4}b_1^{\rm L}(\tau)\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(2)}(\vec{x},\tau)-\frac{\tau}{4}b_1^{\rm L}(\tau)\delta^{(1)}(\vec{x},\tau)\vec{\nabla}\cdot\vec{v}^{(2)}(\vec{x},\tau)\\ &\quad-\frac{\tau}{4}b_1^{\rm L}(\tau)\vec{\nabla}\delta^{(2)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau)-\frac{\tau}{4}b_1^{\rm L}(\tau)\delta^{(2)}(\vec{x},\tau)\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau)\\ &\quad-\left(\frac{\tau}{4}b_2^{\rm L}(\tau)+\frac{\tau}{21}b_1^{\rm L}(\tau)\right)\{\vec{\nabla}[(\delta^{(1)}(\vec{x},\tau))^2]\cdot\vec{v}^{(1)}(\vec{x},\tau)+(\delta^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau)\}\\ &\quad+\frac{\tau}{14}b_1^{\rm L}(\tau)\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau)+\frac{\tau}{14}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau).\end{aligned} \tag{2.21}$$

Now, the mass conservation equation for the DM at third order reads

$$\begin{aligned}\dot{\delta}^{(3)}(\vec{x},\tau)+\vec{\nabla}\cdot\vec{v}^{(3)}(\vec{x},\tau)&=-\vec{\nabla}\delta^{(2)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau)-\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(2)}(\vec{x},\tau)\\ &\quad-\delta^{(2)}(\vec{x},\tau)\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau)-\delta^{(1)}(\vec{x},\tau)\vec{\nabla}\cdot\vec{v}^{(2)}(\vec{x},\tau).\end{aligned} \tag{2.22}$$

Since $\delta^{(3)}(\vec{x},\tau)$ scales like a^3, we can rewrite it as

$$3\mathcal{H}\delta^{(3)}(\vec{x},\tau)+\vec{\nabla}\cdot\vec{v}^{(3)}(\vec{x},\tau)=-[\vec{\nabla}\delta^{(2)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau)+\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(2)}(\vec{x},\tau)\\ +\delta^{(2)}(\vec{x},\tau)\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau)+\delta^{(1)}(\vec{x},\tau)\vec{\nabla}\cdot\vec{v}^{(2)}(\vec{x},\tau)]. \quad (2.23)$$

Equation (2.21) then becomes

$$\delta_{\rm h}^{(3)}(\vec{x},\tau)=\frac{1}{3!}b_3^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^3+\left(\frac{3}{2}b_1^{\rm L}(\tau)+1\right)\delta^{(3)}(\vec{x},\tau)+\frac{1}{2\mathcal{H}}b_1^{\rm L}(\tau)\theta^{(3)}(\vec{x},\tau)\\ -\left(\frac{\tau}{4}b_2^{\rm L}(\tau)+\frac{\tau}{21}b_1^{\rm L}(\tau)\right)\{\vec{\nabla}[(\delta^{(1)}(\vec{x},\tau))^2]\cdot\vec{v}^{(1)}(\vec{x},\tau)+(\delta^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau)\}\\ +\frac{\tau}{14}b_1^{\rm L}(\tau)\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau)+\frac{\tau}{14}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau), \quad (2.24)$$

where $\theta(\vec{x},\tau)=\vec{\nabla}\cdot\vec{v}(\vec{x},\tau)$ satisfies at any order in perturbation theory the DM momentum equation

$$\dot{\theta}(\vec{x},\tau)+\mathcal{H}\theta(\vec{x},\tau)+\partial^j v_i(\vec{x},\tau)\partial^i v_j(\vec{x},\tau)+\vec{v}(\vec{x},\tau)\cdot\vec{\nabla}\theta(\vec{x},\tau)=-\frac{3}{2}\mathcal{H}^2\delta(\vec{x},\tau). \quad (2.25)$$

Following Refs. [8,13], we introduce another nonlocal coefficient

$$t_{ij}(\vec{x},\tau)=-\frac{1}{\mathcal{H}}\left(\partial_i v_j(\vec{x},\tau)-\frac{1}{3}\delta_{ij}\theta(\vec{x},\tau)\right)-s_{ij}(\vec{x},\tau). \quad (2.26)$$

It is traceless and vanishes at first order in perturbation theory as

$$s_{ij}^{(1)}(\vec{x},\tau)=-\frac{1}{\mathcal{H}}\partial_i v_j^{(1)}(\vec{x},\tau)-\frac{1}{3}\delta_{ij}\delta^{(1)}(\vec{x},\tau), \quad (2.27)$$

and therefore

$$t_{ij}^{(1)}(\vec{x},\tau)=\frac{1}{3}\delta_{ij}\left(\frac{1}{\mathcal{H}}\theta^{(1)}(\vec{x},\tau)+\delta^{(1)}(\vec{x},\tau)\right)=0. \quad (2.28)$$

This implies that $t^2(\vec{x},\tau)=t_{ij}(\vec{x},\tau)t^{ij}(\vec{x},\tau)$ is fourth order and we can neglect it here and henceforth. In particular, up to third order,

$$\partial^j v_i(\vec{x},\tau)\partial^i v_j(\vec{x},\tau)=-2\mathcal{H}^2 t(\vec{x},\tau)\cdot s(\vec{x},\tau)\\ +\frac{1}{3}\theta^2(\vec{x},\tau)+\mathcal{H}^2 s^2(\vec{x},\tau). \quad (2.29)$$

The equation for $\theta(\vec{x},\tau)$ then becomes

$$\dot{\theta}(\vec{x},\tau)+\mathcal{H}\theta(\vec{x},\tau)-2\mathcal{H}^2 t(\vec{x},\tau)\cdot s(\vec{x},\tau)+\frac{1}{3}\theta^2(\vec{x},\tau)\\ +\mathcal{H}^2 s^2(\vec{x},\tau)+\vec{v}(\vec{x},\tau)\cdot\vec{\nabla}\theta(\vec{x},\tau)=-\frac{3}{2}\mathcal{H}^2\delta(\vec{x},\tau). \quad (2.30)$$

Since

$$\vec{\nabla}\cdot\vec{v}^{(2)}(\vec{x},\tau)=\frac{\tau^3}{18}\left[-\partial_j\nabla^2\varphi(\vec{x})\partial^j\varphi(\vec{x})-\frac{4}{7}\partial_i\partial_j\varphi(\vec{x})\partial^i\partial^j\varphi(\vec{x})-\frac{3}{7}(\nabla^2\varphi(\vec{x}))^2\right]\\ =\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau)-\frac{4}{7}\mathcal{H}(s^{(1)}(\vec{x},\tau))^2-\frac{13}{21}\mathcal{H}(\delta^{(1)}(\vec{x},\tau))^2, \quad (2.31)$$

we have

$$-\frac{1}{\mathcal{H}}\vec{\nabla}\cdot\vec{v}^{(2)}(\vec{x},\tau)-\delta^{(2)}(\vec{x},\tau)\\ =\frac{2}{7}(s^{(1)}(\vec{x},\tau))^2-\frac{4}{21}(\delta^{(1)}(\vec{x},\tau))^2, \quad (2.32)$$

and it is convenient to define two new nonlocal bias operators [8]

$$\eta(\vec{x},\tau)=-\frac{\theta(\vec{x},\tau)}{\mathcal{H}}-\delta(\vec{x},\tau) \quad (2.33)$$

and

$$\psi(\vec{x},\tau)=\eta(\vec{x},\tau)-\frac{2}{7}s^2(\vec{x},\tau)+\frac{4}{21}\delta^2(\vec{x},\tau). \quad (2.34)$$

