

Finite temperature Casimir interaction between spheres in $(D + 1)$ -dimensional spacetime: Exact computations and asymptotic expansions

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We consider the finite temperature Casimir interaction between two Dirichlet spheres in $(D + 1)$ -dimensional Minkowski spacetime. The Casimir interaction free energy is derived from the zero temperature Casimir interaction energy via the Matsubara formalism. In the high temperature region, the Casimir interaction is dominated by the term with zero Matsubara frequency, and it is known as the classical term since this term is independent of the Planck constant \hbar . Explicit expression of the classical term is derived and it is computed exactly using appropriate similarity transforms of matrices. We then compute the small separation asymptotic expansion of this classical term up to the next-to-leading order term. For the remaining part of the finite temperature Casimir interaction with nonzero Matsubara frequencies, we obtain its small separation asymptotic behavior by applying certain prescriptions to the corresponding asymptotic expansion at zero temperature. This gives us a leading term that is shown to agree precisely with the proximity force approximation at any temperature. The next-to-leading order term at any temperature is also derived and it is expressed as an infinite sum over integrals. To obtain the asymptotic expansion at the low and medium temperature regions, we apply the inverse Mellin transform techniques. In the low temperature region, we obtain results that agree with our previous work on the zero temperature Casimir interaction.

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I. INTRODUCTION

The Casimir effect plays an important role in various areas of physics and mathematics such as quantum field theory, gravitation and cosmology, atomic physics, condensed matter, nanotechnology and mathematical physics (see e.g. [1]). From the perspective of quantum field theory, the Casimir effect is closely related to the one-loop effective action [2]. Since the beginning of the last century, physicists have explored higher-dimensional spacetime models to solve some fundamental problems such as the unification of fundamental forces and the dark energy and cosmological constant problem. This has motivated the study of the Casimir effect in spacetime with arbitrary dimensions [3–14].

Before the turn of this century, the studies of Casimir interaction were concentrated on the parallel plate configuration due to the lack of machineries to compute the Casimir interactions between arbitrary objects. For other configurations, such as the sphere-plate configuration which is a popular experimental setup, proximity force approximation was employed to compute the approximation to the Casimir interaction when the separation between the objects is very small compared to the diameter of the objects. However, the situation changed drastically in the beginning of this century. Various approaches have been

proposed to compute the Casimir interaction between two objects in $(3 + 1)$ -dimensional Minkowski spacetime which can either be classified as the worldline method [15–19], multiple scattering approach [20–34] or mode summation approach [35–38]. Generalizing the prescription in [38] to arbitrary dimensions, we have been able to compute the zero temperature Casimir interaction between a sphere and a plate [39] and between two spheres [40] in $(D + 1)$ -dimensional Minkowski spacetime.

Motivated by the work [41] which shows that the high temperature limit of the Dirichlet Casimir interaction of the sphere-plate and sphere-sphere configurations can be computed exactly, we have computed the high temperature limit of the Dirichlet sphere-plate interaction in [42]. In this work, we generalize both the works [40] and [42] and consider the Casimir interaction between two Dirichlet spheres at finite temperature in $(D + 1)$ -dimensional Minkowski spacetime. In the high temperature limit, we show that the Casimir interaction can also be computed exactly. We then proceed to compute the leading order term and next-to-leading order terms of the small separation asymptotic expansions of the Casimir interaction at any temperature. Explicit formulas were derived for the low temperature, medium temperature and high temperature regions.

We use units where $\hbar = c = k_B = 1$.

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II. THE FINITE TEMPERATURE CASIMIR INTERACTION BETWEEN TWO SPHERES

In this work, we consider the finite temperature Casimir interaction between two spheres with radii R_1 and R_2 respectively. Without loss of generality, we assume that $R_1 \leq R_2$ throughout this article. Let L be the center-to-center distance of the spheres, and let $d = L - R_1 - R_2$ be the distance between the two spheres.

In [40], we showed that when $D \geq 4$, the zero temperature Casimir interaction energy between two Dirichlet spheres with radii R_1 and R_2 can be written as

$$E_{\text{Cas}}^{T=0} = \frac{1}{2\pi} \int_0^\infty d\kappa \sum_{m=0}^\infty \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!} \times \text{Tr} \ln(1 - \mathbb{M}_m(\kappa)), \quad (1)$$

where the elements $M_{m;l,l'}$ of the matrix \mathbb{M}_m are

$$M_{m;l,l'} = T_l^1 \sum_{l''=m}^\infty G_{m;l,l''}^1 T_{l''}^2 G_{m;l'',l'}^2, \quad (2)$$

with

$$T_l^i(\kappa) = \frac{I_{l+\frac{D-2}{2}}(\kappa R_i)}{K_{l+\frac{D-2}{2}}(\kappa R_i)}, \quad (3)$$

and

$$G_{m;l,l'}^1 = G_{m;l',l}^2 = (-1)^{l+m} 2^{2m+D-3} \Gamma\left(m + \frac{D-2}{2}\right)^2 \times \sqrt{\frac{(l+\frac{D-2}{2})(l'+\frac{D-2}{2})(l-m)!(l'-m)!}{(l+m+D-3)!(l'+m+D-3)!}} \times \int_0^\infty d\theta (\sinh \theta)^{2m+D-2} C_{l-m}^{m+\frac{D-2}{2}}(\cosh \theta) \times C_{l'-m}^{m+\frac{D-2}{2}}(\cosh \theta) e^{-\kappa L \cosh \theta}. \quad (4)$$

For fixed m , the trace Tr over $\ln(1 - \mathbb{M}_m(\kappa))$ is

$$\sum_{l=m}^\infty.$$

When $D = 3$, we can also represent the Casimir free interaction energy by (1) provided that the summation $\sum_{m=0}^\infty$ is replaced by the summation $\sum_{m=0}^\infty'$, where the prime $'$ indicates that the term $m = 0$ is summed with weight $1/2$.

Using Matsubara formalism, the finite temperature Casimir free interaction energy between the two spheres can be obtained by replacing κ by the Matsubara frequencies $\kappa_p = 2\pi pT$, and replace the integration over κ by the summation over p . Namely,

$$E_{\text{Cas}} = T \sum_{p=0}^\infty' \sum_{m=0}^\infty \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!} \times \text{Tr} \ln(1 - \mathbb{M}_m(\kappa_p)). \quad (5)$$

Equation (5) can be rewritten as a sum of two terms:

$$E_{\text{Cas}} = E_{\text{Cas}}^{\text{classical}} + E_{\text{Cas}}^{\text{rem}},$$

where the classical Casimir energy $E_{\text{Cas}}^{\text{classical}}$ is given by the $p = 0$ term in (5):

$$E_{\text{Cas}} = \frac{T}{2} \sum_{m=0}^\infty \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!} \times \lim_{\kappa \rightarrow 0} \text{Tr} \ln(1 - \mathbb{M}_m(\kappa)); \quad (6)$$

and the sum of the remaining terms is denoted by $E_{\text{Cas}}^{\text{rem}}$:

$$E_{\text{Cas}}^{\text{rem}} = T \sum_{p=1}^\infty \sum_{m=0}^\infty \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!} \times \text{Tr} \ln(1 - \mathbb{M}_m(\kappa_p)). \quad (7)$$

The classical term of the Casimir interaction (6) is the high temperature (i.e., when $1 \ll dT$) limit of the free energy.

First, we will compute explicitly the classical term. As in [42], we find that as $\kappa \rightarrow 0$,

$$T_l^i(\kappa) \sim \frac{1}{2^{2l+D-3} \Gamma(l+\frac{D-2}{2}) \Gamma(l+\frac{D}{2})} (\kappa R_i)^{2l+D-2}, \quad (8)$$

$$\int_0^\infty d\theta (\sinh \theta)^{2m+D-2} C_{l-m}^{m+\frac{D-2}{2}}(\cosh \theta) C_{l'-m}^{m+\frac{D-2}{2}}(\cosh \theta) e^{-\kappa L \cosh \theta} \sim \frac{1}{\kappa^{l+l'+D-2}} \frac{2^{l+l'-2m}}{(l-m)!(l'-m)!} \frac{\Gamma(l+\frac{D-2}{2}) \Gamma(l'+\frac{D-2}{2}) \Gamma(l+l'+D-2)}{\Gamma(m+\frac{D-2}{2})^2 L^{l+l'+D-2}}. \quad (9)$$

Since the trace of a matrix is not changed if the matrix is replaced by a similar matrix, define

$$\tilde{\mathbb{M}}_m = \mathbb{P}_m^{-1} \mathbb{M}_m \mathbb{P}_m, \quad (10)$$

where \mathbb{P}_m is a diagonal matrix with elements

$$\mathbb{P}_{m;l,l'} = (-1)^l \left(\frac{\kappa L}{2} \right)^l \sqrt{\frac{\Gamma(l+m+D-2)}{(l+\frac{D-2}{2})(l-m)! \Gamma(l+\frac{D-2}{2})}} \frac{1}{\Gamma(l+\frac{D-2}{2})} \delta_{l,l'}. \quad (11)$$

Then

$$\tilde{M}_{m;l,l'}(\kappa) = (-1)^{-l+l'} \left(\frac{\kappa L}{2} \right)^{-l+l'} \sqrt{\frac{(l+\frac{D-2}{2})(l-m)! \Gamma(l'+m+D-2)}{(l'+\frac{D-2}{2})(l'-m)! \Gamma(l+m+D-2)}} \frac{\Gamma(l+\frac{D-2}{2})}{\Gamma(l'+\frac{D-2}{2})} M_{m;l,l'}(\kappa), \quad (12)$$

and one can deduce that

$$\tilde{M}_{m;l,l'}(0) = \sum_{l''=m}^{\infty} \frac{(l+l''+D-3)!(l'+l''+D-3)!}{(l+m+D-3)!(l''-m)!(l''+m+D-3)!(l'-m)!} \left(\frac{R_1}{L} \right)^{2l+D-2} \left(\frac{R_2}{L} \right)^{2l''+D-2}. \quad (13)$$