By construction $\eta(\vec{x},\tau)$ is a second-order quantity and $\psi(\vec{x},\tau)$ is a third-order quantity. Equation (2.24) becomes then

$$\delta_{\rm h}^{(3)}(\vec{x},\tau) = \frac{1}{3!}b_3^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^3 + \left(\frac{3}{2}b_1^{\rm L}(\tau)+1\right)\delta^{(3)}(\vec{x},\tau)$$
$$+\frac{1}{2}b_1^{\rm L}(\tau)\left(-\psi^{(3)}(\vec{x},\tau) - \delta^{(3)}(\vec{x},\tau) - \frac{4}{7}s^{(1)}(\vec{x},\tau)s^{(2)}(\vec{x},\tau) + \frac{8}{21}\delta^{(1)}(\vec{x},\tau)\delta^{(2)}(\vec{x},\tau)\right)$$
$$-\frac{1}{\mathcal{H}}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{21}b_1^{\rm L}(\tau)\right)\{\vec{\nabla}[(\delta^{(1)}(\vec{x},\tau))^2]\cdot\vec{v}^{(1)}(\vec{x},\tau) + (\delta^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau)\}$$
$$+\frac{1}{7\mathcal{H}}b_1^{\rm L}(\tau)\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau) + \frac{1}{7\mathcal{H}}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau). \quad (2.35)$$

Using Eq. (2.15) we can rewrite it as

$$\delta_{\rm h}^{(3)}(\vec{x},\tau) = \frac{1}{3!}b_3^{\rm L}(\tau)(\delta^{(1)}(\vec{x},\tau))^3 + (1+b_1^{\rm L}(\tau))\delta^{(3)}(\vec{x},\tau)$$
$$+\frac{1}{2}b_1^{\rm L}(\tau)\left(-\psi^{(3)}(\vec{x},\tau) - \frac{4}{7}s^{(1)}(\vec{x},\tau)\cdot s^{(2)}(\vec{x},\tau) + \frac{8}{21}\delta^{(1)}(\vec{x},\tau)\delta^{(2)}(\vec{x},\tau)\right)$$
$$-\frac{1}{\mathcal{H}}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{21}b_1^{\rm L}(\tau)\right)\delta^{(1)}(\vec{x},\tau)\{2\vec{\nabla}\delta^{(1)}(\vec{x},\tau)\cdot\vec{v}^{(1)}(\vec{x},\tau) - \mathcal{H}(\delta^{(1)}(\vec{x},\tau))^2\}$$
$$+\frac{1}{7\mathcal{H}}b_1^{\rm L}(\tau)\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau) + \frac{1}{7\mathcal{H}}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2\vec{\nabla}\cdot\vec{v}^{(1)}(\vec{x},\tau), \quad (2.36)$$

or

$$\delta_{\rm h}^{(3)}(\vec{x},\tau) = (1+b_1^{\rm L}(\tau))\delta^{(3)}(\vec{x},\tau) + 2\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{4}{21}b_1^{\rm L}(\tau)\right)\delta^{(1)}(\vec{x},\tau)\delta^{(2)}(\vec{x},\tau)$$
$$+\left[\frac{1}{3!}b_3^{\rm L}(\tau) - \frac{13}{21}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{21}b_1^{\rm L}(\tau)\right)\right](\delta^{(1)}(\vec{x},\tau))^3 + \frac{1}{2}b_1^{\rm L}(\tau)\left(-\psi^{(3)}(\vec{x},\tau) - \frac{4}{7}s^{(1)}(\vec{x},\tau)\cdot s^{(2)}(\vec{x},\tau)\right)$$
$$-\frac{4}{7}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{29}{84}b_1^{\rm L}(\tau)\right)\delta^{(1)}(\vec{x},\tau)(s^{(1)}(\vec{x},\tau))^2 + \frac{1}{7\mathcal{H}}b_1^{\rm L}(\tau)\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau). \quad (2.37)$$

Let us concentrate on the last term of the expression (2.37). We can operate a series of manipulations on it:

$$\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau) = 2s_{ij}^{(1)}(\vec{x},\tau)v_k^{(1)}(\vec{x},\tau)\partial^k s_{ij}^{(1)}(\vec{x},\tau)$$
$$= -\frac{2}{\mathcal{H}}s_{ij}^{(1)}(\vec{x},\tau)v_k^{(1)}(\vec{x},\tau)\partial^k\partial_i v_j^{(1)}(\vec{x},\tau)$$
$$= -\frac{2}{\mathcal{H}}s_{ij}^{(1)}(\vec{x},\tau)[\partial_i(v_k^{(1)}(\vec{x},\tau)\partial^k v_j^{(1)}(\vec{x},\tau)) - \partial_i v_k^{(1)}(\vec{x},\tau)\partial^k v_j^{(1)}(\vec{x},\tau)]$$
$$= -\frac{2}{\mathcal{H}}s_{ij}^{(1)}(\vec{x},\tau)[-\partial_i\partial_j\Phi^{(2)}(\vec{x},\tau) - \partial_i\dot{v}_j^{(2)}(\vec{x},\tau) - \mathcal{H}\partial_i v_j^{(2)}(\vec{x},\tau)]$$
$$+2\mathcal{H}s_{ij}^{(1)}(\vec{x},\tau)\left(s_{ik}^{(1)}(\vec{x},\tau)+\frac{1}{3}\delta_{ik}\delta^{(1)}(\vec{x},\tau)\right)\left(s_{kj}^{(1)}(\vec{x},\tau)+\frac{1}{3}\delta_{kj}\delta^{(1)}(\vec{x},\tau)\right)$$
$$= -\frac{2}{\mathcal{H}}s_{ij}^{(1)}(\vec{x},\tau)\left(-\frac{3\mathcal{H}^2}{2}s_{ij}^{(2)}(\vec{x},\tau)\right) - \frac{2}{\mathcal{H}}s_{ij}^{(1)}(\vec{x},\tau)\left(-\frac{5}{2}\mathcal{H}\partial_i v_j^{(2)}(\vec{x},\tau)\right)$$
$$+2\mathcal{H}s_{ij}^{(1)}(\vec{x},\tau)\left(s_{ik}^{(1)}(\vec{x},\tau)s_{kj}^{(1)}(\vec{x},\tau)+\frac{2}{3}s_{ij}^{(1)}(\vec{x},\tau)\delta^{(1)}(\vec{x},\tau)\right)$$
$$= 3\mathcal{H}s_{ij}^{(1)}(\vec{x},\tau)s_{ij}^{(2)}(\vec{x},\tau) + 5s_{ij}^{(1)}(\vec{x},\tau)\mathcal{H}\left(\frac{1}{3\mathcal{H}}\delta_{ij}\theta^{(2)}(\vec{x},\tau) - t_{ij}^{(2)}(\vec{x},\tau) - s_{ij}^{(2)}(\vec{x},\tau)\right)$$
$$+2\mathcal{H}(s^{(1)}(\vec{x},\tau))^3 + \frac{4}{3}\mathcal{H}(s^{(1)}(\vec{x},\tau))^2\delta^{(1)}(\vec{x},\tau)$$
$$= -2\mathcal{H}s_{ij}^{(1)}(\vec{x},\tau)s_{ij}^{(2)}(\vec{x},\tau) - 5\mathcal{H}s_{ij}^{(1)}(\vec{x},\tau)t_{ij}^{(2)}(\vec{x},\tau) + 2\mathcal{H}(s^{(1)}(\vec{x},\tau))^3 + \frac{4}{3}\mathcal{H}(s^{(1)}(\vec{x},\tau))^2\delta^{(1)}(\vec{x},\tau). \quad (2.38)$$

Therefore

$$\frac{1}{7\mathcal{H}}b_1^{\rm L}(\tau)\vec{\nabla}(s^{(1)}(\vec{x},\tau))^2\cdot\vec{v}^{(1)}(\vec{x},\tau) = -\frac{2}{7}b_1^{\rm L}(\tau)s_{ij}^{(1)}(\vec{x},\tau)s_{ij}^{(2)}(\vec{x},\tau) - \frac{5}{7}b_1^{\rm L}(\tau)s_{ij}^{(1)}(\vec{x},\tau)t_{ij}^{(2)}(\vec{x},\tau) + \frac{2}{7}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^3 + \frac{4}{21}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^2\delta^{(1)}(\vec{x},\tau), \tag{2.39}$$

so that

$$\delta_{\rm h}^{(3)}(\vec{x},\tau) = (1+b_1^{\rm L}(\tau))\delta^{(3)}(\vec{x},\tau) + 2\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{4}{21}b_1^{\rm L}(\tau)\right)\delta^{(1)}(\vec{x},\tau)\delta^{(2)}(\vec{x},\tau) + \left[\frac{1}{3!}b_3^{\rm L}(\tau) - \frac{13}{21}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{21}b_1^{\rm L}(\tau)\right)\right](\delta^{(1)}(\vec{x},\tau))^3 + \frac{1}{2}b_1^{\rm L}(\tau)\left(-\psi^{(3)}(\vec{x},\tau) - \frac{8}{7}s^{(1)}(\vec{x},\tau)\cdot s^{(2)}(\vec{x},\tau)\right) - \frac{4}{7}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{81}b_1^{\rm L}(\tau)\right)\delta^{(1)}(\vec{x},\tau)(s^{(1)}(\vec{x},\tau))^2 - \frac{5}{7}b_1^{\rm L}(\tau)s_{ij}^{(1)}(\vec{x},\tau)t_{ij}^{(2)}(\vec{x},\tau) + \frac{2}{7}b_1^{\rm L}(\tau)(s^{(1)}(\vec{x},\tau))^3. \tag{2.40}$$

Combining the results (2.7), (2.16), and (2.40), we finally get

$$\delta_{\rm h}(\vec{x},\tau) = (1+b_1^{\rm L}(\tau))\delta(\vec{x},\tau) + \left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{4}{21}b_1^{\rm L}(\tau)\right)\delta^2(\vec{x},\tau) - \frac{2}{7}b_1^{\rm L}(\tau)s^2(\vec{x},\tau) + \left[\frac{1}{3!}b_3^{\rm L}(\tau) - \frac{13}{21}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{21}b_1^{\rm L}(\tau)\right)\right]\delta^3(\vec{x},\tau) - \frac{1}{2}b_1^{\rm L}(\tau)\psi - \frac{4}{7}\left(\frac{1}{2}b_2^{\rm L}(\tau)+\frac{2}{81}b_1^{\rm L}(\tau)\right)\delta(\vec{x},\tau)s^2(\vec{x},\tau) - \frac{5}{7}b_1^{\rm L}(\tau)s(\vec{x},\tau)\cdot t(\vec{x},\tau) + \frac{2}{7}b_1^{\rm L}(\tau)s^3(\vec{x},\tau). \tag{2.41}$$

The halo density contrast can thus be written as

$$\delta_{\rm h}(\vec{x},\tau) = b_1\delta(\vec{x},\tau) + \frac{1}{2!}b_2\delta^2(\vec{x},\tau) + \frac{1}{3!}b_3\delta^3(\vec{x},\tau) + \frac{1}{2}b_{s^2}s^2(\vec{x},\tau) + b_\psi\psi(\vec{x},\tau) + b_{st}s(\vec{x},\tau)\cdot t(\vec{x},\tau) + \cdots, \tag{2.42}$$

where the relevant bias coefficients are

$$b_1 = 1+b_1^{\rm L},\qquad b_2 = b_2^{\rm L}+\frac{8}{21}b_1^{\rm L},\qquad b_{s^2} = -\frac{4}{7}b_1^{\rm L},\qquad b_\psi = -\frac{1}{2}b_1^{\rm L},\qquad b_{st} = -\frac{5}{7}b_1^{\rm L}. \tag{2.43}$$

This is the main result of this section.

D. Scale-dependent bias

So far, we have not considered the possibility that the Lagrangian bias factors may be scale dependent. In peak theory for instance, the peak constraint induces k-dependent corrections at all orders in the bias coefficients [9,28–30]. Moreover, peak velocities are statistically biased at linear order, and this bias propagates to higher order owing to gravity mode coupling [9,31]. Even though calculations within the peak formalism are relatively tedious, the modifications brought by the peak constraint can actually be easily implemented in any Lagrangian approach [9,29]: the scale-independent Lagrangian bias factors $b_n^{\rm L}$ should be replaced (in Fourier space) by the scale-dependent functions $c_n^{\rm L}(\vec{k}_1,\ldots\vec{k}_n,\tau_i)$. At the lowest orders,

$$c_1^{\rm L}(k,\tau_i) = b_{10}^{\rm L}(\tau_i) + b_{01}^{\rm L}(\tau_i)k^2, \tag{2.44}$$

$$c_2^{\rm L}(\vec{k}_1,\vec{k}_2,\tau_i) = \left\{b_{20}^{\rm L}(\tau_i) + b_{11}^{\rm L}(\tau_i)(k_1^2+k_2^2) + b_{02}^{\rm L}(\tau_i)k_1^2k_2^2 - 2\chi_{10}^{\rm L}(\tau_i)(\vec{k}_1\cdot\vec{k}_2) + \chi_{01}^{\rm L}(\tau_i)\left[3(\vec{k}_1\cdot\vec{k}_2)^2 - k_1^2k_2^2\right]\right\}. \tag{2.45}$$

As a result, the local Lagrangian bias expansion generalizes to

$$\delta_{\rm h}(\vec{x},\tau_i) = b^{\rm L}_{10}(\tau_i)\delta(\vec{x},\tau_i) - b^{\rm L}_{01}(\tau_i)\vec{\nabla}^2\delta(\vec{x},\tau_i) + \frac{1}{2}b^{\rm L}_{20}(\tau_i)\delta^2(\vec{x},\tau_i) - b^{\rm L}_{11}(\tau_i)\delta(\vec{x},\tau_i)\vec{\nabla}^2\delta(\vec{x},\tau_i) + \frac{1}{2}b^{\rm L}_{02}(\tau_i)[\vec{\nabla}^2\delta(\vec{x},\tau_i)]^2 + \chi^{\rm L}_{10}(\tau_i)(\vec{\nabla}\delta)^2(\vec{x},\tau_i) + \frac{1}{2}\chi^{\rm L}_{01}(\tau_i)[3\partial_i\partial_j\delta - \delta_{ij}\vec{\nabla}^2\delta]^2(\vec{x},\tau_i) + \cdots, \tag{2.46}$$

where the various bias factors $b^{\rm L}_{ij}$ and $\chi^{\rm L}_{ij}$ can be obtained from a peak-background split applied to the halo mass function (see [29] for details).