Let

$$b_i = \frac{R_i}{L}. \quad (14)$$

Then the classical Casimir interaction energy between two spheres in $(D+1)$ -dimensional Minkowski spacetime can be written as

$$E_{\text{Cas}}^{\text{classical}} = \frac{T}{2} \sum_{m=0}^{\infty} \frac{(2m+D-3)(m+D-4)!}{(D-3)!m!} \ln \det(\mathbb{1} - \mathbb{N}_m), \quad (15)$$

where

$$N_{m;l,l'} = \sum_{l''=m}^{\infty} \frac{(l+l''+D-3)!(l'+l''+D-3)!}{(l+m+D-3)!(l''-m)!(l''+m+D-3)!(l'-m)!} b_1^{2l+D-2} b_2^{2l''+D-2}. \quad (16)$$

III. THE PROXIMITY FORCE APPROXIMATION

The proximity force approximation approximates the Casimir interaction force between two objects by summing the local Casimir force density between two planes over the surfaces. In $(D+1)$ -dimensional Minkowski spacetime, the finite temperature Casimir force density between two parallel Dirichlet plates is given by [43]

$$\mathcal{F}_{\text{Cas}}^{\parallel}(d) = -\frac{T}{2^{D-3} \pi^{\frac{D-1}{2}} \Gamma(\frac{D-1}{2})} \sum_{p=0}^{\infty} \int_{\kappa_p}^{\infty} dx (x^2 - \kappa_p^2)^{\frac{D-3}{2}} \frac{x^2}{e^{2dx} - 1}, \quad (17)$$

where d is the distance between the two plates.

As in [40], the proximity force approximation to the finite temperature Casimir interaction force between two Dirichlet spheres in $(D+1)$ -dimensional spacetime is

$$\begin{aligned} F_{\text{Cas}}^{\text{PFA}} &= R_1^{D-1} \int_0^{\pi} d\theta_1 \sin^{D-2} \theta_1 \int_0^{\pi} d\theta_2 \sin^{D-3} \theta_2 \dots \int_0^{\pi} d\theta_{D-2} \sin \theta_{D-2} \int_{-\pi}^{\pi} d\theta_{D-1} \mathcal{F}_{\text{Cas}}^{\parallel}(d(\boldsymbol{\theta})) \\ &= \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} R_1^{D-1} \int_0^{\pi} d\theta_1 \sin^{D-2} \theta_1 \mathcal{F}_{\text{Cas}}^{\parallel}(d(\theta_1)), \end{aligned} \quad (18)$$

where

$$d(\boldsymbol{\theta}) = d(\theta_1) = \sqrt{L^2 - 2R_1L \cos \theta_1 + R_1^2} - R_2 \quad (19)$$

is the distance between a point on the surface of the sphere with radius R_1 to the sphere with radius R_2 . Let $u = d(\theta_1)/d$. Then

$$\begin{aligned} F_{\text{Cas}}^{\text{PFA}} &= \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} R_1^{D-1} \int_1^{\frac{2R_1+d}{d}} du \frac{d(du + R_2)}{R_1L} \left(\frac{d(2R_2 + du + d)(2R_1 + d - du)(2R_1 + 2R_2 + du + d)(u-1)}{(2R_1L)^2} \right)^{\frac{D-3}{2}} \\ &\quad \times \mathcal{F}_{\text{Cas}}^{\parallel}(du) \\ &\sim \frac{(2\pi)^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \left(\frac{dR_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \int_1^{\infty} du (u-1)^{\frac{D-3}{2}} \mathcal{F}_{\text{Cas}}^{\parallel}(du) \\ &= -\frac{T}{2^{\frac{D-3}{2}} \Gamma(\frac{D-1}{2})^2} \left(\frac{dR_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \sum_{p=0}^{\infty} \int_0^{\infty} du u^{\frac{D-3}{2}} \int_{\kappa_p}^{\infty} dx (x^2 - \kappa_p^2)^{\frac{D-3}{2}} \frac{x^2}{e^{2dx(u+1)} - 1} \\ &= -\frac{T}{2^{\frac{D-3}{2}} \Gamma(\frac{D-1}{2})^2} \left(\frac{R_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \sum_{p=0}^{\infty} \int_0^{\infty} du u^{\frac{D-3}{2}} \int_{\kappa_p d}^{\infty} dx (x^2 - \kappa_p^2 d^2)^{\frac{D-3}{2}} \frac{x^2}{e^{2x(u+1)} - 1} \\ &= -\frac{T}{2^{D-3} \Gamma(\frac{D-1}{2})} \left(\frac{R_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \sum_{p=0}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-1}{2}}} \int_{\kappa_p d}^{\infty} dx (x^2 - \kappa_p^2 d^2)^{\frac{D-3}{2}} x^{-\frac{D-5}{2}} e^{-2kx}. \end{aligned} \quad (20)$$

From here, we find that the proximity force approximation to the classical term (i.e., $p = 0$ term) of the Casimir interaction force is

$$\begin{aligned} F_{\text{Cas}}^{\text{classical,PFA}} &= -\frac{T}{2^{D-2} \Gamma(\frac{D-1}{2})} \left(\frac{R_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-1}{2}}} \int_0^{\infty} dx x^{\frac{D-1}{2}} e^{-2kx} \\ &= -\frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}}; \end{aligned} \quad (21)$$

and the proximity force approximation to the zero temperature Casimir interaction force is

$$\begin{aligned} F_{\text{Cas}}^{T=0,\text{PFA}} &= -\frac{1}{2^{D-2} \pi \Gamma(\frac{D-1}{2})} \left(\frac{R_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-1}{2}}} \int_0^{\infty} dk \int_{\kappa d}^{\infty} dx (x^2 - \kappa^2 d^2)^{\frac{D-3}{2}} x^{-\frac{D-5}{2}} e^{-2kx} \\ &= -\frac{\Gamma(\frac{D+3}{2})\zeta(D+1)}{2^{\frac{3D+1}{2}} \sqrt{\pi} \Gamma(\frac{D}{2})} \left(\frac{R_1R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+3}{2}}}. \end{aligned} \quad (22)$$

IV. THE CLASSICAL CASIMIR INTERACTION

In this section, we derive an alternative expression for the classical Casimir interaction energy (15) using similarity transforms of matrices. We then compute the asymptotic behavior of the classical Casimir interaction.

Notice that the matrix \mathbb{N}_m (16) for the sphere-sphere case is more complicated than the corresponding \mathbb{N}_m matrix for the sphere-plate case we obtained in [42]. It can be considered as the multiplication of two different matrices of the type \mathbb{N}_m obtained in [42]. Therefore, the diagonalization of the matrix (16) is also more complicated.

For fixed m , let \mathbb{Q}_m be a lower triangular matrix with elements

$$(\mathbb{Q}_m)_{l,l'} = \begin{cases} \left(\frac{b_1}{b_2}\right)^{-l'} b_1^{2l} \frac{(l-m)!}{(l-l')!(l'-m)!}, & l \geq l' \\ 0, & l < l' \end{cases}. \quad (23)$$

It is easy to check that its inverse matrix is given by

$$(\mathbb{Q}_m^{-1})_{l,l'} = \begin{cases} (-1)^{l-l'} b_1^{-2l'} \left(\frac{b_1}{b_2}\right)^l \frac{(l-m)!}{(l-l')!(l'-m)!}, & l \geq l' \\ 0, & l < l' \end{cases}. \quad (24)$$

It follows that

$$\begin{aligned}
(\mathbb{Q}_m^{-1} \mathbb{N}_m \mathbb{Q}_m)_{l,l'} &= \sum_{l_1=m}^l (-1)^{l-l_1} \left(\frac{b_1}{b_2}\right)^{l-l'} \frac{(l-m)!}{(l-l_1)!(l_1-m)!} \sum_{l_2=m}^{\infty} \sum_{l_3=l'}^{\infty} \frac{(l_1+l_2+D-3)!(l_3+l_2+D-3)!}{(l_1+m+D-3)!(l_2-m)!(l_2+m+D-3)!} \\
&\quad \times b_1^{2l_3+D-2} b_2^{2l_2+D-2} \frac{1}{(l_3-l')!(l'-m)!} \\
&= \sum_{l_2=m}^{\infty} \sum_{l_1=m}^l (-1)^{l-l_1} \left(\frac{b_1}{b_2}\right)^{l-l'} \frac{(l-m)!}{(l-l_1)!(l_1-m)!} \frac{(l_1+l_2+D-3)!(l_2+l'+D-3)!}{(l_1+m+D-3)!(l_2-m)!(l_2+m+D-3)!(l'-m)!} \\
&\quad \times b_1^{2l'+D-2} b_2^{2l_2+D-2} \frac{1}{(1-b_1^2)^{l_2+l'+D-2}}. \tag{25}
\end{aligned}$$

Notice that

$$\begin{aligned}
&\sum_{l_1=m}^l (-1)^{l-l_1} \frac{(l-m)!}{(l-l_1)!(l_1-m)!} \frac{(l_1+l_2+D-3)!}{(l_1+m+D-3)!(l_2-m)!} \\
&= \text{coefficient of } x^{l-l_1} \cdot x^{l_1+m+D-3} \text{ in } (1-x)^{l-m} \cdot \frac{1}{(1-x)^{l_2-m+1}} \\
&= \begin{cases} \frac{(l_2+m+D-3)!}{(l_2-l)!(l+m+D-3)!}, & l_2 \geq l \\ 0, & l_2 < l \end{cases}. \tag{26}
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\mathbb{Q}_m^{-1} \mathbb{N}_m \mathbb{Q}_m)_{l,l'} &= \left(\frac{b_1}{b_2}\right)^{l-l'} \sum_{l_2=l}^{\infty} \frac{(l_2+l'+D-3)!}{(l_2-l)!(l+m+D-3)!(l'-m)!} b_1^{2l'+D-2} b_2^{2l_2+D-2} \frac{1}{(1-b_1^2)^{l_2+l'+D-2}} \\
&= \frac{(l+l'+D-3)!}{(l+m+D-3)!(l'-m)!} \beta^{l'+D-2}, \tag{27}
\end{aligned}$$

where

$$\beta = \frac{b_1 b_2}{1-b_1^2-b_2^2}. \tag{28}$$

Notice that now the matrix $\mathbb{Q}_m^{-1} \mathbb{N}_m \mathbb{Q}_m$ has a form similar to the matrix \mathbb{N}_m in [42] for the sphere-plate case, with β playing the role of $R/(2(R+d))$ in [42], with R the radius of the sphere and d the distance from the sphere to the plane. In fact, in the limit the radius of the second sphere goes to infinity (i.e., the sphere-plane limit), we have

$$\begin{aligned}
\beta &= \frac{R_1 R_2}{L^2 - R_1^2 - R_2^2} \\
&= \frac{R_1 R_2}{(R_1 + R_2 + d)^2 - R_1^2 - R_2^2} \xrightarrow{R_2 \rightarrow \infty} \frac{R_1}{2(R_1 + d)}. \tag{29}
\end{aligned}$$

As in [42], we then find that the classical Casimir interaction energy between two Dirichlet spheres is

$$E_{\text{Cas}}^{\text{classical}} = \frac{T}{2} \sum_{l=0}^{\infty} \frac{(2l+D-2)(l+D-3)!}{(D-2)!l!} \ln(1-y^{2l+D-2}), \tag{30}$$

where $0 < y < 1$ is such that

$$y + y^{-1} = \frac{1}{\beta}. \tag{31}$$

When $D = 3$, we find that

$$E_{\text{Cas}}^{\text{classical}} = \frac{T}{2} \sum_{l=0}^{\infty} (2l+1) \ln(1-y^{2l+1}). \tag{32}$$

Let us compare this to the result obtained in [41] using bispherical coordinates. In [41], it was proved that the classical Casimir interaction energy between two Dirichlet spheres in $(3+1)$ -dimensional Minkowski spacetime is

$$E_{\text{Cas}}^{\text{classical}} = \frac{T}{2} \sum_{l=0}^{\infty} (2l+1) \ln(1-Z^{2l+1}), \tag{33}$$

where

$$Z = \frac{1}{(\lambda_1 + \sqrt{\lambda_1^2 - 1})(\lambda_2 + \sqrt{\lambda_2^2 - 1})}, \tag{34}$$

with

$$\lambda_1 = \frac{1+b_1^2-b_2^2}{2b_1}, \quad \lambda_2 = \frac{1+b_2^2-b_1^2}{2b_2}. \tag{35}$$

Hence,

$$\begin{aligned} Z + Z^{-1} &= (\lambda_1 - \sqrt{\lambda_1^2 - 1})(\lambda_2 - \sqrt{\lambda_2^2 - 1}) + (\lambda_1 + \sqrt{\lambda_1^2 - 1})(\lambda_2 + \sqrt{\lambda_2^2 - 1}) \\ &= 2(\lambda_1\lambda_2 + \sqrt{(\lambda_1^2 - 1)(\lambda_2^2 - 1)}) \\ &= \frac{1 - b_1^2 - b_2^2}{b_1 b_2}. \end{aligned} \quad (36)$$

Since $0 < Z < 1$, Z coincides with the y defined by (31). This shows that the result (32) we obtain here agrees with the result obtained in [41]. But here we only use the similarity transform of matrices, which is much simpler than using bispherical coordinates in [41].

Next we consider the small separation asymptotic expansion. Let

$$\varepsilon = \frac{d}{R_1 + R_2}, \quad a_1 = \frac{R_1}{R_1 + R_2}, \quad a_2 = \frac{R_2}{R_1 + R_2}. \quad (37)$$

Then $a_1 + a_2 = 1$ and

$$\frac{1}{\beta} = \frac{(1 + \varepsilon)^2 - a_1^2 - a_2^2}{a_1 a_2}. \quad (38)$$

Let

$$\begin{aligned} \mu &= -\ln y = \ln \left(\frac{1}{2\beta} + \sqrt{\frac{1}{4\beta^2} - 1} \right) \\ &= \ln \frac{(1 + \varepsilon)^2 - a_1^2 - a_2^2 + \sqrt{(\varepsilon^2 + 2\varepsilon)(2a_1 + \varepsilon)(2a_2 + \varepsilon)}}{2a_1 a_2}. \end{aligned} \quad (39)$$

Then

$$\eta := \frac{\partial \mu}{\partial d} = \frac{2(1 + \varepsilon)}{\sqrt{(2a_1 + \varepsilon)(2a_2 + \varepsilon)(\varepsilon^2 + 2\varepsilon)}} \frac{1}{R_1 + R_2}. \quad (40)$$

On the other hand, we have an expansion of the form (see [42])

$$\frac{(2l + D - 2)(l + D - 3)!}{(D - 2)!!} = \frac{2}{(D - 2)!} \sum_{j=1}^{D-2} x_{D;j} \left(l + \frac{D - 2}{2} \right)^j, \quad (41)$$

where the coefficient $x_{D;j}$ is nonzero only if D and j are both odd or both even.

As in [42], we find that when D is even, the classical Casimir interaction force is given by

$$F_{\text{Cas}}^{\text{classical}} = -\frac{2\eta T}{(D - 2)!} \sum_{j=1}^{D-2} x_{D;j} \left\{ \frac{\Gamma(j + 2)}{2^{j+2} \mu^{j+2}} \zeta(j + 2) + (-1)^j \frac{\Gamma(j + 2)}{2^{j+2} \pi^{j+2}} \zeta(j + 2) + \frac{(-1)^{\frac{j+1}{2}} \pi^{j+2}}{\mu^{j+2}} \sum_{n=1}^{\infty} \frac{n^{j+1}}{e^{\frac{2\pi^2 n}{\mu}} - 1} \right\}; \quad (42)$$

and when D is odd,

$$F_{\text{Cas}}^{\text{classical}} = -\frac{2\eta T}{(D - 2)!} \sum_{j=1}^{D-2} x_{D;j} \left\{ \frac{\Gamma(j + 2)}{2^{j+2} \mu^{j+2}} \zeta(j + 2) + \frac{(-1)^{\frac{j-1}{2}}}{2^{j+2}} \int_0^{\infty} \frac{y^{j+1} \cot \frac{\mu y}{2}}{e^{\pi y} + 1} dy \right\}. \quad (43)$$

Here $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

In principle, one can derive the small separation asymptotic behavior of the classical Casimir interaction force from (42) and (43) up to any order in ε . In the following, we only compute the leading order term and the next-to-leading order term.

As $\varepsilon \ll 1$, one can show that

$$\eta = \frac{1}{R_1 + R_2} \frac{1}{\sqrt{2\varepsilon a_1 a_2}} \left(1 + \frac{3a_1 a_2 - 1}{4a_1 a_2} \varepsilon + \dots \right), \quad (44)$$

$$\mu = \sqrt{\frac{2\varepsilon}{a_1 a_2}} \left(1 + \frac{3a_1 a_2 - 1}{12a_1 a_2} \varepsilon + \dots \right). \quad (45)$$

For the constants $x_{D;j}$,

$$\begin{aligned} x_{D;D-2} &= 1, & x_{D;D-3} &= 0, \\ x_{D;D-4} &= -\frac{(D-2)(D-3)(D-4)}{24}, \dots \end{aligned} \quad (46)$$

On the other hand,

$$\begin{aligned} F_{\text{Cas}}^{\text{classical}} &= -\frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \\ &\times \left\{ 1 - \frac{D-3}{4} \frac{d}{R_1 + R_2} + \left[\frac{D-3}{12} - \frac{(D-3)(D-4)\zeta(D-2)}{3(D-1)\zeta(D)} \right] \left(\frac{d}{R_1} + \frac{d}{R_2} \right) + \dots \right\}. \end{aligned} \quad (49)$$

Notice that the leading term is

$$F_{\text{Cas}}^{\text{classical},0} = -\frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}}, \quad (50)$$

and it agrees with the proximity force approximation (21). When $D = 3$, the next-to-leading order term of the Casimir interaction force is

$$F_{\text{Cas}}^{\text{classical},1} = -\frac{T}{48d}; \quad (51)$$

and when $D \geq 4$,

$$\begin{aligned} F_{\text{Cas}}^{\text{classical},1} &= -\frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \\ &\times \left\{ -\frac{D-3}{4} \frac{d}{R_1 + R_2} + \left[\frac{D-3}{12} - \frac{(D-3)(D-4)\zeta(D-2)}{3(D-1)\zeta(D)} \right] \left(\frac{d}{R_1} + \frac{d}{R_2} \right) \right\}. \end{aligned} \quad (52)$$

In Figs. 1, 2, 3 and 4, we compare the exact classical Casimir interaction force to the two-term asymptotic expansions (48) and (49) when $D = 3, 4, 5$ and 6. Both quantities are normalized by the proximity force approximation.

In Fig. 5, we plot the dependence of the two term asymptotic expansion (49), normalized by the proximity force approximation, on the normalized distance ε and dimension D . It is observed that the correction to the proximity force approximation becomes larger when D is larger. In fact, from (49), we find that when D is large,

$$F_{\text{Cas}}^{\text{classical}} \sim F_{\text{Cas}}^{\text{classical, PFA}} \left\{ 1 - \frac{D}{4} \left(\frac{d}{R_1 + R_2} + \frac{d}{R_1} + \frac{d}{R_2} \right) + \dots \right\}. \quad (53)$$

Hence, in the high temperature region, we find that the next-to-leading order term is proportional to D , which shows a larger deviation from the proximity force approximation when the dimension D becomes larger.