In addition, the Zel'dovich gravity mode-coupling kernels $F^{\rm ZA}_n(\vec{k}_1,\ldots,\vec{k}_n)$ should be replaced by

$$\mathcal{F}^{\rm ZA}_n(\vec{k}_1,\ldots,\vec{k}_n) = F^{\rm ZA}_n(\vec{k}_1,\ldots,\vec{k}_n)\times b_v(k_1)\ldots b_v(k_n). \tag{2.47}$$

Here, $b_v(k)$ is the linear velocity bias. In the peak model for instance, it is $b_v(k) = 1 - R_v^2k^2$ where

$$R_v^2 = \frac{\sigma_0^2}{\sigma_1^2} \tag{2.48}$$

is a characteristic scale that is proportional to the Lagrangian radius of a halo, and $\sigma_j^2 \equiv \langle k^{2j}\rangle$ are moments of the matter power spectrum (smoothed on the Lagrangian halo scale). The k^2 dependence arises from requiring invariance under rotations. In fact, the linear peak velocities can be thought of as arising from the continuous, local relation [9,28]

$$\vec{v}_{\rm h}(\vec{x},\tau_i) = \vec{v}(\vec{x},\tau_i) - R_v^2\vec{\nabla}\delta(\vec{x},\tau_i). \tag{2.49}$$

This development could be generalized to include all the higher-order terms consistent with the symmetry of the problem. However, since we are mainly interested in the first-order scale-dependent corrections that dominate at relatively small k, we will postpone such a study to future work. Equation (2.47) reflects the fact that, in the peak approach, the linear velocity bias remains constant throughout time [9]. By contrast, the fluid approximation to halos and dark matter predicts that any initial velocity bias should decay rapidly to unity [11]. Given that this discrepancy has not been resolved yet (see, however, Ref. [32]), we will assume that the velocity bias remains constant in order to write down expressions more general than those obtained within the fluid approximation.

At the first order, Eq. (2.7) generalizes easily to

$$\delta^{(1)}_{\rm h}(\vec{x},\tau) \simeq (1 + b^{\rm L}_{10}(\tau))\delta^{(1)}(\vec{x},\tau) + (R_v^2 - b^{\rm L}_{01}(\tau))\vec{\nabla}^2\delta^{(1)}(\vec{x},\tau), \tag{2.50}$$

which follows from the linearized continuity equation $\delta^{(1)} \propto -\vec{\nabla}\cdot\vec{v}^{(1)}$. This result reproduces the findings of [9], who explicitly took into account the peak constraint. Note also that $b^{\rm L}_{01}(\tau) = b^{\rm L}_{01}(\tau_i)(a(\tau_i)/a(\tau))$. As can be seen, the amplitude of the contribution proportional to k^2 scales as $R_v^2 - b^{\rm L}_{01}(\tau)$, which is generally nonzero. Therefore, we shall expect this k^2 dependence to appear at sufficiently small scales in the halo bias.

At the second order, the halo overabundance $\delta_{\rm h}(\vec{x},\tau)$ with scale-dependent spatial and velocity bias can be computed by combining the results of Ref. [9] (derived in the Zel'dovich approximation) with those of [10] (derived at higher order in Lagrangian PT). The second-order halo overdensity takes the form

$$\delta_{\rm h}(\vec{x},\tau) = \frac{1}{2}\int\frac{d^3q_1}{(2\pi)^3}\int\frac{d^3q_2}{(2\pi)^3}\left\{\mathcal{F}^{\rm ZA}_2(\vec{q}_1,\vec{q}_2) + \frac{a(\tau_i)}{a(\tau)}[\mathcal{F}^{\rm ZA}_1(\vec{q}_1)c_1(q_2,\tau_i) + \mathcal{F}^{\rm ZA}_1(\vec{q}_2)c_1(q_1,\tau_i)] + \frac{a^2(\tau_i)}{a^2(\tau)}c_2(\vec{q}_1,\vec{q}_2,\tau_i) + \frac{2}{7} - \frac{3}{7}\left(\mu^2 - \frac{1}{3}\right)\right\}\delta^{(1)}(\vec{q}_1,\tau)\delta^{(1)}(\vec{q}_2,\tau)\delta_{\rm D}(\vec{k}-\vec{q}_1-\vec{q}_2), \tag{2.51}$$

where we have momentarily set $\delta^{(1)} \equiv \nabla^2\phi^{(1)}$ so that it resembles the well-known PT expression. The various contributions can be reduced to a combination of local and nonlocal quantities analogously to the calculation performed above. For instance, $(1/2)(a(\tau_i)/a(\tau))[\mathcal{F}^{\rm ZA}_1(\vec{q}_1)c_1(q_2,\tau_i) + 1\leftrightarrow 2]$ includes a term proportional to $(1/2)b^{\rm L}_{01}(\tau)[q_2^2\frac{\vec{k}\cdot\hat{\mathbf{q}}_1}{q_1} + 1\leftrightarrow 2]$ which, after some manipulations, becomes

$$b^{\rm L}_{01}(\tau)\vec{\nabla}(\nabla^2\delta^{(1)}\vec{\nabla}^{-1}\delta^{(1)}) = -b^{\rm L}_{01}(\tau)\nabla^2\delta^{(2)}(\vec{k},\tau) + \frac{9}{7}b^{\rm L}_{01}(\tau)\delta^{(1)}\nabla^2\delta^{(1)} + \frac{18}{7}b^{\rm L}_{01}(\tau)(\vec{\nabla}\delta^{(1)})^2 + \frac{6}{7}b^{\rm L}_{01}(\tau)\left(\frac{2}{3\mathcal{H}^2}\partial_i\partial_j\Phi - \frac{1}{3}\delta_{ij}\delta\right)\left(\partial_i\partial_j\delta - \frac{1}{3}\delta_{ij}\nabla^2\delta\right) + \frac{4}{7}b^{\rm L}_{01}(\tau)\left(\frac{2}{3\mathcal{H}^2}\partial_i\partial_j\partial_k\Phi - \frac{1}{3}\delta_{ij}\partial_k\delta\right)^2. \tag{2.52}$$

We have restored the factors of $3/(2\mathcal{H}^2)$ to be consistent with the units employed throughout the paper. The first term in the right-hand side adds to $-b_{01}^{\rm L}\nabla^2\delta^{(1)}$ in Eq. (2.50) to yield $-b_{01}^{\rm L}\nabla^2\delta$, where δ is the mass density fluctuations at second order. The following two terms contribute to the Eulerian biases b_{11} and b_{02}. The last two terms are new nonlocal bias contributions, which are however heavily suppressed relative to $s^2(\vec{x},\tau)$ as they carry additional powers of $\vec{k}$. The remaining terms in the right-hand side of Eq. (2.51) can be written analogously. For the purpose of the present work however, Eq. (2.50) and its k-dependent contribution at linear order are sufficient. Therefore, we generalize Eq. (2.42) to

$$\begin{aligned}\delta_{\rm h}(\vec{x},\tau) &= b_{10}\delta(\vec{x},\tau) - b_{01}\nabla^2\delta(\vec{x},\tau) + \frac{1}{2!}b_{20}\delta^2(\vec{x},\tau)\\ &\quad + \frac{1}{2}b_{s^2}s^2(\vec{x},\tau) + b_\psi\psi(\vec{x},\tau)\\ &\quad + b_{st}s(\vec{x},\tau)\cdot t(\vec{x},\tau) + \cdots,\end{aligned} \tag{2.53}$$

where

$$\begin{aligned} b_{10} &= 1 + b_{10}^{\rm L}, \qquad b_{01} = -R_\nu^2 + b_{01}^{\rm L},\\ b_{20} &= b_{20}^{\rm L} + \frac{8}{21}b_{10}^{\rm L}, \qquad b_{s^2} = -\frac{4}{7}b_{10}^{\rm L},\\ b_\psi &= -\frac{1}{2}b_{10}^{\rm L}, \qquad b_{st} = -\frac{5}{7}b_{10}^{\rm L}.\end{aligned} \tag{2.54}$$

This is the model we will consider hereafter. As we will see shortly, a scale-dependent bias at linear order appears necessary to explain recent numerical measurements of halo bias with massive neutrinos.