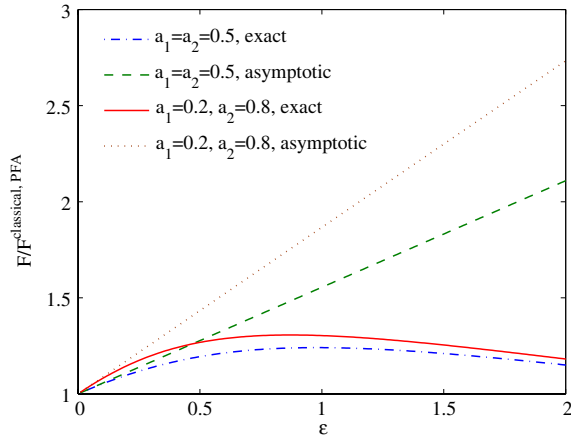


FIG. 1 (color online). The comparison between the exact classical Casimir interaction force with the asymptotic expansion (48) when $D = 3$. Both quantities are normalized by the proximity force approximation.

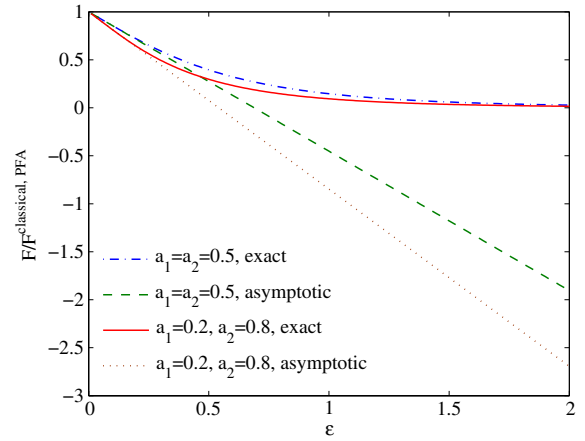


FIG. 4 (color online). The comparison between the exact classical Casimir interaction force with the asymptotic expansion (49) when $D = 6$. Both quantities are normalized by the proximity force approximation.

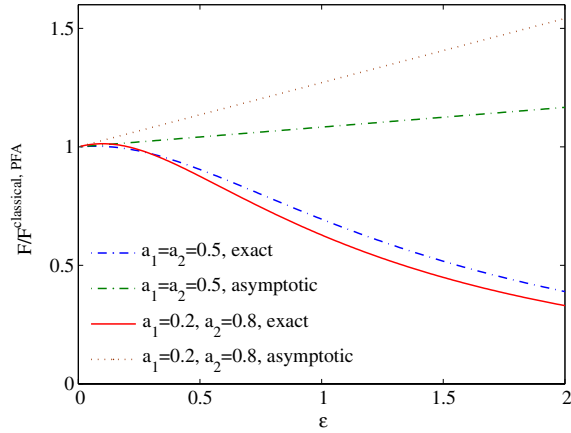


FIG. 2 (color online). The comparison between the exact classical Casimir interaction force with the asymptotic expansion (49) when $D = 4$. Both quantities are normalized by the proximity force approximation.

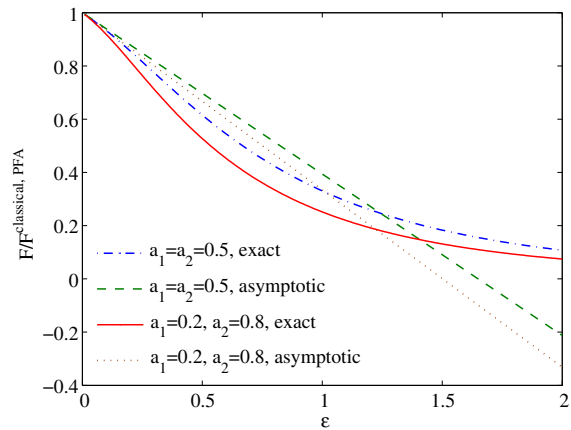


FIG. 3 (color online). The comparison between the exact classical Casimir interaction force with the asymptotic expansion (49) when $D = 5$. Both quantities are normalized by the proximity force approximation.

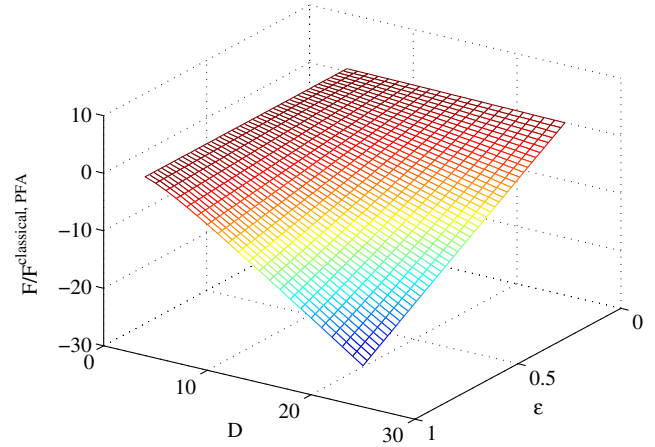


FIG. 5 (color online). Dependence of the asymptotic expansion (49) on ϵ and D for $0 \leq \epsilon \leq 1$ and $6 \leq D \leq 25$. The asymptotic expansion is normalized by the proximity force approximation.

V. SMALL SEPARATION ASYMPTOTIC BEHAVIOR OF THE CASIMIR INTERACTION

The small separation asymptotic expansion of the Casimir interaction is in general not easy to compute. One usually expects that the leading term would coincide with the proximity force approximation. However, the computation of the next-to-leading order term is often tedious, but it captures important information of the Casimir interaction such as the response of the system to the curvature of the surfaces, and it also gives a rough idea of how accurate the leading term is. Using the idea first developed in [29], such computations have been performed for various geometric configurations [29,30,44–52].

In this section, we will compute the small separation asymptotic expansions of the Casimir interaction at any temperature. Since the classical term (6) has been considered in Sec. IV, we first consider the remaining terms (7), and then combine with the results obtained in Sec. IV to obtain the asymptotic expansion of the full interaction.

The small separation asymptotic expansion of the term (7) can be computed in the same way as in [40] for the zero temperature case. In fact, since (7) is obtained from the zero

temperature Casimir interaction energy (1) by replacing κ with κ_p and changing the integration over κ to summation over p , one can use this recipe to obtain the small ε asymptotic expansion of (7) from the small ε asymptotic expansion of the zero temperature Casimir interaction derived in [40].

In [40], we have shown that when $\varepsilon \ll 1$, the leading term of the zero temperature Casimir interaction energy $E_{\text{Cas}}^{T=0,0}$ is given by

$$E_{\text{Cas}}^{T=0,0} = -\frac{a_2^{\frac{D-1}{2}}}{2^{D-1}\pi R_1 \Gamma(\frac{D-1}{2})} \sum_{s=0}^{\infty} \frac{1}{(s+1)^{\frac{D-1}{2}}} \int_0^{\infty} dl l^{\frac{D-1}{2}} \int_0^1 \frac{d\tau \tau^{\frac{D-5}{2}}}{\sqrt{1-\tau^2}} \exp\left(-\frac{2l(s+1)\varepsilon}{a_1\tau}\right); \quad (54)$$

whereas the next-to-leading order term $E_{\text{Cas}}^{T=0,1}$ can be written as a sum of two terms:

$$E_{\text{Cas}}^{T=0,1} = E_{\text{Cas}}^{T=0,1a} + E_{\text{Cas}}^{T=0,1b}. \quad (55)$$

The first term $E_{\text{Cas}}^{T=0,1a}$ vanishes for $D = 3, 4, 5$, and for $D \geq 6$,

$$E_{\text{Cas}}^{T=0,1a} = \frac{(D-3)(D-5)}{3} \frac{a_2^{\frac{D-3}{2}}}{2^{D+1}\pi R_1 \Gamma(\frac{D-1}{2})} \times \sum_{s=0}^{\infty} \frac{1}{(s+1)^{\frac{D-1}{2}}} \int_0^{\infty} dl l^{\frac{D-3}{2}} \int_0^1 \frac{d\tau \tau^{\frac{D-7}{2}}}{\sqrt{1-\tau^2}} \exp\left(-\frac{2l(s+1)\varepsilon}{a_1\tau}\right). \quad (56)$$

The second term $E_{\text{Cas}}^{T=0,1b} = 0$ is

$$E_{\text{Cas}}^{T=0,1b} = -\frac{a_2^{\frac{D-1}{2}}}{2^{D-1}\pi R_1 \Gamma(\frac{D-1}{2})} \sum_{s=0}^{\infty} \frac{1}{(s+1)^{\frac{D-1}{2}}} \int_0^{\infty} dl l^{\frac{D-1}{2}} \int_0^1 \frac{d\tau \tau^{\frac{D-5}{2}}}{\sqrt{1-\tau^2}} \exp\left(-\frac{2l(s+1)\varepsilon}{a_1\tau}\right) \mathcal{A}, \quad (57)$$

where

$$\begin{aligned} \mathcal{A} = & -\frac{(D-1)(5D-7-(9D-15)a_1+(3D-9)a_1^2)}{48a_2l(s+1)}\tau - \frac{(D-1)(D+1)(3a_1-2)}{48a_2l(s+1)}\tau^3 \\ & + \frac{(D-2)^2}{2l} - \frac{(D-1)(D-2)}{4l}\tau^2 + \frac{(D-3)(D-5)(s+1)}{12a_2l}\frac{1}{\tau} - \frac{(D-1)(D-5)\tau}{12a_2l}(s+1) + \frac{(D^2-25)\tau^3}{48a_2l}(s+1) \\ & + \frac{\varepsilon(D-1)(3a_1^2-1)}{6a_1a_2} - \frac{\varepsilon(D-1)(3a_1-2)}{6a_1a_2}\tau^2 - \frac{\varepsilon(D-2)}{a_1}(s+1)\tau - \frac{\varepsilon(D-1)}{3a_1a_2}(s+1)^2 + \frac{\varepsilon(D-1)}{6a_1a_2}(s+1)^2\tau^2 \\ & - \frac{\varepsilon^2l(3a_1-2)}{3a_1^2a_2}(s+1)\tau + \frac{\varepsilon^2l}{3a_1^2a_2}(s+1)^3\tau. \end{aligned} \quad (58)$$

The integration over κ has been changed to the integration over τ where

$$\kappa = \frac{l\sqrt{1-\tau^2}}{R_1\tau}.$$

It follows that

$$d\kappa = -\frac{l}{R_1} \frac{d\tau}{\tau^2\sqrt{1-\tau^2}}.$$

To obtain the corresponding small separation asymptotic expansion for $E_{\text{Cas}}^{\text{rem}}$ (7), we change τ to

$$\frac{l}{\sqrt{l^2 + (R_1 \kappa_p)^2}},$$

and make the replacement

$$\frac{l}{R_1} \int_0^1 \frac{d\tau}{\tau^2 \sqrt{1-\tau^2}} \mapsto 2\pi T \sum_{p=1}^{\infty}. \quad (59)$$

Let

$$z = \frac{\varepsilon l}{a_1}, \quad k = s + 1.$$

With the prescription described above, we find that for the Casimir force

$$F_{\text{Cas}} = -\frac{\partial E_{\text{Cas}}}{\partial d} = F_{\text{Cas}}^{\text{classical}} + F_{\text{Cas}}^{\text{rem}}, \quad (60)$$

the leading term of $F_{\text{Cas}}^{\text{rem}}$ (7) is given by

$$F_{\text{Cas}}^{\text{rem},0} = -\frac{T}{2^{D-3}\Gamma(\frac{D-1}{2})} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-1}{2}}} \sum_{p=1}^{\infty} \int_0^{\infty} dz \frac{z^{D-2}}{(z^2 + (\kappa_p d)^2)^{\frac{D-3}{4}}} \exp\left(-2k\sqrt{z^2 + (\kappa_p d)^2}\right). \quad (61)$$

The next-to-leading order term $F_{\text{Cas}}^{\text{rem},1}$ can be written as a sum of two terms:

$$F_{\text{Cas}}^{\text{rem},1} = F_{\text{Cas}}^{\text{rem},1a} + F_{\text{Cas}}^{\text{rem},1b}. \quad (62)$$

The first term $F_{\text{Cas}}^{\text{rem},1a}$ vanishes for $D = 3, 4, 5$, and for $D \geq 6$,

$$F_{\text{Cas}}^{\text{rem},1a} = \frac{(D-3)(D-5)}{3} \frac{T}{2^{D-1}\Gamma(\frac{D-1}{2})} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-3}{2}} \frac{1}{d^{\frac{D-1}{2}}} \times \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-3}{2}}} \sum_{p=1}^{\infty} \int_0^{\infty} dz \frac{z^{D-4}}{(z^2 + (\kappa_p d)^2)^{\frac{D-5}{4}}} \exp\left(-2k\sqrt{z^2 + (\kappa_p d)^2}\right). \quad (63)$$

The second term $F_{\text{Cas}}^{\text{rem},1b}$ is

$$F_{\text{Cas}}^{\text{rem},1b} = -\frac{T}{2^{D-3}\Gamma(\frac{D-1}{2})} \frac{(R_1 R_2)^{\frac{D-1}{2}}}{(R_1 + R_2)^{\frac{D+1}{2}}} \frac{1}{d^{\frac{D-1}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-1}{2}}} \sum_{p=1}^{\infty} \int_0^{\infty} dz \frac{z^{D-2}}{(z^2 + (\kappa_p d)^2)^{\frac{D-3}{4}}} \exp\left(-2k\sqrt{z^2 + (\kappa_p d)^2}\right) \mathcal{B}, \quad (64)$$

where

$$\begin{aligned} \mathcal{B} = & \frac{(D-3)(D-5)k\sqrt{z^2 + (\kappa_p d)^2}}{12a_1 a_2 z} + \frac{(D-2)^2}{2a_1 z} + \frac{(D-1)(3a_1^2 - 1)}{6a_1 a_2} - \frac{(D-1)k^2}{3a_1 a_2} \\ & + \left\{ -\frac{(D-1)[5D-11-(9D-15)a_1+(3D+3)a_1^2]}{48a_1 a_2 z k} - \frac{(D-1)(D-7)k}{12a_1 a_2 z} - \frac{(D-2)k}{a_1} \right. \\ & \quad \left. - \frac{z(3a_1-2)k}{3a_1 a_2} + \frac{k^3 z}{3a_1 a_2} \right\} \frac{z}{\sqrt{z^2 + (\kappa_p d)^2}} \\ & + \left\{ -\frac{(D-3)(D-2)}{4a_1 z} - \frac{(D-3)(3a_1-2)}{6a_1 a_2} + \frac{(D-3)k^2}{6a_1 a_2} \right\} \frac{z^2}{z^2 + (\kappa_p d)^2} \\ & + \left\{ -\frac{(D-1)(D-3)(3a_1-2)}{48a_1 a_2 z k} + \frac{(D+3)(D-7)k}{48a_1 a_2 z} \right\} \frac{z^3}{(z^2 + (\kappa_p d)^2)^{\frac{3}{2}}}. \end{aligned} \quad (65)$$

Using integration by parts, one can show that the \mathcal{B} in (64) can be replaced by

$$\begin{aligned} \mathcal{B}' = & -\frac{(D-1)(3a_1a_2-1)}{6a_1a_2} + \frac{(D-1)(D+1)(3a_1a_2-1)}{48a_1a_2k} \frac{1}{\sqrt{z^2+(\kappa_p d)^2}} + \frac{(D-3)(D-5)k\sqrt{z^2+(\kappa_p d)^2}}{12a_1a_2 z^2} \\ & - \frac{(D-1)(D-7)k}{12a_1a_2} \frac{1}{\sqrt{z^2+(\kappa_p d)^2}} + \frac{(D+3)(D-7)k}{48a_1a_2} \frac{z^2}{(z^2+(\kappa_p d)^2)^{\frac{3}{2}}} - \frac{k^3}{3a_1a_2} \frac{z^2}{\sqrt{z^2+(\kappa_p d)^2}}. \end{aligned}$$

Together with the results from Sec. IV, we find that the small separation leading term of the Casimir interaction force is

$$\begin{aligned} F_{\text{Cas}}^0 = & -\frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1R_2}{R_1+R_2}\right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \\ & - \frac{T}{2^{D-3}\Gamma(\frac{D-1}{2})} \left(\frac{R_1R_2}{R_1+R_2}\right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \sum_{k=1}^{\infty} \frac{1}{k^{\frac{D-1}{2}}} \sum_{p=1}^{\infty} \int_0^{\infty} dz \frac{z^{D-2}}{(z^2+(\kappa_p d)^2)^{\frac{D-3}{4}}} \exp\left(-2k\sqrt{z^2+(\kappa_p d)^2}\right). \end{aligned} \quad (66)$$

When $D = 3$, the next-to-leading order term is

$$F_{\text{Cas}}^1 = -\frac{T}{48d} + F_{\text{Cas}}^{\text{rem},1b}; \quad (67)$$

when $D = 4$,

$$F_{\text{Cas}}^1 = -\frac{3T\zeta(4)}{32\sqrt{2}} \left(\frac{R_1R_2}{R_1+R_2}\right)^{\frac{3}{2}} \frac{1}{d^{\frac{5}{2}}} \left\{ -\frac{1}{4} \frac{d}{R_1+R_2} + \frac{1}{12} \left(\frac{d}{R_1} + \frac{d}{R_2}\right) \right\} + F_{\text{Cas}}^{\text{rem},1b}; \quad (68)$$

and when $D \geq 5$,

$$\begin{aligned} F_{\text{Cas}}^1 = & -\frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1R_2}{R_1+R_2}\right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \\ & \times \left\{ -\frac{D-3}{4} \frac{d}{R_1+R_2} + \left[\frac{D-3}{12} - \frac{(D-3)(D-4)\zeta(D-2)}{3(D-1)\zeta(D)} \right] \left(\frac{d}{R_1} + \frac{d}{R_2}\right) \right\} + F_{\text{Cas}}^{\text{rem},1a} + F_{\text{Cas}}^{\text{rem},1b}. \end{aligned} \quad (69)$$

Compare the leading term (66) to the proximity force approximation (20); it is easy to check that under the change of variables

$$x = \sqrt{z^2 + (\kappa_p d)^2},$$

and (61) is equal to the sum of the $p \neq 0$ terms in (20). Together with (50) and (21), we find that the small separation leading term of the Casimir interaction always agrees with the proximity force approximation. This is true at any temperature. In fact, for $D = 3$, we have proved this in the work [50]. We would like to stress that this is a remarkable result since it holds at *any* temperature.