III. HALO BIAS IN THE PRESENCE OF MASSIVE NEUTRINOS

References [22–24] investigated the impact of massive neutrinos on the spatial distribution of dark matter halos and galaxies using large box-size N-body simulations that incorporate massive neutrinos as an extra set of particles. They found that massive neutrinos generate an additional scale-dependence in the halo power spectrum on mildly nonlinear scales, in agreement with previous theoretical predictions [33]. In this section, we will compare the scale dependence induced by massive neutrinos with that generated by gravitational mode coupling and Lagrangian halo bias. We will show that the latter is substantially steeper, so that it should be relatively easy to isolate the contribution of massive neutrinos from a measurement of the halo power spectrum.

A. Perturbative approach

To model the impact of massive neutrinos on the clustering of dark matter halos, we follow [33–36] and assume that the latter trace the cold dark matter (CDM) plus baryons fluctuation field, with a linear growth rate suppressed in a scale-dependent way by the massive neutrinos. This approximation is motivated by the smallness of the neutrino masses we consider. For $\sum m_\nu < 0.6$ eV, the neutrino free-streaming scale is sufficiently large that the neutrino perturbations remain in the linear regime up to late time. It has been shown to work well in Refs. [22–24]. We thus write the total dark matter perturbation as a weighted sum of cold dark matter (we will ignore the effect of baryons in what follows, except for the fact that our CDM transfer function is a weighted sum of the baryons + CDM transfer functions) and neutrino fluctuations,

$$\delta_{\rm m} = (1-f_\nu)\delta_{\rm c} + f_\nu\delta_\nu, \tag{3.1}$$

where the neutrino overdensity δ_ν is in the linear regime, and the neutrino fraction f_ν is

$$f_\nu = \frac{\Omega_\nu}{\Omega_c + \Omega_\nu}. \tag{3.2}$$

Replacing $\delta(\vec{x},\tau)$ by $\delta_c(\vec{x},\tau)$ in the right-hand side of Eq. (2.53), the halo-mass cross-power spectrum reads

$$P_{\rm hm}(k) = (b_{10} + b_{01}k^2)P_{\rm cm}^{\rm NL}(k) + \Delta P_{\rm hm}(k) + P_{\rm cc}(k)I_3(k), \tag{3.3}$$

where

$$P_{\rm cm}^{\rm NL} = \frac{P_{\rm mm}^{\rm NL} - f_\nu P_{\nu\rm m}}{1-f_\nu} = \frac{P_{\rm mm}^{\rm NL} - f_\nu(1-f_\nu)P_{\rm c\nu} - f_\nu^2 P_{\nu\nu}}{1-f_\nu}, \tag{3.4}$$

and

$$\begin{aligned}\Delta P_{\rm hm}(k) &= (1-f_\nu)b_{20}\int\frac{{\rm d}^3q}{(2\pi)^3}P_{\rm cc}(q)P_{\rm cc}(|\vec{k}-\vec{q}|)F_2(\vec{q},\vec{k}-\vec{q})\\ &\quad + (1-f_\nu)b_{s^2}\int\frac{{\rm d}^3q}{(2\pi)^3}P_{\rm cc}(q)P_{\rm cc}(|\vec{k}-\vec{q}|)F_2(\vec{q},\vec{k}-\vec{q})S(\vec{q},\vec{k}-\vec{q}).\end{aligned} \tag{3.5}$$

Here, $P_{\rm cc}$, $P_{\nu\nu}$ and $P_{\rm c\nu}$ are the linear CDM, neutrinos power spectrum and the CDM-neutrinos cross-power spectrum, respectively. We have adopted the notation of Refs. [8,37] for our definition of

$$I_3(k) = \frac{32}{105}(1-f_\nu)\left(b_{st} - \frac{5}{2}b_{s^2} + \frac{16}{21}b_\psi\right) \times \int \mathrm{d}\ln r \Delta^2_{\mathrm{cc}}(kr) I_R(r), \quad (3.6)$$

being

$$I_R(r) = I(r) + \frac{5}{6},$$
$$I(r) = \frac{105}{32}\int_{-1}^{1} \mathrm{d}\mu D_2(\vec{q},\vec{k}) S(\vec{q},\vec{k}-\vec{q}) \quad (3.7)$$

and

$$F_2(\vec{q}_1,\vec{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\vec{q}_1\cdot\vec{q}_2}{q_1q_2}\left(\frac{q_1}{q_2}+\frac{q_2}{q_1}\right) + \frac{2}{7}\left(\frac{\vec{q}_1\cdot\vec{q}_2}{q_1q_2}\right)^2,$$
$$S(\vec{q}_1,\vec{q}_2) = \left(\frac{\vec{q}_1\cdot\vec{q}_2}{q_1q_2}\right)^2 - \frac{1}{3},$$
$$D_2(\vec{q}_1,\vec{q}_2) = \frac{2}{7}\left[S(\vec{q}_1,\vec{q}_2) - \frac{2}{3}\right]. \quad (3.8)$$

The latter are the second-order perturbative kernels. Note that, below the neutrino free-streaming scale, the cross-power spectrum $P^{\mathrm{NL}}_{\mathrm{cm}}$ is enhanced by a factor of $(1-f_\nu)^{-1}$ relative to $P^{\mathrm{NL}}_{\mathrm{mm}}$. Following Ref. [8], one should interpret b_{st} etc. as "renormalized" bias parameters. Equation (2.43) leads to the following relation:

$$\frac{32}{105}\left(b_{st} - \frac{5}{2}b_{s^2} + \frac{16}{21}b_\psi\right) = \frac{32}{315}b^{\mathrm{L}}_{10} \equiv b_{\mathrm{NL}}. \quad (3.9)$$

Further details regarding the evaluation of these integrals can be found in the Appendix.

The nonlinear mass power spectrum $P^{\mathrm{NL}}_{\mathrm{mm}}$ and the linear P_{cc}, $P_{\mathrm{c}\nu}$ and $P_{\nu\nu}$, Eqs. (3.3), (3.4) and (3.5), in the presence of massive neutrinos are obtained from the CLASS code [38] (see also Ref. [39] for an overview). We refer the reader to [40] for details about this implementation.

B. Bias parameters

To evaluate the halo-matter power spectrum Eq. (3.3), we need predictions for the values of the bias parameters b^{L}_{10}, b^{L}_{01}, b^{L}_{20} together with the scale R_v that quantifies the magnitude of the k-dependent velocity bias.

For this purpose, we compute all the Lagrangian bias factors using the excursion set peak (ESP) mass function [41,42], which has been shown to agree well with simulated halo mass functions constructed with a spherical overdensity (SO) criterion [43–45]. Following [24,35], we replace the average mass density ρ_{m} by ρ_{cdm}, and the variance of mass fluctuations σ_{m} by σ_{cc} in the ESP halo mass function. While we refer the reader to the aforementioned references for details, it is worth stressing the following points:

(1) The bias parameters depend both on redshift and halo mass. To follow the analysis of [22–24] as closely as possible, we average the Lagrangian bias factors over a suitable range of halo mass,

$$b^{\mathrm{L}}_{ij} = \frac{\int_{M_{\min}}^{M_{\max}} b^{\mathrm{L}}_{ij}(M) n_{\mathrm{ESP}}(M)\mathrm{d}M}{\int_{M_{\min}}^{M_{\max}} n_{\mathrm{ESP}}(M)\mathrm{d}M}, \quad (3.10)$$

where n_{ESP} is the ESP halo mass function. The mass range is $2\times 10^{13}h^{-1}M_\odot < M < 3\times 10^{15}h^{-1}M_\odot$. In the following Table I some typical values for the bias parameters for different neutrino masses are given.