Define

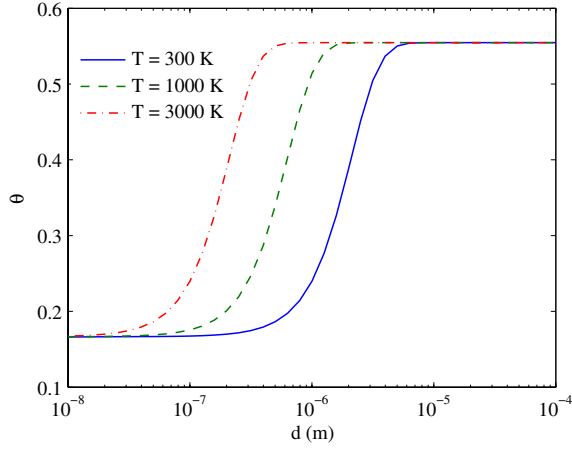
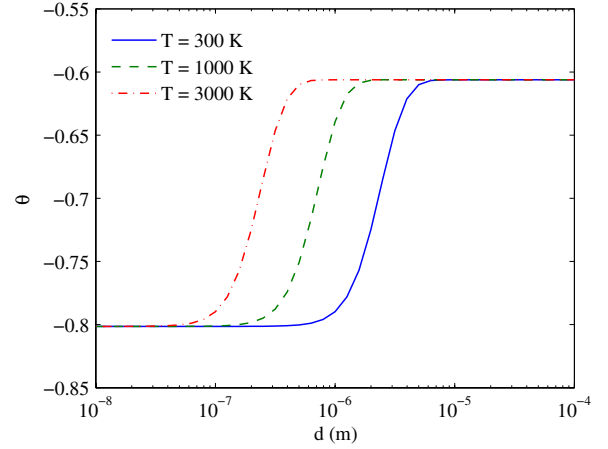
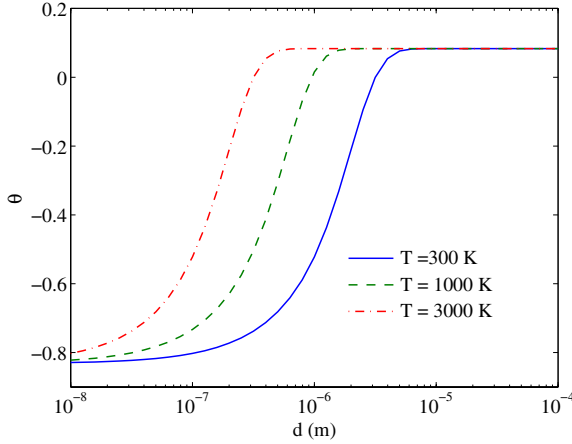
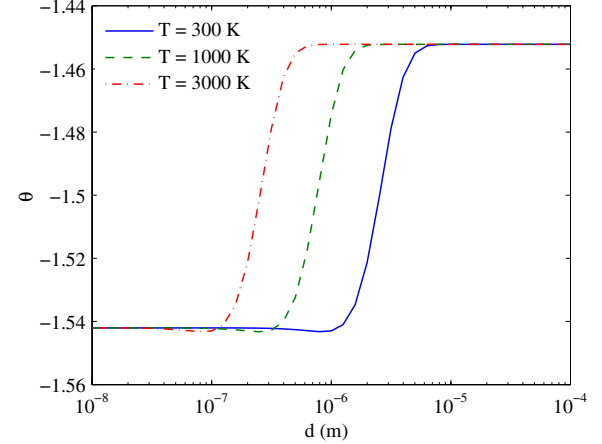
$$\theta = \frac{1}{\varepsilon} \frac{F_{\text{Cas}}^1}{F_{\text{Cas}}^0} \quad (70)$$

to be the ratio of the next-to-leading order term to the leading order term divided by ε . In Figs. 6, 7, 8 and 9, we plot the dependence of θ on the distance between the spheres when $D = 3, 4, 5$ and 6. We take the radius of both spheres to be $R = 1$ mm.

Let us now obtain the explicit small separation asymptotic expansions in different temperature regions. As in [51], we consider the following three regions:

- (1) High temperature: $1 \ll dT \ll R_1 \leq R_2 T$,
- (2) Low temperature: $dT \ll R_1 T \leq R_2 T \ll 1$,
- (3) Medium temperature: $dT \ll 1 \ll R_1 T \leq R_2 T$.

In the high temperature regime, it is easy to see that the dominating term is the classical term considered in Sec. IV since (61), (63) and (64) go to zero exponentially fast when $dT \gg 1$. In considering the Casimir interaction at finite temperature, we have separated out the classical term which corresponds to $p = 0$. The main reason is that the classical

FIG. 6 (color online). Dependence of θ on the distance between the spheres when $D = 3$.FIG. 8 (color online). Dependence of θ on the distance between the spheres when $D = 5$.FIG. 7 (color online). Dependence of θ on the distance between the spheres when $D = 4$.FIG. 9 (color online). Dependence of θ on the distance between the spheres when $D = 6$.

term can in fact be computed exactly. However, we have seen [Eq. (21)] that by putting $p = 0$ in the leading term (61), we indeed obtain the leading term of the classical Casimir interaction force (50). For $D \geq 5$, one can also check that putting $p = 0$ in (63) and (64) will give exactly the same next-to-leading order term of the classical Casimir interaction force (52).

Let us now consider the low temperature and medium temperature regions. Using the inverse Mellin transform formula

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \Gamma(w) z^{-w}, \quad (71)$$

where c is a positive constant, we find that

$$\begin{aligned} F_{\text{Cas}}^{\text{rem},0} &= -\frac{T}{2^{D-3}\Gamma(\frac{D-1}{2})} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \Gamma(w) 2^{-w} \zeta\left(w + \frac{D-1}{2}\right) \sum_{p=1}^{\infty} \int_0^{\infty} dz \frac{z^{D-2}}{(z^2 + (\kappa_p d)^2)^{\frac{w+D-3}{2}}} \\ &= -\frac{T}{2^{D-2}} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \\ &\quad \times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dw \Gamma(w) 2^{-w} \zeta\left(w + \frac{D-1}{2}\right) \frac{\Gamma(\frac{w}{2} - \frac{D+1}{4})}{\Gamma(\frac{w}{2} + \frac{D-3}{4})} d^{-w} (2\pi T)^{\frac{D+1}{2}-w} \zeta\left(w - \frac{D+1}{2}\right). \end{aligned} \quad (72)$$

The pole of the integrand is at $w = \frac{D+3}{2}, \frac{D+1}{2}$ and $w = 0, -1, -2, \dots$. Evaluating the residues at these poles, we find that as $dT \ll 1$,

$$F_{\text{Cas}}^{\text{rem},0} \sim -\frac{T}{2^{D-2}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \left\{ \frac{\zeta(D+1)}{2^{\frac{D+5}{2}} \sqrt{\pi} T d^{\frac{D+3}{2}}} \frac{\Gamma(\frac{D+3}{2})}{\Gamma(\frac{D}{2})} - \frac{(D-1)\zeta(D)}{2^{\frac{D+3}{2}} d^{\frac{D+1}{2}}} \right. \\ \left. + \frac{\zeta(\frac{D-1}{2})}{\Gamma(\frac{D-3}{4})} \frac{(2T)^{\frac{D+1}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{D+3}{4}\right) \zeta\left(\frac{D+3}{2}\right) - \frac{\zeta(\frac{D-3}{2})}{\Gamma(\frac{D-5}{4})} \frac{2^{\frac{D+5}{2}} T^{\frac{D+3}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{D+5}{4}\right) \zeta\left(\frac{D+5}{2}\right) d + \dots \right\}. \quad (73)$$

When $D = 3$, it is understood that

$$\frac{\zeta(\frac{D-1}{2})}{\Gamma(\frac{D-3}{4})} = \lim_{D \rightarrow 3} \frac{\zeta(\frac{D-1}{2})}{\Gamma(\frac{D-3}{4})} = \frac{1}{2}. \quad (74)$$

Similarly, when $D = 5$,

$$\frac{\zeta(\frac{D-3}{2})}{\Gamma(\frac{D-5}{4})} = \lim_{D \rightarrow 5} \frac{\zeta(\frac{D-3}{2})}{\Gamma(\frac{D-5}{4})} = \frac{1}{2}. \quad (75)$$

Combining with the leading term of the classical term (50), we find that when $dT \ll 1$, the leading term of the Casimir interaction force is

$$F_{\text{Cas}}^0 \sim -\frac{\zeta(D+1) \Gamma(\frac{D+3}{2})}{2^{\frac{3D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+3}{2}}} - \frac{\zeta(\frac{D-1}{2})}{\Gamma(\frac{D-3}{4})} \frac{T^{\frac{D+3}{2}}}{2^{\frac{D+5}{2}} \sqrt{\pi}} \Gamma\left(\frac{D+3}{4}\right) \zeta\left(\frac{D+3}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \\ + \frac{\zeta(\frac{D-3}{2})}{\Gamma(\frac{D-5}{4})} \frac{T^{\frac{D+5}{2}} d}{2^{\frac{D+9}{2}} \sqrt{\pi}} \Gamma\left(\frac{D+5}{4}\right) \zeta\left(\frac{D+5}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} + \dots \quad (76)$$

The first term

$$F_{\text{Cas}}^{T=0,0} = -\frac{\zeta(D+1) \Gamma(\frac{D+3}{2})}{2^{\frac{3D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+3}{2}}} \quad (77)$$

is the zero temperature leading term (22). The next two terms in (76) come from finite temperature contributions. Namely, the leading terms of the temperature correction are

$$\Delta_T F_{\text{Cas}}^0 \sim -\frac{\zeta(\frac{D-1}{2})}{\Gamma(\frac{D-3}{4})} \frac{T^{\frac{D+3}{2}}}{2^{\frac{D+5}{2}} \sqrt{\pi}} \Gamma\left(\frac{D+3}{4}\right) \zeta\left(\frac{D+3}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \\ + \frac{\zeta(\frac{D-3}{2})}{\Gamma(\frac{D-5}{4})} \frac{T^{\frac{D+5}{2}} d}{2^{\frac{D+9}{2}} \sqrt{\pi}} \Gamma\left(\frac{D+5}{4}\right) \zeta\left(\frac{D+5}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} + \dots \quad (78)$$

They become significant when $dT \sim 1$.