(2) Here and henceforth, we will use the word "local" when we refer to the simplest bias prescription without any scale dependence. In this case, we set $b_{s^2} = b_\psi = b_{st} = b_{01} = 0$. Conversely, our scale-dependent bias prescription is summarized by Eq. (2.54) with R_v computed from Eq. (2.48). We only retain the nonlocal peak bias b^{L}_{01} since (i) it is the only nonlocal Lagrangian term for which we have the time evolution and (ii) several second-order peak bias factors induce k^2 corrections (see e.g. [9,10]) that are at least partly degenerate with $(b^{\mathrm{L}}_{01} - R_v^2)k^2$ over the range of wave numbers considered here.

(3) The scale dependencies induced by the peak constraint also propagate to b_{s^2}, b_ψ, b_{st} etc. for which the Lagrangian to Eulerian mapping can be derived upon analyzing terms like Eq. (2.52). Since b_{s^2} etc. are small however, we do also not expect significant corrections on the scales of interest.

To compare our predictions with the numerical results of [22–24], we use a set of $(\Omega_{\mathrm{m}}, \Omega_\nu)$ such that the total mass density $\Omega_{\mathrm{m}} = \Omega_{\mathrm{c}} + \Omega_\nu$ is held fixed to 0.2708 while the CDM and neutrino density are varied. The sum of neutrino masses is $\sum m_\nu = 0$, 0.1, 0.2, 0.3 and 0.6 eV, such that Ω_ν varies between 0 and 0.0131. We have also adopted

TABLE I. Lagrangian bias factors as computed from the ESP mass function for the cosmological models considered here (they are labeled according to the sum of neutrino masses). The bias parameters, defined relative to the linear density field extrapolated at $z=0$, are weighted over the mass range $2\times 10^{13}h^{-1}M_\odot < M < 3\times 10^{15}h^{-1}M_\odot$.

$\sum_i m_{\nu_i}$ (eV)	b^{L}_{10}	b^{L}_{01}	R_v^2	b^{L}_{20}
0	0.68	12.32	10.41	−0.32
0.1	0.73	12.43	10.38	−0.28
0.2	0.80	12.55	10.33	−0.22
0.3	0.87	12.68	10.29	−0.15
0.6	1.15	13.13	10.17	0.24

$h = 0.7$ for the Hubble rate, $n_s = 1$ for the scalar spectral index, and $A_s = 2.43 \times 10^{-9}$ for the amplitude of primordial scalar perturbations. The normalization amplitude σ_8 changes in accordance with the sum of neutrino masses [46].

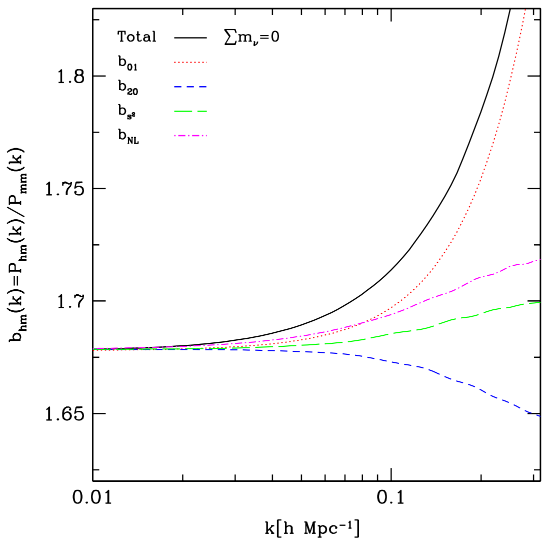

FIG. 1 (color online). Halo bias at redshift $z = 0$ as a function of wave number in the case $\sum m_\nu = 0$. The different terms that contribute to the scale-dependence bias in Eq. (3.11) are labeled according to the bias parameter they are proportional to (see text). The solid black curve represents the sum of all the contributions, Eq. (3.11). All the bias factors have been computed consistently from the ESP halo mass function and the relations Eq. (2.54).

C. Results

The quantity we focus on is the halo bias defined as the ratio of the halo-matter cross-power spectrum to the matter auto-power spectrum,

$$b_{\rm hm} = \frac{P_{\rm hm}}{P_{\rm mm}^{\rm NL}} = \frac{(b_{10} + b_{01}k^2)P_{\rm cm}^{\rm NL}(k) + \Delta P_{\rm hm}(k) + P_{\rm cc}(k)I_3(k)}{P_{\rm mm}^{\rm NL}(k)}. \tag{3.11}$$

It is expected to be constant (matching the value of $1 + b_{10}^{\rm L}$) on large scales, with a scale dependence arising at smaller scales. The origin of this scale dependence is twofold: the nonlocality discussed in Sec. II and the suppression of the linear growth rate induced by the presence of massive neutrinos. Therefore, we begin by exploring the contributions generated by the nonlocality arising from gravity and the peak constraint. Figure 1 displays the various scale-dependent contributions to the halo bias Eq. (3.11) when the neutrinos are massless, i.e. $\sum m_\nu = 0$. We have labeled the curves according to the bias parameters that weight each scale-dependent contribution. Namely, b_{20} and b_{s^2} denote the two terms of Eq. (3.5) while $b_{\rm NL}$, defined in Eq. (3.9), indicates the term proportional to $I_3(k)$ in Eq. (3.6). Note that $P_{\rm cm}^{\rm NL}(k) = P_{\rm mm}^{\rm NL}(k)$ in this case. Clearly, the dominant contribution arises from the peak constraint through the curvature term $(b_{01}^{\rm L} - R_v^2)k^2$, which is of positive sign for the mass considered here (see Table I). It is only partly compensated by the other ones, which are negative or close to zero.

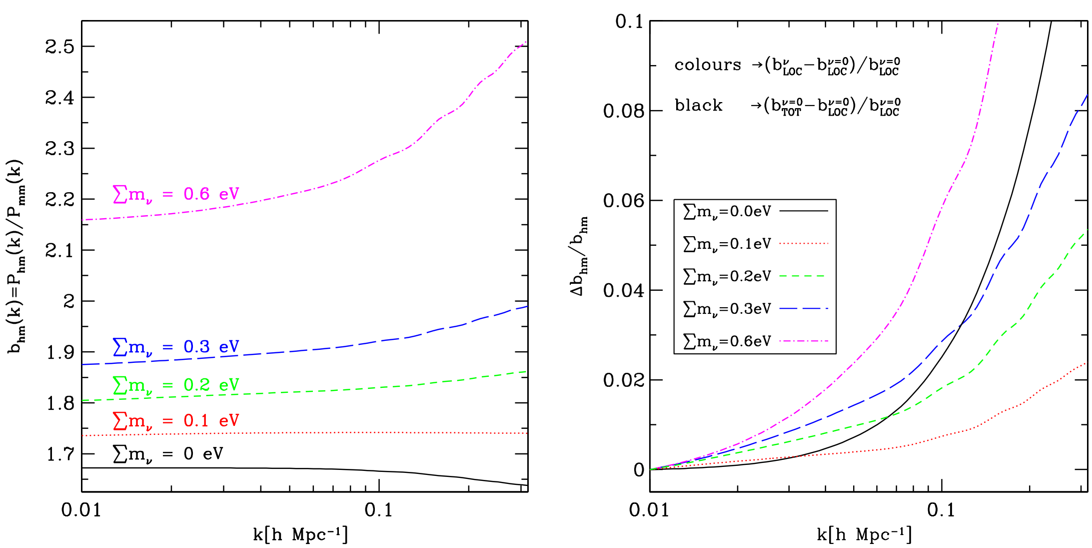

FIG. 2 (color online). Halo bias (left) and fractional scale dependence (right) at $z = 0$ as a function of wave number for values of $\sum_i m_{\nu_i} = 0$, 0.1, 0.2, 0.3 and 0.6 eV. In the left panel, only the local bias terms are included in the predictions. In the right panel, the models with nonzero neutrino masses still assume local bias, whereas the solid (black) curve represents the case in which massive neutrinos are absent but all nonlocal terms are accounted for. The nonlocal bias contributions induced by gravity and by the peak constraint generate a sharp rise beyond $k \sim 0.1h\,{\rm Mpc}^{-1}$ substantially steeper than the effect of nonzero neutrino mass.

The left panel of Fig. 2 displays the scale dependence of the halo bias at $z = 0$ as a function of neutrino masses. For the sake of illustration, we have assumed the simplest local bias model specified above, in which only $b_{10} = 1 + b_{10}^{\rm L}$ and $b_{20} = b_{20}^{\rm L} + 8/21 b_{10}^{\rm L}$ are different from zero. In the right panel of Fig. 2, we show the scale dependence relative to the $\sum m_\nu = 0$ case. This should be compared to the solid (black) curve, which represents the scale dependence obtained when massive neutrinos are absent but all the nonlocal terms are included. These terms generate a sharp rise beyond $k \sim 0.1 h\,{\rm Mpc}^{-1}$ which is much steeper than the effect of varying the sum of neutrino masses. Therefore, a measurement of the scale dependence of bias over a range of wave numbers should help disentangle the contribution of massive neutrinos from that induced by the nonlocal terms.

In Figs. 3 and 4, we compare our predictions for the halo bias at $z = 0$ and 0.5 with the N-body measurements of [22]. In both figures, the left panel displays the case of a neutrino sum of 0.3 eV. The dashed (blue) curve is the prediction in the simplest local bias model, which falls short of explaining the rise on scales $k \gtrsim 0.1 h\,{\rm Mpc}^{-1}$. Our full nonlocal prediction shown as the solid (red) curve agrees with the numerical data within 3% for $k \lesssim 0.3 h\,{\rm Mpc}^{-1}$. Most of the difference with the local bias prediction arises from including the peak constraint

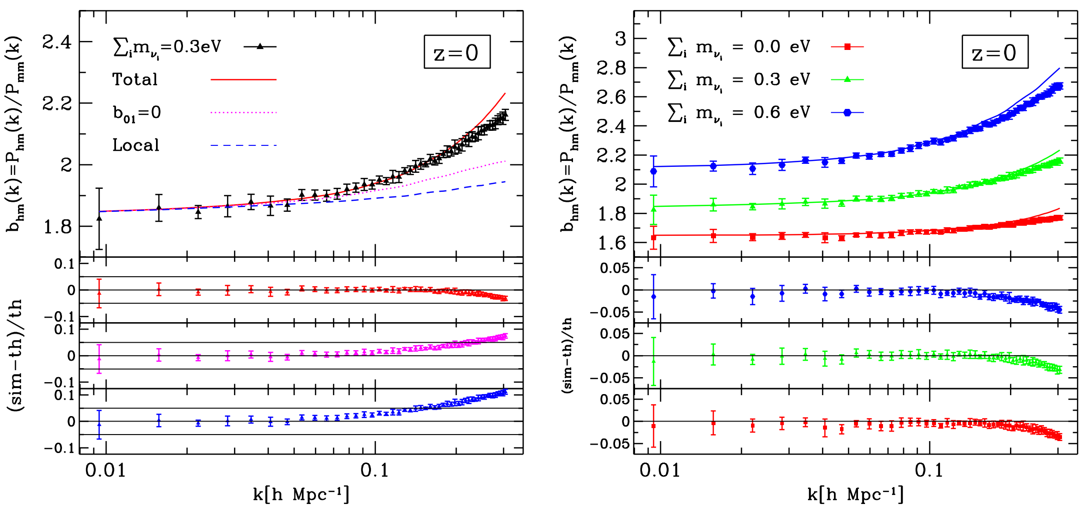

FIG. 3 (color online). Halo bias at $z = 0$ as a function of wave number. Data points are from [22]. In the upper left panel, we show for $\sum_i m_{\nu_i} = 0.3$ eV the local bias prediction as the dashed (blue) curve, and our full nonlocal model as the solid (red) curve. The difference between the solid (red) and the dotted (magenta) curve represents the effect of turning off the contribution $b_{01}k^2$ arising from the peak constraint. In the upper right panel, we compare our nonlocal prediction with the numerical data for $\sum_i m_{\nu_i} = 0$, 0.3 and 0.6 eV. The lower panels show the fractional deviation between theory and simulations. Note that these predictions have no free parameter.

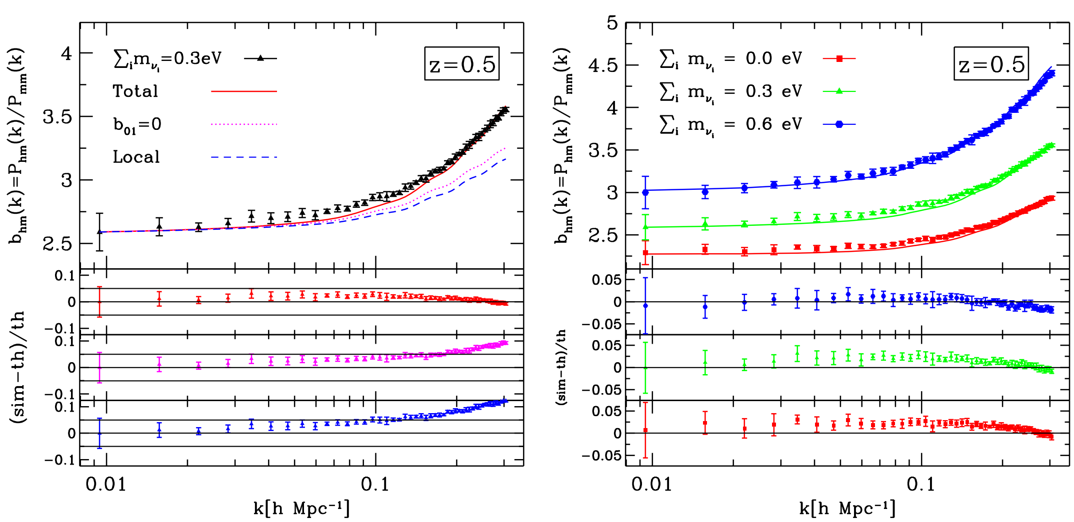

FIG. 4 (color online). Same as Fig. 3 but at redshift $z = 0.5$.

through the contribution $b_{01}k^2$, which is the difference between the solid (red) and dotted (magenta) curve. The right panel compares our full nonlocal model with the numerical results for $\sum_i m_{\nu_i} = 0$, 0.3 and 0.6 eV. In all cases the agreement is at the ∼3% level down to $k = 0.3h\,\mathrm{Mpc}^{-1}$. We emphasize again that our theoretical predictions have no free parameter: all the bias factors are consistently determined from the ESP halo mass function and from the relations Eq. (2.54). However, the fact that the magnitude of the k^2 correction is consistent with $b_{01}^{\mathrm{L}} - R_v^2$ may be coincidental as we expect similar contributions from higher-order Lagrangian bias. Since we do not have as yet Eulerian expressions for these additional bias contributions, we defer a thorough discussion of this issue to future work.