Using the same method, we can compute the asymptotic expansion of the next-to-leading order term $F_{\text{Cas}}^{\text{rem},1}$ when $dT \ll 1$. When $D = 3, 4, 5$, $F_{\text{Cas}}^{\text{rem},1a} = 0$. For $D \geq 6$,

$$F_{\text{Cas}}^{\text{rem},1a} = \frac{(D-5)}{3} \frac{1}{2^{\frac{3D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-3}{2}} \frac{\Gamma(\frac{D+1}{2}) \zeta(D-1)}{\Gamma(\frac{D-2}{2}) d^{\frac{D+1}{2}}} - \frac{T}{2^{\frac{3D-1}{2}}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-3}{2}} \frac{\zeta(D-2) (D-3)(D-5)}{d^{\frac{D-1}{2}} 3} \\ + \frac{(D-5)}{3} \frac{T^{\frac{D+1}{2}}}{2^{\frac{D-1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-3}{2}} \frac{\Gamma(\frac{D+1}{4})}{\Gamma(\frac{D-5}{4})} \zeta\left(\frac{D-3}{2}\right) \zeta\left(\frac{D+1}{2}\right) \\ - \frac{(D-5)}{3} \frac{T^{\frac{D+3}{2}} d}{2^{\frac{D-3}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-3}{2}} \frac{\Gamma(\frac{D+3}{4})}{\Gamma(\frac{D-7}{4})} \zeta\left(\frac{D-5}{2}\right) \zeta\left(\frac{D+3}{2}\right) + \dots \quad (79)$$

When $D = 3$,

$$F_{\text{Cas}}^{\text{rem},1b} = -\frac{R_1 R_2 \zeta(4)}{16\pi(R_1 + R_2)d^3} \left[\frac{1}{3} \left(\frac{d}{R_1} + \frac{1}{R_2} \right) - \frac{d}{R_1 + R_2} \right] + \frac{T}{48d} \\ + \frac{\pi T^2}{12} \frac{R_1 R_2}{R_1 + R_2} \left[\frac{1}{3} \left(\frac{\pi^2}{6} - 1 \right) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{\pi^2}{6} \frac{1}{R_1 + R_2} \right] - \frac{dT^3 R_1 R_2 \zeta(3)}{(R_1 + R_2)} \left[\frac{1}{4} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{1}{R_1 + R_2} \right]. \quad (80)$$

For $D \geq 4$,

$$F_{\text{Cas}}^{\text{rem},1b} = \frac{1}{2^{\frac{3D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{\Gamma(\frac{D+1}{2}) \zeta(D+1)}{\Gamma(\frac{D}{2}) d^{\frac{D+3}{2}}} \left\{ \frac{(D^2-1)}{8} \frac{d}{R_1 + R_2} - \left[\frac{D^2-1}{24} \frac{\zeta(D-1)(D+3)(D-2)}{\zeta(D+1)} \frac{1}{6D} \right] \left(\frac{d}{R_1} + \frac{d}{R_2} \right) \right\} \\ + \frac{(D-1)T\zeta(D)}{2^{\frac{3D-1}{2}}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+1}{2}}} \left\{ -\frac{D-3}{4} \frac{d}{R_1 + R_2} + \left[\frac{D-3}{12} - \frac{(D-3)\zeta(D-2)}{3(D-1)\zeta(D)} \right] \left(\frac{d}{R_1} + \frac{d}{R_2} \right) \right\} \\ - \frac{T^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{\Gamma(\frac{D+1}{4})}{\Gamma(\frac{D+1}{4})} \zeta\left(\frac{D+1}{2}\right)^2 \left\{ \frac{(D^2-1)}{4} \frac{1}{R_1 + R_2} - \left[\frac{(D^2-1)}{12} - \frac{\zeta(\frac{D-3}{2})D-5}{\zeta(\frac{D+1}{2})} \frac{1}{3} \right] \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\} \\ + \frac{T^{\frac{D+3}{2}} d}{2^{\frac{D-3}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{D-1}{2}} \frac{\Gamma(\frac{D+3}{4})}{\Gamma(\frac{D-3}{4})} \zeta\left(\frac{D+3}{2}\right) \zeta\left(\frac{D-1}{2}\right) \left\{ \frac{(D-1)(D+5)}{4} \frac{1}{R_1 + R_2} \right. \\ \left. - \left[\frac{(D-1)(D+5)}{12} - \frac{\zeta(\frac{D-5}{2})(D-7)(D+3)}{\zeta(\frac{D-1}{2})} \frac{1}{3(D-3)} \right] \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\}. \quad (81)$$

From (67) and (80), we find that when $D = 3$, the next-to-leading order term of the small separation asymptotic expansion of the Casimir interaction force is

$$F_{\text{Cas}}^1 \sim -\frac{R_1 R_2 \zeta(4)}{16\pi(R_1 + R_2)d^3} \left[\frac{1}{3} \left(\frac{d}{R_1} + \frac{1}{R_2} \right) - \frac{d}{R_1 + R_2} \right] + \frac{\pi T^2}{12} \frac{R_1 R_2}{R_1 + R_2} \left[\frac{1}{3} \left(\frac{\pi^2}{6} - 1 \right) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{\pi^2}{6} \frac{1}{R_1 + R_2} \right] \\ - \frac{dT^3 R_1 R_2 \zeta(3)}{(R_1 + R_2)} \left[\frac{1}{4} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{1}{R_1 + R_2} \right]. \quad (82)$$

The first term is the zero temperature part, whereas the remaining terms are temperature corrections. Combining with (66), we find that when $d \ll R_1 \leq R_2$, the zero temperature Casimir interaction force behaves as

$$F_{\text{Cas}}^{T=0} \sim -\frac{R_1 R_2 \zeta(4)}{8\pi(R_1 + R_2)d^3} \left\{ 1 - \frac{1}{2} \frac{d}{R_1 + R_2} + \frac{1}{6} \left(\frac{d}{R_1} + \frac{d}{R_2} \right) + \dots \right\}. \quad (83)$$

This is the small separation asymptotic expansion in the low temperature regime.

For the temperature correction, we have

$$\Delta_T F_{\text{Cas}} \sim -\frac{\zeta(3)T^3}{2} \frac{R_1 R_2}{R_1 + R_2} + \frac{\pi^3 T^4 d}{45} \frac{R_1 R_2}{R_1 + R_2} + \dots + \frac{\pi T^2}{12} \frac{R_1 R_2}{R_1 + R_2} \left[\frac{1}{3} \left(\frac{\pi^2}{6} - 1 \right) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{\pi^2}{6} \frac{1}{R_1 + R_2} \right] \\ - \frac{dT^3 R_1 R_2 \zeta(3)}{(R_1 + R_2)} \left[\frac{1}{4} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) - \frac{1}{R_1 + R_2} \right] + \dots \quad (84)$$

This, together with (83), gives the small separation asymptotic expansion in the medium temperature regime.

When $D = 4$, (66), (68) and (81) give

$$F_{\text{Cas}}^{T=0} = -\frac{15\zeta(5)}{512\sqrt{2}} \left(\frac{R_1 R_2}{R_1 + R_2} \right)^{\frac{3}{2}} \frac{1}{d^{\frac{3}{2}}} \left\{ 1 - \frac{3}{4} \frac{d}{R_1 + R_2} + \left[\frac{1}{4} - \frac{7}{30} \frac{\zeta(3)}{\zeta(5)} \right] \left(\frac{d}{R_1} + \frac{d}{R_2} \right) + \dots \right\}, \quad (85)$$

$$\begin{aligned}
\Delta_T F_{\text{Cas}} = & -\frac{\pi^2 T}{576\sqrt{2}} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{1}{2}} \frac{1}{d^{\frac{3}{2}}} - \sqrt{2} \frac{\Gamma(\frac{7}{4}) T^{\frac{7}{2}}}{\Gamma(\frac{1}{4}) \sqrt{\pi}} \zeta\left(\frac{3}{2}\right) \zeta\left(\frac{7}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{3}{2}} \\
& + 4\sqrt{2} \frac{\Gamma(\frac{9}{4}) T^{\frac{9}{2}} d}{\Gamma(-\frac{1}{4}) \sqrt{\pi}} \zeta\left(\frac{1}{2}\right) \zeta\left(\frac{9}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{5}{2}} \\
& - \frac{1}{4\sqrt{2}} \frac{\Gamma(\frac{5}{4}) T^{\frac{5}{2}}}{\Gamma(\frac{3}{4}) \sqrt{\pi}} \zeta\left(\frac{5}{2}\right)^2 \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{3}{2}} \left\{ \frac{15}{4} \frac{1}{R_1 + R_2} - \left[\frac{5}{4} + \frac{1}{3} \frac{\zeta(\frac{1}{2})}{\zeta(\frac{3}{2})} \right] \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\} \\
& + \frac{1}{\sqrt{2}} \frac{\Gamma(\frac{7}{4}) T^{\frac{7}{2}} d}{\Gamma(\frac{1}{4}) \sqrt{\pi}} \zeta\left(\frac{7}{2}\right) \zeta\left(\frac{3}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{3}{2}} \left\{ \frac{27}{4} \frac{1}{R_1 + R_2} - \left[\frac{9}{4} + 7 \frac{\zeta(-\frac{1}{2})}{\zeta(\frac{3}{2})} \right] \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\} + \dots \quad (86)
\end{aligned}$$

When $D \geq 5$, (66), (69), (79), and (81) give

$$\begin{aligned}
F_{\text{Cas}}^{T=0} = & -\frac{\zeta(D+1) \Gamma(\frac{D+3}{2})}{2^{\frac{3D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \frac{1}{d^{\frac{D+3}{2}}} \left\{ 1 - \frac{D-1}{4} \frac{d}{R_1 + R_2} \right. \\
& \left. + \left[\frac{D-1}{12} - \frac{(D-1)(D-2)(D-3) \zeta(D-1)}{3D(D+1)} \right] \left(\frac{d}{R_1} + \frac{d}{R_2} \right) + \dots \right\}. \quad (87)
\end{aligned}$$

This gives the small separation asymptotic expansion in the low temperature regime.