IV. CONCLUSION

In light of the expected accuracy of upcoming galaxy surveys, it is of great importance to characterize as well as possible the bias between the matter distribution and the luminous tracers. The relation between the halo overdensity and the underlying DM fluctuations is not local owing to the nonlinear gravitational evolution and to the nature of the initial overdensities which halos collapse from. In the first part of this paper, we have taken another step towards this goal by computing the nonlocal terms induced by gravity at third order in perturbation theory. We have computed the coefficients of the nonlocal operators by expanding the halo density contrast in terms of the DM quantities. We have generalized our expressions to include a nonlocal, linear term induced by a peak constraint in the initial conditions. In the second part of the paper, we have applied our results to model the scale dependence of bias that arises in cosmologies with nonzero neutrino masses. For the range of halo mass considered here ($M \sim 10^{13}h^{-1}M_\odot$), the various nonlocal bias terms conspire to create a steep rise beyond $k \sim 0.1h\,\mathrm{Mpc}^{-1}$ which is quite distinct from the gradual scale dependence generated by the massive neutrinos. We have shown that the inclusion of the nonlocal bias terms, especially the linear k^2 correction induced by e.g. a peak constraint in the initial conditions, is crucial for reproducing N-body data if the Eulerian bias factors satisfy the relations Eq. (2.54). Using the ESP halo mass function, we have been able to fit the N-body measurement of [22] to within ∼3% up to $k \sim 0.3h\,\mathrm{Mpc}^{-1}$ without any free parameter. One could envisage further improvements to our computation, e.g. the inclusion of additional Lagrangian bias parameters, to extend the agreement to higher wave numbers. Clearly however, a more detailed comparison including e.g. several halo mass bins or bispectrum measurements is in order to test the validity of this approach.

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Note added.—On a final note, we recently became aware of a similar work by Saito *et al.* [47]. Our findings agree with theirs wherever there is overlap [basically Eq. (2.43)]. In contrast to our approach, Ref. [47] does not include k^2 bias. However, they treat the amplitude of b_{NL} as a free parameter and include additional constraints from the bispectrum, so there is no real contradiction. Furthermore, their best-fit values of b_{NL} are somewhat larger than the expectation $(32/315)b_{10}^{\mathrm{L}}$ for $b_{10} \lesssim 2$, a discrepancy which could be partly resolved with the inclusion of k^2 bias.

APPENDIX: THE HALO-MASS CORRELATOR

The typical integral we need to calculate is

$$\mathcal{I}_{ij}(k) = \int \frac{\mathrm{d}^3q}{(2\pi)^3} P_i(q) P_j(|\vec{k}-\vec{q}|) F_2(\vec{q}, \vec{k}-\vec{q}), \quad \text{(A1)}$$

where

$$F_2(\vec{q}_1, \vec{q}_2) = \frac{5}{7} + \frac{1}{2}\frac{\vec{q}_1\cdot\vec{q}_2}{q_1 q_2}\left(\frac{q_1}{q_2} + \frac{q_2}{q_1}\right) + \frac{2}{7}\left(\frac{\vec{q}_1\cdot\vec{q}_2}{q_1 q_2}\right)^2. \quad \text{(A2)}$$

Furthermore

$$S(\vec{q}_1, \vec{q}_2) = \left(\frac{\vec{q}_1\cdot\vec{q}_2}{q_1 q_2}\right)^2 - \frac{1}{3}. \quad \text{(A3)}$$

We now define the following variables

$$\mu = \frac{\vec{k}\cdot\vec{q}}{kq}, \qquad r = \frac{q}{k}, \quad \text{(A4)}$$

such that

$$\begin{aligned}
\int \frac{\mathrm{d}^3 q}{(2\pi)^3} &= \frac{k^3}{(2\pi)^2}\int_0^\infty r^2 \mathrm{d}r \int_{-1}^{1}\mathrm{d}\mu,\\
\eta k \equiv |\vec{k}-\vec{q}| &= \sqrt{q^2+k^2-2\vec{k}\cdot\vec{q}} = k\sqrt{r^2+1-2r\mu},\\
F_2(\vec{q},\vec{k}-\vec{q}) &= \frac{5}{7}+\frac{1}{2}[(\vec{k}\cdot\vec{q})-q^2]\left(\frac{1}{q^2}+\frac{1}{k^2\eta^2}\right)+\frac{2}{7}\frac{[(\vec{k}\cdot\vec{q})-q^2]^2}{q^2k^2\eta^2}\\
&= \frac{5}{7}+\frac{1}{2}(\mu r-r^2)\frac{2r^2+1-2r\mu}{r^2\eta^2}+\frac{2}{7}\frac{(\mu r-r^2)^2}{r^2\eta^2}\\
&= \frac{3r+7\mu-10\mu^2 r}{14r(1+r^2-2\mu r)}\\
S(\vec{q},\vec{k}-\vec{q}) &= \frac{(\mu r-r^2)^2}{r^2\eta^2}-\frac{1}{3}.
\end{aligned} \tag{A5}$$

We finally obtain

$$\mathcal{I}_{ij}(k) = \frac{k^3}{56\pi^2}\int_0^\infty \mathrm{d}r\int_{-1}^{1}\mathrm{d}\mu P_i(kr)P_j\left(k\sqrt{r^2+1-2r\mu}\right)\frac{r(3r+7\mu-10\mu^2 r)}{(1+r^2-2\mu r)}. \tag{A6}$$

Notice that, in order to avoid the IR divergence when $|\vec{k}-\vec{q}|$ goes to zero, by exploring the symmetry of the integral, we can rewrite it as

$$\begin{aligned}
\mathcal{I}_{ij}(k) &= \int\frac{\mathrm{d}^3 q}{(2\pi)^3}P_i(q)P_j(|\vec{k}-\vec{q}|)F_2(\vec{q},\vec{k}-\vec{q})\Theta(|\vec{k}-\vec{q}|-q)+(i\leftrightarrow j)\\
&= \frac{k^3}{56\pi^2}\int_0^\infty \mathrm{d}r\int_{-1}^{1}\mathrm{d}\mu P_i(kr)P_j\left(k\sqrt{r^2+1-2r\mu}\right)\frac{r(3r+7\mu-10\mu^2 r)}{(1+r^2-2\mu r)}\Theta(1-2r\mu)+(i\leftrightarrow j),
\end{aligned} \tag{A7}$$

where $\Theta(x)$ is the Heaviside step function. Similarly

$$\begin{aligned}
\int\frac{\mathrm{d}^3 q}{(2\pi)^3}P_i(q)P_j(|\vec{k}-\vec{q}|)F_2(\vec{q},\vec{k}-\vec{q})S(\vec{q},\vec{k}-\vec{q}) &= \frac{k^3}{168\pi^2}\int_0^\infty \mathrm{d}r\int_{-1}^{1}\mathrm{d}\mu P_i(kr)P_j\left(k\sqrt{r^2+1-2r\mu}\right)\\
&\quad\times\frac{(3r+7\mu-10\mu^2 r)(1-3\mu^2+4\mu r-2r^2)}{r(1+r^2-2\mu r)^2}.
\end{aligned} \tag{A8}$$

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