For the temperature correction, we have

$$\begin{aligned}
\Delta_T F_{\text{Cas}} = & -\frac{\zeta(\frac{D-1}{2}) T^{\frac{D+3}{2}}}{\Gamma(\frac{D-3}{4}) 2^{\frac{D-5}{2}} \sqrt{\pi}} \Gamma\left(\frac{D+3}{4}\right) \zeta\left(\frac{D+3}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \\
& + \frac{\zeta(\frac{D-3}{2}) T^{\frac{D+5}{2}}}{\Gamma(\frac{D-5}{4}) 2^{\frac{D-9}{2}} \sqrt{\pi}} \Gamma\left(\frac{D+5}{4}\right) \zeta\left(\frac{D+5}{2}\right) \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} d + \dots \\
& - \frac{T^{\frac{D+1}{2}}}{2^{\frac{D+1}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \frac{\Gamma(\frac{D+1}{4})}{\Gamma(\frac{D-1}{4})} \zeta\left(\frac{D+1}{2}\right)^2 \left\{ \frac{(D^2-1)}{4} \frac{1}{R_1 + R_2} \right. \\
& \quad \left. - \left[\frac{(D^2-1)}{12} + \frac{(D-5)(D-7) \zeta(\frac{D-3}{2})}{6 \zeta(\frac{D+1}{2})} \right] \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\} \\
& + \frac{T^{\frac{D+3}{2}} d}{2^{\frac{D-3}{2}} \sqrt{\pi}} \left(\frac{R_1 R_2}{R_1 + R_2}\right)^{\frac{D-1}{2}} \frac{\Gamma(\frac{D+3}{4})}{\Gamma(\frac{D-3}{4})} \zeta\left(\frac{D+3}{2}\right) \zeta\left(\frac{D-1}{2}\right) \left\{ \frac{(D-1)(D+5)}{4} \frac{1}{R_1 + R_2} \right. \\
& \quad \left. - \left[\frac{(D-1)(D+5)}{12} + \frac{(D-1)(D-7)(D-9) \zeta(\frac{D-5}{2})}{6(D-3) \zeta(\frac{D-1}{2})} \right] \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \right\} + \dots \quad (88)
\end{aligned}$$

(87) together with (88) give the small separation asymptotic expansion of the Casimir interaction force in the medium temperature regime.

One can check that the zero temperature asymptotic expansions (83), (85) and (87) agree with the result we derived in [40].

From the graphs Figs. 6, 7, 8 and 9, we find that when $dT \ll 1$, θ approaches a limiting value given by the zero temperature limit

$$\theta^{T=0} = \begin{cases} -\frac{1}{2} + \frac{1}{6a_1 a_2}, & D = 3, \\ -\frac{3}{4} + \left[\frac{1}{4} - \frac{7}{30} \frac{\zeta(3)}{\zeta(5)} \right] \frac{1}{a_1 a_2}, & D = 4, \\ -\frac{D-1}{4} + \left[\frac{D-1}{12} - \frac{(D-1)(D-2)(D-3) \zeta(D-1)}{3D(D+1)} \right] \frac{1}{a_1 a_2}, & D \geq 5; \end{cases} \quad (89)$$

whereas as $dT \gg 1$, θ approaches a limiting value given by the high temperature limit

$$\theta^{\text{classical}} = \begin{cases} \frac{1}{6\zeta(3)a_1a_2}, & D = 3, \\ -\frac{D-3}{4} + \left[\frac{D-3}{12} - \frac{(D-3)(D-4)\zeta(D-2)}{3(D-1)\zeta(D)} \right] \frac{1}{a_1a_2}, & D \geq 4. \end{cases} \quad (90)$$

VI. COMPARISON TO DERIVATIVE EXPANSION

First, let us consider the sphere-plate limit by taking $R_1 = R$, and $R_2 \rightarrow \infty$. From (83), (85) and (87), we find that at zero temperature, the Casimir interaction force is given by

$$F_{\text{Cas}}^{T=0} = \begin{cases} -\frac{\pi^3 R}{720d^3} \left\{ 1 + \frac{1}{6} \frac{d}{R} + \dots \right\}, & D = 3, \\ -\frac{15\zeta(5)R^3}{512\sqrt{2}d^5} \left\{ 1 + \left[\frac{1}{4} - \frac{7\zeta(3)}{30\zeta(5)} \right] \frac{d}{R} + \dots \right\}, & D = 4, \\ -\frac{\zeta(D+1)\Gamma(\frac{D+3}{2})R^{\frac{D-1}{2}}}{2^{\frac{3D+1}{2}}\sqrt{\pi}\Gamma(\frac{D}{2})d^{\frac{D+3}{2}}} \left\{ 1 + \left[\frac{D-1}{12} - \frac{(D-1)(D-2)(D-3)\zeta(D-1)}{3D(D+1)\zeta(D+1)} \right] \frac{d}{R} + \dots \right\}, & D \geq 5. \end{cases} \quad (91)$$

From (48) and (49), we find that the classical term of the Casimir interaction force is given by

$$F_{\text{Cas}}^{\text{DE, classical}} = \begin{cases} -\frac{TR}{8d^2}\zeta(3) \left\{ 1 + \frac{1}{6\zeta(3)} \frac{d}{R} + \dots \right\}, & D = 3, \\ -\frac{(D-1)T\zeta(D)R^{\frac{D-1}{2}}}{2^{\frac{3D-1}{2}}d^{\frac{D+1}{2}}} \left\{ 1 + \left[\frac{D-3}{12} - \frac{(D-3)(D-4)\zeta(D-2)}{3(D-1)\zeta(D)} \right] \frac{d}{R} + \dots \right\}, & D \geq 4. \end{cases} \quad (92)$$

In a series of papers [53–55], Fosco *et al.* used derivative expansion to compute the Casimir interaction energy between a curved surface and a plane up to the next-to-leading order term. In [54], they showed that at zero temperature, the derivative expansion of the zero temperature Casimir interaction energy is given by

$$E_{\text{Cas}}^{\text{DE}, T=0} = \int_S d^{D-1} \mathbf{x}_\perp \left(b_0(D) \frac{1}{|\psi(\mathbf{x}_\perp)|^D} + b_2(D) \frac{\|\nabla\psi\|^2}{|\psi(\mathbf{x}_\perp)|^D} + \dots \right).$$

Here $\mathbf{x}_\perp = (x_2, \dots, x_D)$, $x_1 = \psi(\mathbf{x}_\perp)$, $\mathbf{x}_\perp \in S$ defines the position of the curved surface with respect to the plane at $x_1 = 0$, and

$$b_0(D) = -\frac{\Gamma(\frac{D+1}{2})\zeta(D+1)}{2^{D+1}\pi^{\frac{D+1}{2}}},$$

$$b_2(D) = -\frac{1}{3 \times 2^{D+2}\pi^{\frac{D+1}{2}}} \left\{ -\frac{(D-3)(D-1)(D-2)}{2D} \Gamma\left(\frac{D-1}{2}\right)\zeta(D-1) + (D+1)\Gamma\left(\frac{D+1}{2}\right)\zeta(D+1) \right\}.$$

In [54] and [55], it was shown that in the high temperature limit, the classical term of the Casimir interaction energy is

$$E_{\text{Cas}}^{\text{DE, classical}} = T \int_S d^{D-1} \mathbf{x}_\perp \left(b_0(D-1) \frac{1}{|\psi(\mathbf{x}_\perp)|^{D-1}} + b_2(D-1) \frac{\|\nabla\psi\|^2}{|\psi(\mathbf{x}_\perp)|^{D-1}} + \dots \right).$$

In our case the center of the sphere is of distance $L = R + d$ from the plane $x_1 = 0$. We can take

$$\psi(\mathbf{x}_\perp) = L - \sqrt{R^2 - \mathbf{x}_\perp^2}.$$

Then straightforward computations show that at zero temperature,

$$E_{\text{Cas}}^{\text{DE}, T=0} = -\frac{\Gamma(\frac{D+1}{2})\zeta(D+1)R^{\frac{D-1}{2}}}{2^{\frac{3D+1}{2}}\sqrt{\pi}\Gamma(\frac{D}{2})d^{\frac{D+1}{2}}} \left(1 + \left[\frac{D+1}{12} - \frac{(D-2)(D-3)\zeta(D-1)}{3D\zeta(D+1)} \right] \frac{d}{R} + \dots \right). \quad (93)$$

In the high temperature limit,

$$E_{\text{Cas}}^{\text{DE, classical}} = -\frac{T\zeta(D)R^{\frac{D-1}{2}}}{2^{\frac{3D-3}{2}}d^{\frac{D-1}{2}}}\left(1 + \left[\frac{D-1}{12} - \frac{D-4}{3}\frac{\zeta(D-2)}{\zeta(D)}\right]\frac{d}{R} + \dots\right) \quad (94)$$

when $D \geq 4$, and

$$E_{\text{Cas}}^{\text{DE, classical}} = -\frac{T\zeta(3)R}{8d}\left(1 - \frac{1}{6\zeta(3)}\frac{d}{R}\ln\frac{d}{R} + \dots\right) \quad (95)$$

for $D = 3$.

Taking the derivatives of (93), (94) and (95) with respect to d , one finds that except for the zero temperature case when $D = 4$, our results (91) and (92) agree with the results obtained using derivative expansion.

VII. CONCLUSION

In this work, we have considered the finite temperature Casimir interaction between two spheres subject to Dirichlet boundary conditions in $(D + 1)$ -dimensional Minkowski spacetime. Starting with the TGTG formula for the zero temperature Casimir interaction energy we derived in [40], we use Matsubara formalism to obtain the finite temperature Casimir free interaction energy. The term corresponding to zero Matsubara frequency is singled out. It gives the high temperature limit of the Casimir interaction and is known as the classical term. This term can be computed exactly by some similarity transformations of matrices. We then use the Abel-Plana summation formula to obtain an alternative expression that can be used to deduce the small separation asymptotic expansions.

For the remaining part of the Casimir interaction, we use our results in the zero temperature case [40] together with Matsubara formalism to derive its small separation leading and next-to-leading order terms which are valid at any temperature. Combining with the classical term, we find that the leading order term agrees with the proximity force approximation at any temperature. This is a remarkable result. An inverse Mellin transform is then used to compute the analytic expression of the leading and next-to-leading order terms in the low and medium temperature regions. In the low temperature region, the dominating term is the zero temperature term. In the medium temperature region, we have to take into account the contribution from the finite temperature corrections.

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